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Pricing in a competitive stochastic insurance market

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ABSTRACT

This paper studies a one-period stochastic game to determine the optimal premium strategies of non-life insurers in a competitive market. Specifically, the optimal premium strategy is determined by the Nash equilibrium of an n-player game, in which each player is assumed to maximise the expected utility of terminal wealth. The terminal wealth is stochastic, since the number of policies and the size of claims are considered to be random variables. The total loss of each insurer is described by the collective risk model. The expected number of policies is affected by all the premiums in the market and further investigated by two distinct demand functions. Both models have an exponential functional form, that is characterised by market and price sensitivity parameters. The demand in the first model is zero for premiums above a given threshold, whereas the second model does not include such restriction. The pure strategy Nash equilibrium premiums are given as solutions to constrained optimisation problems. For the first model we prove the existence and uniqueness of a pure strategy Nash equilibrium, whereas for the second model we provide a formula when it exists. Two numerical examples are provided to illustrate the applicability of our findings.

1. Introduction

1.1. Motivation

In the insurance literature, there is a continuous and strong interest in the modelling of insurance premiums in a competitive market. Insurers determine their premiums in response to the premiums that are being offered by the competitor companies.

Standard actuarial approaches, particularly in non-life insurance, are almost exclusively devoted on calculating the distribution of the underlying risk, following the underlying trends in distribution instead of formulating the underwriting strategies (Emms et al., 2007). Such approaches ignore the actions made by the competitors which affect the movement of premium rates and, eventually, the exposure of the company into the corresponding market (Taylor, 1986). Due to the complex nature of price competition, several modelling and pricing challenges arise, and this is the motivation of our paper.

1.2. Literature review

The present paper formulates a stochastic model in competitive non-life insurance pricing. A standard approach is to let the premium exclusively depend on the distribution of the underlying risk (see Kaas et al., 2008). However, in this approach one does not assume that the insurance markets are competitive.

Taylor (1986) addresses the subjective nature of the insurance underwriting in a deterministic discrete-time framework. He proposes a demand function to capture how the number of policies sold by the insurer is affected by the premium choices compared to other competitors' premiums. Under this demand function, an insurer selects the optimal response to movements of premiums in the market. In a subsequent work, Taylor (1987) shows that significant changes in the optimal premiums may happen when non-constant expense rates occur. Emms and Haberman (2005), Emms et al. (2007), Emms (2007) extend the literature with a stochastic continuous-time framework. They calculate the optimal premium strategy, which maximises the expected terminal wealth of an insurer, considering a demand law and applying optimal control theory. Pantelous and Passalidou (2013, 2015) embed the insurer's reputation and a stochastic disturbance into their stochastic discrete-time model and explicitly calculate the optimal premiums of an insurer as solutions of a polynomial. All the aforementioned models consist of single optimisation problems, in which competitors' premiums are unaffected by the individual insurer's premium strategy. In Table 1, we classify the papers, where the break-even and market's premiums are considered either as deterministic or stochastic, into static/dynamic and deterministic/stochastic settings.

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Table 1
Single-optimisation papers' segmentation regarding time frame and randomness.

Papers	Discrete	Continuous	Deterministic	Stochastic
Taylor (1986, 1987)	√		√	
Emms and Haberman (2005)		√		√
Emms et al. (2007)		√		√
Emms (2007)		√		√
Pantelous and Passalidou (2013, 2015)	√			√

Table 2
Non-cooperative papers' segmentation regarding time frame and randomness.

Papers	Static	Dynamic	Deterministic	Stochastic
Emms (2012)		√	√	√
Dutang et al. (2013)	√			√
Wu and Pantelous (2017)	√		√	
Boonen et al. (2018)		√	√	
Asmussen et al. (2019)		√		√
This paper	√			√

Non-cooperative game theory provides tools to embark from individual optimisation procedures and aim to simultaneous optimisation solutions for all participants in the market. First cooperative game theoretical approaches in insurance can be found in Borch (1962, 1974), Bühlmann (1980, 1984), and Lemaire (1984, 1991). The non-cooperative game models that have been applied to non-life insurance markets can be distinguished into two categories: (i) the Bertrand oligopoly in which insurers set premiums and (ii) the Cournot oligopoly in which insurers choose the volume of business. Emms (2012) studies the Nash equilibria of an n-player non-cooperative differential game in which each insurer aims to maximise her expected utility of terminal wealth under a demand function. He proposes both a deterministic and a stochastic differential game regarding whether or not the break-even premium was considered uncertain. Dutang et al. (2013) use a loss model for the claims and a lapse model to describe the probabilistic behaviour of individuals as a function of insurers' premiums. By solving a maximisation problem that consisted of a quadratic equation and a solvency constraint function, they prove the existence and uniqueness of the Nash equilibrium, as well as the existence of the Stackelberg equilibrium of a one-period non-cooperative game. Wu and Pantelous (2017) determine the optimal premium strategy by calculating and proving the existence of Nash equilibria in an n-player potential game with non-linear aggregation. Instead of applying a demand function to describe the exposure of insurers, they aggregate all the paired competitions in the market. Boonen et al. (2018) apply optimal control theory to determine the open-loop Nash equilibrium premium strategies in an n-player differential game, by including the solvency ratio in the demand function. Recently, Asmussen et al. (2019) consider the customer's problem with market frictions and formulate stochastic differential game between two insurance companies that have a different size. Table 2 classifies the papers with non-cooperative game theoretical orientation into static/dynamic and deterministic/stochastic settings.

1.3. Our contribution

In actuarial practice, the resulting premium offered to insureds usually diverges from the actuarial (net) premium, producing a positive (or sometimes negative) premium loading. Many factors may contribute to this deviation, such as the customers' affordability, insurers' claims and expenses, past experience, and various market conditions.

As shown in Table 2, we propose a static and a stochastic model in this paper. In this sense, this paper is closest to Dutang et al. (2013). Like Dutang et al. (2013), this paper proposes a model in which the terminal wealth of each insurer is characterised by the collective risk model, which depends on the premiums in the market. We differ from Dutang et al. (2013) in the following three ways. First, in our paper, the demand function is related to how many policies are expected to be sold in a reference period. We assume that the number of policies is either Poisson-distributed, whose intensity is driven by the competition in the market, or follows a Negative Binomial distribution. Second, we propose two distinct exponential demand functions: the first one is concave, and considers that there is a premium in the market under which no policy will be sold. The second demand function, proposed by Taylor (1986), is convex and does not assume an upper bound on the premium. Third, we assume that the insurers are strictly risk-averse, and all of them maximise exponential utility functions (see, Emms, 2012). Under this setting, we calculate the best response strategies and show the existence of Nash equilibria. Moreover, we consider two numerical examples to illustrate the applicability of our treatment.

This paper is set out as follows. In Section 2, we provide the necessary notations and assumptions. Assuming a loss model for the total claim amount, Poisson or Negative Binomial distributions for the number of policies, and insurers who are exponential utility maximisers, we obtain the lower bound for the premium domain and the cost function of each insurer. In Sections 3.1 and 3.2, we introduce two families of exposure functions and find the candidate optimal premiums of each individual insurer. In Section 3.3, two results for optimality are presented. In Sections 4.1 and 4.2, we define two exponential exposure functions, with different curvatures, which we use to derive optimal premiums for each insurer. Section 5 provides a Nash equilibrium premium strategy for each model. In the first one the existence and uniqueness of a Nash equilibrium is shown, whereas in the second one there is a structure when it exists. Section 6 presents a numerical example for both models to illustrate our main results. Section 7 concludes, and further suggestions of future research are provided. The proofs are delegated to the Appendix.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, be the finite set of insurers in the market. Every insurer selects a certain premium p_i per policy. Given the premium of the insurer and the premiums of all other insurers in the market, the insurer underwrites a stochastic number of insurance policies, which are realised at a fixed future period in time. The key concept of our model is that premium choices affect directly the intensity of the number of policies which flow among insurers. Rather than finding the premium strategy of one insurer, our objective is to calculate the premium strategies followed by all insurers in the market in their attempt to maximise the expected utility of terminal wealth.

2.1. Basic notations

The definitions of key parameters that we use in this paper are listed as follows:

w_i^-	*Initial wealth* of insurer $i \in N$,
p_i	*Premium value* (per policy) charged by insurer i for the period under consideration,
p_{-i}	*Strategy profile* of premiums charged by all insurers except insurer i,
$\bar{p}_{-i}$	*Average* premium charged by all insurers except insurer i,
$q_i(p_i, p_{-i})$	*Exposure volume* of insurer i for the period under consideration. The exposure is treated as a function of the strategy profile $(p_i, p_{-i}) = \{p_j\}_{j=1}^n$,
q_i^-	Exposure volume of insurer i from the previous period,
N_i	Actual number of policies underwritten by insurer i within the period under consideration,
λ_i	*Risk aversion parameter* of insurer i,
h	Risk aversion parameter of the most risk-averse individual in the market,
b	*Market scale parameter*,
a_i	*Price sensitivity parameter* of insurer i.

2.2. Loss model

Let us now define the collective risk model adopted in our analysis. The total claim amount faced by insurer i, $S_i(N_i)$, for the period under consideration, is defined as

$$S_i(N_i) = \begin{cases} 0, & N_i = 0 \\ \sum_{j=1}^{N_i} X_{ij}, & N_i \geq 1, \end{cases}$$

where X_{ij} is a random variable which represents the claim amount associated to claimant j, for insurer i, for the period under consideration. Claims $\{X_{ij}\}_{j=1}^{N_i}$ are non-negative random variables, and N_i reflects the number of policies sold by insurer i, and this is a random variable which follows either Poisson distribution with parameter $q_i(p_i, p_{-i})$, or Negative Binomial distribution with parameters $q_i(p_i, p_{-i})$ and $k \in (0, 1)$. Thus, $q_i(p_i, p_{-i})$ is related to the expected number of insurance policies that are sold. This parameter depends on all the premiums in the insurance market, and will be specified later in Section 3. Moreover, elements of the set $\{\{X_{ij}\}_{j=1}^{N_i}\}_{i=1}^n$ are independent and identically distributed (i.i.d.) random variables, as well independent from N_i. For convenience, we denote the common distribution of $\{\{X_{ij}\}_{j=1}^{N_i}\}_{i=1}^n$ with X, i.e., $X_{ij} \sim X$, for all $j = 1, 2, \ldots, N_i$, $i = 1, 2, \ldots, n$. Throughout this paper, we make the following assumption.

Assumption 1. We deal with a non-negative random variable X whose moment generating function in λ_i, $M_X(\lambda_i)$, is finite for all $i = 1, \ldots, n$.

An implication of Assumption 1 in our analysis is that the distribution of claims X does not have a heavy tail. Moreover, it is equivalent to assume that $M_X(\lambda) < +\infty$, where $\lambda = \max\{\lambda_1, \ldots, \lambda_n\}$.

2.3. Preferences of the insurers

Emms (2012) proposes a stochastic game that adopts a utility function with constant absolute risk aversion parameter which is the same for all insurers. We expand his idea and assume that all insurers are exponential expected utility maximisers with their own risk aversion parameter. Thus, we adopt the following utility function

$$u_i(x) = \begin{cases} -e^{-\lambda_i x}, & \lambda_i > 0, \\ x, & \lambda_i = 0. \end{cases} \tag{1}$$

Throughout this paper, we assume that all insurers are risk-averse, i.e., $\lambda_i > 0$ for all $i \in N$, and the case with $\lambda_i = 0$ is only used to compare our results with the literature.

2.4. Lower bound on the premiums

Let each insurer i have a utility function of the form (1), with risk aversion parameter $\lambda_i > 0$. Then, insurer i is willing to charge a premium p_i, in order to cover a potential loss X, only if the expected utility of this action is greater than or equal to not undertake this risk. Hence, the premium p_i charged by insurer i should satisfy the following individual rationality constraint,

$$E\left[u_i\left(w_i^- + N_i p_i - S_i(N_i)\right)\right] \geq u_i\left(w_i^-\right), \tag{2}$$

where w_i^- is the initial capital of insurer i.

Lemma 2. *For any distribution of N_i independent of X_{ij} and such that $\Pr[N_i > 0] > 0$, the premium charged by insurer i, p_i, satisfies (2) if and only if*

$$p_i \geq \frac{1}{\lambda_i} \log\left(M_X\left(\lambda_i\right)\right), \tag{3}$$

where λ_i is the risk aversion parameter of insurer i.

The indifference premium for insurer i is defined by

$$p_i^L = \frac{1}{\lambda_i} \log\left(M_X\left(\lambda_i\right)\right), \tag{4}$$

and via the following lemma, we present some main properties of it.

Lemma 3 (Kaas et al., 2008)**.** *For the non-negative random variable X, define the indifference premium p_i^L as a function of λ_i by $p(\lambda_i)$. It satisfies the following properties:*

(1) It is an increasing function of λ_i.
(2) $\lim_{\lambda_i \to 0^+} p(\lambda_i) = E[X]$.
(3) Assume that X is a bounded non-negative random variable with $\max\{X\} = c$, i.e., $\Pr[X \leq c] = 1$ but for all $\varepsilon > 0$, $\Pr[X > c - \varepsilon] > 0$. Then it holds $\lim_{\lambda_i \to +\infty} p(\lambda_i) = c$.

2.5. Objective of the insurers

The objective of each insurer $i \in N$ is to set a premium p_i per policy such that to maximise their expected utility of terminal wealth. From this point up to Section 5, we focus only on maximisation procedure regarding one insurer, conditioning on other insurers' premiums, although we have assumed that all insurers' premiums are unknown. In Section 5 we apply the notion of a Nash equilibrium strategy profile, and how it is calculated through simultaneous individual maximisation procedures. Recalling the necessary notation presented in Section 2.1, next we state the wealth of insurer i. The wealth of insurer i, for the period under investigation, is given by

$$w_i(p_i, p_{-i}) = w_i^- + p_i N_i(q_i(p_i, p_{-i})) - S_i(N_i(q_i(p_i, p_{-i}))). \tag{5}$$

Note that the number of policies that are underwritten is realised after the prices are set, and the distribution of the insurance claims does depend on this amount. It is well-known that for exponential utilities, the initial wealth does not affect the risk preferences. Since the premium profile (p_i, p_{-i}) is generally clear from the context, we simplify our notation by writing $w_i = w_i(p_i, p_{-i})$, $S_i = S_i(N_i(q_i(p_i, p_{-i})))$, and $q_i = q_i(p_i, p_{-i})$.

Now, we observe that the random variable w_i for the wealth is a function of the random variables N_i and S_i, whereas S_i depends on N_i. Thus, we invoke the law of iterated expectations to calculate the expected utility of the wealth of insurer i.

Proposition 4. *Given that insurer i's risk aversion parameter is $\lambda_i > 0$ and N_i follows a Poisson distribution with parameter $q_i(p_i, p_{-i})$, the expected utility of insurer i's wealth is*

$$E\left[u_i\left(w_i\right)\right] = -\exp\left\{-\lambda_i w_i^- + q_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right)\right\}, \tag{6}$$

where $M_X(\lambda_i)$ is the moment generating function of X evaluated at λ_i.

We continue considering the case where the actual number of policies underwritten by insurer i, N_i, follows a Poisson distribution with intensity $q_i(p_i, p_{-i}) \geq 0$. Each insurer i would set a premium p_i such that to maximise her expected utility of wealth. We observe from (6) that $E\left[u_i\left(w_i\right)\right] < 0$, for all p_i. This enables us to convert the maximisation problem into a minimisation procedure, due to the following equivalent characterisation

$$\begin{aligned} \max_{p_i}\left\{E\left[u_i\left(w_i\right)\right]\right\} \text{ or } \min_{p_i}\left\{-E\left[u_i\left(w_i\right)\right]\right\} \\ \text{or } \min_{p_i}\left\{\log(-E\left[u_i\left(w_i\right)\right])\right\}. \end{aligned} \tag{7}$$

Therefore, (6) and (7) lead us to the cost (objective) function $C_i : P \to \mathbb{R}$, which is defined by

$$\begin{aligned} C_i(p_i, p_{-i}) &= \log(-E[u_i(w_i(p_i, p_{-i}))]) \\ &= -\lambda_i w_i^- + q_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right), \end{aligned} \tag{8}$$

for all $i = 1, 2, \ldots, n$, where $P = P_1 \times P_2 \times \cdots \times P_n$ is the set of all strategy profiles, $p_i \in P_i$ is a strategy of insurer i, and $p_{-i} = (p_j : j \neq i) \in P_{-i}$ is a strategy profile of all insurers apart from i.

Remark 5. Let N_i follow a Negative Binomial distribution[1] with parameters $q_i(p_i, p_{-i})$ and $0 < k < 1$. Instead of (8), the cost function is now equal to

$$C_i(p_i, p_{-i}) = -\lambda_i w_i^- + q_i(p_i, p_{-i}) \log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right). \tag{9}$$

3. Model and individual price optimisation

The actual number of policies that insurer i underwrites, N_i, follows a Poisson distribution (or a Negative Binomial distribution later in some remarks), whose expectations depend on $q_i(p_i, p_{-i})$, which we will specify in this section. We assume that the volume of exposure $q_i(p_i, p_{-i})$ is driven by the competition in the market and affected directly by changes in premiums. We capture the interaction among insurers in the market by defining for each insurer i the exposure function as in Taylor (1986), who explained that exposure in the current period should be proportional to exposure in the previous period. Define

$$q_i(p_i, p_{-i}) = f\left(p_i, \bar{p}_{-i}\right) q_i^-, \tag{10}$$

where $\bar{p}_{-i} = \frac{1}{n-1}\sum_{j\neq i} p_j$[2] (e.g., Taylor, 1986, 1987), f is a twice continuously differentiable function of $p_i \in P_i$ and $\bar{p}_{-i}$, and $q_i^- > 0$. Here, q_i^- is the exposure volume of insurer i from the previous period, and $f\left(p_i, \bar{p}_{-i}\right)$ is the relative change of the expected number of policies sold by insurer i. Moreover, the first partial derivative of f with respect to p_i is related to the price-elasticity of demand, which is given by

$$e(p_i) = -\frac{\dfrac{\partial q_i(p_i, p_{-i})}{\partial p_i}}{\dfrac{q_i(p_i, p_{-i})}{p_i}} = -\frac{\partial f\left(p_i, \bar{p}_{-i}\right)}{\partial p_i}\frac{p_i}{f\left(p_i, \bar{p}_{-i}\right)}. \tag{11}$$

Hereafter we consider two distinctive classes regarding the component, $f\left(p_i, \bar{p}_{-i}\right)$. In Section 3.1 we introduce the *exposure function with market restriction*, which contains an upper bound denoted by p^U and achieves the value zero when the premium of insurer i is set equal to p^U. Then, in Section 3.2 we introduce the *exposure function without market restriction*, which does not contain an upper bound, and is such that there is always a positive insurance demand no matter how high the premium is.

3.1. Optimisation problem of exposure function with market restriction

Apart from insurers' utility function, we also consider the utility of an individual in the market. In this regard, we identify one potential policyholder that is the most risk-averse one in the market. This is defined via a parameter h, as in the following assumption.

Assumption 6. The utility function u_M is of the same type as in (1), with risk aversion parameter denoted by $h > 0$. Moreover, the moment generating function of X in h, $M_X(h)$, is finite.

If this agent with utility function u_M is not willing to pay a premium p_i, then nobody in the market wants to buy a policy. Thus, in order to find a policyholder in the market, the premium p_i is assumed to satisfy the following participation constraint:

$$E\left[u_M\left(c - X\right)\right] \leq u_M\left(c - p_i\right), \tag{12}$$

where c is the initial capital of the individual and the utility function u_M satisfies Assumption 6.

Via similar techniques as in Lemma 2, this yields the following upper bound on the premiums:

$$p_i \leq \frac{1}{h}\log\left(M_X\left(h\right)\right) := p^U. \tag{13}$$

The upper premium bound p^U can be interpreted as the maximum premium that can be charged in order to find at least one individual to buy the policy.

If $p_i^L \geq p^U$, insurer i has no incentive to offer any insurance in the market. That is, her risk aversion parameter is so high that her indifference premium is larger to the highest premium that individuals are willing to pay, and so the insurer cannot generate any profit. We assume that $p_i^L < p^U$. From Lemma 3 and (13), this yields $\lambda_i < h$. The feasible set of premiums of insurer i is then defined by

$$P_i = \left[p_i^L,\ p^U\right], \tag{14}$$

where p_i^L and p^U are given by (4) and (13), respectively.

To distinguish the model in the present section from the model in Section 3.2, we introduce separate notation for the model with market restriction. Denote the relative change of the expected number of policies f as $\tilde{f}$ and the volume of exposure q_i as $\tilde{q}_i$.

[1] Detailed calculations for the Negative Binomial are omitted as they are similar to those for the Poisson case.

[2] However, in our mathematical formulation, $\bar{p}_{-i}$ can be any continuous and differentiable function.

We propose the following basic properties for $\tilde{f}$:

$$\frac{\partial \tilde{f}\,(p_i, \bar{p}_{-i})}{\partial p_i} < 0 \quad \text{for all } p_i \in \left[p_i^L, p^U\right), \bar{p}_{-i} < p^U, \tag{15}$$

$$\tilde{f}\,(p_i, \bar{p}_{-i}) = 0 \quad \text{at } p_i = p^U, \text{ for all } \bar{p}_{-i}. \tag{16}$$

In particular, for $\bar{p}_{-i} < p^U$, it holds that $\tilde{f}\,(p_i, \bar{p}_{-i}) \geq 0$ for all $p_i \in P_i$ and $\tilde{f}\,(p_i, \bar{p}_{-i}) > 0$ for all $p_i \in \left[p_i^L, p^U\right)$.

Regarding (11), the property in (15) implies positive price-elasticity of demand. Therefore, any increase in insurer i's premium leads to a decrease in the volume of exposure (see, for instance, Taylor, 1986; Emms et al., 2007; Pantelous and Passalidou, 2015). Moreover, we assume that $\tilde{q}_i(p_i, p_{-i})$ is a non-negative and decreasing function of p_i on P_i, which achieves its lowest value, i.e., zero, at the highest possible value of p_i on P_i.

We next solve the individual optimisation problem, given by

$$\min_{p_i \in P_i} C_i(p_i, p_{-i}), \tag{17}$$

where $p_{-i} \in P_{-i}$.

Lemma 7. *Given the premium profile $p_{-i} \in P_{-i}$ such that $\bar{p}_{-i} \neq p^U$ and let N_i follow a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$, then candidate optimal premiums of (17) for insurer i, $\tilde{p}_i^*$, are the solutions to the equation*

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i \tilde{q}_i(p_i, p_{-i}) M_X(\lambda_i) e^{-\lambda_i p_i} = 0, \tag{18}$$

and belong to the interior of the premium domain P_i, i.e., $\tilde{p}_i^* \in \left(p_i^L, p^U\right)$.

3.2. Optimisation problem of exposure function without market restriction

We next introduce the *exposure function without market restriction*. Here, we denote the relative change of the expected number of policies f as $\hat{f}$ and the volume of exposure q_i as $\hat{q}_i$. We propose the following basic properties for $\hat{f}$:

$$\hat{f}\,(p_i, \bar{p}_{-i}) > 0 \quad \text{for all } p_i \geq p_i^L, \tag{19}$$

$$\frac{\partial \hat{f}\,(p_i, \bar{p}_{-i})}{\partial p_i} < 0 \quad \text{for all } p_i \geq p_i^L. \tag{20}$$

Specifically, Taylor (1986) proposed the demand function as in (10), which satisfies properties (19)–(20). The difference with the previous demand function is that the exposure volume of insurer i will be positive even for large values of her premium p_i. Additionally, regarding (11), the property (20) implies positive price-elasticity of demand.

In this setting, the set of all available premiums of insurer i is equal to

$$P_i = \left[p_i^L, +\infty\right). \tag{21}$$

In the remaining of the paper, it will be clear of the context which set is used, and when necessary, we distinguish them. For P_i as in (21), we next solve the individual optimisation problem given by

$$\min_{p_i \in P_i} C_i(p_i, p_{-i}), \tag{22}$$

where $p_{-i} \in P_{-i}$.

Lemma 8. *Given the premium profile $p_{-i} \in P_{-i}$ and N_i follows a Poisson distribution with parameter $\hat{q}_i(p_i, p_{-i})$, candidate optimal premiums to (22) for insurer i, $\hat{p}_i^*$, are the solutions to the equation*

$$\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i \hat{q}_i(p_i, p_{-i}) M_X(\lambda_i) e^{-\lambda_i p_i} = 0, \tag{23}$$

and satisfy $\hat{p}_i^* > p_i^L$.

Remark 9. Let N_i follow a Negative Binomial distribution with parameters $q_i(p_i, p_{-i})$ and $0 < k < 1$. Using similar arguments as in Lemmas 7 and 8, candidate optimal premiums are the solutions to

$$\begin{aligned} &\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} \log\left(\frac{k M_X(\lambda_i) e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right) \\ &- \frac{\lambda_i q_i(p_i, p_{-i})}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}} = 0, \end{aligned} \tag{24}$$

and belong to the interior of the corresponding premium domain, where $q_i = \tilde{q}_i$ when the market restriction is present, or $q_i = \hat{q}_i$ otherwise. Moreover, the optimal premiums are denoted by $\tilde{p}_i^*$ and $\hat{p}_i^*$ for the restricted and unrestricted models, respectively.

3.3. Convexity analysis

Let us consider the case of N_i following a Poisson distribution with parameter $q_i(p_i, p_{-i})$. Given that $q_i(p_i, p_{-i})$ is twice continuously, partially differentiable with respect to p_i, in our attempt to minimise the cost function $C_i(p_i, p_{-i})$ with respect to p_i, we find all the critical points of C_i by solving the first order condition, and using the second order condition we establish the extremum nature of them. In Sections 3.1 and 3.2, we showed that the optimal solutions must be in the interior of P_i. Generally, by Lemmas 7 and 8, we get that both of them are the solutions of an equation of the form

$$\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i q_i(p_i, p_{-i}) M_X(\lambda_i) e^{-\lambda_i p_i} = 0, \tag{25}$$

where $q_i = \tilde{q}_i$ when the market restriction is present, or $q_i = \hat{q}_i$ otherwise. Now, we investigate the convexity of the objective function $C_i(p_i, p_{-i})$ with respect to p_i. Without any loss of generality, we denote the candidate optimal solutions of (25) with p_i^s and whenever it is necessary we distinguish if $p_i^s \in (p_i^L, p^U)$ or $p_i^s > p_i^L$. The following two results concern both exposure functions, so we keep the general notation of q_i.

From (8), the second partial derivative of $C_i(p_i, p_{-i})$ with respect to p_i, is equal to

$$\begin{aligned} \frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} = &\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) \\ &- 2\lambda_i \frac{\partial q_i(p_i, p_{-i})}{\partial p_i} M_X(\lambda_i) e^{-\lambda_i p_i} \\ &+ \lambda_i^2 q_i(p_i, p_{-i}) M_X(\lambda_i) e^{-\lambda_i p_i}. \end{aligned} \tag{26}$$

Observing that the factors $M_X(\lambda_i)e^{-\lambda_i p_i} - 1$ and $\partial q_i(p_i, p_{-i})/\partial p_i$ in (26) are non-positive, whereas the rest of them are positive, we state two results for the convexity of the cost function C_i, based on $\partial^2 q_i(p_i, p_{-i})/\partial p_i^2$. The first one is global, over all the premium domain of p_i, whereas the second is local, at the solution p_i^s of (25).

Theorem 10. *Let $p_{-i} \in P_{-i}$, p_i^s be a solution which satisfies (25) and N_i follow a Poisson distribution with parameter $q_i(p_i, p_{-i})$. Then, p_i^s*

is a global minimum for the cost function $C_i(p_i, p_{-i})$, if for all $p_i \in P_i$ it holds

$$\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \leq 0. \tag{27}$$

Theorem 11. *Let $p_{-i} \in P_{-i}$, p_i^s be a solution which satisfies (25) and N_i follow a Poisson distribution with parameter $q_i(p_i, p_{-i})$. Then, p_i^s is a local minimum for the cost function $C_i(p_i, p_{-i})$, if*

$$\left.\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2}\right|_{p_i=p_i^s} < \left(\frac{M_X(\lambda_i)e^{-\lambda_i p_i^s}+1}{M_X(\lambda_i)e^{-\lambda_i p_i^s}}\right)\left(\left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i=p_i^s}\right)^2 \left(q_i(p_i^s, p_{-i})\right)^{-1}, \tag{28}$$

and

$$\left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i=p_i^s} < 0. \tag{29}$$

Remark 12. When the insurer i is risk-neutral, i.e., $\lambda_i = 0$, then the result in Theorem 11 is equivalent to the condition $g''(p_k) < [g'(p_k)]^2$ of Theorem 5.3.1. in Taylor (1986) (see also Section 3 in Emms, 2012), where $q_k = f(p_k, \bar{p}_k)q_{k-1}$ and $g(p_k) = \log(f(p_k, \bar{p}_k))$. Then,

$$g'(p_k) = \frac{1}{q_k}\frac{\partial q_k}{\partial p_k},$$

$$g''(p_k) = -\frac{1}{q_k^2}\left(\frac{\partial q_k}{\partial p_k}\right)^2 + \frac{1}{q_k}\frac{\partial^2 q_k}{\partial p_k^2}.$$

Now, the inequality $g''(p_k) < [g'(p_k)]^2$ can be rewritten in terms of the exposure volume q_k as

$$\frac{\partial^2 q_k}{\partial p_k^2} < 2\left(\frac{\partial q_k}{\partial p_k}\right)^2 (q_k)^{-1}.$$

Remark 13. If $N_i \sim \mathrm{NB}(q_i(p_i, p_{-i}), k)$, $0 < k < 1$, the second partial derivative of the cost function with respect to p_i is equal to

$$\begin{aligned}\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} &= \frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2}\log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right)\\ &\quad -2\lambda_i\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^{-1}\\ &\quad +\lambda_i^2 q_i(p_i, p_{-i})\frac{(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}{\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^2}.\end{aligned} \tag{30}$$

Thus, for this case, Theorem 10 provides the same condition for global optimality of premiums that solve the first order condition.

Using the first and second order conditions for optimality of the cost function, and denoting by p_i^s the solutions to the first order condition, we obtain the corresponding result to Theorem 11, which is to replace (28) with

$$\left.\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2}\right|_{p_i=p_i^s} < A\left(\left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i=p_i^s}\right)^2 \left(q_i(p_i^s, p_{-i})\right)^{-1}, \tag{31}$$

where

$$A = 2-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i^s}\log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i^s}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i^s}}\right).$$

4. Exponential exposure function

The exposure function $q_i(p_i, p_{-i})$ describes how the expected number of policies will be sold by insurer i while competing with other insurers. Section 3 provides in general the properties that the exposure functions have to satisfy in our modelling. In this section we define two specific forms of exponential exposure function $q_i(p_i, p_{-i})$, and we prove that the minimisation problems in Sections 3.1 and 3.2 have solutions for both exposure functions.

4.1. Exponential exposure function with market restriction

We introduce a specific exposure function that satisfies the conditions in (15)–(16). The exponential exposure function with market restriction of insurer i is defined by

$$\tilde{q}_i(p_i, p_{-i}) = b\left[1-\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\right]q_i^- \tag{32}$$

for $p_i \in P_i$ and $\bar{p}_{-i} < p^U$. Here, $b > 1$ is a market scale parameter, which represents the available potentials of the market. Larger values of b indicate greater amount of individuals looking to be insured. Moreover, $a_i > 0$ is a price sensitivity parameter, and larger values of it are associated to insurers who possess greater market power. Our interpretation of (32) is that changes in small values of premiums have greater impact on $\tilde{q}_i(p_i, p_{-i})$ than changes in larger premium values. Therefore, the rate of decrease of $\tilde{q}_i$ with respect to p_i is assumed a decreasing function of p_i.

The first and second partial derivatives of $\tilde{q}_i(p_i, p_{-i})$ with respect to p_i are

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} = -b\frac{a_i}{p^U-\bar{p}_{-i}}\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}q_i^- < 0, \quad \text{for all } p_i \in P_i, \tag{33}$$

and

$$\frac{\partial^2 \tilde{q}_i(p_i, p_{-i})}{\partial p_i^2} = -b\left(\frac{a_i}{p^U-\bar{p}_{-i}}\right)^2\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}q_i^- < 0, \quad \text{for all } p_i \in P_i. \tag{34}$$

Proposition 14. *Given the exponential exposure function in (32), the average premium $\bar{p}_{-i} < p^U$ and N_i follows a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$, there is unique optimal solution, $\tilde{p}_i^*$, of the minimisation problem (17), which is given by*

$$\begin{aligned}&\frac{a_i}{p^U-\bar{p}_{-i}}\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right)\\ &\quad +\lambda_i\left[1-\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\right]M_X(\lambda_i)e^{-\lambda_i p_i} = 0,\end{aligned} \tag{35}$$

and satisfies $\tilde{p}_i^* \in \left(p_i^L, p^U\right)$.

Remark 15. When $N_i \sim \mathrm{NB}(\tilde{q}_i(p_i, p_{-i}), k)$, $0 < k < 1$, substituting (32) and (33) into (24), the unique optimal premium $\tilde{p}_i^* \in \left(p_i^L, p^U\right)$ is the solution to

$$\begin{aligned}&\frac{a_i}{p^U-\bar{p}_{-i}}\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right)\\ &+\lambda_i\left[1-\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\right]\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^{-1}\\ &= 0.\end{aligned} \tag{36}$$

4.2. Exponential exposure function without market restriction

In this section, we study a well-known exponential functional form, which is introduced by Taylor (1986). The *exponential exposure function without market restriction* is defined as

$$\hat{q}_i(p_i, p_{-i}) = \exp\left\{-a_i \frac{p_i - \bar{p}_{-i}}{\bar{p}_{-i}}\right\} q_i^-, \tag{37}$$

where $p_i \in P_i$ and $\bar{p}_{-i} > 0$.

We observe that $\hat{q}_i(p_i, p_{-i}) > 0$, for all $p_i \geq p_i^L$, and the first and second partial derivatives with respect to p_i are equal to

$$\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} = -\frac{a_i}{\bar{p}_{-i}} \hat{q}_i(p_i, p_{-i}), \tag{38}$$

$$\frac{\partial^2 \hat{q}_i(p_i, p_{-i})}{\partial p_i^2} = (\frac{a_i}{\bar{p}_{-i}})^2 \hat{q}_i(p_i, p_{-i}). \tag{39}$$

Proposition 16. *Given the exposure function in (37), the premium profile $p_{-i} \in P_{-i}$ and N_i follows a Poisson distribution with parameter $\hat{q}_i(p_i, p_{-i})$, there is unique optimal solution, $\hat{p}_i^*$, of the minimisation problem (22), which is given by*

$$\hat{p}_i^* = -\frac{1}{\lambda_i} \log\left[\frac{a_i}{(a_i + \lambda_i \bar{p}_{-i})\, M_X(\lambda_i)}\right], \tag{40}$$

and satisfies $\hat{p}_i^* > p_i^L$.

Remark 17. Expression (39) implies that $\hat{q}_i(p_i, p_{-i})$ is a convex function in p_i, and the parameter a_i is related to the price-elasticity of demand (Malueg, 1994). Indeed, regarding (11), the price-elasticity of demand is equal to

$$e(p_i) = -\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} \frac{p_i}{\hat{q}_i(p_i, p_{-i})} = a_i \frac{p_i}{\bar{p}_{-i}}.$$

We observe that the price-elasticity of demand is an increasing function of a_i, and as Taylor (1986) indicated, a decrease in price-elasticity is associated to an increase in optimal premiums. This can be proved by showing that the first derivative of (40) with respect to a_i is negative. Indeed, it derives in a straightforward manner that

$$\frac{\partial \hat{p}_i^*}{\partial a_i} = \frac{\partial}{\partial a_i}\left[-\frac{1}{\lambda_i} \log\left(\frac{a_i}{(a_i + \lambda_i \bar{p}_{-i})\, M_X(\lambda_i)}\right)\right]$$
$$= -\frac{\bar{p}_{-i}}{a_i(a_i + \lambda_i \bar{p}_{-i})} < 0, \text{ for all } a_i > 0.$$

Remark 18. When $N_i \sim \mathrm{NB}(\hat{q}_i(p_i, p_{-i}), k)$, $0 < k < 1$, substituting (37) and (38) into (24), optimal premiums $\hat{p}_i^* > p_i^L$ are the solutions to

$$\frac{a_i}{\bar{p}_{-i}} \log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right) + \lambda_i\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^{-1} = 0. \tag{41}$$

Moreover, substituting (38) and (39) into (31), we verify that (31) is satisfied.

5. Nash equilibrium premiums

In the previous sections, we have presented our analysis for a single optimisation problem. However, we are interested in finding the best choice of premiums for all insurers in the market who do not cooperate and act selfishly. To do so, we next study the Nash equilibrium. In a Nash equilibrium, no insurer has an incentive to unilaterally deviate from the premium strategy. The formal definition is provided as follows.

Definition 19. A strategy profile $p^* = (p_1^*, p_2^*, \ldots, p_n^*) \in P$ is a pure strategy Nash equilibrium if for all $i \in N$, the premium p_i^* solves the minimisation problem

$$\min_{p_i \in P_i} C_i(p_i, p_{-i}^*). \tag{42}$$

Now, the next results state the existence of a pure strategy Nash equilibrium in the case of both exposure functions, and how it is calculated.

Theorem 20. *Let N_i follow a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$. There exists unique pure strategy Nash equilibrium $\tilde{p}^*$ in the case of the exponential exposure function with market restriction (32), in which for all $i = 1, 2, \ldots, n$, $\tilde{p}_i^*$ solves*

$$\frac{a_i}{p^U - \bar{\tilde{p}}_{-i}^*} \exp\left\{-a_i \frac{p^U - p_i}{p^U - \bar{\tilde{p}}_{-i}^*}\right\}\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) + \lambda_i\left[1 - \exp\left\{-a_i \frac{p^U - p_i}{p^U - \bar{\tilde{p}}_{-i}^*}\right\}\right] M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \tag{43}$$

and $\tilde{p}_i^* \in (p_i^L, p^U)$.

Remark 21. Let $N_i \sim \mathrm{NB}(\tilde{q}_i(p_i, p_{-i}), k)$, $0 < k < 1$. Similar to Theorem 20, there exists unique pure strategy Nash equilibrium $\tilde{p}^*$ in the case of the exponential exposure function with market restriction (32), in which for all $i = 1, 2, \ldots, n$, $\tilde{p}_i^* \in (p_i^L, p^U)$ solves

$$\frac{a_i}{p^U - \bar{\tilde{p}}_{-i}^*} \exp\left\{-a_i \frac{p^U - p_i}{p^U - \bar{\tilde{p}}_{-i}^*}\right\} \log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right) + \lambda_i\left[1 - \exp\left\{-a_i \frac{p^U - p_i}{p^U - \bar{\tilde{p}}_{-i}^*}\right\}\right]\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^{-1} = 0. \tag{44}$$

Proposition 22. *Let N_i follow a Poisson distribution with parameter $\hat{q}_i(p_i, p_{-i})$. A pure strategy Nash equilibrium $\hat{p}^*$ in the case of the exponential exposure function without market restriction (37) is obtained by solving for all $i = 1, 2, \ldots, n$*

$$p_i = -\frac{1}{\lambda_i} \log\left[\frac{a_i}{\left(a_i + \lambda_i \bar{\hat{p}}_{-i}^*\right) M_X(\lambda_i)}\right], \tag{45}$$

where $\hat{p}_i^* > p_i^L$.

Remark 23. Let $N_i \sim \mathrm{NB}(\hat{q}_i(p_i, p_{-i}), k)$, $0 < k < 1$. A pure strategy Nash equilibrium $\hat{p}^*$ in the case of the exponential exposure function without market restriction (37) is obtained by solving for all $i = 1, 2, \ldots, n$

$$\frac{a_i}{\bar{\hat{p}}_{-i}^*} \log\left(\frac{kM_X(\lambda_i)e^{-\lambda_i p_i}}{1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}}\right) + \lambda_i\left[1-(1-k)M_X(\lambda_i)e^{-\lambda_i p_i}\right]^{-1} = 0, \tag{46}$$

and $\hat{p}_i^* > p_i^L$.

Remark 24. In Theorem 20 and Proposition 22, it is found that both equilibria are affected by insurers' risk aversion parameters, the claims distribution and the sensitivity parameter of the demand functions. On the other hand, the initial wealth does not affect insurers' premium strategies.

Table 3
Model parameters used in Section 6.

Parameters	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
q_i^-	1000	2000	3000	2000	500
w_i^-	10000	10000	10000	10000	10000
λ_i	0.003	0.004	0.006	0.005	0.001
a_i	1.6	1.7	1.8	1.7	1.5
$b = 1.2$					
$m = 100$					
$h = 0.007$					

6. Numerical application

In this section, we present a hypothetical insurance market to illustrate our theoretical results. We assume that there are five insurers in the market competing with each other and we call them Insurer i, for $i = 1, 2, \ldots, 5$. Claims X follow an exponential distribution with mean $m = 100$, and hence the moment generating function is given by

$$M_X(t) = \frac{1}{1 - mt}, \; t < \frac{1}{m}, \tag{47}$$

and the number of policies, N_i, follows a Poisson distribution with intensity $q_i(p_i, p_{-i})$. In the following two subsections, we investigate separately the Nash equilibrium of our two models, as well as we perform a sensitivity analysis of the underlying model parameters.

6.1. Model with market restriction

Firstly, we adopt the exposure function (32), with risk aversion parameter h. Based on Assumptions 1 and 6, the insurers' risk aversion parameters λ_i, $i = 1, 2, \ldots, 5$, and h should be less than $(1/m) = 0.01$, and it holds that $\lambda_i < h$, $i = 1, 2, \ldots, 5$. The market power is illustrated by the shape parameter a_i. In our scenario, Insurer 3 possesses a leading position in the market, Insurers 2 and 4 share the same portion of the market, whereas Insurer 5 is considered the "weakest" player in the market. The set of all the necessary model parameters is summarised in Table 3. Since the insurers use exponential utilities, it is well-known that the value of w_i^- does not affect the risk attitude of the insurer, and thus it does not affect the insurance prices in equilibrium.

The upper bound for all insurers' premiums is given by (13) and equals $p^U = 172.00$, whereas the insurers' indifference premiums is given by (4) and depicted in Table 4. Note that $p_i^L > m = 100$, $i = 1, 2, \ldots, 5$, i.e., the indifference premium is larger than the net premium, producing positive loading for all insurers.

According to Theorem 20, a unique pure strategy Nash equilibrium $\tilde{p}^* = (\tilde{p}_1^*, \ldots, \tilde{p}_5^*)$ exists, and is obtained by solving simultaneously, for all $i = 1, 2, \ldots, 5$, Eq. (43). The equilibrium premiums are illustrated in Table 4. We verify that for each insurer $i = 1, 2, \ldots, 5$ the condition $\tilde{p}_i^* \in \left(p_i^L, p^U\right)$ is satisfied. Table 5 depicts the exposure volume at Nash equilibrium.

Since Insurer 3 possesses the largest proportion of the market, she is more vulnerable to uncertain risks, and hence her risk aversion parameter is the largest one. For this reason, the resulting Nash equilibrium premium for Insurer 3 is the highest one in the market, which leads to significant reduction in her exposure volume for the period under consideration. On the other hand, the Nash equilibrium premiums of Insurers 1 and 5 are lower than the others, in order to gain more share of the market in the upcoming period. Finally, Insurer 4's Nash equilibrium premium is larger than Insurer 2's, since she has higher risk aversion, resulting to larger reduction in her exposure volume.

Table 5
Exposure volume at Nash equilibrium.

Exposure volume	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
$\tilde{q}_i^*$	1025.75	1996.70	2281.58	1851.96	526.87
q_i^-	1000	2000	3000	2000	500

Table 6
Nash equilibrium premiums and exposure volumes for three values of the risk aversion parameter h. The second column presents the upper premium bound p^U.

Parameter h	p^U	$\tilde{p}_1^*$	$\tilde{p}_2^*$	$\tilde{p}_3^*$	$\tilde{p}_4^*$	$\tilde{p}_5^*$
$h = 0.007$	172.00	158.29	159.70	164.75	161.44	156.63
$h = 0.008$	201.18	177.95	179.63	184.51	181.36	176.02
$h = 0.009$	255.84	214.07	215.91	220.71	217.45	212.38
		$\tilde{q}_1^*$	$\tilde{q}_2^*$	$\tilde{q}_3^*$	$\tilde{q}_4^*$	$\tilde{q}_5^*$
$h = 0.007$		1025.75	1996.70	2281.58	1851.96	526.87
$h = 0.008$		998.99	1973.16	2654.98	1893.67	506.32
$h = 0.009$		981.53	1966.01	2833.35	1929.22	488.18

Now, maintaining all other the parameters in Table 3, we increase only the risk aversion parameter h. The upper premium bound is depicted in Table 6 for each value of h. Moreover, this table also shows the pure strategy Nash equilibrium and the corresponding exposure volumes.

As the risk aversion parameter h increases, insureds are willing to pay more for their insurance coverage, which yields the upper premium bound to increase as well. Therefore, all insurers take advantage of this opportunity and increase their premiums in order to produce larger profits. Moreover, we observe that insurers follow the same reasoning in their premium strategy as in the case of $h = 0.007$, i.e weaker insurers set lower premiums to gain market power, whereas larger insurers charge relatively high premiums. For the exposure volumes when $h = 0.008$ or 0.009, it holds that only weaker insurers' exposure decreases, in contrast to Insurers 3 and 4, whose exposure volume increases (see Table 6). On the other hand, the losses of exposure for Insurers 1, 2 and 5 are still lower than the losses for Insurers 3 and 4. This may be due to the fact that insureds are capable of paying more money for their insurance, so they trust insurers with greater market power, which is usually accompanied by better reputation, as mentioned in Pantelous and Passalidou (2013, 2015).

Next, maintaining all the parameters as in Table 3, we vary only the mean m of the exponentially distributed claims X. The upper premium bound and the indifference premiums are shown in Table 7. The corresponding Nash equilibrium and exposure volumes appear in Table 8. When the size of the underlying risk increases, the lower and upper premium bounds increase, too. The corresponding Nash equilibrium premiums increase as well, but following the same pattern as in the case of $m = 100$ and leading to a similar structure for the exposure volume. However,

Table 4
Nash equilibrium premiums.

Premium	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
Upper bound: p^U	172.00	172.00	172.00	172.00	172.00
Nash equilibrium: $\tilde{p}_i^*$	158.29	159.70	164.75	161.44	156.63
Indifference premium: p_i^L	118.89	127.71	152.72	138.63	105.36

Table 7
Indifference premiums and upper bound p^U for three values of the mean m.

Expected claim	p^U	p_1^L	p_2^L	p_3^L	p_4^L	p_5^L
$m = 90$	142.04	104.90	111.57	129.42	119.57	94.31
$m = 100$	172.00	118.89	127.71	152.72	138.63	105.36
$m = 120$	261.80	148.76	163.48	212.16	183.26	127.83

Table 8
Nash equilibrium premiums and exposure volumes for three values of the mean m.

Expected claim	$\tilde{p}_1^*$	$\tilde{p}_2^*$	$\tilde{p}_3^*$	$\tilde{p}_4^*$	$\tilde{p}_5^*$
$m = 90$	132.66	133.69	137.27	134.95	131.39
$m = 100$	158.29	159.70	164.75	161.44	156.63
$m = 120$	230.45	233.02	243.24	236.29	227.86
	$\tilde{q}_1^*$	$\tilde{q}_2^*$	$\tilde{q}_3^*$	$\tilde{q}_4^*$	$\tilde{q}_5^*$
$m = 90$	1028.69	1995.48	2232.14	1839.38	530.74
$m = 100$	1025.75	1996.70	2281.58	1851.96	526.87
$m = 120$	1016.73	1999.01	2422.41	1885.67	514.99

Table 9
Nash equilibrium premiums and exposure volumes for two values of the risk aversion parameter λ_3. Moreover, the second column presents the indifference premium of Insurer 3.

Parameter	p_3^L	$\tilde{p}_1^*$	$\tilde{p}_2^*$	$\tilde{p}_3^*$	$\tilde{p}_4^*$	$\tilde{p}_5^*$
$\lambda_3 = 0.006$	152.72	158.29	159.70	164.75	161.44	156.63
$\lambda_3 = 0.005$	138.63	157.53	159.07	161.20	160.99	155.70
		$\tilde{q}_1^*$	$\tilde{q}_2^*$	$\tilde{q}_3^*$	$\tilde{q}_4^*$	$\tilde{q}_5^*$
$\lambda_3 = 0.006$		1025.75	1996.70	2281.58	1851.96	526.87
$\lambda_3 = 0.005$		1004.46	1949.01	2731.28	1792.63	517.75

the increase of exposure volumes of the smaller Insurers 1 and 5 decreases for larger values of m, whereas the exposure volumes of the larger Insurers 2, 3 and 4 increase. This may indicate that individuals feel more secure with insurers who possess large market power when the underlying risk increases.

Next, we maintain all the parameters as in Table 3 and decrease only the risk aversion parameter of Insurer 3. The indifference premium of Insurer 3 is given in Table 9, for $\lambda_3 = 0.005, 0.006$. The corresponding Nash equilibrium premiums and exposure volumes are presented in Table 9. According to Lemma 3, smaller values of the risk aversion parameter yield smaller values for the indifference premium, as verified in Table 9. Moreover, we observe a general reduction in the Nash equilibrium premiums for all insurers, with the highest decrease in Insurer 3's premium. Since the premium of the largest Insurer 3 is now more appealing than before, her exposure volume presents an increase, whereas all the other insurers' exposure volumes decrease.

Finally, maintaining all the parameters as in Table 3, we decrease the market shape parameter of Insurer 3, a_3, and present the corresponding Nash equilibrium premiums and exposure volumes in Table 10. We observe that the optimal premium for Insurer 3 increases with respect to a_3. Moreover, an increase in Insurer 3's optimal premium yields an increase in the premiums of the other insurers in the Nash equilibrium. However, the relative premium strategies are not affected, since the weaker Insurers 1 and 5 still charge lower premiums than the stronger Insurers 2, 3 and 4.

6.2. Model without market restriction

Now, we adopt the exposure function (37). Based on Assumption 1, insurers' risk aversion parameters λ_i, $i = 1, 2, \ldots, 5$ are less than $(1/m) = 0.01$. It is assumed that less market power implies a greater price sensitivity parameter a_i, as indicated in Lerner (1934), Wu and Pantelous (2017), and Boonen et al. (2018). In particular, let $a_1 = 2.7,\ a_2 = 2.6,\ a_3 = 2.5,\ a_4 = 2.6$, and $a_5 = 2.8$. All other model parameters coincide with the parameters in Table 3. For all cases that we study in this section, the pure strategy Nash equilibrium exists and is unique.

Table 10
Nash equilibrium premiums and exposure volumes for two values of the market shape parameter a_3.

Market shape parameter	$\tilde{p}_1^*$	$\tilde{p}_2^*$	$\tilde{p}_3^*$	$\tilde{p}_4^*$	$\tilde{p}_5^*$
$a_3 = 1.8$	158.29	159.70	164.75	161.44	156.63
$a_3 = 1.6$	158.24	159.66	164.53	161.41	156.57
	$\tilde{q}_1^*$	$\tilde{q}_2^*$	$\tilde{q}_3^*$	$\tilde{q}_4^*$	$\tilde{q}_5^*$
$a_3 = 1.8$	1025.75	1996.70	2281.58	1851.96	526.87
$a_3 = 1.6$	1024.34	1993.53	2161.57	1847.97	526.27

The indifference premiums for every insurer are the same as in Section 6.1, (see Table 4). The pure strategy Nash equilibrium $\hat{p}^* = (\hat{p}_1^*, \ldots, \hat{p}_5^*)$ is obtained by solving simultaneously for all $i = 1, 2, \ldots, 5$ Eq. (45) and illustrated in Table 11. We observe that for each insurer $i = 1, 2, \ldots, 5$ the condition $\hat{p}_i^* > p_i^L$ is satisfied. Table 12 depicts the exposure volumes at Nash equilibrium.

As in Section 6.1, Insurer 3 possesses the largest proportion of the market which makes her more vulnerable to uncertain risks, and hence her risk aversion parameter is the largest one. For this reason, the resulting Nash equilibrium premium for Insurer 3 is the highest one in the market, which leads to significant reduction in her exposure volume for the period under consideration. On the other hand, the Nash equilibrium premiums of Insurers 1 and 5 are lower than the others in order to gain more share of the market in the upcoming period. Finally, Insurer 4's Nash equilibrium premium is larger than the Nash equilibrium premium of Insurer 2, since Insurer 4 has higher risk aversion, resulting to reduction in her exposure volume.

Next, while keeping all other the parameters the same as above, we now vary the mean m of the exponentially distributed claims X. The indifference premiums coincide with the ones in Table 7. The corresponding Nash equilibrium and exposure volumes are shown in Table 13. When the size of the underlying risk increases, the indifference premiums increase too. The corresponding Nash equilibria increase as well, but following the same reasoning as in the initial case of $m = 100$ and leading to an equivalent structure for the exposure volume. Under this model, we observe that insurers 3 and 4, with greater market power, face a reduction in their exposure volume, whilst the rest of them face an increase in their exposure volume. For larger values of m, this may happen because policyholders lapse from insurers 3 and 4 because they are charging high premiums, and buy policies from the other insurers that charge relatively lower premiums.

While keeping all other the parameters the same as above, we decrease only the risk aversion parameter of Insurer 3 (the largest insurer). The indifference premium of Insurer 3 coincides with the ones in Table 9, for each value of λ_3. The corresponding Nash equilibrium and exposure volumes are presented in Table 14. According to Lemma 3, smaller values of the risk aversion parameter yield smaller values for the indifference premium. Moreover, we observe a general reduction in the Nash equilibrium premiums for all insurers, with the highest decrease in the premium of Insurer 3. Since the premium of Insurer 3 is now more appealing than before, the exposure volume of Insurer 3 increases, whereas the exposure volumes of all other insurers decrease.

Finally, while keeping all the other parameters the same as above, we decrease the price sensitivity parameter of Insurer 3, a_3, and present the corresponding Nash equilibrium and exposure volumes in Table 15. Regarding Remark 17, the parameter a_3 is related to the price-elasticity of demand which is an increasing

Table 11
Nash equilibrium premiums.

Premium	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
Nash equilibrium: $\hat{p}_i^*$	184.48	192.89	214.82	201.34	173.81
Indifference premium: p_i^L	118.89	127.71	152.72	138.63	105.36

Table 12
Exposure volume at the Nash equilibrium.

Exposure volume	Insurer 1	Insurer 2	Insurer 3	Insurer 4	Insurer 5
$\hat{q}_i^*$	1167.62	2019.58	2104.31	1749.9	707.23
q_i^-	1000	2000	3000	2000	500

Table 13
Nash equilibrium premiums and exposure volumes for three values of the mean m.

Expected claim	$\hat{p}_1^*$	$\hat{p}_2^*$	$\hat{p}_3^*$	$\hat{p}_4^*$	$\hat{p}_5^*$
$m = 90$	162.94	169.51	185.19	175.56	154.36
$m = 100$	184.48	192.89	214.82	201.34	173.81
$m = 120$	231.01	244.42	287.41	260.34	215.25
	$\hat{q}_1^*$	$\hat{q}_2^*$	$\hat{q}_3^*$	$\hat{q}_4^*$	$\hat{q}_5^*$
$m = 90$	1138.44	2000.2	2231.52	1779.18	678.95
$m = 100$	1167.62	2019.58	2104.31	1749.9	707.23
$m = 120$	1250.45	2087.25	1779.89	1690.33	779.29

Table 14
Nash equilibrium premiums and exposure volumes for two values of the risk aversion parameter λ_3.

Parameter	$\hat{p}_1^*$	$\hat{p}_2^*$	$\hat{p}_3^*$	$\hat{p}_4^*$	$\hat{p}_5^*$
$\lambda_3 = 0.006$	184.48	192.89	214.82	201.34	173.81
$\lambda_3 = 0.005$	183.23	191.67	202.14	200.18	172.45
	$\hat{q}_1^*$	$\hat{q}_2^*$	$\hat{q}_3^*$	$\hat{q}_4^*$	$\hat{q}_5^*$
$\lambda_3 = 0.006$	1167.62	2019.58	2104.31	1749.9	707.23
$\lambda_3 = 0.005$	1125.32	1941.45	2446.05	1674.45	685.04

Table 15
Nash equilibrium premiums and exposure volumes for two values of the price sensitivity parameter a_3.

Price sensitivity	$\hat{p}_1^*$	$\hat{p}_2^*$	$\hat{p}_3^*$	$\hat{p}_4^*$	$\hat{p}_5^*$
$a_3 = 2.5$	184.48	192.89	214.82	201.34	173.81
$a_3 = 2$	185.75	194.12	227.70	202.51	175.19
	$\hat{q}_1^*$	$\hat{q}_2^*$	$\hat{q}_3^*$	$\hat{q}_4^*$	$\hat{q}_5^*$
$a_3 = 2.5$	1167.62	2019.58	2104.31	1749.9	707.23
$a_3 = 2$	1210.4	2098.78	2001.87	1826.67	729.55

function of a_3. As mentioned in Taylor (1986), any decrease in a_3, i.e., in the price elasticity, causes an increase in the optimal premiums, because the loss of the number of policies due to high premiums is relatively small. This phenomenon is clearly illustrated by Table 15.

7. Conclusion

In this paper, we derive the optimal premium strategy for non-life insurance products in a competitive market. The optimal premiums are determined by the Nash equilibrium of an n-player, one-stage game, in which each player is assumed to maximise the expected utility of terminal wealth. For this purpose, a stochastic model is applied to investigate how the randomness of both the number of policies and the size of claims impact the pricing process. The competition among insurers is modelled via two exponential demand functions with different curvatures, and the market power of each insurer is affected by price and market sensitivity parameters. The first model is restricted from above and below, whereas the second model only from below. The pure strategy Nash equilibrium premiums are given as solutions to constrained optimisation problems. We prove that no insurer has an incentive to set premiums equal to the boundaries of the strategy domain. The lower premium bound is the indifference premium of the insurer, whereas the upper bound leads to no insurance buyers. For the first model we prove the existence and uniqueness of a pure strategy Nash equilibrium, whereas for the second we provide a structure when it exists. A hypothetical insurance market with five insurers is constructed to illustrate our findings and analyse the effect of our model parameters in the Nash equilibrium premium profile.

For future work, we are interested in expanding our stochastic model into a dynamic framework and apply dynamic games to investigate how competition may result in premium cycles. Further, we wish to derive mixed strategy Nash equilibria, in which insurers are allowed to randomise their premium strategy.

Appendix. Proofs

Proof of Lemma 2. From (1) and (2) we obtain the following inequality

$$E\left[u_i\left(w_i^- + N_i p_i - S_i(N_i)\right)\right] = E\left[E\left[u_i\left(w_i^- + N_i p_i - S_i(N_i)\right)|N_i\right]\right]$$
$$= E\left[E\left[-e^{-\lambda_i\left(w_i^- + \sum_{j=1}^{N_i}(p_i - X_{ij})\right)}\middle| N_i\right]\right]$$
$$= -e^{-\lambda_i w_i^-} E\left[E\left[\prod_{j=1}^{N_i} e^{-\lambda_i p_i} e^{\lambda_i X}\middle| N_i\right]\right]$$
$$= -e^{-\lambda_i w_i^-} E\left[\prod_{j=1}^{N_i} e^{-\lambda_i p_i} E\left[e^{\lambda_i X}\right]\right] = -e^{-\lambda_i w_i^-} E[(e^{-\lambda_i p_i} M_X(\lambda_i))^{N_i}].$$

From this and (2), we get

$$E[(e^{-\lambda_i p_i} M_X(\lambda_i))^{N_i}] \leq 1. \tag{48}$$

This inequality reads as $E[c^{N_i}] \leq 1$ for some $c \geq 0$, and since $\Pr[N_i > 0] > 0$ this implies $c \leq 1$. Hence, (48) leads to $e^{-\lambda_i p_i} M_X(\lambda_i) \leq 1$. From this, we readily derive (3), which concludes the proof. ■

Proof of Proposition 4. Applying (5) to (1) and taking expectations, we obtain

$$E\left[u_i\left(w_i\right)\right] = E\left[-e^{-\lambda_i w_i}\right] = -e^{-\lambda_i w_i^-} E\left[e^{-\lambda_i(p_i N_i - S_i)}\right]. \tag{49}$$

Considering that N_i follows a Poisson distribution with parameter $q_i(p_i, p_{-i})$ and applying the law of iterated expectations, the second factor on the right-hand side of (49) can be written as

$$E\left[e^{-\lambda_i(p_i N_i - S_i)}\right] = E\left[E\left[e^{-\lambda_i(p_i N_i - S_i)} \mid N_i\right]\right]$$
$$= \sum_{m=0}^{+\infty} e^{-\lambda_i p_i m} E\left[e^{\lambda_i(X_{i1} + X_{i2} + \cdots + X_{im})}\right] \Pr\left[N_i = m\right].$$

Since $X_{i1}, X_{i2}, \ldots, X_{im}$ are independent and identically distributed as X, and the moment generating function of X is finite, we derive

$$E\left[e^{-\lambda_i(p_iN_i-S_i)}\right] = \sum_{m=0}^{+\infty}\left[M_X(\lambda_i)e^{-\lambda_i p_i}\right]^m \Pr\left[N_i=m\right] = P_{N_i}\left(M_X(\lambda_i)e^{-\lambda_i p_i}\right). \tag{50}$$

The probability generating function of N_i is defined by $P_{N_i}(t) = E\left[t^{N_i}\right]$, and it always converges at least for $|t| \leq 1$. From (48), we have that the argument of the probability generating function in (50) is positive and less than or equal to 1, and hence we obtain

$$E\left[e^{-\lambda_i(p_iN_i-S_i)}\right] = \exp\left\{q_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right)\right\}. \tag{51}$$

Substituting (51) to (49), we obtain Eq. (6). ■

Proof of Lemma 7. Given the premium profile p_{-i} charged by all insurers except insurer i, the premium p_i charged by insurer i is the solution to the constrained minimisation problem

$$\min_{p_i\in P_i}\left\{-\lambda_i w_i^- + \tilde{q}_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right)\right\}, \tag{52}$$

subject to

$$-\tilde{q}_i(p_i, p_{-i}) \leq 0, \tag{53}$$

$$M_X(\lambda_i)e^{-\lambda_i p_i} - 1 \leq 0. \tag{54}$$

This is a minimisation problem with inequality constraints and we solve it using the Karash–Kuhn–Tucker (KKT) conditions, which are necessary conditions for optimality. The KKT stationarity condition is

$$\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i}\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right) - \lambda_i\tilde{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} - \mu\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i} - \nu\lambda_i M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \tag{55}$$

and the complementary slackness conditions are

$$-\mu\tilde{q}_i(p_i,p_{-i}) = 0,\ \nu(M_X(\lambda_i)e^{-\lambda_i p_i}-1) = 0, \tag{56}$$

$$\mu \geq 0,\ \nu \geq 0. \tag{57}$$

Any premium p_i which satisfies the KKT conditions (55)–(57) is only a candidate optimal solution. There exist four cases we have to investigate:

Case	$-\tilde{q}_i(p_i, p_{-i})$	$M_X(\lambda_i)e^{-\lambda_i p_i} - 1$
I	$= 0$	$= 0$
II	$= 0$	< 0
III	< 0	$= 0$
IV	< 0	< 0

Case I. Equation $\tilde{q}_i(p_i, p_{-i}) = 0$, and requirements (15) and (16) yield $p_i = p^U$. From equation $M_X(\lambda_i)e^{-\lambda_i p_i} - 1 = 0$ and Lemma 2, we obtain $p_i = p_i^L$. Now, we should show that there exist non-negative multipliers μ and ν which satisfy the KKT stationarity condition (55). From (55), we have that

$$-\mu\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i} - \nu\lambda_i = 0. \tag{58}$$

From (15), and the fact that λ_i is positive, we can find non-negative multipliers μ and ν to satisfy Eq. (58). So, it must hold that $\tilde{p}_i^* = p_i^L$ and $\tilde{p}_i^* = p^U$ is a candidate optimal solution to minimisation problem (52)–(54). Thus, the premium strategy set of insurer i is the point set $P_i = \{\tilde{p}_i^*\}$. This is in contradiction to $\lambda_i < h$. Insurer i is not going to generate profits by charging $\tilde{p}_i^* = p_i^L$ or $\tilde{p}_i^* = p^U$.

Case II. From (56)–(57), we obtain $\nu = 0$. Since $\tilde{q}_i(p_i, p_{-i}) = 0$, we have $p_i = p^U$. Eq. (55) yields

$$\left(M_X(\lambda_i)e^{-\lambda_i p^U} - 1 - \mu\right)\left.\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i}\right|_{p_i=p^U} = 0. \tag{59}$$

There are two sub-cases:

(i) If $\left.\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i}\right|_{p_i=p^U} < 0$, we obtain $\mu = M_X(\lambda_i)e^{-\lambda_i p^U} - 1 < 0$, which is in contradiction with (57).

(ii) If $\left.\frac{\partial\tilde{q}_i(p_i,p_{-i})}{\partial p_i}\right|_{p_i=p^U} = 0$, then (59) holds for any nonnegative μ, and a candidate optimal solution is given by $\tilde{p}_i^* = p^U$, where $p_i^L < p^U$.

Now, we observe that $\tilde{q}_i(p^U, p_{-i}) = 0$, and from (8) we obtain $C_i(p^U, p_{-i}) = -\lambda_i w_i^-$. However, for any $p_i \in \left(p_i^L, p^U\right)$, we have $\tilde{q}_i(p_i, p_{-i}) > 0$, and $M_X(\lambda_i)e^{-\lambda_i p_i} - 1 < 0$. Therefore, from (8), and for any $p_i \in \left(p_i^L, p^U\right)$, we find

$$\begin{aligned} C_i(p_i,p_{-i}) &= -\lambda_i w_i^- + \tilde{q}_i(p_i,p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right) \\ &< -\lambda_i w_i^- = C_i(p^U, p_{-i}). \end{aligned}$$

Thus, $\tilde{p}_i^* = p^U$ is not a local solution to minimisation problem (52)–(54).

Case III. From (56)–(57), we obtain $\mu = 0$. Equation $M_X(\lambda_i)e^{-\lambda_i p_i} - 1 = 0$ yields $p_i = p_i^L$. Eq. (55) implies $-\lambda_i\tilde{q}_i(p_i^L, p_{-i}) - \nu\lambda_i = 0$. From this, $\lambda_i > 0$ and (15)–(16), we get $\nu = -\tilde{q}_i(p_i^L, p_{-i}) < 0$, which is a contradiction to (57).

Case IV. From (56)–(57), we obtain $\mu = \nu = 0$, and substituting them into Eq. (55), we obtain (18).

Therefore, solutions $\tilde{p}_i^*$ to (18) are only candidate optimal solutions, and belong to the interior of the premium domain P_i, i.e. $\tilde{p}_i^* \in \left(p_i^L, p^U\right)$.

In conclusion of the proof, only Case IV provides candidate optimal solutions, $\tilde{p}_i^*$, which are given by Eq. (18), and belong to the open interval $\left(p_i^L, p^U\right)$. ■

Proof of Lemma 8. Given the premium profile p_{-i} charged by all insurers except insurer i, the premium p_i charged by insurer i is the solution to the constrained minimisation problem

$$\min_{p_i\in P_i}\left\{-\lambda_i w_i^- + \hat{q}_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right)\right\}, \tag{60}$$

subject to

$$M_X(\lambda_i)e^{-\lambda_i p_i} - 1 \leq 0. \tag{61}$$

This is a minimisation problem with inequality constraint and we solve it using the KKT conditions. The KKT stationarity condition is

$$\frac{\partial\hat{q}_i(p_i,p_{-i})}{\partial p_i}\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right) - \lambda_i\hat{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} - \nu\lambda_i M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \tag{62}$$

and the complementary slackness conditions are

$$\nu(M_X(\lambda_i)e^{-\lambda_i p_i} - 1) = 0,\ \nu \geq 0. \tag{63}$$

Any premium p_i which satisfies the KKT conditions (62)–(63) is only a candidate optimal solution.

Case I. If $M_X(\lambda_i)e^{-\lambda_i p_i} - 1 = 0$, then it follows from (4) that $p_i = p_i^L$. From the KKT stationarity condition (62) and the fact that $\lambda_i > 0$, we obtain

$$-\lambda_i\hat{q}_i(p_i^L, p_{-i}) - \nu\lambda_i = 0,$$

which implies

$$\nu = -\hat{q}_i(p_i^L, p_{-i}) < 0,$$

where the inequality is due to (19). This is in contradiction with (63).

Case II. If $M_X(\lambda_i)e^{-\lambda_i p_i} - 1 < 0$, then it follows from (63) that $\nu = 0$, and substituting it into the KKT stationarity condition (62), we obtain the candidate optimal solutions, $\hat{p}_i^*$, by (23) and $\hat{p}_i^* > p_i^L$. ■

Proof of Theorem 10. Given inequality (27), we obtain from (26) that for all p_i in the corresponding premium domain

$$\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} \geq 0.$$

Therefore, $C_i(p_i, p_{-i})$ is a convex function in p_i. Now, (27) implies that $q_i(p_i, p_{-i})$ is concave in p_i, and hence $-q_i(p_i, p_{-i})$ is convex in p_i. Moreover, $M_X(\lambda_i)e^{-\lambda_i p_i} - 1$ is convex in p_i, too. Thus, the constraints (53) and (54) of the minimisation problem (52), and (61) of the minimisation problem (60), are convex functions of p_i. Since the cost function $C_i(p_i, p_{-i})$, and the constraints are convex in p_i, the KKT conditions (55)–(57) and (62)–(63) are not only necessary, but also sufficient for global optimality. Therefore, any solution p_i^s to Eq. (25) is a global minimum. ■

Proof of Theorem 11. The second partial derivative of $C_i(p_i, p_{-i})$ with respect to p_i, given by (26), can be rewritten as

$$\begin{aligned}\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} = {}& \frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2}\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) \\ & - \lambda_i \frac{\partial q_i(p_i, p_{-i})}{\partial p_i} M_X(\lambda_i)e^{-\lambda_i p_i} \\ & - \lambda_i \Bigg[\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} M_X(\lambda_i)e^{-\lambda_i p_i} \\ & - \lambda_i q_i(p_i, p_{-i}) M_X(\lambda_i)e^{-\lambda_i p_i}\Bigg]. \end{aligned} \tag{64}$$

Given (29), Eq. (25) evaluated at p_i^s yields

$$M_X(\lambda_i)e^{-\lambda_i p_i^s} - 1 = \left(\left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i = p_i^s}\right)^{-1} \lambda_i q_i(p_i^s, p_{-i}) M_X(\lambda_i)e^{-\lambda_i p_i^s}, \tag{65}$$

and we also derive from (25) that

$$\begin{aligned}&\left.\left[\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} M_X(\lambda_i)e^{-\lambda_i p_i} - \lambda_i q_i(p_i, p_{-i}) M_X(\lambda_i)e^{-\lambda_i p_i}\right]\right|_{p_i = p_i^s} \\ &= \left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i = p_i^s}.\end{aligned} \tag{66}$$

Substituting (65) and (66) into (64), we obtain

$$\begin{aligned}&\left.\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2}\right|_{p_i = p_i^s} \\ &= \left[\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2}\left(\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right)^{-1} \lambda_i q_i(p_i, p_{-i}) M_X(\lambda_i)e^{-\lambda_i p_i} \right. \\ &\quad \left.\left. - \lambda_i \frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\left(M_X(\lambda_i)e^{-\lambda_i p_i} + 1\right)\right]\right|_{p_i = p_i^s}.\end{aligned} \tag{67}$$

In order for p_i^s to be a local minimum for C_i, the second partial derivative of C_i with respect to p_i, evaluated at p_i^s, has to be positive, i.e.

$$\left.\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2}\right|_{p_i = p_i^s} > 0. \tag{68}$$

For interior solutions p_i^s, we have that $\tilde{q}_i(p_i^s, p_{-i})$ and $\hat{q}_i(p_i^s, p_{-i})$ are positive, and thus it is allowed to invert $q_i(p_i^s, p_{-i})$. Now, $\left.\frac{\partial q_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i = p_i^s} < 0$ together with (67) and (68) lead to inequality (28). ■

Proof of Proposition 14. According to Lemma 7, the candidate optimal premiums, $\tilde{p}_i^*$, in the case of the exponential exposure function with market restriction, are given by Eq. (18) and satisfy $\tilde{p}_i^* \in (p_i^L, p^U)$. Substituting the exposure function (32) and its partial derivative (33) into (18), we obtain Eq. (35). Since $\tilde{q}_i(p_i, p_{-i}) \geq 0$, $p_i \in P_i$, and inequalities (33) and (34) are strictly on P_i, we obtain from (26) that

$$\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} > 0 \text{ for all } p_i \in P_i.$$

Therefore, $C_i(p_i, p_{-i})$ is a strictly convex function in p_i. Now, (34) implies that $-\tilde{q}_i(p_i, p_{-i})$ is strictly convex in p_i. Moreover, $M_X(\lambda_i)e^{-\lambda_i p_i} - 1$ is convex in p_i too. Thus, the constraints (53) and (54) of the minimisation problem (52) are convex functions of p_i. Since the constraints are convex in p_i, and the objective function $C_i(p_i, p_{-i})$ is strictly convex in p_i, the optimal solution $\tilde{p}_i^*$ to Eq. (35) is a unique global minimum. ■

Proof of Proposition 16. According to Lemma 8, the candidate optimal premiums, $\hat{p}_i^*$, in the case of the exponential exposure function without market restriction, are given by (23) and satisfy $\hat{p}_i^* > p_i^L$. Substituting (37) and (38) into (23) we obtain

$$-\frac{a_i}{\bar{p}_{-i}}\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \tag{69}$$

which has unique solution $\hat{p}_i^*$ given in (40).

Now, from (4) and (8) we observe that the value of the cost function at $p_i = p_i^L$ equals $C_i(p_i^L, p_{-i}) = -\lambda_i w_i^-$. Since $\hat{q}_i(p_i, p_{-i})$ is positive and $(M_X(\lambda_i)e^{-\lambda_i p_i} - 1)$ is negative for all $p_i > p_i^L$, it holds

$$\begin{aligned}C_i(p_i, p_{-i}) &= -\lambda_i w_i^- + \hat{q}_i(p_i, p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) \\ &< -\lambda_i w_i^- = C_i(p_i^L, p_{-i}).\end{aligned}$$

Therefore, the lower bound p_i^L is not optimal solution to minimisation problems (60)–(61), and $\hat{p}_i^* > p_i^L$.

Now it remains to prove that the solution of (23), given by (40), is indeed a global minimum point for the cost function $C_i(p_i, p_{-i})$. From Theorem 11, substituting (38) and (39) into (28), we verify that (28) is satisfied. Indeed, since (38) satisfies condition (29), we obtain

$$\begin{aligned}\left.\frac{\partial^2 \hat{q}_i(p_i, p_{-i})}{\partial p_i^2}\right|_{p_i = \hat{p}_i^*} &< \left(\frac{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*} + 1}{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*}}\right)\left(\left.\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i}\right|_{p_i = \hat{p}_i^*}\right)^2 \\ &\quad \times \left(\hat{q}_i(\hat{p}_i^*, p_{-i})\right)^{-1}, \\ (\frac{a_i}{\bar{p}_{-i}})^2 \hat{q}_i(\hat{p}_i^*, p_{-i}) &< \left(\frac{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*} + 1}{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*}}\right)\left(\frac{a_i}{\bar{p}_{-i}}\right)^2 \\ &\quad \times \left(\hat{q}_i(\hat{p}_i^*, p_{-i})\right)^2 \left(\hat{q}_i(\hat{p}_i^*, p_{-i})\right)^{-1}, \\ 1 &< \left(\frac{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*} + 1}{M_X(\lambda_i)e^{-\lambda_i \hat{p}_i^*}}\right),\end{aligned}$$

which always holds. Therefore, Theorem 11 yields that the solution $\hat{p}_i^*$ of (69) is a local minimum of $C_i(p_i, p_{-i})$. Moreover, it is unique global minimum, since (69) has unique solution $\hat{p}_i^*$, given by (40). ■

Proof of Theorem 20. We observe that the set of all strategy profiles, $P = P_1 \times P_2 \times \cdots \times P_n$ is convex, non-empty and compact set, since $P_i = \left[p_i^L, p^U\right]$, $i = 1, 2, \ldots, n$. Moreover, in the proof of

Proposition 14, we show that if we substitute (32)–(34) into (26), we obtain

$$\begin{aligned}\frac{\partial^2 C_i(p_i,p_{-i})}{\partial p_i^2} &= -b\left(\frac{a_i}{p^U-\bar{p}_{-i}}\right)^2 \exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\} q_i^- \\ &\times \left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right) \\ &+2\lambda_i b\frac{a_i}{p^U-\bar{p}_{-i}} \exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\} q_i^- \\ &\times M_X(\lambda_i)e^{-\lambda_i p_i} \\ &+\lambda_i^2 b\left[1-\exp\left\{-a_i\frac{p^U-p_i}{p^U-\bar{p}_{-i}}\right\}\right] q_i^- M_X(\lambda_i)e^{-\lambda_i p_i} \\ &> 0 \text{ for all } (p_i,p_{-i})\in P.\end{aligned}$$

Therefore, the cost function $C_i(p_i, p_{-i})$ is strictly convex and continuously differentiable with respect to p_i for all $(p_i, p_{-i}) \in P$ and every insurer $i \in N$. Thus, from Corollary 5.1 in Friesz (2010), a Nash equilibrium exists. We next prove uniqueness in a similar way as Dutang et al. (2013). Theorem 2 in Rosen (1965) states that the Nash equilibrium is unique if, for all $x, y \in P$,

$$\sum_{i=1}^{n} r_i \left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=y}(x_i-y_i)+\sum_{i=1}^{n} r_i \left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=x}(y_i-x_i)<0, \tag{70}$$

for some $r \in \mathbb{R}^n$, with $r_i > 0$, for all $i = 1, 2, \ldots, n$. Considering that $C_i(p_i, p_{-i})$ is strictly convex in p_i, we obtain

$$\left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=y}(x_i-y_i) < C_i(x)-C_i(y), \tag{71}$$

and

$$\left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=x}(y_i-x_i) < C_i(y)-C_i(x). \tag{72}$$

Thus, from (71) and (72), we get

$$\left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=y}(x_i-y_i) + \left.\frac{\partial C_i(p_i,p_{-i})}{\partial p_i}\right|_{p=x}(y_i-x_i) < 0. \tag{73}$$

Setting $r_i = 1$, $i = 1, 2, \ldots, n$, and summing (73) for all $i = 1, 2, \ldots, n$, we obtain (70), verifying the uniqueness of the Nash equilibrium. ■

Proof of Proposition 22. From (37), we observe that $p_i \to +\infty$ yields $\hat{q}_i(p_i, p_{-i}) \to 0$. Therefore, the strategy set of each insurer should be a compact set.

Regarding (26), $C_i(p_i, p_{-i})$ is continuously differentiable with respect to p_i for all $p = (p_i, p_{-i}) \in P$ and every $i \in N$. Using Corollary 5.1 in Friesz (2010) for existence of Nash equilibrium, we can determine under which conditions $C_i(p_i, p_{-i})$ is convex in p_i. Substituting (38) and (39) into (26), we obtain

$$\begin{aligned}\frac{\partial^2 C_i(p_i,p_{-i})}{\partial p_i^2} &= \left(\frac{a_i}{\bar{p}_{-i}}\right)^2\hat{q}_i(p_i,p_{-i})\left(M_X(\lambda_i)e^{-\lambda_i p_i}-1\right) \\ &+2\lambda_i\frac{a_i}{\bar{p}_{-i}}\hat{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} \\ &+\lambda_i^2\hat{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} \\ &= \left[\left(\frac{a_i}{\bar{p}_{-i}}\right)^2+2\lambda_i\frac{a_i}{\bar{p}_{-i}}+\lambda_i^2\right]\hat{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} \\ &-\left(\frac{a_i}{\bar{p}_{-i}}\right)^2\hat{q}_i(p_i,p_{-i}) \\ &= \left(\frac{a_i}{\bar{p}_{-i}}+\lambda_i\right)^2\hat{q}_i(p_i,p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} \\ &-\left(\frac{a_i}{\bar{p}_{-i}}\right)^2\hat{q}_i(p_i,p_{-i}).\end{aligned}$$

In order for $C_i(p_i, p_{-i})$ to be convex in p_i, it needs to hold that

$$p_i < -\frac{1}{\lambda_i}\log\left[\left(\frac{a_i}{a_i+\lambda_i\bar{p}_{-i}}\right)^2\frac{1}{M_X(\lambda_i)}\right]. \tag{74}$$

From (45) it follows that the equilibrium premium $\hat{p}_i^*$ always satisfies (74). ■

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