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# General finite-size effects for zero-entropy states in one-dimensional quantum integrable models

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## Abstract

We present a general derivation of the spectrum of excitations for gapless states of zero entropy density in Bethe ansatz solvable models. Our formalism is valid for an arbitrary choice of bare energy function which is relevant to situations where the Hamiltonian for time evolution differs from the Hamiltonian in a (generalized) Gibbs ensemble, i.e. out of equilibrium. The energy of particle and hole excitations, as measured with the time-evolution Hamiltonian, is shown to include additional contributions stemming from the shifts of the Fermi points that may now have finite energy. The finite-size effects are also derived and the connection with conformal field theory discussed. The critical exponents can still be obtained from the finite-size spectrum, however the velocity occurring here differs from the one in the constant Casimir term. The derivation highlights the importance of the phase shifts at the Fermi points for the critical exponents of asymptotes of correlations. We generalize certain results known for the ground state and discuss the relation to the dressed charge (matrix). Finally, we discuss the finite-size corrections in the presence of an additional particle or hole, which are important for dynamical correlation functions.

Keywords: Bethe ansatz, conformal field theory, finite-size scaling

## 1. Introduction

The combination of Bethe ansatz (BA) and conformal field theory (CFT) is a strong set of tools in the study of quantum mechanical systems in one space dimension. To get insight into the correlations, a routinely employed technique is to compute general expressions for

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correlation asymptotes from CFT, fixing the critical exponents from the finite size spectrum compared to the BA solution. This works well for static correlations by taking the ground state as a reference state [1, 2], while for time-dependent correlations also contributions from certain impurity configurations should be included [3–5]. This set of approaches thus provide a rather complete picture of (asymptotics of) equilibrium correlations in one-dimensional systems.

One of the outstanding benefits of the BA solution is the description it provides of the full Hilbert space and the possibility to study out-of-equilibrium states. The exact solvability can be attributed to the existence of an infinite collection of local charges  $\hat{Q}_n$  commuting with the Hamiltonian  $H$ ,

$$[H, \hat{Q}_n] = 0, \quad n \in \mathbb{N}. \quad (1)$$

Out-of-equilibrium problems have attracted a lot of attention recently regarding the question when and how unitary quantum systems do or do not thermalize. Important in this respect is the idea that correlations at late times can be computed in a generalized Gibbs ensemble (GGE) [6, 7] defined not just by the Hamiltonian, but rather by all [8, 9] conserved (quasi) local quantities

$$\hat{\rho}_{\text{GGE}} = Z_{\text{GGE}}^{-1} \exp \left\{ - \sum_n \beta_n \hat{Q}_n \right\}. \quad (2)$$

Equivalently, correlations can be computed on a single representative eigenstate which can be determined by reasonings paralleling (generalizing) the thermodynamic Bethe ansatz (gTBA) or by the quench action method [8–27]. In the latter, one constructs a free-energy functional straight from the overlaps of the initial state with the eigenstates of the time evolution Hamiltonian  $H$  [15, 27]. The GGE reasoning underscores the double role the Hamiltonian has in equilibrium quantum mechanics in determining both the statistical ensemble as well as the time evolution. Out of equilibrium, these two roles are separated, at least in the presence of nontrivial local conserved quantities.

A simple class of out-of-equilibrium states in BA solvable models corresponds to the zero-temperature equivalent of a GGE with nonmonotonic effective bare free energy (in gTBA language: driving function). Such states can be specified by consecutive blocks of filled quantum numbers in the Bethe ansatz solution, and in many ways resemble the ground-state Fermi sea or a simple boosted version of it, although now it combines several of such Fermi-sea blocks with different mean momentum. It has been shown that even in such cases, the description of correlation asymptotics is provided by multiple CFTs and that the finite-size corrections to the spectrum can again be used to obtain the critical exponents provided that the appropriate GGE energy function  $\epsilon_{\text{GGE}}(\lambda)$  is used [28].

The point of the present paper is to draw attention to a slightly uncanny feature of the standard derivation of the finite-size spectrum from Bethe ansatz [1, 2, 28], namely that it requires the dressed energy function to vanish for excitations at the Fermi points. This is done in equilibrium by subtracting the appropriate chemical potential. In other words, this requirement naturally follows when we use the Hamiltonian that defines the statistical ensemble in a grand canonical or GGE sense to measure energies, but out of equilibrium, one may question the naturalness of this assumption. In particular, when discussing dynamical correlations it is important to use the time-evolution Hamiltonian to measure energies. This suggests that one should be able to derive the relation between critical exponents and the finite-size spectrum for states of zero entropy density using  $H$ —or any combination of the conserved quantities for that matter—and the corresponding energy function, also when this is not in line with the statistical ensemble. This has indeed been verified numerically in

studies of dynamical correlations in out-of-equilibrium zero entropy states in the Lieb–Liniger and XXZ models [29, 30].

We therefore revisit the derivation of the energy of zero-entropy states and excitations in the limit of large system size and show that many of the well known relations between the spectrum and CFT hold for arbitrary energy functions, but with essential modifications. In terms of applications, the simplest example of such a situation occurs when we choose to measure energies with respect to a different chemical potential while still fixing a certain filling in a microcanonical sense. This would of course change the energy of excitations, but should not change the physics in an essential way. Another simple application is that of a boosted state observed in the lab frame. We however here present a general treatment, applicable to any (multiply) split Fermi sea in an integrable model.

## 2. Bethe ansatz and finite-size corrections

To be specific, consider the repulsive Lieb–Liniger model defined by the Hamiltonian

$$H = \int dx [\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)], \quad c > 0. \quad (3)$$

The coordinate Bethe ansatz provides exact expressions for all eigenstates  $|\{\lambda_j\}\rangle$  of the system with  $N$  particles in a box of size  $L$  in terms of the rapidities  $\lambda_j$  satisfying the Bethe equations [1]

$$Lp_0(\lambda_j) + \sum_{k=1}^N \theta(\lambda_j - \lambda_k) = 2\pi I_j. \quad (4)$$

Here  $p_0(\lambda) = \lambda$  is the bare momentum of particles and  $\theta(\lambda) = 2 \arctan(\lambda/c)$  is the two-particle scattering phase and  $I_j$  are (half-odd) integers depending on whether  $N$  is (even) odd. All states are classified by specifying  $N$  filled quantum numbers  $I_j$ . The momentum and energy of a state are

$$P = \sum_j \frac{2\pi}{L} I_j = \sum_j p_0(\lambda_j), \quad E = \sum_j \epsilon_0(\lambda_j) \quad (5)$$

with  $\epsilon_0(\lambda) = \lambda^2$ . Note that the energy does not include a chemical potential term and is really the eigenvalue of the operator  $H$ . The conserved charges  $\hat{Q}_n$  of the Lieb–Liniger model can be taken to represent the monomials in the Bethe basis

$$\hat{Q}|\{\lambda_j\}\rangle = Q_n|\{\lambda_j\}\rangle, \quad Q_n = \sum_j \lambda_j^2 \quad (6)$$

such that  $Q_0 = N$ ,  $Q_1 = P$  and  $Q_2 = E$ . Using the conserved charges we can in principle construct a Hamiltonian for any bare energy function  $\epsilon_0(\lambda) = \sum_n \beta_n \lambda^n$  by matching the  $\beta_n$  in the GGE.

Let us now consider a state  $|\{k_{ia}\}\rangle$  which corresponds to  $n$  disjoint Fermi seas specified by left and right Fermi momenta,

$$k_{ia}, \quad a = R, L, \quad i = 1, \dots, n. \quad (7)$$

These determine intervals of filled quantum numbers between extrema  $I_{ia} = (2\pi)^{-1} L k_{ia}$ . We take the  $I_{ia}$  to lie halfway between allowed quantum-number slots such that the filled quantum numbers correspond to

$$\{I_j\} = \bigcup_{i=1}^n \{I_{iL} + 1/2, I_{iL} + 3/2, \dots, I_{iR} - 1/2\}. \quad (8)$$

To take the thermodynamic limit  $N, L \rightarrow \infty$  with  $N/L$  fixed, we introduce the rapidity density

$$\rho(\lambda_j) = \frac{1}{L(\lambda_{j+1} - \lambda_j)}. \quad (9)$$

Using the Euler–Maclaurin formula, one can show that to order  $1/L$  the density satisfies

$$\rho(\lambda) = \frac{p'_0(\lambda)}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \rho(\nu) + \frac{1}{24L^2} \sum_{ia} \frac{s_a K'(\lambda - \lambda_{ia})}{2\pi \rho(\lambda_{ia})} \quad (10)$$

where  $K(\lambda) = \theta'(\lambda)$  and we introduced  $s_{R,L} = \pm 1$ , and  $\lambda_{ia}$  as the image of  $I_{ia}$  in rapidity space under the Bethe equations. The energy similarly becomes (to order  $1/L$ )

$$E = L \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\lambda \epsilon_0(\lambda) \rho(\lambda) - \frac{1}{24L} \sum_{ia} \frac{s_a \epsilon'_0(\lambda)}{\rho(\lambda_{ia})}. \quad (11)$$

The remainder of this paper is largely concerned with the analysis of these expressions.

We note that the solutions to other integrable models follow similar lines with appropriate definitions of the functions  $\theta(\lambda)$ ,  $p_0(\lambda)$  and  $\epsilon_0(\lambda)$ . For the XXZ model for instance,

$$H = \sum_{j=1}^L [S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta (S_j^z S_{j+1}^z - 1/4)] \quad (12)$$

with  $\Delta = \cos \zeta \in (-1, 1)$ , we have

$$p_0(\lambda) = 2 \arctan \left[ \frac{\tanh(\lambda)}{\tan(\zeta/2)} \right], \quad \theta(\lambda) = 2 \arctan \left[ \frac{\tanh(\lambda)}{\tan(\zeta)} \right] \quad (13)$$

and

$$\epsilon_0(\lambda) = \frac{-2 \sin^2 \zeta}{\cosh(2\lambda) - \cos \zeta}. \quad (14)$$

A complicating factor in XXZ is that solutions to the Bethe equations can be complex. Using the string hypothesis the reasoning can easily be generalized to these string states, but for simplicity we will assume that quantum numbers and parameters are chosen such that we deal with real rapidities. As is often the case in BA solvable models, the specific definitions do not matter much in the later derivations, but the relations between the functions do. This also means that  $\epsilon_0(\lambda)$  can be chosen essentially at will.

### 3. The energy of zero-entropy states

Our first task is the evaluation of equations (10) and (11). This follows standard practice [1, 28], but we include it for completeness. We expand the solution to equation (10) in powers of  $1/L$  as

$$\rho(\lambda) = \rho_\infty(\lambda) + \sum_{ia} \frac{s_a \rho_{ia}(\lambda)}{24L^2 \rho_\infty(\lambda_{ia})} \quad (15)$$

which results in the defining integral equations

$$\rho_\infty(\lambda) = \frac{p'_0(\lambda)}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \rho_\infty(\nu), \quad (16)$$

$$\rho_{ia}(\lambda) = \frac{K'(\lambda - \lambda_{ia})}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \rho_{ia}(\nu). \quad (17)$$

The equation for  $\rho(\lambda)$  is the straightforward generalization of the standard Lieb equation [31]. The second equation shows that  $\rho_{ia}(\lambda)$  is related to the two-parameter function  $L(\lambda|\lambda')$  defined by

$$L(\lambda|\lambda') = \frac{K(\lambda - \lambda')}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) L(\nu|\lambda'). \quad (18)$$

Equation (18) shows that, considered as integration kernels on the domain specified by the Fermi rapidities  $\lambda_{ia}$ , the operator  $(\mathbb{1} + L)$  is the inverse of  $(\mathbb{1} - \frac{K}{2\pi})$ . Using this fact we obtain

$$E = L \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\lambda \epsilon_0(\lambda) \rho_\infty(\lambda) - \sum_{ia} \frac{s_a \epsilon'(\lambda_{ia})}{24L\rho_\infty(\lambda_{ia})} \quad (19)$$

to order  $1/L$ , where the function  $\epsilon(\lambda)$  is defined by the integral equation

$$\epsilon(\lambda) = \epsilon_0(\lambda) + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \epsilon(\nu). \quad (20)$$

This definition is the direct analog of the dressed energy function in equilibrium settings which specifies the energy of the single particle and hole excitations on the ground state, but, as we will see later, this is not the case anymore. The true single particle dispersion, which we will denote by  $\tilde{\epsilon}(\lambda)$ , will in fact pick up additional contributions from the Fermi points  $\lambda_{ia}$  due to their nonzero energy.

CFT predicts a universal  $1/L$  energy correction in terms of the velocities of right and left moving modes of the form in equation (19). However, here the velocity

$$v_{ia} = \frac{\epsilon'(\lambda_{ia})}{2\pi\rho_\infty(\lambda_{ia})} \quad (21)$$

differs from the dynamic velocity  $\tilde{v}_{ia}$  from the dispersion  $\tilde{\epsilon}(\lambda)$  which governs the propagation of correlations.

From here on we will drop the subscript  $\infty$  and denote by  $\rho(\lambda)$  the density in the thermodynamic limit.

#### 4. The shift function

As it turns out, the shift function  $F(\lambda|\lambda')$  determined by the integral equation

$$F(\lambda|\lambda') = \frac{\theta(\lambda - \lambda')}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) F(\nu|\lambda') \quad (22)$$

plays an important role. Its definition can be obtained by considering a particle-hole excitation with rapidity  $\lambda_p$  for the particle and  $\lambda_h$  for the hole, as is discussed in standard textbooks [1]. Denoting  $\lambda_j$  for the solution of the Bethe equations for the state  $|\{k_{ia}\}\rangle$  and  $\tilde{\lambda}_j$  for the excited state, we can define the shift function for the particle-hole excitation as

$$F(\lambda_j|\lambda_p, \lambda_h) = \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_{j+1} - \lambda_j}. \quad (23)$$

From the Bethe equations one can show that  $F(\lambda|\lambda_p, \lambda_h) = F(\lambda|\lambda_p) - F(\lambda|\lambda_h)$  with definitions according to (22).

In this section we collect various results on the shift function for zero entropy states which are quite useful. Especially for the case of the ground state this is all known, but a discussion of the generality seems unavailable in the literature or is at least hard to find.

Since  $\partial_{\lambda'}\theta(\lambda - \lambda') = -K(\lambda - \lambda')$  we easily see that

$$\partial_{\lambda'}F(\lambda|\lambda') = -L(\lambda|\lambda'). \quad (24)$$

It is worth noting that  $L(\lambda|\lambda') = L(\lambda'|\lambda)$ , but

$$\partial_{\lambda}F(\lambda|\lambda') = L(\lambda|\lambda') - \sum_{ia} s_a L(\lambda|\lambda_{ia})F(\lambda_{ia}|\lambda') \quad (25)$$

which follows from equation (22) by using a partial integration. Another, very useful, relation is

$$F(\lambda|\lambda') + F(\lambda'|\lambda) = \sum_{ia} s_a F(\lambda_{ia}|\lambda)F(\lambda_{ia}|\lambda'). \quad (26)$$

This in particular implies

$$\sum_{kc} [\delta_{ia,kc} - s_c F(\lambda_{kc}|\lambda_{ia})][\delta_{jb,kc} - s_b F(\lambda_{kc}|\lambda_{jb})] = \delta_{ia,jb} \quad (27)$$

hence we have found a matrix-inverse pair

$$U_{ia,jb} = \delta_{ia,jb} - s_b F(\lambda_{jb}|\lambda_{ia}), \quad [U^{-1}]_{ia,jb} = \delta_{ia,jb} - s_b F(\lambda_{ia}|\lambda_{jb}). \quad (28)$$

Finally,

$$\sum_{ia} s_a F(\lambda|\lambda_{ia})F(\lambda'|\lambda_{ia}) = \sum_{ia} s_a F(\lambda_{ia}|\lambda)F(\lambda_{ia}|\lambda'). \quad (29)$$

Strictly speaking, the function  $F(\lambda|\lambda')$  does not encode the shift of rapidities when a single particle or hole is created in bosonic models such as Lieb–Liniger and XXZ due to the 1/2 shift in the quantum number lattice when we change particle-number parity. Rather,  $F(\lambda|\lambda')$  represents the phase shifts in the fermionic dual which is the Cheon–Shigehara model [32, 33] for Lieb–Liniger and spinless lattice fermions for XXZ.

Using the conventions in which adding a particle shifts the occupied quantum numbers to the left while adding a hole shifts them to the right, we can define the bosonic shift function

$$F_B(\lambda|\lambda') = \frac{\theta(\lambda - \lambda') - \pi}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu)F_B(\nu|\lambda'). \quad (30)$$

The relation between  $F(\lambda|\lambda')$  and  $F_B(\lambda|\lambda')$  may be expressed as

$$F_B(\lambda|\lambda') = F(\lambda|\lambda') - \frac{1}{2}Z(\lambda), \quad (31)$$

where  $Z(\lambda)$  is the analog of the dressed charge

$$Z(\lambda) = 1 + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu)Z(\nu) \quad (32)$$

which is related to critical exponents in the case of the ground state, but a similar interpretation is not present in the general case. Note that derivatives with respect to  $\lambda$  or  $\lambda'$  of  $F(\lambda|\lambda')$  and  $F_B(\lambda|\lambda')$  coincide.

From hereon we will implicitly assume that we deal with the fermionic version of the models. This makes the connection with the (fermionic) effective field theory most transparent. The difference is only important for single particle or hole excitations.

## 5. Energy and momentum of excitations

In order to determine the single-particle dispersion function, let us consider again a particle-hole excitation on top of the state  $|\{k_{ia}\}\rangle$  with particle rapidity  $\lambda_p$  and hole rapidity  $\lambda_h$ . Let  $\lambda_j$  and  $\tilde{\lambda}_j$  again denote the solution to the Bethe equations before and after excitation. The energy difference

$$\Delta E(\lambda_p, \lambda_h) = \epsilon_0(\lambda_p) - \epsilon_0(\lambda_h) + \sum_j [\epsilon_0(\tilde{\lambda}_j) - \epsilon_0(\lambda_j)] \quad (33)$$

can be expressed in the thermodynamic limit as

$$\Delta E(\lambda_p, \lambda_h) = \tilde{\epsilon}(\lambda_p) - \tilde{\epsilon}(\lambda_h) \quad (34)$$

with

$$\tilde{\epsilon}(\lambda) = \epsilon_0(\lambda) - \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\nu \epsilon'_0(\nu) F(\nu|\lambda). \quad (35)$$

By a partial integration we obtain

$$\tilde{\epsilon}(\lambda) = \epsilon_0(\lambda) - \sum_{ia} s_a \epsilon_0(\lambda_{ia}) F(\lambda_{ia}|\lambda) + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\nu \epsilon_0(\nu) \partial_\nu F(\nu|\lambda) \quad (36)$$

from where equation (25) expresses the actual single-particle dispersion as

$$\tilde{\epsilon}(\lambda) = \epsilon(\lambda) - \sum_{ia} s_a \epsilon(\lambda_{ia}) F(\lambda_{ia}|\lambda). \quad (37)$$

Note that this indeed differs from  $\epsilon(\lambda)$  when  $\epsilon(\lambda_{ia}) \neq 0$  and we have nontrivial backflow  $F(\lambda_{ia}|\lambda) \neq 0$ .

The momentum of a particle is defined by the equation

$$k(\lambda) = p_0(\lambda) - \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\nu p'_0(\nu) F(\nu|\lambda) \quad (38)$$

from which it is easy to see that  $k'(\lambda) = 2\pi\rho(\lambda)$  as in the equilibrium case.

Of particular interest is the energy and velocity of particles close to the Fermi points  $k_{ia}$ . We note that equations (37) and (27) imply

$$\tilde{\epsilon}(\lambda_{ia}) = \sum_{jb} [\delta_{ia,jb} - s_b F(\lambda_{jb}|\lambda_{ia})] \epsilon(\lambda_{jb}), \quad (39)$$

$$\epsilon(\lambda_{ia}) = \sum_{jb} [\delta_{ia,jb} - s_b F(\lambda_{ia}|\lambda_{jb})] \tilde{\epsilon}(\lambda_{jb}). \quad (40)$$

The Fermi velocity for the Fermi point  $k_{ia}$  is defined as

$$\tilde{v}_{ia} = \left. \frac{\partial \tilde{\epsilon}}{\partial k} \right|_{k=k_{ia}} = \frac{\tilde{\epsilon}'(\lambda_{ia})}{2\pi\rho(\lambda_{ia})}. \quad (41)$$



The relation between  $\epsilon(\lambda)$  and  $\tilde{\epsilon}(\lambda)$  can also be expressed as

$$\epsilon(\lambda) = \tilde{\epsilon}(\lambda) - \sum_{ia} s_a \tilde{\epsilon}(\lambda_{ia}) F(\lambda|\lambda_{ia}). \quad (42)$$

## 6. Finite-size spectrum and critical exponents

Now that we have established the energy of zero-entropy states to order  $1/L$  [equation (11)] and the energy of particle and hole excitations in the thermodynamic limit (equation (37)) we will ask the usual question: What is the change in energy upon adding or removing particles very close to the Fermi points  $I_{ia}$ ? Let us consider a state defined by  $I_{ia} \rightarrow I_{ia} + s_a N_{ia}$ , i.e.  $N_{ia}$  denotes the number of particles added or removed at the Fermi point  $k_{ia}$ .

In terms of the quantum numbers

$$N_i = L \int_{\lambda_{iL}}^{\lambda_{iR}} d\lambda \rho(\lambda), \quad D_i = L \left\{ \int_{-\infty}^{\lambda_{iL}} - \int_{\lambda_{iR}}^{\infty} \right\} d\lambda \rho(\lambda) \quad (43)$$

we have

$$N_{ia} = \frac{\Delta N_i + s_a \Delta D_i}{2}, \quad (44)$$

where  $\Delta N_i$ ,  $\Delta D_i$  denotes the change in  $N_i$ ,  $D_i$ . We can also express the variation of the state in terms of the change in the Fermi rapidities  $\lambda_{jb} \rightarrow \lambda_{jb} + \delta\lambda_{jb}$ . The definitions in equation (43) allow us to compute the Jacobian

$$\frac{\partial N_{ia}}{\partial \lambda_{jb}} = s_a L \left\{ \rho(\lambda_{ia}) \delta_{ia,jb} + \frac{1}{2} \int_{-\infty}^{\infty} d\lambda s_a \operatorname{sgn}(\lambda_{ia} - \lambda) \frac{d\rho}{d\lambda_{jb}}(\lambda) \right\}. \quad (45)$$

Using that

$$\frac{\partial \rho}{\partial \lambda_{jb}}(\lambda) = s_b \rho(\lambda_{jb}) L(\lambda_{jb}|\lambda) = -s_b \rho(\lambda_{jb}) \partial_\lambda F(\lambda_{jb}|\lambda) \quad (46)$$

and a partial integration one finds

$$\frac{\partial N_{ia}}{\partial \lambda_{jb}} = L s_b \rho(\lambda_{jb}) [\delta_{ia,jb} - s_a F(\lambda_{jb}|\lambda_{ia})]. \quad (47)$$

We recognize the matrix  $[U^{-1}]_{jb,ia}$  from equation (28), which immediately gives

$$\frac{\partial \lambda_{ia}}{\partial N_{jb}} = \frac{1}{L s_a \rho(\lambda_{ia})} [\delta_{ia,jb} - s_a F(\lambda_{ia}|\lambda_{jb})] \quad (48)$$

and therefore we can express

$$\delta \lambda_{ia} = \sum_{jb} \frac{\delta_{ia,jb} - s_a F(\lambda_{ia}|\lambda_{jb})}{L s_a \rho(\lambda_{ia})} N_{jb}. \quad (49)$$

Since the Fermi momenta are directly related to the numbers  $I_{ia}$ , the change in Fermi momentum is

$$\delta k_{ia} = \frac{L s_a N_{ia}}{2\pi} \quad (50)$$

which can also be obtained from the definition of  $k(\lambda)$  in equation (38). Hence also the relations

$$\frac{\partial k_{ia}}{\partial \lambda_{jb}} = [\delta_{ia,jb} - s_b F(\lambda_{jb}|\lambda_{ia})] 2\pi \rho(\lambda_{jb}), \quad (51)$$

$$\frac{\partial \lambda_{ia}}{\partial k_{jb}} = \frac{1}{2\pi \rho(\lambda_{ia})} [\delta_{ia,jb} - s_b F(\lambda_{ia}|\lambda_{jb})] \quad (52)$$

are valid.

Let us consider corrections to the energy  $E$  in equation (11) to order  $1/L$  when  $k_{ia} \rightarrow k_{ia} + \delta k_{ia}$ . We express

$$\delta E = \sum_{ia} \frac{\partial E}{\partial \lambda_{ia}} \delta \lambda_{ia} + \frac{1}{2} \sum_{ia,jb} \frac{\partial^2 E}{\partial \lambda_{ia} \partial \lambda_{jb}} \delta \lambda_{ia} \delta \lambda_{jb} \quad (53)$$

or equivalently

$$\delta E = \sum_{ia} \frac{\partial E}{\partial k_{ia}} \delta k_{ia} + \frac{1}{2} \sum_{ia,jb} \frac{\partial^2 E}{\partial k_{ia} \partial k_{jb}} \delta k_{ia} \delta k_{jb}. \quad (54)$$

Note that these corrections can only come from the extensive contribution to  $E$  since  $\delta \lambda_{ia}$  and  $\delta k_{ia}$  are of order  $1/L$ .

From equations (11) and (40) we obtain

$$\frac{\partial E}{\partial \lambda_{ia}} = L s_a \rho(\lambda_{ia}) \epsilon(\lambda_{ia}) = \sum_{jb} L s_a \rho(\lambda_{ia}) [\delta_{ia,jb} - s_b F(\lambda_{ia}|\lambda_{jb})] \tilde{\epsilon}(\lambda_{jb}) \quad (55)$$

which together with equation (52) shows

$$\frac{\partial E}{\partial k_{ia}} = \frac{s_a L}{2\pi} \tilde{\epsilon}(\lambda_{ia}) \quad (56)$$

so that

$$\delta E^{(1)} = \sum_{ia} \frac{\partial E}{\partial k_{ia}} \delta k_{ia} = \sum_{ia} \tilde{\epsilon}(\lambda_{ia}) N_{ia} \quad (57)$$

(where we have introduced the notation  $\delta E^{(n)}$  for the order  $L^{-n}$  term in  $\delta E$ ).

Next, consider the second order correction

$$\delta E^{(2)} = \frac{1}{2} \sum_{ia,jb} \frac{\partial^2 E}{\partial k_{ia} \partial k_{jb}} \delta k_{ia} \delta k_{jb}. \quad (58)$$

From equation (35) we find that

$$\frac{\partial \tilde{\epsilon}}{\partial \lambda_{jb}}(\lambda) = -s_b \epsilon'_0(\lambda_{jb}) F(\lambda_{jb}|\lambda) - \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\nu \epsilon'_0(\nu) \frac{\partial F}{\partial \lambda_{jb}}(\nu|\lambda), \quad (59)$$

which together with

$$\frac{\partial F}{\partial \lambda_{jb}}(\lambda|\lambda') = s_b L(\lambda|\lambda_{jb}) F(\lambda_{jb}|\lambda') \quad (60)$$

can be used to show that

$$\frac{\partial \tilde{\epsilon}}{\partial \lambda_{jb}}(\lambda) = -s_b \tilde{\epsilon}'(\lambda_{jb}) F(\lambda_{jb}|\lambda). \quad (61)$$

For the derivation it is useful to note

$$\epsilon'(\lambda) = \epsilon'_0(\lambda) + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} d\nu \partial_\lambda L(\lambda|\nu) \epsilon_0(\nu), \quad (62)$$

$$\tilde{\epsilon}'(\lambda) = \epsilon'(\lambda) + \sum_{ia} s_a \epsilon(\lambda_{ia}) L(\lambda_{ia}|\lambda). \quad (63)$$

Computing

$$\frac{\partial}{\partial \lambda_{jb}} \left( \frac{\partial E}{\partial k_{ia}} \right) = \frac{s_a L}{2\pi} [\delta_{ia,jb} - s_b F(\lambda_{jb}|\lambda_{ia})] \tilde{\epsilon}'(\lambda_{jb}) \quad (64)$$

we thus find

$$\delta E^{(2)} = \frac{1}{2} \sum_{ia,jb,kc} \frac{\partial \lambda_{kc}}{\partial k_{jb}} \frac{\partial}{\partial \lambda_{kc}} \left( \frac{\partial E}{\partial k_{ia}} \right) \quad (65)$$

$$= \frac{1}{L} \sum_{ia,jb,kc} \frac{\tilde{\epsilon}'(\lambda_{kc})}{2\rho(\lambda_{kc})} [\delta_{ia,kc} - s_c F(\lambda_{kc}|\lambda_{ia})] [\delta_{jb,kc} - s_c F(\lambda_{kc}|\lambda_{jb})] N_{ia} N_{jb}. \quad (66)$$

Now it is easy to also incorporate the number of particle–hole excitations corresponding to a total number of momentum quanta  $n_{ia}$  close to the Fermi point  $k_{ia}$  and arrive at the general result for the spectrum

$$\delta E = \sum_{ia} \tilde{\epsilon}(\lambda_{ia}) N_{ia} + \frac{2\pi}{L} \sum_{ia} s_a \tilde{v}_{ia} \left[ n_{ia} + \frac{1}{2} \left( \sum_{jb} U_{jb,ia} N_{jb} \right)^2 \right]. \quad (67)$$

This is valid for general zero-entropy states  $|\{k_{ia}\}\rangle$  and general energy functions  $\epsilon_0(\lambda)$  with

$$U_{ia,jb} = \delta_{ia,jb} - s_b F(\lambda_{ia}|\lambda_{jb}), \quad [U^{-1}]_{ia,jb} = s_a s_b U_{jb,ia}. \quad (68)$$

Note that the velocity  $s_a \tilde{v}_{ia}$  can be negative in the current setup.

The matrix  $U_{ia,jb}$  is identified with the matrix of the Bogoliubov transformation diagonalizing the multi-component Tomonaga–Luttinger Hamiltonian describing the state [30]. These parameters determine the exponents of critical correlations, i.e. the conformal dimensions of scaling fields in the language of CFT.

## 7. The symmetric case

In the case of a symmetric quantum number configuration,  $I_{iL} = -I_{n+1-iR}$ , we have the equalities

$$\tilde{v}_{iL} = -\tilde{v}_{n+1-iR} \quad \text{and} \quad U_{ia,jb} = U_{n+1-i\bar{a},n+1-j\bar{b}} \quad (69)$$

(with  $\bar{L} = R$  and  $\bar{R} = L$ ). Define the matrices

$$Z_{ij} = U_{iR,jR} - U_{n+1-iL,jR} = \delta_{ij} - F(\lambda_{jR}|\lambda_{iR}) + F(\lambda_{jR}|\lambda_{n+1-iL}), \quad (70)$$

$$Y_{ij} = U_{iR,jR} + U_{n+1-iL,jR} = \delta_{ij} - F(\lambda_{jR}|\lambda_{iR}) - F(\lambda_{jR}|\lambda_{n+1-iL}). \quad (71)$$

Using that in the symmetric case  $F(-\lambda|-\lambda') = -F(\lambda|\lambda')$  and  $\lambda_{iL} = -\lambda_{n+1-iR}$ , equation (26) gives

$$\sum_k Z_{ik} Y_{jk} = \delta_{ij} \quad (72)$$

and so  $Z^{-1} = Y^T$  which is closely related to the general relation  $[U^{-1}]_{ia,jb} = s_a s_b U_{jb,ia}$ .

The finite-size correction to the energy can then be written as

$$\delta E = \sum_i \tilde{\epsilon}_i \tilde{N}_i + \frac{2\pi}{L} \sum_i \frac{\tilde{v}_i}{2} \left[ \left( \sum_j [Z^{-1}]_{ij} \tilde{N}_j \right)^2 + \left( \sum_j Z_{ji} \tilde{D}_j \right)^2 \right], \quad (73)$$

where  $\tilde{\epsilon}_i = \tilde{\epsilon}(\lambda_{iR})$ ,  $\tilde{v}_i = \tilde{v}_{iR}$  and

$$\tilde{N}_i = N_{iR} + N_{n+1-iL}, \quad \tilde{D}_i = N_{iR} - N_{n+1-iL}. \quad (74)$$

We can write

$$Z_{ij} = \delta_{ij} + \int_{\lambda_{n+1-iL}}^{\lambda_{iR}} d\nu L(\lambda_{jR}|\nu). \quad (75)$$

We can also obtain this matrix from as  $Z_{ij} = \xi_{ij}(\lambda_{jR})$  where  $\xi_{ij}(\lambda)$  is defined by

$$\xi_{ij}(\lambda) = \delta_{ij} + \sum_k \int_{\lambda_{n+1-iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \xi_{kj}(\nu) \quad (76)$$

which is straightforward to derive using the relation  $\partial_{\lambda'} F(\lambda|\lambda') = -L(\lambda|\lambda')$  from equation (75). Hence, in the symmetric case we reach the same conclusion as [28], namely that the critical exponents can equivalently be expressed in terms of a dressed charge matrix  $\xi_{ij}(\lambda)$  similar to models solvable by nested Bethe ansatz [34–44].

## 8. Impurity configurations

Let us consider an impurity configuration defined by one hole with  $\lambda_h$  in, or one particle with  $\lambda_p$  outside of the Fermi-sea blocks and ask again what the spectrum of excitations at the Fermi points is to order  $1/L$ . Here, the energy of the state  $|\{k_{ia}\}\rangle$  still serves as the reference. We restrict the analysis to the particle case, as the case of a hole just introduces appropriate minus signs. Note that we assume to work in the fermionic dual here such that  $F(\lambda|\lambda')$  encodes the shift of rapidities for a single-particle excitation.

In the case of an impurity we have to go back to the derivation of for the root density in section 3 to order  $1/L^2$ . From the Bethe equations we find

$$\rho(\lambda) = \frac{p'_0(\lambda)}{2\pi} + \sum_i \int_{\lambda_{iL}}^{\lambda_{iR}} \frac{d\nu}{2\pi} K(\lambda - \nu) \rho(\nu) + \frac{K(\lambda - \lambda_p)}{2\pi L} + \frac{1}{24L^2} \sum_{ia} \frac{s_a K'(\lambda - \lambda_{ia})}{2\pi \rho(\lambda_{ia})} \quad (77)$$

in this case. The solution for  $\rho(\lambda)$  thus has an extra contribution due to the impurity

$$\rho(\lambda) = \rho_\infty(\lambda) + \frac{\rho_{\text{imp}}(\lambda|\lambda_p)}{L} + \sum_{ia} \frac{\rho_{ia}(\lambda)}{24L^2 \rho_\infty(\lambda_{ia})}, \quad (78)$$

where clearly

$$\rho_{\text{imp}}(\lambda|\lambda_p) = L(\lambda|\lambda_p). \quad (79)$$

Going back to the definitions of  $N_i$  and  $D_i$ , we find that

$$N_i = n_i^{\text{imp}} + L \int_{\lambda_{iL}}^{\lambda_{iR}} d\lambda \rho_{\infty}(\lambda), \quad D_i = d_i^{\text{imp}} + L \left\{ \int_{-\infty}^{\lambda_{iL}} - \int_{\lambda_{iR}}^{\infty} \right\} d\lambda \rho_{\infty}(\lambda) \quad (80)$$

with

$$n_i^{\text{imp}} = \int_{\lambda_{iL}}^{\lambda_{iR}} d\lambda L(\lambda|\lambda_p) = -F(\lambda_{iR}|\lambda_p) + F(\lambda_{iL}|\lambda_p), \quad (81)$$

$$d_i^{\text{imp}} = \left\{ \int_{-\infty}^{\lambda_{iL}} - \int_{\lambda_{iR}}^{\infty} \right\} d\lambda L(\lambda|\lambda_p) = -F(\lambda_{iR}|\lambda_p) - F(\lambda_{iR}|\lambda_p). \quad (82)$$

Considering the energy difference of the state  $|\{k_{ia}\}\rangle$  and the state defined by the addition of particles at the Fermi points according to the numbers  $\{N_{ia}\}$  and the additional particle impurity with quantum number  $I_p$ , leads to

$$\delta E = \tilde{\epsilon}(\lambda_p) + \sum_{ia} \tilde{\epsilon}(\lambda_{ia}) [N_{ia} - n_{ia}^{\text{imp}}] + \frac{2\pi}{L} \sum_{ia} s_a \tilde{v}_{ia} \left[ n_{ia} + \frac{1}{2} \left( \sum_{jb} U_{jb,ia} [N_{jb} - n_{jb}^{\text{imp}}] \right)^2 \right] \quad (83)$$

with

$$n_{ia}^{\text{imp}} = \frac{n_i^{\text{imp}} + s_a d_i^{\text{imp}}}{2} = -s_a F(\lambda_{ia}|\lambda_p) \quad (84)$$

which follows by the same reasoning as leading up to equation (67) but using equation (80).

A hole impurity just replaces  $F(\lambda_{ia}|\lambda_p) \rightarrow -F(\lambda_{ia}|\lambda_h)$ . The generalization to multiple impurities is straightforward.

## 9. Conclusion

We have considered the energy of excitations on states of zero entropy density in the Lieb–Liniger and other Bethe ansatz solvable models. These states can be considered as the zero-temperature limit of a statistical ensemble defined by a generalized Hamiltonian in the spirit of the GGE. We explicitly allowed the energies to be measured with a different Hamiltonian which generically would correspond to the physical Hamiltonian of the model. We have shown that the dispersion function is not necessarily determined by a single integral equation, but includes contributions from the generalized Fermi points that may have finite energy in the situation under consideration. We derived a generalization of the expression for finite-size corrections to the spectrum. This derivation is valid for arbitrary bare energy functions  $\epsilon_0(\lambda)$  constructed from the eigenvalues of local charges on the Bethe basis and also for arbitrary configurations of Fermi seas. The energy corrections related to addition or subtraction of particles at the generalized Fermi points, which are directly related to critical exponents, are expressed in terms of the shift function and only for a symmetric configuration can this be expressed in terms of a dressed charge matrix. Similar expressions are derived in the presence of an additional particle and hole impurity.

Our results are interesting in the light of recent developments in the correspondence between Bethe ansatz solvable models and effective field theory methods. The characteristic

power-law behavior of correlations well known from the correspondence with CFT can be interpreted in terms of the Anderson orthogonality catastrophe due to the phase shift of the modes at the Fermi points. While for static correlations one only considers Umklapp-like configurations, time dependent correlations include additional contributions from certain impurity configurations, but the logic in both cases is remarkably similar. The point is that the power law exponents are completely determined by the phase shifts (static data) while the characteristic frequencies of oscillations in space and time are determined by the momentum and energy differences of the reference state with the Umklapp and impurity excitations. Our work suggests that this decomposition of effects can be extended to out-of-equilibrium correlations of zero-entropy states and the power-law exponents depend only on the scattering data of the theory and are Hamiltonian independent.

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