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COORDINATE-FREE LOGIC

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Abstract. A new logic is presented without predicates—except equality. Yet its expressive power is the same as that of predicate logic, and relations can faithfully be represented in it. In this logic we also develop an alternative for set theory. There is a need for such a new approach, since we do not live in a world of sets and predicates, but rather in a world of things with relations between them.

§1. How standard logic misleads us. *4 is less than 6 and 6 is greater than 4.* A child of five knows this. We have both a greater-than predicate and a less-than predicate. But this is quite something different than saying that there are 'out there' a less-than relation and a greater-than relation. In my view, people who think there are really two such relations are misled by language.

It seems hard to deny that 4's being less than 6 is the very same fact as 6's being greater than 4. In English and in many other languages we have different ways of expressing this fact efficiently by exploiting the linear order of words, but we should not confuse representational features with the real things.

Our standard logical apparatus reinforces the widespread misconception that the arguments of a relation always come in a certain order. It interprets the predicate '$<$' as a collection of ordered pairs $\langle x, y\rangle$, where x is less than y, and it interprets the predicate '$>$' as a collection of ordered pairs $\langle x, y\rangle$, where x is greater than y. The logic does not offer an easy way to express in a neutral way how things stand to each other.

Another misleading aspect of standard logic concerns the existence of things. In standard logic, predicates are often interpreted as relations, and because predicates have argument-places, standard logic suggests that the corresponding relations are universals having argument-places as well. This, however, is not a view on relations shared by everyone. According to a nominalistic view on relations, there exist no universal things as 'relations', but only facts and complexes. It would be better if the logic was unbiased in this matter.

A third weakness of standard logic to be noted concerns symmetry. We can express that a (binary) relation R is symmetrical by saying that for all x, y we have $R(x, y) \leftrightarrow R(y, x)$. But there is another, often overlooked, notion of symmetry: a relation is said to be *strictly symmetric* if for all x, y the complex $R(x, y)$ is identical to the complex $R(y, x)$. About strict symmetry, standard logic is and can only be silent.

We would like to have is a logic that is more supportive of different views on the structure of the world than standard logic. In this paper, we aim to develop such a logic.[1] The new logic will have the following features: (i) it allows us to express relations without an artifactual ordering of the arguments, (ii) it accommodates different views on the existence of universals, and (iii) it can deal with strict symmetry.

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[1] Parts of §2 and §3 have been published earlier in Leo (2014). However, there are some important differences. In particular, structures and relational complexes are defined in a new way.

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We will take a ‘layered’ approach. We start in §2 with developing a new logical framework in which relations do not directly play a role; it is all about entities with an input–output functionality. The logic looks simpler than predicate logic, but it will be shown to have the same proof-theoretic strength. Then, in §3, a formalization of relations is given within the new framework. In particular, relational complexes are viewed as input–output entities. For example, in the complex of Adam’s loving Eve we may substitute Romeo for Adam and Julia for Eve and this results in Romeo’s loving Julia. The love relation itself is a network of such complexes interrelated by substitutions. The conception of relations as networks of connections is essentially the *antipositionalist view* on relations developed by Kit Fine (2000).

Because the design of the new logic is qua structures radically different from predicate logic, we can no longer use set theory as a foundational system. Therefore, in §4, we develop a new foundation of mathematics as well. Unlike standard foundations, it has no axiom of extensionality and no axiom of foundation. In the concluding section, we give a brief evaluation of the new logic.

It should be noted that, in this paper, for the mathematical results only elementary finitistic reasoning will be used. If we would use for our reasoning an existing foundation like set theory, then we would continue to be stuck with the old foundation.

§2. Defining a new logic.

I present a new logic that has no predicates—except equality. Furthermore, the logic has no constants and variables as distinct types of symbols. Yet its expressive power is the same as that of predicate logic.

The starting point is to view the world as consisting of first-class entities like people, numbers, facts, etc. These entities may have an input–output functionality or behavior. For example, if we have a blender, then putting a tomato in it gives tomato juice as output. In our new logic, only purely logical assertions about entities and their input–output functionality are made.

For the new logic, I will use the name *coordinate-free logic*.

2.1. Syntax.

A language of coordinate-free logic has the following symbols:

(i) simple terms: $a, b, x, \ldots,$
(ii) application symbol: $\cdot(\cdot)$,
(iii) equality symbol: $=$,
(iv) connectives: $\wedge, \neg, \forall$.

We do not have constants and variables. Instead, we have *simple terms*, and we allow their number to be finite or infinite (of any cardinality).[2] The *application symbol* will be used for partial function application. The details will be given shortly.

Like predicate logic, coordinate-free logic has terms and formulas. The terms will represent entities. Their definition is straightforward:

DEFINITION 2.1. *The collection of* terms *is defined inductively as follows:*

(i) *every simple term is a term,*
(ii) *if t, t' are terms, then $t(t')$ is a term.*

For convenience, we will often write tt' for $t(t')$.

[2] At this point—due to our finitistic reasoning—the notion of infinite cardinalities is meaningless, but it will make sense within the context of the foundational theory developed in §4.

All atomic formulas are equalities of terms; all other formulas are built up from them by using only logical connectives and quantifiers:

DEFINITION 2.2. *The collection of* formulas *is defined inductively as follows:*

(i) *if t, t' are terms, then $t = t'$ is a formula,*
(ii) *if φ, ψ are formulas, then $(\varphi \wedge \psi), \neg\varphi$ are formulas,*
(iii) *if φ is a formula and x is a simple term, then $\forall x\, \varphi$ is a formula.*

We will assume that $\vee, \rightarrow, \leftrightarrow, \exists$ are defined in an obvious way. For example, $\exists x\, \varphi$ denotes $\neg\forall x\, \neg\varphi$.

As the definitions show, we have no terms with more than one argument-place, and no predicates, except equality. Therefore, an artifactual ordering of arguments does not occur in coordinate-free logic, except in equality statements. The ordering in equality statements, however, is innocuous since equality is strictly symmetric. Of course, this does not yet mean that coordinate-free logic is adequate for reasoning about natural relations like the love relation. In §3, we will see how it can be used as a basis for a general account of relations.

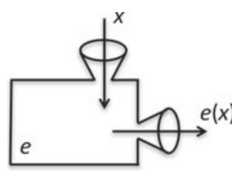
2.2. Semantics. The semantics of coordinate-free logic is based on the following ideas:

1. A term does not necessarily have an interpretation.
2. If terms t, t' are interpreted as entities e, e', then we interpret $t(t')$ as the output of the entity e for input e', if such an output exists.
3. A formula $t = t'$ will be interpreted as true iff t and t' are interpreted as the same entity.

Structures. We define structures that are different from the traditional ones:

DEFINITION 2.3. *A* coordinate-free structure *is a (possibly empty) collection E of entities that may have input–output functionality, where the inputs and outputs of the entities belong to E as well.*

If x is an input of an entity e, then we denote the output as $e(x)$ or as ex.
We may depict entities as input–output machines:



We allow that different entities may have the same input–output functionality and we allow that entities may have themselves as inputs.

Coordinate-free structures clearly deviate from set-theoretic structures. In set-theoretic structures we have individuals, functions, and relations, where the individuals themselves are passive, powerless objects that only play a role as arguments or results of functions and relations. In coordinate-free structures, any entity can fulfill an active role.[3]

In classical model theory, the properties of the underlying objects are completely irrelevant for the structure. The only thing that counts is the cardinality of the collection of

[3] Coordinate-free structures resemble applicative structures $\langle X, \cdot \rangle$ (see, e.g., (Barendregt, 1984, p. 88)). However, there is a significant difference. Coordinate-free structures can be translated in a trivial way to applicative structures, but the translation in the reverse direction is not trivial:

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objects. This is in stark contrast to the current approach where the objects themselves fix the structure.

REMARK 2.4. *Instead of coordinate-free structures we could have chosen structures of the form* $\langle E, \mathsf{App}\rangle$ *with* App *a total function from* E *to the collection of partial functions on* E*. Actually, I did this in Leo (2014); it seemed logical to use structures that are similar to the set-theoretic structures of predicate logic. However, I now consider such structures as less natural: we read a term* $t(t')$ *as the result of* t *applied to* t'*, and* not *as the result of first applying a function* App *to* t *and then applying the output to* t'*.*

As we will see in §4, the choice we made for the structures has consequences for a choice of an adequate mathematical foundation. In particular, the new proposed foundation will have no axiom of extensionality and no axiom of foundation.

Interpreting terms and formulas. A language of coordinate-free logic is determined by its simple terms, which we take as its *signature*. For a fixed collection of simple terms, we give an interpretation of terms and formulas by means of a partial assignment of entities in E to the simple terms of our language.

Let E be a coordinate-free structure and let g : simple terms $\rightarrow E$ be a partial function. Then terms are interpreted as follows:

$$\begin{aligned} [t]_g &= g(t) \quad \text{if } t \text{ is a simple term,} \\ [t(t')]_g &= [t]_g([t']_g). \end{aligned}$$

Note that a term may have no interpretation for two reasons: (i) g is a partial function, (ii) an entity in E is not necessarily an input of another given entity in E.

For interpreting formulas, let $V_{E,g}$: formulas $\rightarrow \{0, 1\}$ be a total function such that:

$$\begin{aligned} V_{E,g}(t = t') &= 1 \ \text{ iff } \ [t]_g \text{ and } [t']_g \text{ are both defined and } [t]_g = [t']_g, \\ V_{E,g}(\varphi \wedge \psi) &= 1 \ \text{ iff } \ V_{E,g}(\varphi) = 1 \text{ and } V_{E,g}(\psi) = 1, \\ V_{E,g}(\neg\varphi) &= 1 \ \text{ iff } \ V_{E,g}(\varphi) = 0, \\ V_{E,g}(\forall x\, \varphi) &= 1 \ \text{ iff } \ V_{E,g[x:e]}(\varphi) = 1 \text{ for every } e \in E, \end{aligned}$$

where $g[x : e]$ maps x to e, and any other simple term y to $g(y)$.[4]

Using this valuation, we define the notions of satisfiability, models, and logical consequence in a straightforward way:

DEFINITION 2.5. *If* $V_{E,g}(\varphi) = 1$*, then we say that* E satisfies φ with g*, and write* $E, g \vDash \varphi$*.*

Let Γ *be a collection of formulas. If* $E, g \vDash \varphi$ *for every formula* φ *of* Γ*, then we call* E *with* g *a* model *for* Γ*.*

If any valuation $V_{E,g}$ *that satisfies all formulas in* Γ *also satisfies* φ*, then we say that* Γ logically implies φ*, and write* $\Gamma \vDash \varphi$*. If* Γ *is empty, then we say that* φ *is* logically valid*, and write* $\vDash \varphi$*.*

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the *inert* elements of X have to be translated to entities which have themselves an input–output functionality. In fact, for the reverse translation we need an axiom of universality as presented in §4.

4 It is a design choice to let $V_{E,g}$ be a total two-valued function. Alternatively, we could, e.g., have chosen for a partial two-valued function, where we leave the value of $t = t'$ undefined if $[t]_g$ or $[t']_g$ is undefined.

Note that $\forall x\, x = x$ is logically valid, but $x = x$ is not. We have $E, g \not\models x = x$ for any g that is undefined for x.

Caveat. At this points we have not made any assumptions on the applicative powers of entities. This means that, for all we know now, a sentence like $\forall x\, (x(x) = x)$ could be logically valid. It is only in the light of the foundational theory that we develop in §4 that we can refute the validity of such a sentence.

Because in coordinate-free logic simple terms may play the role of constants and variables, we define a *theory* less restrictive than usual:

DEFINITION 2.6. *A* theory *is a collection of formulas.*

The simple terms that occur free in the formulas of a theory can fulfill a similar role as constants in predicate logic. A main difference is that coordinate-free structures themselves do not give an interpretation to simple terms, whereas in predicate logic the structures have as an ingredient an interpretation function that assigns elements of the structure to the constants.

In what follows, we will frequently use the following notions:

DEFINITION 2.7. A *term t has* existential import *if t has an interpretation:*

$$\mathsf{E}\, t \;=_{\text{df}}\; \exists x\; x = t, \textit{ with } x \textit{ not in } t.$$

Terms t, t' are weakly equal *if existential import of one of them implies that their interpretations are the same:*

$$t \simeq t' \;=_{\text{df}}\; \mathsf{E}\, t \vee \mathsf{E}\, t' \to t = t'.$$

Terms t, t' are functionally equivalent *if they have the same input–output functionality:*

$$t \equiv t' \;=_{\text{df}}\; \forall u\; tu \simeq t'u.$$

Alternatively, $\mathsf{E}\, t$ could be defined as $t = t$, but then some of the axioms given in §2.3 should be replaced by other axioms.

2.3. Axiomatization. We give an axiomatization of coordinate-free logic that is quite similar to a standard axiomatization of free predicate logic (cf. (Oliver and Smiley, 2013, p. 191)).

For a given language of coordinate-free logic, the logical axioms are all generalizations of formulas of the following forms, i.e., all instances prefaced by zero or more universal quantifications:

1. Tautologies,
2. $\forall x\, (\varphi \to \psi) \to (\forall x\, \varphi \to \forall x\, \psi)$,
3. $\varphi \to \forall x\, \varphi$, where x does not occur free in φ,
4. $\forall x\, \varphi(x) \to (\mathsf{E}\, t \to \varphi(t))$, where t is free for x in φ,
5. $\forall x\; x = x$,
6. $t = t' \to (\varphi \leftrightarrow \varphi')$, where φ' is obtained from φ by zero or more substitutions of t' for t where both t and t' occur free,
7. $t = t \to \mathsf{E}\, t$,
8. $\mathsf{E}\, tt' \to \mathsf{E}\, t \wedge \mathsf{E}\, t'$.

I use here the term *tautologies* for generalizations of the tautologies of propositional logic obtained by replacing each propositional symbol by a formula of the language of coordinate-free logic.

We have one rule of inference:

$$\text{from } \varphi \text{ and } \varphi \rightarrow \psi, \text{ infer } \psi.$$

As usual, we write $\Gamma \vdash \varphi$ for deducibility.

We have—of course—the following soundness theorem for coordinate-free logic.

THEOREM 2.8. $\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$.

Proof. The theorem can be proved along the lines of the soundness proof given in (Oliver and Smiley, 2013, p. 201). □

2.4. Proof-theoretic strength. We make a comparison of the proof-theoretic strength of coordinate-free logic and of predicate logic. We will show that, under mild conditions, theories of coordinate-free logic are faithfully interpretable in theories of free predicate logic, and vice versa.

We assume that the reader is familiar with the notion of interpretability of theories in first-order predicate logic (see, e.g., Tarski et al. (1953); Visser (2006)). The idea is to define a translation τ from the language of a theory U to the language of a theory V, such that for all U-sentences A, $U \vdash A$ implies $V \vdash A^\tau$. If the converse implication also holds, then the interpretation is called *faithful*.

Interpretations between theories of predicate logic and theories of coordinate-free logic can be defined analogously to interpretations of theories in predicate logic. However, we have to take into consideration that in coordinate-free logic we do not have constants and variables as distinct types of symbols. We will handle this as follows.

Recall that in Definition 2.6 we defined a theory as just a collection of formulas. We will regard the simple terms that occur free in a formula of a theory U as constants, and we will call any formula whose free simple terms all occur free in a formula that belongs to U a *U-sentence*.

We start with defining translations from predicate logic to coordinate-free logic. The definition closely resembles the one given in (Visser, 2009, pp. 3–4). It is, however, a bit simpler because we restrict ourselves to one-dimensional parameter-free translations.

The definition we give assumes that the language for (free) predicate logic is relational, but the definition can be extended in an obvious way to languages with terms by using a standard translation to relational languages.

Let Σ be the signature of a relational language for (free) predicate logic, and let Ξ be the signature of a language for coordinate-free logic. We assume that Ξ has an infinite number of simple terms. We fix a sequence containing all variables $u_1, u_2, \ldots$ for Σ, and we fix three disjoint collections of simple terms, namely a collection D, a sequence $v_1, v_2, \ldots$, and a sequence $z_1, z_2, \ldots$ for Ξ. The idea is to let the terms in D play the role of constants.

A *translation* $\tau : \Sigma \rightarrow \Xi$ is given by a pair $\langle \delta, F \rangle$. Here, δ is a Ξ-formula with bound terms among the z_i, and free terms among v_1 and the terms in D. It defines the domain of the interpretation. The F is a mapping that associates to each n-ary relational symbol R of Σ a Ξ-formula $F(R)$ with bound terms among the z_i, and free terms among $v_1, \ldots, v_n$ and the terms in D.

We translate Σ-formulas to Ξ-formulas as follows:

- $(R(u_{j_1}, \ldots, u_{j_n}))^\tau := F(R)(v_{j_1}, \ldots, v_{j_n})$, where the formula $F(R)(v_{j_1}, \ldots, v_{j_n})$ is the result of simultaneous substitution of v_{j_i} for v_i, where $1 \leq i \leq n$;
- $(\cdot)^\tau$ commutes with the propositional connectives;
- $(\forall u_k A)^\tau := \forall v_k \, (\delta(v_k) \rightarrow A^\tau)$.

If there is no domain restriction, then we say that we have a *unrelativized translation* τ given by F. If, in addition, the translation is *identity preserving*, i.e., it translates identity to identity, then we call the translation *direct*.[5]

Translations can be composed. If $\tau_1 = \langle \delta_1, F_1 \rangle$ and $\tau_2 = \langle \delta_2, F_2 \rangle$, then the composition $\tau_2 \circ \tau_1 = \langle \delta, F \rangle$ is defined by $\delta := \delta_2 \wedge (\delta_1)^{\tau_2}$ and $F(R) := (F_1(R))^{\tau_2}$.

This definition of a composition can be applied to all translations discussed in this section.

We say that a translation τ *supports an interpretation of a theory* U *in a theory* V if for each U-sentence A, $U \vdash A$ implies $V \vdash A^\tau$. We call an interpretation *unrelativized* if the translation is unrelativized, and we call it *direct* if the translation is direct.

We define U^τ as the theory consisting of all formulas A^τ, with A in U.

The interpretations that we give involve a predicate logic that is universally free and negative, where *universally free* means that the domains of structures may be empty and that terms may be undefined, and *negative* means that all atomic formulas with at least one undefined term are evaluated false.

As logical axioms for negative, universally free predicate logic with identity we may choose axioms similar to the axioms 1 to 7 given in §2.3 plus for each n-ary predicate symbol P, $P(t_1, \dots, t_n) \rightarrow \mathsf{E}\, t_1 \wedge \dots \wedge \mathsf{E}\, t_n$, and for each n-ary function symbol F, $\mathsf{E}\, F(t_1, \dots, t_n) \rightarrow \mathsf{E}\, t_1 \wedge \dots \wedge \mathsf{E}\, t_n$.

The next theorem is our main result about the relative proof-theoretic strength of predicate logic and coordinate-free logic.

THEOREM 2.9. *Let U be a theory of negative, universally free predicate logic with identity. Then U is faithfully interpretable in a theory of coordinate-free logic.*

Proof. Because function symbols can be eliminated in a standard way, we may assume that U has a relational language. Let Σ be its signature.

We define a coordinate-free signature Ξ consisting of the following simple terms:

- a sequence of simple terms $\upsilon_1, \upsilon_2, \dots$,
- a simple term ind for representing the individuals,
- for each predicate letter P in Σ, a unique simple term p.

We define a translation $\tau = \langle \delta, F \rangle$ from Σ to Ξ as follows.

Define δ as

$$\mathsf{E}\,\mathsf{ind}(\upsilon_1).$$

Define $F(=)$ as $\upsilon_1 = \upsilon_2$, and for each other n-ary predicate symbol P, define $F(P)$ as

$$\mathsf{E}\, p\upsilon_1 \dots \upsilon_n.$$

It is easy to see that τ supports an interpretation of U in U^τ, i.e., in the theory consisting of all formulas A^τ, with A in U. That the interpretation is *faithful* is less trivial. We will prove it in Appendix A by making use of so-called *piecewise interpretations*. □

We now define converse translations, namely from coordinate-free logic to predicate logic.

Let again Σ be the signature of a relational language for (free) predicate logic. Let Ξ be the signature of a language for coordinate-free logic, and let D be a subcollection of its

[5] What we call here a *translation* is often called a *relative translation*, and what we call an *unrelativized translation* is often called a *translation*.

simple terms. A *translation* $\tau : \Xi \to \Sigma$-formulas is defined as the composition $\tau_2 \circ \tau_1$, where:

1. $\tau_1 : \Xi \to \Theta$ is a translation from Ξ to a signature Θ of predicate logic that preserves identity and that maps the application symbol $\cdot(\cdot)$ to the term $\mathsf{App}(v_1, v_2)$, and any d from D to a unique constant c_d.
2. $\tau_2 : \Theta \to \Sigma$ is just any translation from Θ to Σ.

We translate Ξ-formulas to Θ-formulas in an obvious way. We have, e.g., $(u_{j_1}(u_{j_2}) = u_{j_3})^{\tau_1} := \mathsf{App}(v_{j_1}, v_{j_2}) = v_{j_3}$.

The translation τ may support an interpretation of a theory U in a theory V if all the terms that occur free in a formula of U belong to the collection D.

The next theorem shows that coordinate-free theories of infinite signature can be faithfully interpreted in predicate logic.

THEOREM 2.10. *Let U be a theory of coordinate-free logic with an infinite number of simple terms in its language. Then U is faithfully and directly interpretable in a theory of negative, universally free predicate logic with identity.*

Proof. Let D be the collection of simple term that occur free in a formula of U. Let L be a language with no predicate symbols except equality, a function symbol App of arity 2, and a collection of constants c_d where d belongs to D. As variables of L, choose a sequence $v_1, v_2, \ldots$ of symbols disjoint from the constants.

Let $\tau = F$ be the translation given by:

$$\begin{aligned} F(=) &:= v_1 = v_2, \\ F(\cdot(\cdot)) &:= \mathsf{App}(v_1, v_2), \\ F(d) &:= c_d \text{ for all simple terms } d \text{ that occur free in a formula of } U. \end{aligned}$$

It is straightforward to verify that τ supports a faithful interpretation of U in U^τ. □

Expressive power. At this point, we may not yet make claims about the expressive power of coordinate-free logic, because we have not yet said anything about possible models for a theory. In §4, we will define a foundation for coordinate-free logic for which completeness holds. Then it follows from the previous interpretation theorems that the expressive power of coordinate-free logic *within this foundation* is similar to the expressive power of predicate logic *within* ZFC. More precisely, it then follows that validity in negative, universally free first-order predicate logic is effectively reducible to validity in coordinate-free logic, and vice versa if the language of coordinate-free logic has infinitely many simple terms.

§3. Defining relations.

At the beginning of the 20th century, Bertrand Russell developed a substitutional theory of classes and relations (Russell, 1973). The theory has one type of primitive formula denoted as

$$p \, \tfrac{x}{a}! \, q,$$

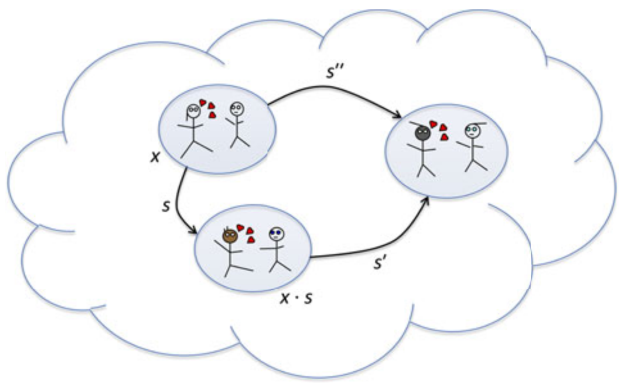
which is read as "q results from p by substituting x for a in all those places (if any) where a occurs in p" (Russell, 1973, p. 168). The theory seemed to offer a solution for Russell's paradox of classes—which was troubling the foundations of mathematics at the time—but the theory soon turned out to be inconsistent as well (Landini, 2003, pp. 271–278).

My goal here is more modest than Russell's with his substitutional theory. Russell's theory was intended as a foundational system for mathematics; here I only want to show

how relational structures can adequately be defined within the framework of coordinate-free logic.

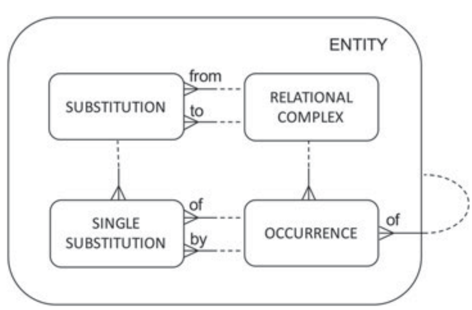
3.1. Depicting the structure of relations. Before developing a formal account of relations, let us first take a closer look at the structure of relations.

Kit Fine developed a *neutral* view on relations, which he called *antipositionalism* Fine (2000). The basic idea is that a relation is a network of relational complexes interrelated by substitutions:



In this picture, x is, for example, the complex of Adam's loving Eve, and $x \cdot s$ the complex obtained by substituting Romeo for Adam and Juliet for Eve.

In fact, substitutions do not work directly on objects, but on *occurrences* of objects. If we substitute Narcissus for the occurrence of Adam and also for the occurrence of Eve in Adam's loving Eve, then we get a complex with two occurrences of Narcissus.[6] The diagram below depicts antipositionalism with a refined substitution mechanism:



In this so-called *entity–relationship diagram*, the relationship between RELATIONAL COMPLEX and OCCURRENCE should be read as: each relational complex may contain one or more occurrences of entities and each occurrence belongs to exactly one relational complex. Other relationships should be read in a similar way. The diagram also contains

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6 In my view, not only expressions but also complexes may contain more than one occurrence of an entity. I consider occurrences of entities as basic because without them the interrelatedness of the complexes of certain relations could not be expressed (Leo, 2010a, pp. 168–169). My approach here is in line with Fine's claim that for a *general theory of constituent structure* we must posit an ontology of occurrences of entities, in addition to the entities themselves (Fine, 1989, pp. 235–236). In Fine (2000), however, he ignored the fact that substitution is properly done on occurrences for the sake of simplicity (Leo, 2008, p. 358).

subtypes, e.g., RELATIONAL COMPLEX and OCCURRENCE are subtypes of ENTITY. We do not assume that every instance of ENTITY belongs to one of the given subtypes.[7]

The antipositionalist view has been criticized (MacBride, 2007; Wieland, 2010; Orilia, 2011; Gaskin and Hill, 2012; MacBride, 2013), but there are good reasons to consider it as the superior view:

1. In the standard view on relations the arguments in a relation come in a certain order, but it does not make sense to say that Adam occurs first in Adam's loving Eve. The standard view lacks the resources to express facts in a neutral manner.
2. In the positionalist view on relations each relation comes with argument-places, e.g., the love relation comes with the argument-places *lover* and *beloved*. So, it offers a neutral account of relations. However, a weakness of the view is that it cannot handle strictly symmetric relations very well. For instance, for the adjacency relation it gives *two* complexes for the state of a's being adjacent to b (Fine, 2000, pp. 17–18).
3. Antipositionalism does not have these shortcomings, and, from a technical perspective, it essentially has no limitations compared to the other two views. In particular, building mathematical models for the views showed that antipositionalism has the same expressive power as positionalism (Leo, 2010b, pp. 181–183). Moreover, substitution—the central notion in antipositionalism—is a primitive kind of operation (or of a low logical type).

We proceed to give a formalization for the antipositionalist view on relations.

3.2. A characterization of complexes. Instead of directly defining a relational complex within coordinate-free logic, I first characterize a complex in a more independent way.

Let x be a relational complex. We introduce the following denotations:

$x \cdot s$ denotes the complex obtained by substitution s in x,
$s[\alpha]$ denotes the occurrence that s substitutes for α,
$\widehat{\alpha}$ denotes the entity of which α is an occurrence.

Then x has the following fundamental properties:[8]

RC-1. Identity substitution.

$$\exists! s\, (x \cdot s = x \wedge \forall \alpha\, (\mathsf{E}\, s[\alpha] \to s[\alpha] = \alpha)).$$

RC-2. Composition of substitutions.[9]

$$\mathsf{E}\, (x \cdot s) \cdot s' \to \exists! s''\, ((x \cdot s) \cdot s' = x \cdot s'' \wedge \forall \alpha\ s'[s[\alpha]] \simeq s''[\alpha]).$$

RC-3. Inverse substitution.

$$\mathsf{E}\, x \cdot s \to \exists! s'\, ((x \cdot s) \cdot s' = x \wedge \forall \alpha\, (\mathsf{E}\, s[\alpha] \to s'[s[\alpha]] = \alpha)).$$

RC-4. Definedness of occurrences.

$$\mathsf{E}\, x \cdot s \to \forall \alpha\, (\mathsf{E}\, s[\alpha] \to \mathsf{E}\, \widehat{\alpha}).$$

RC-5. Determinedness of substitutions.

$$\mathsf{E}\, x \cdot s \wedge \mathsf{E}\, x \cdot s' \wedge \forall \alpha\ \widehat{s[\alpha]} \simeq \widehat{s'[\alpha]}\ \to\ x \cdot s = x \cdot s'.$$

[7] For more details about entity–relationship modeling, see Barker (1989) and Thalheim (2000).
[8] The formulas are understood to be universally quantified over s, s', and s''.
[9] Recall that $\simeq$ denotes weak equality (see Definition 2.7).

Property RC-1 says that any complex has a unique identity substitution. Property RC-2 says that substitutions can be composed like functions, and that the resulting substitution is unique. Property RC-3 says that any substitution has a unique inverse. This implies that substitutions are injective. One could consider to weaken the property by adding injectivity of s as a condition. RC-4 says that any occurrence is an occurrence of an entity, and RC-5 says that the result of a substitution is uniquely determined by what entities are substituted for the occurrences in the original complex.

The conditions contain some redundancy: the uniqueness of the identity substitution (in RC-1) and the inverse substitution (in RC-3) follows from property RC-2. It is important, however, that each property can be considered in it own right.

3.3. A formal definition of complexes. Complexes can be defined in a variety of ways in coordinate-free logic. The key requirement is that the whole network of substitutions of a relation can be reconstructed. We present here a rather straightforward definition that satisfies the characterization given in the previous section.

We formally define $x \cdot s$, $s[\alpha]$, and $\widehat{\alpha}$ as follows:

$$\begin{aligned} x \cdot s &=_{\text{df}} xs, \\ s[\alpha] &=_{\text{df}} s\alpha, \\ \widehat{\alpha} &=_{\text{df}} \alpha\alpha. \end{aligned}$$

Note that if an entity x fulfills Properties RC-1 to RC-5, then not necessarily $x \cdot s$ fulfills them as well. For example, $x \cdot s$ may not have an identity substitution, and composition of substitutions starting in $x \cdot s$ may fail as well. But if x is a genuine relational complex, then $x \cdot s$ (if it exists) also has these properties. We take this into account in the formal definition of a complex:

DEFINITION 3.1. *We call x a* complex *if x and any $x \cdot s_0$ with $\mathsf{E}\, x \cdot s_0$ satisfy the Properties RC-1 to RC-5.*

We say that α is an occurrence in *a complex x if there is a substitution in x that is defined for α:*

$$\alpha \text{ in } x \;=_{\text{df}}\; \exists s\, (\mathsf{E}\, x \cdot s \wedge \mathsf{E}\, s[\alpha]).$$

We use $\forall\alpha$ in x ... as an abbreviation for $\forall\alpha\, (\alpha \text{ in } x \rightarrow \ldots)$, and $\exists\alpha$ in x ... as an abbreviation for $\exists\alpha\, (\alpha \text{ in } x \wedge \ldots)$.

Of course, defining a term as a complex in this way does not imply that its interpretations are ‘real’ relational complexes; it only says that its interpretations have certain structural properties. The aim is to talk in the new logic as directly as possible about relational complexes. In that context, one may ask whether relational complexes literally have substitutions as inputs. If one has reservations, then there are alternatives. One could, for example, formally define a relation as a collection of substitutions mapping complexes to complexes.

In the proof of the next lemma we only make use of Definition 3.1. The proof is independent of a particular formal definition and the results can be seen as properties of ‘real’ complexes.

LEMMA 3.2. *Let x be a complex. Then:*

1. *for any s, if $\mathsf{E}\, x \cdot s$, then $x \cdot s$ is a complex,*
2. *substitutions in x have the same domains, i.e., $\mathsf{E}\, x \cdot s \rightarrow \forall\alpha \text{ in } x\; \mathsf{E}\, s[\alpha]$,*

3. *substitutions are bijective, i.e.,*

$$\mathsf{E}\, x \cdot s \rightarrow \forall \alpha, \alpha' \,(s[\alpha] = s[\alpha'] \rightarrow \alpha = \alpha') \wedge \forall \beta \text{ in } x \cdot s\, \exists \alpha\; s[\alpha] = \beta,$$

4. *substitutions are extensional, i.e.,*

$$\mathsf{E}\, x \cdot s \wedge \mathsf{E}\, x \cdot s' \wedge \forall \alpha\; s[\alpha] \simeq s'[\alpha] \rightarrow s = s'.$$

Proof. To prove Property 1, we have to show that any entity $(x \cdot s) \cdot s'$ with existential import satisfies Properties RC-1 to RC-5. So, assume $\mathsf{E}\,(x \cdot s) \cdot s'$. By Property RC-2, there is an s'' such that $(x \cdot s) \cdot s' = x \cdot s''$. So, by the definition of a complex, $(x \cdot s) \cdot s'$ satisfies RC-1 to RC-5. Because this is the case for any s' with $\mathsf{E}\,(x \cdot s) \cdot s'$, it follows that $x \cdot s$ is a complex as well.

To prove Property 2, assume $\mathsf{E}\, x \cdot s$. By RC-3, s has an inverse substitution s'. By RC-2, there is a unique composition of s and s', which has the same domain as s. By RC-1 this composition is the unique identity substitution of x. So it follows that s has the same domain as this unique identity substitution.

To prove Property 3, assume $\mathsf{E}\, x \cdot s$. Injectivity of s follows directly from Property RC-3. To prove surjectivity, let β be an occurrence in $x \cdot s$. By RC-3, s has an inverse s', which is injective as well. So, by Property 2 and RC-2, $s[s'[\beta]] = \beta$.

To prove Property 4, assume $\mathsf{E}\, x \cdot s \wedge \mathsf{E}\, x \cdot s' \wedge \forall \alpha\; s[\alpha] \simeq s'[\alpha]$. Let id_x be the identity substitution, which exists by RC-1. By RC-2 and by Property 2, there is exactly one s_1 such that $x \cdot s_1 = (x \cdot \mathsf{id}_x) \cdot s \wedge \forall \alpha\; s[\alpha] \simeq s_1[\alpha]$. Obviously, s itself could be chosen for s_1. Because, by RC-5, $x \cdot s = x \cdot s'$, we see that s' could also be chosen for s_1. So, by the uniqueness of s_1, it follows that $s = s'$. □

3.4. Theories with complexes. Let us take a look at some applications of our formalization of complexes.

EXAMPLE 3.3. *For the love relation, we may define a theory with simple terms:*

$$\mathsf{a\_loving\_b},\ \mathsf{oc_a},\ \mathsf{oc_b},\ \mathsf{adam},\ \mathsf{eve}, \ldots.$$

As axioms, we may choose the axioms of coordinate-free logic plus:

$$\begin{aligned}
&\mathsf{E}\,\mathsf{a_loving_b} \rightarrow \mathsf{a_loving_b} \text{ is a complex},\\
&\mathsf{E}\,\mathsf{a_loving_b} \rightarrow \mathsf{E}\,\mathsf{oc_a} \wedge \mathsf{E}\,\mathsf{oc_b} \wedge \mathsf{oc_a} \neq \mathsf{oc_b},\\
&\mathsf{E}\,\mathsf{a_loving_b} \rightarrow \forall x (x \text{ in } \mathsf{a_loving_b} \leftrightarrow x = \mathsf{oc_a} \vee x = \mathsf{oc_b}).
\end{aligned}$$

The term $\mathsf{a\_loving\_b}$ *represents an arbitrary love state with* oc_a *an occurrence of an entity in the role of lover and* oc_b *an occurrence of an entity in the role of beloved. The axioms are as stated; they should not be prefixed by universal quantifications. The terms* $\mathsf{a\_loving\_b}$*,* $\mathsf{oc_a}$*, and* $\mathsf{oc_b}$ *function like constants.*

To express that Adam loves Eve, we could write:

$$\exists s\, (\mathsf{E}\,\mathsf{a\_loving\_b} \cdot s \wedge \widehat{s[\mathsf{oc_a}]} = \mathsf{adam} \wedge \widehat{s[\mathsf{oc_b}]} = \mathsf{eve}).$$

Note that in the example it is not excluded that nobody loves anybody. Because simple terms do not necessarily have existential import, this case is handled well by the axioms. It is also not excluded that we have only people who love themselves: $\mathsf{oc_a}$ and $\mathsf{oc_b}$ may be occurrences of the same object.

In predicate logic, we define a symmetric binary relation R as a relation for which $\forall x, y\, (R(x, y) \leftrightarrow R(y, x))$ holds. But this does not say whether or not the relation is *strictly symmetric*, i.e., whether the complexes $R(x, y)$ and $R(y, x)$ are the same. With our approach, we may define strict symmetry in a natural way:

EXAMPLE 3.4. *To define a theory for the adjacency relation, we first introduce as simple terms:*

$$\mathsf{a\_adj\_b},\ \mathsf{oc}_\mathsf{a},\ \mathsf{oc}_\mathsf{b},\ \ldots.$$

In a straightforward manner, we can express that $\mathsf{a\_adj\_b}$ *is a complex with exactly two occurrences* oc_a *and* oc_b. *Furthermore, the next formula expresses that* $\mathsf{a\_adj\_b}$ *is strictly symmetric:*

$$\exists s\,(\mathsf{a\_adj\_b} \cdot s = \mathsf{a\_adj\_b} \wedge s[\mathsf{oc}_\mathsf{a}] = \mathsf{oc}_\mathsf{b} \wedge s[\mathsf{oc}_\mathsf{b}] = \mathsf{oc}_\mathsf{a}).$$

Then, by properties RC-2 and RC-3, it follows that all complexes of the adjacency relation are strictly symmetric.

Formulations in coordinate-free logic may be more cumbersome than in predicate logic. However, we may introduce notations to make things more convenient. For example, we may use

$$x(\alpha_1 \twoheadrightarrow a_1, \ldots, \alpha_n \twoheadrightarrow a_n)$$

to denote $x \cdot s$, where x is a complex with n occurrences $\alpha_1, \ldots, \alpha_n$, and s such that $\widehat{s[\alpha_1]} = a_1 \wedge \cdots \wedge \widehat{s[\alpha_n]} = a_n$.

To justify this notation, note that $x \cdot s$ is uniquely defined by property RC-5.

With our account of complexes, we can also define multisets and sets in a natural way. A *multiset* can be defined as a complex x such that for each s, if $x \cdot s$ exists, then $x \cdot s = x$, and for each permutation s of the occurrences of objects in x, there is a functionally equivalent s' with $x \cdot s' = x$. A *set* can be defined as a multiset whose objects have only one occurrence.[10]

EXAMPLE 3.5. *A theory for natural numbers with ordering can be defined in terms of complexes as follows.*
Simple terms:

$$\mathsf{S},\ \mathsf{ord},\ \alpha_\mathsf{ord},\ \alpha'_\mathsf{ord},\ \mathsf{N},\ 0,\ \ldots.$$

Definitions:

$$\begin{aligned} x < y &: \ \mathsf{E}\,\mathsf{ord}(\alpha_\mathsf{ord} \twoheadrightarrow x, \alpha'_\mathsf{ord} \twoheadrightarrow y), \\ x \in \mathsf{N} &: \ \exists\alpha\,(x = \widehat{\alpha} \wedge \alpha \text{ in } \mathsf{N}). \end{aligned}$$

Axioms:

$$\begin{aligned} &\mathsf{N} \text{ is a set} \wedge \exists x\ x \in \mathsf{N}, \\ &\forall y \in \mathsf{N}\,(y \neq 0 \rightarrow \exists x \in \mathsf{N}\ \mathsf{S}x = y), \\ &\forall x, y \in \mathsf{N}\,(x < \mathsf{S}y \leftrightarrow x < y \vee x = y), \\ &\forall x \in \mathsf{N}\ x \not< 0, \\ &\forall x, y \in \mathsf{N}\,(x < y \vee x = y \vee y < x), \\ &\forall x, y \in \mathsf{N}\,(x < y \rightarrow y \not< x), \\ &\forall x, y, z \in \mathsf{N}\,(x < y \wedge y < z \rightarrow x < z). \end{aligned}$$

The axioms of the example look like formulas of ordinary predicate logic, but there is a fundamental difference. Here the order disappears when the formulas are written in their basic form. With the new logic, the essence of the relations may be preserved more faithfully.

---

[10] There are alternative definitions for sets and multisets possible. It may be preferable to define them as equivalence classes in the foundational theory developed in §4. Another thing to be noticed is that conceiving of sets as unimultisets only works when operations like union are defined in a specific way.

I do not claim that we should use complexes for everything. Consider, for example, the addition operation. We can define it in coordinate-free logic in different ways: (i) as a curried function that takes a number as input and returns a function that takes another number as input and then returns the sum of the two, (ii) as a function that takes unordered pairs of numbers as input, (iii) as a function that takes ordered pairs of numbers as input, or (iv) as a network of ternary complexes, which is the same as the network for subtraction. Which implementation is the most appropriate one may depend on the context.

Note that we did not explicitly define *relations* in the example theories. Nominalists about universals may feel no need to do so or they may choose to define within the theories a relation as a set of complexes obtained by substitutions from a given complex. For realists about universals this may not be satisfactory. They may prefer to include a relation as a part or constituent of relational complexes.[11] This approach requires only minor changes in the definition of a complex. We might, for example, define the result of a complex x applied to itself as the relation R in the complex, and the result of R applied to x as x.

If one takes a relation to be a part of a complex, then *variable polyadicity* can be accounted for in a natural way. For example, the relation of *being surrounded by* may have complexes with different numbers of occurrences. Such relations correspond to a disconnected collection of networks of complexes interrelated by substitutions. What the complexes have in common is simply that they have the same relation as a (universal) part.

REMARK 3.6. *In the proof of Theorem 2.9 we translated predicate logic into coordinate-free logic. A perhaps more natural interpretation of predicates can be given in terms of complexes:*

$$\textit{Interpret}\ \ P(x_1, \ldots, x_n)\ \ \textit{as}\ \ \mathsf{E}\, p(\alpha_1 \twoheadrightarrow x_1, \ldots, \alpha_n \twoheadrightarrow x_n).$$

By interpreting predicates in this way, predicate logic is not in conflict with the antipositionalist view. We could even regard the interpretation as a justification for its use. However, I think we still need another logic—like coordinate-free logic—to make the underlying structure of relations visible.

§4. A foundation for the new logic.

Since set theory makes use of a predicate, namely the ∈-predicate, it does not correspond in a natural way to the framework of coordinate-free logic. Our goal is to develop a better fitting mathematical foundation. We do not start entirely from scratch, but we use Zermelo–Fraenkel Set Theory and Jan Kuper's axiomatic theory of partial functions (Kuper, 1993) as sources of inspiration.

A desideratum for the new foundation is that it provides any (syntactically) consistent theory with a model consisting of entities in the new foundation. This has far-reaching consequences:

1. We cannot have an axiom of extensionality, because in consistent theories different entities may have the same input–output functionality.
2. We cannot have an axiom of foundation, because in consistent theories entities may have themselves as inputs or outputs.

We call the new foundational theory CFF, *Coordinate-free Foundation*.

[11] In (Fine, 2000, p. 30) an object is said to be a *constituent of* a complex only if it is open to substitution, but according to Fine no well-defined meaning can be given to a substitution of one neutral relation for another (for example, to the substitution of the relation of vertical placement for the love relation in the complex of a's loving b). Therefore, he does not regard a neutral relation in a complex as one of its constituents.

4.1. Axiomatization. We give a formal definition of CFF, followed by an informal explanation of the axioms.

First of all, we assume that we have a countably infinite number of simple terms:

$$x_1, x_2, x_3, \ldots.$$

DEFINITION 4.1.

$$\begin{aligned} x \sqsubseteq y &=_{\text{df}} \mathsf{E}\, yx, \\ x \sqsubseteq y &=_{\text{df}} \forall z \sqsubseteq x\ xz = yz, \\ x \text{ inj} &=_{\text{df}} \forall y, z\,(xy = xz \to y = z), \\ x \text{ in field of } y &=_{\text{df}} x \sqsubseteq y \lor \exists z\, x = yz. \end{aligned}$$

So, x is in the field of y iff x is an input or an output of y.

We use $\forall x \sqsubseteq y \ldots$ as an abbreviation for $\forall x\,(x \sqsubseteq y \to \ldots)$, and $\exists x \sqsubseteq y \ldots$ as an abbreviation for $\exists x\,(x \sqsubseteq y \land \ldots)$.

CFF consists of the following axiom schema and axioms:[12]

1. Axiom Schema of Selection.
For each formula φ, without z free, and with zx free for y,

$$\exists z\,(\forall x\,(x \sqsubseteq z \leftrightarrow x \text{ in field of } a \land \exists y\,\varphi(y)) \land \forall x \sqsubseteq z\ \varphi(zx)).$$

2. Axiom of Union.

$$\exists z\,\forall y \sqsubseteq a\,\forall x \sqsubseteq y\ \ x \sqsubseteq z.$$

3. Axiom of Power.[13]

$$\exists z\,\forall x\,(x \sqsubseteq a \to \exists! y \sqsubseteq z\ y \equiv x).$$

4. Axiom of Infinity.

$$\exists z\,(z \text{ inj} \land \exists x \sqsubseteq z\,\forall y\ x \neq zy \land \forall x \sqsubseteq z\ zx \sqsubseteq z).$$

5. Axiom of Universality.

$$\exists z\,(z \text{ inj} \land \forall x \sqsubseteq a\,(x \sqsubseteq z \land ax \overset{z}{\underset{\sim}{\to}} zx) \land \forall x \sqsubseteq z\,(x \not\sqsubseteq a \to zx = x)),$$

where $f \overset{z}{\underset{\sim}{\to}} g$ is defined (for z inj) as

$$\forall x \sqsubseteq f\ z(fx) = g(zx) \land \forall y \sqsubseteq g\,\exists x \sqsubseteq f\ zx = y.$$

Let us take a closer look at the axioms.

The Axiom Schema of Selection

For any formula φ and any entity a there is a z with inputs restricted by a and φ, and such that for any input x, $\varphi(zx)$:

$$\exists z\,(\forall x\,(x \sqsubseteq z \leftrightarrow x \text{ in field of } a \land \exists y\,\varphi(y)) \land \forall x \sqsubseteq z\ \varphi(zx)).$$

[12] The free simple terms are understood to be universally quantified.

[13] Recall that $\equiv$ denotes functional equivalence (see Definition 2.7).

The axiom schema combines aspects of the axiom schema of separation, the axiom schema of replacement and the axiom of choice in set theory. Because in coordinate-free logic any entity is like a function, we do not need in our foundation separate counterparts for these set-theoretical principles. For proofs, however, it might be interesting to distinguish explicitly between instances of the schema where for each x there is at most one y such that φ and instances where a choice for y has to be made.

By applying Selection to the formula $\varphi \wedge y = ax$, with y, z not free in φ, we get the following axiom schema of separation:

$$\exists z\, (\forall x\, (x \sqsubseteq z \leftrightarrow x \sqsubseteq a \wedge \varphi) \wedge z \sqsubseteq a).$$

The Axiom of Union

For any a there is a z containing as inputs all inputs of inputs of a:

$$\exists z\, \forall y \sqsubseteq a\, \forall x \sqsubseteq y\;\; x \sqsubseteq z.$$

The axiom resembles its namesake in set theory. Obviously, z is not unique. By using the Axiom Schema of Selection, we can get an entity whose inputs are exactly the inputs of the inputs of a, but as we will see later, in CFF nothing can be defined in a unique way.

The Axiom of Power

For any a there is a z containing as inputs unique representatives of subentities of a:

$$\exists z\, \forall x\, (x \sqsubseteq a \rightarrow \exists!\, y \sqsubseteq z\;\; y \equiv x).$$

The formulation of the axiom is more complicated than the set-theoretic axiom of power set. But suppose we would choose as axiom $\exists z\, \forall x\, (x \sqsubseteq a \rightarrow x \sqsubseteq z)$. Then this would imply that all entities with the same input–output functionality as a would be inputs of some entity z. This is a restriction I do not want to impose on the foundation. On the contrary, it is excluded by the Axiom of Universality.

The Axiom of Infinity

There is an entity z with an infinite number of inputs:

$$\exists z\, (z \text{ inj} \wedge \exists x \sqsubseteq z\, \forall y\;\, x \neq zy \wedge \forall x \sqsubseteq z\;\, zx \sqsubseteq z).$$

We need infinity in our foundation to prove that any consistent theory has a model. The actual formulation says that there is an entity with the following properties: (1) it gives different outputs for different inputs, (2) it has an input that is not an output, and (3) any output is an input.

The Axiom of Universality

Any indexed family of deterministic graphs depicts the input–output functionality of a collection of entities:[14]

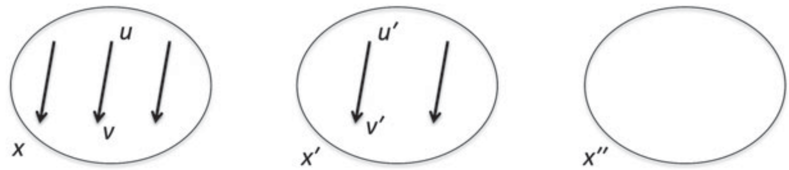
$$\exists z\, (z \text{ inj} \wedge \forall x \sqsubseteq a\, (x \sqsubseteq z \wedge ax \xrightarrow[\sim]{z} zx) \wedge \forall x \sqsubseteq z\, (x \not\sqsubseteq a \rightarrow zx = x)),$$

where $ax \xrightarrow[\sim]{z} zx$ means that z isomorphically maps ax (seen as a deterministic graph) onto zx.

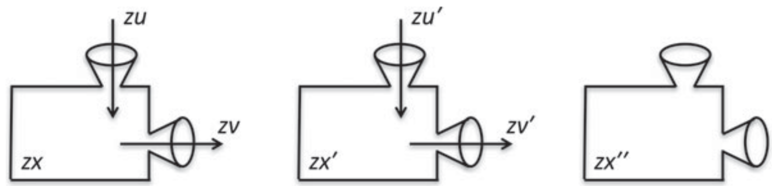
[14] When talking about a *collection* within the context of CFF, I will always mean the totality of inputs of some entity.

The axiom makes it possible for entities to have themselves as inputs and outputs. The idea is as follows.

Start with an indexed family of deterministic graphs:



Then there is an entity z with its outputs isomorphic to the graphs:



Furthermore, z is injective and z maps an input x to itself if x is not an index of one of the graphs.

The reason for demanding that z maps x to itself if x is not an index of one of the graphs is to let inputs and outputs of the arrows stand for themselves if they are not indexes.

It is easy to see that the axiom gives us entities that have themselves as input or outputs. Assume, for example, that in the graph above $u = x$. Then zx has itself as input. If we look at the formulation of the axiom, this only requires that we start with an entity a with $x \sqsubseteq ax$.

Another point worth mentioning is that by the injectivity of z, for each entity there exists a plenitude of functionally equivalent entities. Each entity has even more functionally equivalent variants than the number of inputs of any entity, as shown in the next example.

EXAMPLE 4.2. *In CFF, there is for any p, q an entity z with more inputs than q and with each input functionally equivalent to p.*

To see this, let r be an entity with more inputs than q. By using the Axiom Schema of Selection, we get an entity a with the same number of inputs as r and such that for each $x \sqsubseteq a$, x is not in the field of p, and $ax = p$. Then the Axiom of Universality gives in one step an injective entity z with the property that for any $x \sqsubseteq a$, $zx \equiv p$.

Because we allow the number of simple terms to be of any cardinality, and because we may define a theory with for any pair of distinct simple terms x, y as axiom $x \equiv y \wedge x \neq y$, our desideratum that any consistent theory has a model requires that the foundation provides for at least some entities proper-class many functionally equivalent variants.

REMARK 4.3. *The Axiom of Union is in fact redundant because it follows immediately from the Axiom of Universality. The reason for presenting it as a separate axiom is that I consider it as an unintended consequence of the formulation of the Axiom of Universality. If we would drop Universality, then we would probably not want to give up Union as well.*

It is generally accepted that every finitistic fact can be formalized and proved in ZFC (see, e.g., (Kunen, 2011, p. 10)). We can give similar arguments for the thesis that CFF validates finitistic reasoning as well. A more indirect argument can be given by using the fact that ZFC is interpretable in CFF, as we will show in §4.3.

4.2. Completeness. We have the following completeness theorem.

THEOREM 4.4. *Assume* CFF. *Then* $\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$ *if the number of simple terms in the language of* Γ *is infinite.*

Proof. We only sketch the proof since it is similar to completeness proofs for predicate logic in which a Henkin model is constructed (see, e.g., (Oliver and Smiley, 2013, pp. 201–205)). In step (iii), however, we have something new: there we transform lifeless entities into entities with input–output functionality.

Suppose that the language of the formulas in Γ has a countably infinite number of simple terms, and suppose $\Gamma \nvdash \varphi$. We will construct a structure with a valuation of the simple terms that satisfies all formulas of Γ, but not φ.

(i) *Define Henkin terms*:
Extend the language of our logic with simple terms $h_1, h_2, h_3, \ldots$, which we shall call *Henkin terms*.

(ii) *Define a maximal extension* Δ *of* Γ *with* $\Delta \nvdash \varphi$:
Let $A_1, A_2, A_3, \ldots$ be an enumeration of the resulting formulas after extending the language with the Henkin terms. Define a sequence Γ_n as follows:

$$\begin{aligned}\Gamma_1 &= \Gamma\\ \Gamma_{n+1} &= \begin{cases}\Gamma_n & \text{if } \Gamma_n, A_n \vdash \varphi;\\ \Gamma_n, A_n & \text{if } \Gamma_n, A_n \nvdash \varphi \text{ and } A_n \text{ is not } \exists x\, A(x);\\ \Gamma_n, A_n, A(h), \mathsf{E}\, h & \text{if } \Gamma_n, A_n \nvdash \varphi \text{ and } A_n \text{ is } \exists x\, A(x),\end{cases}\end{aligned}$$

where h is the first Henkin term which does not occur in Γ_n or in A_n.

Define Δ as the union of the Γ_n. Then $\Delta \nvdash \varphi$ and Δ is maximal in this respect. (In the proof of $\Gamma \nvdash \varphi$ in the extended language, it is used that the number of simple terms in the original language is infinite.)

(iii) *Define a coordinate-free structure* E *based on* Δ:
Call a Henkin term h_n *initial* if $h_n = h_n$ belongs to Δ, but for all $j < n$, $h_j = h_n$ is not in Δ.

For any term a, let a^* be the initial Henkin term with $a^* = a$ in Δ, if it exists.

Let App be an entity with $\mathsf{App}(a^*)$ always defined, with $\mathsf{App}(a^*)(x)$ defined only if x is some b^*, and with

$$\mathsf{App}(a^*)(b^*) = \begin{cases} a(b)^* & \text{if } a(b)^* \text{ exists;}^{15}\\ \text{undefined} & \text{otherwise.}\end{cases}$$

By the Axiom of Universality, we have an injective mapping $a^* \mapsto \overline{a^*}$ such that

$$\overline{a^*}(\overline{b^*}) = \begin{cases} \overline{a(b)^*} & \text{if } a(b)^* \text{ exists;}\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Let E be the coordinate-free structure consisting of all entities $\overline{a^*}$.

[15] To be clear, $a(b)^*$ is the result of * applied to the *term* $a(b)$ itself.

(iv) *For some valuation, E satisfies Γ, but not φ*:
Let val be a valuation of the simple terms such that

$$\mathsf{val}(a) \quad = \quad \begin{cases} \overline{a^*} & \text{if } a^* \text{ exists;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then the structure E with this valuation satisfies Γ, but it does not satisfy φ.

In case the language of Γ contains an uncountable number of simple terms, then the proof is slightly more complicated. If we have κ simple terms, then in step (i) we add κ Henkin terms, and in step (ii) we well-order the resulting formulas.[16] □

REMARK 4.5. *In part (iii) of the proof we used the Axiom of Universality. The injectivity of the mapping $a^* \mapsto \overline{a^*}$ is of importance in part (iv): for E with the given valuation to satisfy Γ, it is necessary that the formula $a = b$ belongs to Δ iff $\overline{a^*}$ and $\overline{b^*}$ are identical.*

REMARK 4.6. *As we will show in §4.4, nothing can be defined in CFF uniquely. It may seem that we ignored this in the proof. However, the definitions given are unique modulo functional equivalence, which is enough in this case.*

4.3. Proof-theoretic strength. Dana Scott showed that dropping the axiom of extensionality in set theory can decrease its essential power (Scott 1961). CFF does not have an axiom of extensionality, as a consequence of the Axiom of Universality. Nevertheless, as we will show in this section, the proof-theoretic strength of CFF is the same as that of ZFC.[17] Furthermore, we will show that the Axiom of Universality does not contribute to the proof-theoretic strength of CFF—despite its fundamental role in the proof of the completeness theorem 4.4.

First, we will prove that CFF is interpretable in ZFA, Peter Aczel's non-well-founded set theory (Aczel 1988).[18] ZFA consists of the axioms of ZFC minus the Axiom of Foundation plus the Anti-Foundation Axiom (AFA). Because ZFA is interpretable in ZFC (Aczel, 1988, pp. 36–37), it follows that CFF is interpretable in ZFC as well.

Next, we will prove that ZFC minus Foundation in interpretable in CFF minus Universality. Then, because ZFC is interpretable in ZFC minus Foundation (Fraenkel et al., 1973, pp. 98–102), it follows that ZFC is also interpretable in CFF minus Universality, and because CFF is interpretable in ZFC, CFF is also interpretable in CFF minus Universality.

It should be pointed out that in the proofs only elementary finitistic reasoning will be used.

THEOREM 4.7. CFF *is interpretable in* ZFA.

Proof (*Sketch*). We define a translation τ from the language of CFF to the language of set theory by the following mapping F:[19]

$$\begin{aligned} F(=) \quad &:= \quad v_1 = v_2, \\ F(\cdot(\cdot)) \quad &:= \quad \forall z_1\,(\langle v_2, z_1\rangle \in v_1 \leftrightarrow v_3 = z_1). \end{aligned}$$

16 In this paper, we did not completely develop the necessary machinery; we would also have to prove that in CFF any collection of entities can be well-ordered.

17 We say that two theories U and V have the *same proof-theoretic strength* if they are interpretable in each other.

18 In fact, the interpretation is direct, but in this section we will ignore such refinements.

19 We use here the naming convention for variables and simple terms given in §2.4.

So, in sloppy notation, τ maps $x = y$ to itself, and it maps $xy = z$ to a formula provably equivalent with

$$\forall z' \, (\langle y, z' \rangle \in x \leftrightarrow z = z').$$

The idea behind this translation is that we think of each set x as mapping y to z iff x contains the ordered pair $\langle y, z \rangle$ and no other ordered pair of the form $\langle y, z' \rangle$. Obviously, in this way, for each set there is a plenitude of other sets with the same input–output functionality.

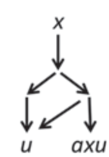
To prove that for all CFF-sentences A, CFF $\vdash A \Rightarrow$ ZFA $\vdash A^\tau$, it is sufficient to prove this implication for the axioms of coordinate-free logic and for the axioms of CFF. This is straightforward, except for the Axiom Schema of Selection and the Axiom of Universality.

For dealing with the Axiom Schema of Selection, we need in ZFA the following generalization of the axiom of choice:

> Let φ be a formula such that for each $x \in I$ there is an y such that $\varphi(x, y)$. Then there exists a function f on I such that, for each $x \in I$, $\varphi(x, f(x))$.

The proof is given in Appendix B. Using this result, the translation of the Axiom Schema of Selection follows easily in ZFA.

To deal with the Axiom of Universality, let a be an arbitrary set in ZFA, and let ax denote y with $\forall y' \, (\langle x, y' \rangle \in a \leftrightarrow y = y')$. Define a labeled graph $\mathcal{A}$ that contains for each ax a node x, and for each axu an edge from x to a subgraph representing the ordered pair $\langle u, axu \rangle$:



Note that this representation corresponds with the standard definition of an ordered pair $\langle u, v \rangle$ as the set $\{\{u\}, \{u, v\}\}$.

We label each node x for which ax does not exist by itself, and we label each other node with a unique singleton set.

Now by the labelled anti-foundation axiom (Aczel, 1988, p. 10), we get for $\mathcal{A}$ a decoration z that fulfills the translation of CFF's Axiom of Universality. □

Sets in ZFC are much less refined than entities in CFF. For an interpretation of ZFC in CFF we need a suitable equivalence relation in CFF. What might come to mind first is functional equivalence, but this is too refined. What we need is the notion of *bisimulation*.

DEFINITION 4.8. *Two pointed graphs Ga and Hb are called* bisimilar *if there is a relation R between the nodes of G and H such that $R(a, b)$ and for all x, y,*

$$R(x, y) \Rightarrow \forall u \text{ with } x \to_G u \; \exists v \text{ with } y \to_H v \;\; R(u, v) \text{ and}$$
$$\forall v \text{ with } y \to_H v \; \exists u \text{ with } x \to_G u \;\; R(u, v).$$

Graphs are normally defined in terms of sets, but also in coordinate-free logic we may define them and related notions in a straightforward way.

THEOREM 4.9. ZFC *minus the Axiom of Foundation is interpretable in* CFF *minus the Axiom of Universality.*

Proof (*Sketch*). The key idea for the interpretation is to treat entities as equal if their input pictures are bisimilar, where an *input picture* of an entity a is defined as a pointed graph Ga with the following properties:

1. for each node x in G, there is a path from a to x,
2. for each node x, y in G, there is an edge (x, y) iff y is an input of x.

It is not difficult to prove in CFF minus Universality that any entity has an input picture and that its input pictures are all bisimilar.

Let a, b be entities.

$$\text{Define } a \doteq b \;\text{ as }\; a \text{ and } b \text{ have bisimilar input pictures.}$$

$$\text{Define } a \mathrel{\dot{\sqsubseteq}} b \;\text{ as }\; \exists x\,(a \doteq x \wedge x \sqsubseteq b).$$

Now we define a translation τ from the formulas of ZFC to the formulas of CFF by replacing the symbol $=$ by $\doteq$, and the symbol $\in$ by $\dot{\sqsubseteq}$. More formally, we define τ by the following mapping F:

$$\begin{aligned} F(=) &:= v_1 \doteq v_2, \\ F(\in) &:= v_1 \mathrel{\dot{\sqsubseteq}} v_2. \end{aligned}$$

We have to prove that for all ZFC-sentences A, ZFC minus Foundation $\vdash A \Rightarrow$ CFF minus Universality $\vdash A^\tau$. It suffices to show that the translation of the logical axioms and the translation of the axioms of ZFC minus Foundation are provable in CFF minus Universality.

Let us consider the logical axioms first. Because by the Axiom of Infinity in ZFC we have at least one set, we may assume that the underlying logic is a free logic. As logical axioms for ZFC we may take the axioms 1 to 7 of coordinate-free logic given in §2.3 plus the following axiom 8′:

$$t \in t' \rightarrow \mathsf{E}\,t \wedge \mathsf{E}\,t'.$$

For the translation, we only need to be careful about the axioms involving identity, which are the axioms 4 to 7, and axiom 8′.

The translation of axiom 4 is

$$\forall x\, \varphi(x)^\tau \rightarrow (\exists x\; x \doteq t \rightarrow \varphi(t)^\tau),$$

where t is free for x in φ^τ. The provability of this translation follows from the observation that t has existential import if it has an input picture.

The translation of axiom 5 is

$$\forall x\; x \doteq x.$$

As remarked above, in CFF minus Universality any entity has an input picture. It is obvious that an input picture is bisimilar to itself, and thus the translation of axiom 5 is provable in CFF minus Foundation.

The translation of axiom 6 is

$$(\exists x\; x \doteq t \vee \exists x\; x \doteq t' \rightarrow t \doteq t') \rightarrow (\varphi^\tau \leftrightarrow \varphi'^\tau),$$

where φ^τ and φ'^τ are the translations of the formulas φ and φ', and where φ' is obtained from φ by zero or more substitutions of t' for t where both t and t' occur free. The provability is reducible to showing that $\doteq$ is antisymmetric and transitive, and to checking the formulas:

$$a \doteq b \wedge a \mathrel{\dot{\sqsubseteq}} c \rightarrow b \mathrel{\dot{\sqsubseteq}} c,$$

$$a \doteq b \wedge c \mathrel{\dot{\sqsubseteq}} a \rightarrow c \mathrel{\dot{\sqsubseteq}} b.$$

This follows immediately from the definition of $\doteq$ and $\dot{\sqsubseteq}$.

The translation of axiom 7 is

$$t \doteq t \rightarrow \exists x\, x \doteq t.$$

The provability of this translation is a direct consequence of the fact that a term can only have an input picture if the term has existential import.

The translation of axiom 8′ is

$$t \dot{\sqsubseteq} t' \rightarrow \exists x\, x \doteq t \wedge \exists x\, x \doteq t'.$$

Suppose $t \dot{\sqsubseteq} t'$. Then t has an input picture and t' has an input. So, both have existential import. Thus, the translation of axiom 8′ is provable.

We will consider each of the axioms of ZFC minus Foundation separately. The formulation of the axioms is close to the formulations given in Kunen (2011) and Aczel (1988).

(i) The translation of the Axiom of Extensionality is

$$\forall x\, (x \dot{\sqsubseteq} a \leftrightarrow x \dot{\sqsubseteq} b) \rightarrow a \doteq b.$$

By the definition of bisimilarity, we have

$$\forall x \sqsubseteq a\, \exists y \sqsubseteq b\; x \doteq y \wedge \forall y \sqsubseteq b\, \exists x \sqsubseteq a\; x \doteq y \;\rightarrow\; a \doteq b.$$

Furthermore, by the definition of $\dot{\sqsubseteq}$, we have

$$\forall x\, (x \dot{\sqsubseteq} a \rightarrow x \dot{\sqsubseteq} b) \rightarrow \forall x \sqsubseteq a\, \exists y \sqsubseteq b\; x \doteq y,$$
$$\forall x\, (x \dot{\sqsubseteq} b \rightarrow x \dot{\sqsubseteq} a) \rightarrow \forall y \sqsubseteq b\, \exists x \sqsubseteq a\; x \doteq y.$$

From these facts the translation of Extensionality follows immediately.

(ii) The translation of the Axiom of Comprehension is

$$\exists z\, \forall x\, (x \dot{\sqsubseteq} z \leftrightarrow x \dot{\sqsubseteq} a \wedge \varphi^{\tau}(x))$$

with z not free in φ. Let ψ be the formula $x \sqsubseteq a \wedge \varphi^{\tau}(x)$. Then, by the Axiom Schema of Selection, there is a z such that $\forall x\, (x \sqsubseteq z \leftrightarrow \psi(x))$. Because $x \doteq x' \rightarrow (\varphi^{\tau}(x) \leftrightarrow \varphi^{\tau}(x'))$, we also have $x \dot{\sqsubseteq} z \leftrightarrow x \dot{\sqsubseteq} a \wedge \varphi^{\tau}(x)$.

(ii) The translation of the Axiom of Pairing is

$$\exists z\, (a \dot{\sqsubseteq} z \wedge b \dot{\sqsubseteq} z).$$

By the Axiom of Infinity in CFF, there are u, v, w such that $u \neq v \wedge u \sqsubseteq w \wedge v \sqsubseteq w$. By the Axiom Schema of Selection, there is an x such that $xu = a \wedge xv = b$. By again applying this axiom schema, we get a z such that $a \sqsubseteq z \wedge b \sqsubseteq z$.

(iii) The translation of the Axiom of Union is

$$\exists z\, \forall y \dot{\sqsubseteq} a\, \forall x \dot{\sqsubseteq} y\; x \dot{\sqsubseteq} z.$$

For a given a, let z be as in the Axiom of Union in CFF. Suppose $y \dot{\sqsubseteq} a \wedge x \dot{\sqsubseteq} y$. Then there is an $y' \sqsubseteq a$ and $x' \sqsubseteq y$ such that $y \doteq y' \wedge y' \sqsubseteq a \wedge x \doteq x' \wedge x' \sqsubseteq y'$. So, by the Axiom of Union in CFF, $y' \sqsubseteq z$, and because $x \doteq x'$, $u \dot{\sqsubseteq} z$.

(iv) The translation of the Axiom of Power Set is

$$\exists z\, \forall x\, (\forall u\, (u \dot{\sqsubseteq} x \rightarrow u \dot{\sqsubseteq} a) \rightarrow x \dot{\sqsubseteq} z).$$

For a given a, let z be as in the Axiom of Power. Suppose that x is such that $\forall u\,(u \mathrel{\dot{⋿}} x \rightarrow u \mathrel{\dot{⋿}} a)$. Then $\forall u ⋿ x\, \exists u' ⋿ a\; u \doteq u'$. By the Axiom Schema of Selection, there is an x' such that $x \doteq x'$ and $\forall u ⋿ x'\; u ⋿ a$. So, by the Axiom of Power, $\exists y ⋿ z\; y \equiv x'$. Because $y \equiv x' \rightarrow y \doteq x'$, it follows that $x \mathrel{\dot{⋿}} z$.

(v) The translation of the Axiom Schema of Replacement is

$$\forall x \mathrel{\dot{⋿}} a\, \exists y\, \forall y'\, (\varphi^{\tau}(x, y') \leftrightarrow y \doteq y') \rightarrow \exists z\, \forall x \mathrel{\dot{⋿}} a\, \exists y \mathrel{\dot{⋿}} z\; \varphi^{\tau}(x, y)$$

with z not free in φ.

By the Axiom Schema of Selection, we get

$$\forall x ⋿ a\, \exists y\; \varphi^{\tau}(x, y) \rightarrow \exists u\, \forall x ⋿ a\; \varphi^{\tau}(x, ux).$$

By again applying Selection, we get

$$\forall x ⋿ a\, \exists y\; \varphi^{\tau}(x, y) \rightarrow \exists z\, \forall x ⋿ a\, \exists y ⋿ z\; \varphi^{\tau}(x, y).$$

Because $x \doteq x' \rightarrow (\varphi^{\tau}(x) \leftrightarrow \varphi^{\tau}(x'))$, we get

$$\forall x \mathrel{\dot{⋿}} a\, \exists y\, \varphi^{\tau}(x, y') \rightarrow \exists z\, \forall x \mathrel{\dot{⋿}} a\, \exists y \mathrel{\dot{⋿}} z\; \varphi^{\tau}(x, y).$$

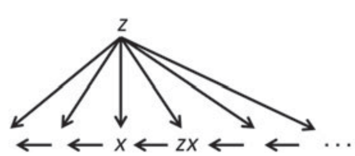
So, we see that the translation of Replacement is provable in CFF.

(vi) The translation of the Axiom of Infinity is

$$\exists z\, (\exists x \mathrel{\dot{⋿}} z\, \forall y\, \neg\, y \mathrel{\dot{⋿}} x \wedge \forall x \mathrel{\dot{⋿}} z\, (\exists y \mathrel{\dot{⋿}} z\; x \mathrel{\dot{⋿}} y \wedge \forall u, v \mathrel{\dot{⋿}} x\; u \doteq v)).$$

The provability of this formula is slightly more complicated. We only sketch the idea. Let z be as in the Axiom of Infinity in CFF. We may assume that z is well-ordered, i.e., for any nonempty $y \sqsubseteq z$, there exists a $u ⋿ y$ such that $\neg\exists v ⋿ y\, (u = yv)$.

Let G be a graph with for each $x ⋿ z$, arrows (z, x) and (zx, x):



It is easy to check that G is well-founded. In CFF minus Universality we can prove that any well-founded graph has a *decoration*, i.e., an assignment of an entity to each node such that the inputs of the entity assigned to a node are the entities assigned to the children of that node.[20] So, we may assume that $\forall y ⋿ z\, \forall x\, (x ⋿ y \leftrightarrow zx = y)$. It then follows easily that z is as in the translation above.

(vii) The translation of the Axiom of Choice is

$$\forall x \mathrel{\dot{⋿}} a\, \exists y\; y \mathrel{\dot{⋿}} x \wedge \forall x, y \mathrel{\dot{⋿}} a\, \exists u\, (u \mathrel{\dot{⋿}} x \wedge u \mathrel{\dot{⋿}} y \rightarrow x = y)$$
$$\rightarrow \exists z\, \forall x \mathrel{\dot{⋿}} a\, \exists y \mathrel{\dot{⋿}} x\, \forall u\, (u \mathrel{\dot{⋿}} z \leftrightarrow u \doteq y).$$

Suppose that the antecedent of this formula is valid. Then by the Axiom Schema of Selection, there is a v such that $\forall x ⋿ a\, \exists y ⋿ x\; vx = y$. Hence there is a z such that $\forall u\, (u ⋿ z \leftrightarrow \exists x ⋿ a\, \exists y ⋿ x\, (vx = y \wedge u = y))$. It is easy to verify that $\forall x \mathrel{\dot{⋿}} a\, \exists y \mathrel{\dot{⋿}} x\, \forall u\, (u \mathrel{\dot{⋿}} z \leftrightarrow u \doteq y)$. □

[20] It can be proved in a similar way as Mostowski's Collapsing Lemma in ZFC.

COROLLARY 4.10. CFF *and* ZFC *are interpretable in each other.*

Proof. In Theorem 4.7 we proved that CFF is interpretable in ZFA. In (Aczel, 1988, pp. 36–37) it is proved that ZFA is interpretable in ZFC. So, by transitivity of interpretations, CFF is interpretable in ZFC.

In (Fraenkel et al., 1973, pp. 98–102) it is proved that ZFC is interpretable in ZFC minus Foundation. By Theorem 4.9, ZFC minus Foundation is interpretable in CFF. So, ZFC is interpretable in CFF. □

COROLLARY 4.11. CFF *is interpretable in* CFF *minus the Axiom of Universality.*

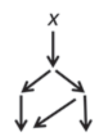
Proof. We have the following facts: (i) CFF is interpretable in ZFC (Corollary 4.10), (ii) ZFC is interpretable in ZFC minus Foundation (Fraenkel et al., 1973, pp. 98–102), and (iii) ZFC minus Foundation is interpretable in CFF minus Universality (Theorem 4.9). So, by transitivity of interpretations, CFF is interpretable in CFF minus Universality. □

4.4. Dealing with indiscernibles. In metaphysics, physics, and mathematics the notion of indiscernibility plays a significant role. We will show that CFF, in contrast to ZFC, fully accommodates indiscernible entities.

In CFF we have an ultra-refined notion of entity. By the Axiom of Universality, each entity has functionally equivalent variants. In particular, it follows that an axiom of extensionality does not hold. Moreover, nothing in CFF can be defined uniquely. We will prove this in Theorem 4.13, after introducing the notion of a *picture* of an entity:

DEFINITION 4.12. *The* picture Ga *of an entity* a *is a graph* G *with* a *as a distinguished node and with the following properties:*

1. *each node is either a basic node or an auxiliary node,*
2. *the distinguished node* a *is a basic node,*
3. *for each node* x*, there is a path from* a *to* x*,*
4. *for each basic node* x *and* $u \sqsubseteq x$*, there is an edge from* x *to a subgraph representing* (u, xu)*:*



where the intermediate nodes of the subgraph and no others are unique auxiliary nodes,

5. *each basic node* x *has no other outgoing edges.*

We say that entities a, b *have* isomorphic pictures *if there is a bijection* τ *from the basic nodes in a picture of* a *to the basic nodes in a picture of* b *such that* $\tau a = b$*, and for each basic node* x *in the picture of* a*, and for each* u, v*,*

$$xu = v \leftrightarrow \tau x(\tau u) = \tau v.$$

We now show that in CFF nothing can be defined uniquely:

THEOREM 4.13. *In* CFF*, for each formula* φ *with exactly one free simple term* a*, and with* x *free for* a*,*

$$\mathsf{E}\, a \wedge \varphi(a) \rightarrow \exists x\, (x \neq a \wedge \varphi(x)).$$

Proof. Let a be a simple term with $\mathsf{E}\, a$. By using the Axiom of Universality, we get an $x \neq a$ whose picture is isomorphic to a picture of a. Then x is indistinguishable from a in the sense that for any formula φ with a as its only free simple term and with x free for a, we have $\varphi(a) \leftrightarrow \varphi(x)$. This can be proved by induction:

By induction on the structure of formulas, it is not difficult to see that for any formula φ with all free simple terms among $u_1, \ldots, u_n$, and with $v_1, \ldots, v_n$ free for $u_1, \ldots, u_n$, if $\langle u_1, \ldots, u_n \rangle$ and $\langle v_1, \ldots, v_n \rangle$ have isomorphic pictures, then $\varphi(u_1, \ldots, u_n) \leftrightarrow \varphi(v_1, \ldots, v_n)$.[21] □

Note that in the proof we defined x independent from φ. Also note that the theorem does not always hold for an x that is functionally equivalent to a. For example, if φ is $\forall u\, (u \sqsubseteq a \leftrightarrow u = a)$, then $\mathsf{E}\, a \wedge \varphi(a) \wedge \varphi(x) \wedge x \equiv a \rightarrow x = a$.

Not being able to define anything uniquely, may look like a serious impediment. There are, however, possibilities to deal with it in a convenient way. First of all, there are many situations in which we are not interested in uniqueness, but only in an entity having a certain property. For example, it may be useful to say things like:

$$\text{Let } \varnothing \text{ be an entity } x \text{ such that } \neg\exists u\; u \sqsubseteq x.$$

In a similar way, we could introduce natural numbers and other structures that can be modeled with a single entity. But for large structures, like the class of ordinal numbers, this approach does not work.

We would like to be able to abstract from certain differences between entities. What comes to mind, of course, is to work with equivalence classes of entities. A problem, however, is that equivalence classes can be large. As we have seen in Example 4.2, for each entity its functionally equivalent variants outnumber the inputs of any entity.

A rather simple solution is to work with equivalence classes of entities in a similar way as we do with classes in set theory: we may treat a class of entities as an informal notion.

In different contexts different equivalence relations of entities may be of interest. We will briefly discuss here one such relation, namely bisimulation of pictures of entities.

We already defined bisimilar graphs in Definition 4.8, and we applied bisimilarity to input pictures of entities. Now we want to apply it to full pictures of entities.

Let $[x]_{\mathsf{bis}}$ denote the class of entities that have a picture which is bisimilar to a picture of x.

Of particular interest are *function-like* equivalence classes $[x]_{\mathsf{bis}}$, which have the property that for any $u, u' \sqsubseteq x$, if $[u]_{\mathsf{bis}} = [u']_{\mathsf{bis}}$, then $[x(u)]_{\mathsf{bis}} = [x(u')]_{\mathsf{bis}}$. For these classes $[x]_{\mathsf{bis}}$ we may define function application:

$$[x]_{\mathsf{bis}}([u]_{\mathsf{bis}}) \;=_{\mathrm{df}}\; [x(u)]_{\mathsf{bis}}.$$

We call $[x]_{\mathsf{bis}}$ a *pure function* if for each u in a transitive closure of x, the class $[u]_{\mathsf{bis}}$ is function-like.[22]

[21] A tuple $\langle u_1, \ldots, u_n \rangle$ may, e.g., be defined as a term u such that $\forall x\, ((x \sqsubseteq u \leftrightarrow x = u_1) \wedge (x \sqsubseteq u(u_1) \leftrightarrow x = u_2) \wedge \ldots \wedge (x \sqsubseteq u(u_1) \ldots (u_{n-1}) \leftrightarrow x = u_n) \wedge \neg(x \sqsubseteq u(u_1) \ldots (u_n)))$.

[22] Similar as for sets, we call an entity x *transitive* if $u \sqsubseteq v$ and $v \sqsubseteq x$ imply $u \sqsubseteq x$. We talk about *a* transitive closure of x and not about *the* transitive closure because it cannot be defined uniquely in CFF.

A nice property of pure functions is that they are extensional. We can formulate this concisely in an extended language of CFF:

$$\forall \underline{x}\ \underline{f}\,\underline{x} \simeq \underline{g}\,\underline{x} \rightarrow \underline{f} = \underline{g},$$

where $\underline{f}, \underline{g}, \underline{x}$ denote pure functions.

This approach makes it possible to work with pure functions and other kinds of classes in a very convenient way. But it is important to realize that all we say in an extended language of CFF can always be expressed in the original language of CFF. The new formulas are essentially just abbreviations.

§5. Conclusions. The paper started with a critique of predicate logic. Subsequently, I tried to develop a better alternative. But is the new logic really better? In this concluding section we look at this question from two angles: (1) Does the new logic correspond better with reality? (2) Is the accompanying new foundational system superior to set theory?

Before diving into these questions, let me first recapitulate the main results:

1. We defined a logic with no constants, no predicates except equality, and with structures consisting of entities with an input–output functionality. The logic has terms of the form $t(t')$, which are interpreted as the result of an interpretation of t applied to an interpretation of t'.
2. In the new logic, we gave a characterization of relations based on the idea that a relation is a network of complexes interrelated by substitutions.
3. In the new logic, we also defined a foundational theory that provides any consistent theory with a model consisting of a collection of entities of the foundational theory itself.

Correspondence with reality. My objections in §1 to standard logic concerned the artifactual ordering of arguments, the implicit suggestion that relations are universals, and the inability to express strict symmetry. The new logic does not have these problems, but, of course, it may have other weaknesses. Let us look at some potential concerns.

A starting-point for the new logic was to see the world as consisting of entities that may have an input–output functionality. We defined structures simply as collections of such potent entities. This is obviously different from standard logic where we always have a domain of 'lifeless' entities and a collection of relations and functions. Furthermore, in first-order logic only the 'lifeless' entities are treated as first-class citizens, whereas in the new logical framework, relational complexes are first-class citizens as well. We defined complexes as entities that may take substitutions as inputs. But how natural is it to view Adam's loving Eve as an entity with input–output functionality?

Although one might perhaps argue that relational complexes have by definition substitutions as inputs, I prefer viewing substitutions as more abstract correspondences between relational complexes. Substitutions can be *modeled* as inputs of entities that *represent* the relational complexes 'out there'. Instead of saying that Adam's loving Eve *is* an entity with input–output functionality, I would say that it can naturally be modeled as such.[23]

Another point I would like to mention here is that in the new logic we do not have variables and constants. It is not a big issue, but I see no reason why names for certain

[23] There is a certain similarity with ordered pairs in set theory: we may model an ordered pair $\langle x, y\rangle$ as the set $\{\{x\}, \{x, y\}\}$, but it does not make sense to say that $\{x, y\}$ is an element of the ordered pair.

entities would be part of the structures themselves. In my view it is more natural to let names of entities be part of the interpretation of the terms of the language. We can let simple terms that occur free in a formula fulfill the role of constants. Besides, not having variables and constants as distinct types of symbols makes the semantics of the logic somewhat simpler.

If the new logic corresponds better with reality than predicate logic, should we get rid of predicate logic? I think predicate logic is clearly not the 'ultimate' logic and from a purely theoretical point of view unnecessary. From a practical perspective; however, the situation is different.

The new logic in unabbreviated form is in some cases quite cumbersome and tedious, but, as we argued in §3.4, we can introduce notations to make the logic more convenient. New notations can also lead to new formal systems. Predicate logic itself can be seen as a higher-level formal language which can be translated in a pretty straightforward way into the new logic. When we are aware that such a reduction is possible, the use of predicate logic is not only harmless, but will continue to be appropriate for various applications.

Foundational considerations. Let us make a comparison of ZFC and CFF. As criteria for evaluating these foundational systems we may, for example, consider proof-theoretic strength, simplicity, and naturalness. As we have seen in §4.3, CFF has the same proof-theoretic strength as ZFC, but we will argue that the systems differ with respect to simplicity and naturalness.

Simplicity. An important difference between CFF and ZFC is that CFF does not have an axiom of extensionality. CFF embodies an ultra-refined conception of entities. This makes CFF in certain ways simpler, in particular, when indiscernibility plays a role.

In CFF indiscernible things can be modeled straightforwardly (see §4.4). In ZFC this is not the case. Take, for example, the structure of the complex numbers in which the numbers i and $-i$ are not absolutely discernible. In any implementation of the complex numbers in ZFC, the representations of the numbers are all absolutely discernible. We can of course enrich ZFC with atoms, but this is only a half-hearted solution. For modeling indiscernible things with an internal structure or with an input–output functionality, we still face in ZFC with atoms the same problem.

A strong feature of ZFC is that many things can be defined in a unique way, whereas in CFF nothing can be defined uniquely. However, as explained in §4.4, this is not a major obstacle because we may work in CFF conveniently with (informal) equivalence classes, and we may define things uniquely modulo some equivalence relation.

One could argue that sets are conceptually simpler than functions. Yet this does not mean that sets should be taken as the building-blocks of a foundational theory. As von Neumann said, it is formally easier to base the notion of set on that of function than conversely (von Neumann, 1925, p. 222).

Naturalness. For the formulation of any foundational theory, the use of entities with a functional character seems unavoidable. Moreover, interpretations of logical systems are always defined in terms of functions. Therefore, it is natural to take functions or function-like entities as primitive for a foundational system. In set theory, however, we have to code functions in an artificial way.[24]

[24] In set theory, a function is defined as a set of ordered pairs, where an ordered pair $\langle a, b\rangle$ may, e.g., be defined as follows: (1) Kuratowski's definition: $\{\{a\}, \{a, b\}\}$, (2) Wiener's definition: $\{\{\{a\}, \varnothing\}, \{\{b\}\}\}$, (3) Hausdorff's definition: $\{\{a, 1\}, \{b, 2\}\}$.

Our objective to create a foundation that provides each consistent theory in the new logic with a straightforward model that does not require any artificial coding guided us in choosing the axioms. It limited our options: we could not choose an axiom of extensionality and not an axiom of foundation.

For set theory the situation is different. The choices for some of its axioms are more arbitrary. For example, the motivation often given for choosing Foundation is that it simplifies the discussion of models, but theories of non-well-founded sets turn out to have several interesting applications (Aczel, 1988; Barwise and Etchemendy, 1987).[25]

In CFF each entity has an abundance of variants with the same input–output functionality. This can be exploited in defining mathematical objects. It allows us, for example, to represent natural numbers and ordinal numbers as entities without internal structure. In ZFC this is not possible because sets are only unequal when their internal structures are different; we have to say things like $2 = \{0, 1\}$, but it is dubious whether this really makes sense (cf. Benacerraf (1965)).

A worry that one might have is that in CFF relations inevitably have multiple realizations. Two things can be said about this. First, it is indeed true that we cannot uniquely define in CFF a network of complexes interrelated by substitutions. However—on an intensional conception of relations—different relations 'out there' may be structurally indiscernible. Hence, it could be viewed as a strength of a foundational system if it accommodates indiscernible networks of complexes. Second, in CFF relations are not necessarily underspecified. In fact, *from the perspective of an entity within the universe of* CFF, it might very well be possible to uniquely identify certain complexes.[26]

In conclusion, I submit, CFF is more natural than ZFC or any other set theory. We do not live in a world of sets but rather in a world of entities with relations between them. In set theory almost everything can be coded, but the coding is in some cases arbitrary and artificial.

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Appendices.

§A. A faithful interpretation. In Theorem 2.9 the claim was made that any theory U of negative, universally free predicate logic with identity is faithfully interpretable in a theory of coordinate-free logic. The proof, however, was not yet complete. We still have to show that the interpretation supported by the given translation τ is indeed *faithful*. This will be done in this appendix.

[25] As an aside, I would like to say that I have some difficulty with the idea of a set containing itself as element. I picture a set as a box containing things and it seems strange to have a box containing itself. With entities having themselves as inputs I do not have such problems.

[26] The relevance of internal perspectives for the way we perceive the world will be treated in detail in a forthcoming paper. The basic idea is that an entity ξ can be uniquely identified from the perspective of an entity X if any automorphism σ on the picture of X that maps X to itself also maps ξ to itself.

We first show that we may assume without loss of generality that the signature of the language of U is finite. Then we define a *piecewise* interpretation $\langle U^\tau, \tau', U\rangle$ with the property that any sentence A in the language of U is provably equivalent with $(A^\tau)^{\tau'}$, i.e., $\vdash A \leftrightarrow (A^\tau)^{\tau'}$. From this it follows immediately that the interpretation $\langle U, \tau, U^\tau\rangle$ is faithful.

1. Reduction to languages of finite signature.

We prove that faithful interpretability of U in U^τ follows from faithful interpretations of subtheories with finite signature.

CLAIM A.1. *Suppose that any restriction of τ to languages of finite signature supports a faithful interpretation of the corresponding restriction of U in the corresponding restriction of U^τ. Then it follows that τ itself supports a faithful interpretation of U in U^τ.*

Proof. Assume that $U \vdash A$. If we restrict the signature Σ of the original language of U to a signature Σ' consisting of the relational symbols occurring in a given deduction of A from U, then Σ' is obviously finite. So we may apply our supposition about restrictions of τ to languages of finite signature. It follows that $U^\tau \vdash A^\tau$.

Conversely, assume that $U^\tau \vdash A^\tau$. Then by a similar reasoning it follows that $U \vdash A$. □

2. An inverse piecewise interpretation.

We now assume that the language of U is of finite signature Σ. We will define a piecewise interpretation $\langle U^\tau, \tau', U\rangle$ such that for any Σ-sentence A, $\vdash A \leftrightarrow (A^\tau)^{\tau'}$.

The notion of *piecewise interpretations* comes from Harvey Friedman and Albert Visser. Before dealing with coordinate-free logic, we give Visser's definition of piecewise translations for predicate logic (Visser, 2009), where we restrict ourselves to parameter-free translations.

The definition is rather technical. Let me therefore first say a few words about the idea of piecewise translations.

Within a theory, the variables in a formula can fulfill different roles. For example, in the formula $x = y$, the variable x may stand for a natural number or for an integer. Now we would get great flexibility if we could make the translation of a formula dependent on the roles played by the variables. This is exactly what is done in piecewise translations.

Let Σ and Ξ be signatures for (free) first-order predicate logic. We fix a sequence containing all variables $u_1, u_2, \ldots$ for Σ, and we fix $\ell + 1$ disjoint sequences of variables $v^1_1, v^1_2, \ldots,$ and $\ldots$ and $v^\ell_1, v^\ell_2, \ldots,$ and $z_1, z_2, \ldots$ for Ξ. We write $\vec{v}^{\,j}_i$ for $v^j_{\nu^j(i-1)+1}, \ldots, v^j_{\nu^j i}$.

A *piecewise translation* $\tau : \Sigma \to \Xi$ is given by a quadruple $\langle \ell, \nu, \delta, F\rangle$, where

- ℓ is a natural number that stands for the pieces $1, \ldots, \ell$;
- ν is a function that assigns to each piece j a dimension ν^j;
- δ is a function from pieces j to domains δ^j of dimension ν^j, where δ^j is a Ξ-formula with bound variables among the z_i, and at most $\vec{v}^{\,j}_1$ free;
- F is a mapping that associates to each pair (R, f) a Ξ-formula with bound variables among the z_i and free variables among $\vec{v}^{\,f1}_1, \ldots, \vec{v}^{\,fn}_n$.
  Here R is an n-ary relational symbol of Σ, and f is a mapping that assigns to each argument-place of R a piece.

We translate Σ-formulas to Ξ-formulas as follows. The basic form of translation is $A^{\tau,g}$, where g is a function from the free variables of A to pieces.[27]

- $(R(u_{j_1},\ldots,u_{j_n}))^{\tau,g} := F(R,f)(\vec{v}^{\,f1}_{j_1},\ldots,\vec{v}^{\,fn}_{j_n})$, where $f(s) := g(u_{j_s})$;
- $(A \wedge B)^{\tau,g} := A^{\tau,g\restriction \mathsf{fv}(A)} \wedge B^{\tau,g\restriction \mathsf{fv}(B)}$;
  similarly for the other propositional connectives;
- $(\forall u_k\, A)^{\tau,g} := \bigwedge_{j\le\ell} \forall \vec{v}^{\,j}_k\, (\delta^j(\vec{v}^{\,j}_k) \to A^{\tau,g[u_k:j]})$, where $g[u_k : j]$ is the result of setting g at u_k to j.

A coordinate-free language with an infinite number of terms can be translated in a trivial way into a predicate language with terms (see Theorem 2.10). Furthermore, predicate languages with terms can be translated in a standard way into relational languages. Therefore, we may define piecewise translations from coordinate-free logic to predicate logic in an obvious way.

In the standard translation of a predicate language with terms to a relational language, each n-ary function symbol corresponds with an $(n+1)$-ary relation symbol. For the simple terms that play the role of constants we therefore have to take one auxiliary argument-place into account, and for the application symbol $\cdot(\cdot)$ two normal argument-places plus an auxiliary one.

Let's get to work and describe an inverse piecewise translation τ' for the translation given the proof of Theorem 2.9.

In the proof of the theorem we introduced for the domain of the interpretation a simple term ind, and for each predicate letter P a simple term p. The translation τ was defined by

$$\begin{aligned} \delta &:= \mathsf{E}\,\mathsf{ind}(v_1), \\ F(=) &:= v_1 = v_2, \\ F(P) &:= \mathsf{E}\, p v_1 \ldots v_n, \text{ for any other } n\text{-ary predicate } P. \end{aligned}$$

For the translation τ' we define the following pieces:

- a piece s_{ind_0} of dimension 0;
- a piece s_{ind_1} of dimension 1;
- for each n-ary predicate letter P, a collection of pieces $s_{p_0},\ldots,s_{p_n}$, where s_{p_i} has dimension i.

The piece s_{ind_0} stands for ind itself, s_{ind_1} stands for x in $\mathsf{ind}(x)$, s_{p_0} stands for p itself, and s_{p_i} with $i \ge 1$ stands for $x_1,\ldots,x_i$ in $px_1\ldots x_i$.

Note that because the language of U is of finite signature, we have a finite number of pieces.

For τ' we define for the domains a function δ' as follows:

$$\delta'(i) := \begin{cases} P(\vec{v}^{\,i}_1) & \text{if } i \text{ is piece } s_{p_n}; \\ \top & \text{otherwise.} \end{cases}$$

Furthermore, we define for τ' a mapping F' for identity, ind, the simple terms p, and the application symbol.

[27] I deviate here slightly from Visser (2009), where g is a function from the indices of the free variables of A to pieces.

Identity is preserved, but we require that the arguments belong to the same piece:

$$F'(=, ij) := \begin{cases} \vec{v}_1^{\,i} = \vec{v}_2^{\,i} & \text{if } i = j; \\ \bot & \text{otherwise.} \end{cases}$$

Here $\vec{v}_1^{\,i} = \vec{v}_2^{\,i}$ stands for $\bigwedge_{1 \le m \le \nu^i} v_m^i = v_{m+\nu^i}^i$.

For the simple term ind, we require that the auxiliary argument belongs to the piece s_{ind_0}:

$$F'(\mathsf{ind}, i) := \begin{cases} \top & \text{if } i \text{ is piece } s_{\mathsf{ind}_0}; \\ \bot & \text{otherwise.} \end{cases}$$

Similarly, for each simple term p corresponding to a predicate letter P, we require that the auxiliary argument belongs to the piece s_{p_0}. We treat 0-ary predicate letters in a special way:

$$F'(p, i) := \begin{cases} \top & \text{if } i \text{ is piece } s_{p_0}; \\ \bot & \text{otherwise.} \end{cases}$$

For the application symbol $\cdot(\cdot)$, the mapping F' is more complicated. We have two normal argument-places and one auxiliary one. The first argument-place can stand for ind or for a term $pu_1 \ldots u_m$ with $m < n$, where n is the arity of the predicate symbol P that corresponds to p. The second argument-place always stands for an individual that belongs to the domain. When the first argument-place stands for $pu_1 \ldots u_m$, then the third argument-place stands for $pu_1 \ldots u_{m+1}$, and $F'(\cdot(\cdot), ijk)$ becomes

$$\vec{v}_3^{\,k} = (v_1^i, \ldots, v_m^i, v_2^j),$$

where i is the piece representing $pu_1 \ldots u_m$, j is the piece representing u_{m+1}, and k is the piece representing $pu_1 \ldots u_{m+1}$.

So, we get for the application symbol:

$$F'(\cdot(\cdot), ijk) := \begin{cases} \top & \text{if } i \text{ is piece } s_{\mathsf{ind}_0},\ j \text{ is the piece } s_{\mathsf{ind}_1}, \\ & \text{and } k \text{ is piece } s_{\mathsf{ind}_0}; \\ \vec{v}_3^{\,k} = (\vec{v}_1^{\,i}, v_2^j) & \text{if } i \text{ is piece } s_{p_m},\ j \text{ is piece } s_{\mathsf{ind}_1}, \\ & \text{and } k \text{ is piece } s_{p_{m+1}}; \\ \bot & \text{otherwise.} \end{cases}$$

Let us look at an example. Suppose A is $\forall x\, \exists y\, P(x, y)$. Then A^τ is

$$\forall x\, (\mathsf{E}\,\mathsf{ind}(x) \to \exists y\, (\mathsf{E}\,\mathsf{ind}(y) \wedge \mathsf{E}\, pxy)).$$

What $(A^\tau)^{\tau'}$ is depends also on the signature Σ. The formula can be large and contain quite a number of equalities and $\top$'s and $\bot$'s, but it is easy to verify that the formula will be provably equivalent with A.

More generally, we can prove the following claim:

CLAIM A.2. *Any Σ-sentence A is provably equivalent with $(A^\tau)^{\tau'}$.*

Proof. We first make a few observations. Note that

$$(\forall u_k\, (\mathsf{E}\,\mathsf{ind}(u_k) \to B))^{\tau', g} = \bigwedge_{j \le \ell} \forall \vec{v}_k^{\,j}\, (\delta^j(\vec{v}_k^{\,j}) \to (\mathsf{E}\,\mathsf{ind}(u_k) \to B)^{\tau', g[u_k : j]})$$

and verify that this is provably equivalent with $\forall v_k^{s_{\mathsf{ind}_1}}\ B^{\tau', g[u_k : s_{\mathsf{ind}_1}]}$.

Also note that for any n-ary predicate P,

$$(\mathsf{E}\, pu_1 \ldots u_n)^{\tau',g} = (\exists u_k\ pu_1 \ldots u_n = u_k)^{\tau',g}, \text{ with } k > n$$
$$= \bigvee_{j \leq \ell} \exists \vec{v}^{\,j}_k\, (\delta^j(\vec{v}^{\,j}_k) \wedge (pu_1 \ldots u_n = x)^{\tau', g[u_k:j]})$$

and verify that if for each $i \leq n$, $g(u_i) = s_{\mathsf{ind}_1}$, then this formula is provably equivalent with $P(v_1^{g(u_1)}, \ldots, v_n^{g(u_n)})$.

Now we syntactically identify the variables $u_1, u_2, \ldots$ of Σ with $v_1^j, v_2^j, \ldots$, where j is the piece s_{ind_1}. Then, using the observations, it is easy to prove by induction that any Σ-formula A is provably equivalent with $(A^\tau)^{\tau',g}$, where g assigns to each free variable of A^τ the piece s_{ind_1}. □

CLAIM A.3. *The translation τ' supports an interpretation of U^τ in U.*

Proof. Suppose $U^\tau \vdash B$, where B is a sentence with respect to U^τ, i.e., all free terms of B occur free in some formula of U^τ. The deduction is a sequence $B_1, B_2, \ldots, B_m$ such that B_m is B, and for each $i \leq m$, B_i is a logical axiom, or B_i is in U^τ, or for some $j, k < i$, B_i is obtained by modus ponens from B_j and B_k.

We translate this sequence $B_1, B_2, \ldots, B_m$ to a sequence of Σ-formulas by translating each formula B_i to a subsequence of formulas $B_i^{\tau',g}$, where g is any function from the free terms in B_i that do not occur free in a formula of U to the pieces of τ'.

It is straightforward to prove that:

1. if B_i is a logical axiom, then $\vdash B_i^{\tau',g}$,
2. if B_i is A^τ with A in U, then $\vdash B^{\tau'} \leftrightarrow A$,
3. if B_i is obtained by modus ponens from B_j and B_k, then $B_i^{\tau',g}$ can be obtained by modus ponens from $B_j^{\tau',g}$ and $B_k^{\tau',g}$.

From this it follows that the sequence of Σ-formulas is a deduction of $B^{\tau'}$ from U. So, we have an interpretation $\langle U^\tau, \tau', U\rangle$. □

This concludes the proof of Theorem 2.9.

§B. A generalization of the Axiom of Choice. In standard set theory as well as in non-well-founded set theory, we have a schematic generalization of the Axiom of Choice:

THEOREM B.1. *Assume* ZFC *or* ZFA. *Let φ be a formula such that for each $x \in z$ there is an y such that $\varphi(x, y)$. Then there exists a function f on z such that, for each $x \in z$, $\varphi(x, f(x))$.*

Proof.
Assume ZFC.
We prove the claim by using the Axiom of Foundation.[28]

By Foundation, every set x has a *rank*, defined as the least α such that $x \subseteq V_\alpha$. For any nonempty class X, let v be the class of elements of X of minimal rank. Then v is obviously a nonempty set. So, we have a class function G that assigns to X the set v.

[28] The proof given here for ZFC is similar to the one in (Vaught, 1985, p. 94).

Now let g be the function that maps to each $x \in z$ the set $G(\{y \mid \varphi(x, y)\})$. Then, by the Axiom of Choice, there is a mapping f on z such that for each $x \in z$, $f(x) \in G(\{y \mid \varphi(x, y)\})$.

Assume ZFA.

Since in ZFA not every set has a rank, the proof is more complicated; we need some extra steps to map each nonempty class to a nonempty subset.

By using the Axiom of Choice and the Power Set Axiom, it can be proved that every set can be well-ordered (cf. (Kunen, 2011, p. 68). Furthermore, every well-ordered set is isomorphic to some ordinal (Kunen, 2011, p. 39). Thus, every set is in bijection with some ordinal.

In ZFA every set has a picture, i.e., a graph that represents the set (Aczel, 1988, p. 5). Because every set is in bijection with some ordinal, every graph is isomorphic to a graph in the well-founded part of our universe. It follows that every set in ZFA has a picture in the well-founded part of our universe.

We will now show how every nonempty class X can be mapped unambiguously to a nonempty subset υ.

Let P_X be the class of pictures of the elements of X in the well-founded part of our universe. Then Q_X, the class of pictures in P_X of minimal rank, is obviously a nonempty set. Let υ be the class of sets represented by the pictures in Q_X. Then, by the Axiom Schema of Replacement, υ is a nonempty set as well. So, we have constructed a class function G that assigns to X the set υ.

We may finish the proof as above for ZFC. □

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