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THE FOURTEEN VICTORIA DELFINO PROBLEMS AND THEIR STATUS IN THE YEAR 2019

ANDRÉS EDUARDO CAICEDO, BENEDIKT LÖWE

§1. Introduction. The Victoria Delfino Problems played an important role in the development of descriptive set theory in the context of the Cabal. The first set of problems (#1 to #5) were announced during one of the *Very Informal Gatherings of Logicians* (VIG) at UCLA in 1978. They were subsequently published as an Appendix [KM78A] in [CABAL i] with the following explanations and rules:

The following list of problems was distributed during a very informal gathering of logicians at UCLA in January 1978. We are reproducing it here because of its obvious relevance to the contents of this volume.

A cash prize of \$100 is offered by the logicians in the Los Angeles area for the solution of each of the following five problems. This competition is financed by the Victoria Delfino Fund for the Advancement of Logic which was established by a generous contribution from Miss Victoria Delfino.

Employees of UCLA and Caltech and their immediate families (other than students) are ineligible for these prizes; competition is open to everyone else. All decisions by the judges are final. Multiple entries are allowed.

1.1. Victoria Delfino. Victoria Delfino was a realtor in the Los Angeles area who helped Yiannis Moschovakis buy his house.¹ When Tony Martin moved to UCLA, Moschovakis referred him to Delfino, who also became Martin's realtor and found the house where Martin still lives. Two weeks after the sale was finalised, Delfino gave Moschovakis an amount of money as commission for the referral, and did not accept his attempts to reject it.

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¹Most of this section is based on recollections shared by Moschovakis with the first author during a telephone conversation.

As a result, Moschovakis decided instead to use the money to help fund the series of Very Informal Gatherings, the first of which had taken place in the fall of 1975.² The second Very Informal Gathering, in 1978, started a new tradition: with a single exception, all subsequent VIGs taken place on Super Bowl weekend, in late January or early February. Moschovakis comments:

The time of the only exception, there was an earthquake! A clear sign that moving the date was a mistake.

Together with the funding of the Very Informal Gatherings, the money was also set to cover the prizes for the solutions of the five original Delfino Problems. (Contrary to popular belief, no monetary prize was attached to further problems.)

When Moschovakis introduced these five problems (in what he described as one of the most significant Very Informal Gatherings to date), and mentioned the *Victoria Delfino Fund*, Martin, taken by surprise, exclaimed “That’s my broker!” Not all in attendance heard this, and Moschovakis offered no further explanation for the name of the fund. This led for a short while to a variety of conjectures trying to find appropriate interpretations to explain the name.

Originally, the fund was kept in a joint account by Alexander Kechris, Martin, and Moschovakis. It was supplemented by occasional donations from other logicians in the area. Martin reports (personal communication) that by 1998, “all the money in the fund had been used and we had stopped asking people to contribute to it.” Eventually, it became so low that it made sense to use it all and close the account. Nowadays, the Very Informal Gatherings are typically funded through support of the NSF.

As for Delfino, she eventually retired, moved out of state to take care of an ill relative, and her trail disappears there. It is unknown whether she ever found out that her name was associated with the problems or with the Cabal.

1.2. The problems. After the first announcement of the Victoria Delfino Problems, progress reports were published in [CABAL ii] and [CABAL iii]. In 1985, three of the original problems had been solved, and seven new problems (# 6 to # 12) were added and published as [KMS88A] in [CABAL iv], preceded by the following comment:

At the “Very Informal Gathering” of January 1984, the Cabal announced the addition of seven problems to the Victoria Delfino list. We are happy (and not at all embarrassed) to report that

²There is some uncertainty about the date of the first VIG; in preparation for the twentieth VIG in February 2019, the original organisers discussed this question and concluded that “our best recollection now is that the first VIG was in the fall of 1975” (Kechris, personal communication, 2018).

since then four of these problems have been solved. Below we list the new problems, beginning with #6 since there were five problems on the original list. For each we describe briefly what was known when it was added to the list, and what has been its fate since.

In the years following the publication of the final original Cabal volume, there were two more problems announced at one of the Very Informal Gatherings in the late 1980s or early 1990s (the precise date could not be identified), but they were never published as Victoria Delfino Problems. We include these problems as #13 and #14.

Today, two of the problems remain open. The first one is better known under the name of *Martin's Conjecture* (#5), the other one has now been embedded into Woodin's theory AD^+ (#14). In Table 1, the reader can find a synoptic list of the problems with their current status.

	Published in	Status
# 1	[KM78A]	Solved by Steve Jackson (1983)
# 2	[KM78A]	Solved by Yiannis Moschovakis (1981)
# 3	[KM78A]	Solved by Howard Becker & Alexander Kechris (1983)
# 4	[KM78A]	Solved by John Steel (1993)
# 5	[KM78A]	Open
# 6	[KMS88A]	Solved by John Steel (1984)
# 7	[KMS88A]	Solved by Steve Jackson (1985)
# 8	[KMS88A]	Solved by John Steel (1994)
# 9	[KMS88A]	Solved by Tony Martin & John Steel (1985)
# 10	[KMS88A]	Solved by W. Hugh Woodin & Saharon Shelah (1985)
# 11	[KMS88A]	Solved by John Steel (1994)
# 12	[KMS88A]	Solved by John Steel (1997)
# 13	unpublished	Solved by W. Hugh Woodin (1999)
# 14	unpublished	Open

TABLE 1. List of the Victoria Delfino Problems and their current status

This paper is organised as follows: Each problem is presented in its own section that, except for the last two problems, starts with a quote from the original Cabal volumes under the headline *Original problem*. The quotation is essentially literal, although we have followed the general practice of modernising and homogenising notation and writing style. For the first five problems, the original formulation is followed by one or several subsections entitled *Progress report* where we reproduce the text from subsequent Cabal volumes providing updates on the problem. We then proceed with a brief discussion of the current state of knowledge in a subsection entitled *2019 comments*.

1.3. Acknowledgements. We should like to thank Kai Hauser, Daisuke Ikegami, Antonio Montalbán, Jan Reimann, Ralf Schindler, Ted Slaman, John Steel, Simon Thomas, Hugh Woodin, and Yizheng Zhu for detailed remarks and comments. Particular thanks are due to Yiannis Moschovakis and Tony Martin for their recollections on Victoria Delfino, the Delfino Fund, and the early history of the Very Informal Gatherings.

1. Projective Ordinals.

Original problem [KM78A]. For each positive integer n , let δ_n^1 be the least nonzero ordinal not the length of a Δ_n^1 prewellordering of the reals. Assume AD + DC. It is known that $\delta_1^1 = \omega_1$, $\delta_2^1 = \omega_2$, $\delta_3^1 = \omega_{\omega+1}$, $\delta_4^1 = \omega_{\omega+2}$, $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^+$, and δ_{2n+1}^1 is always the successor (cardinal) of a cardinal of cofinality ω .

PROBLEM # 1. Compute δ_5^1 .

Kunen has some partial results on this problem, results which suggest the answer ω_{ω^3+1} .

The problem is related to that of whether $\delta_3^1 \rightarrow (\delta_3^1)^{\delta_3^1}$. Kunen has shown that $\delta_3^1 \rightarrow (\delta_3^1)^\alpha$ for each $\alpha < \delta_3^1$. Results of Kleinberg imply that δ_3^1 has exactly three normal measures. It is likely that the regular cardinals between δ_3^1 and δ_5^1 are exactly the ultrapowers of δ_3^1 with respect to these normal measures. This would be important in getting an upper bound on δ_5^1 from Choice plus $\text{AD}^{\mathbf{L}(\mathbb{R})}$, the hypothesis that every set of reals in $\mathbf{L}(\mathbb{R})$ is determined.

(Needless to say, the decision of the judges as to what constitutes a “computation” of δ_5^1 will be final.)

Progress report [KMM81A]. Martin has established the conjectured lower bound for δ_5^1 by proving (from AD + DC) that

$$\delta_5^1 \geq \aleph_{\omega^3+1};$$

moreover Martin showed (from AD) that the ultrapowers of $\delta_3^1 = \aleph_{\omega+1}$ under the three normal measures on δ_3^1 are exactly $\delta_4^1 = \aleph_{\omega+2}$ (this was known to Kunen), $\aleph_{\omega \cdot 2+1}$ and \aleph_{ω^2+1} and that these three cardinals are measurable (and hence regular), so that (in particular), δ_5^1 is not the first regular cardinal after δ_4^1 . We still have no upper bounds for δ_5^1 from AD.

Progress report [KMM83A]. It was announced in [KMM81A] that Martin had shown $\delta_5^1 \geq \aleph_{\omega^3+1}$ and that the ultrapowers of δ_3^1 with respect to the three normal measures on δ_3^1 are $\aleph_{\omega+2}$, $\aleph_{\omega \cdot 2+1}$ and \aleph_{ω^2+1} . The proof of part of the last assertion, that the ultrapower by the ω_2 -cofinal measure is $\leq \aleph_{\omega^2+1}$, was incorrect. Actually this ultrapower is larger ($\aleph_{\omega^\omega+1}$).

Steve Jackson has completely solved the first problem. He first proved that $\delta_5^1 \leq \aleph_{\omega^{(\omega^\omega)+1}}$. This result will appear in his UCLA Ph.D. Thesis. He next used the machinery for getting this upper bound to analyze all measures on δ_3^1 and to get a good representation of functions with respect to these measures. Martin observed that this representation and ideas of Kunen allow one to show $\delta_3^1 \rightarrow (\delta_3^1)^{\delta_3^1}$. From this it follows by a result of

Martin that the ultrapower of δ_3^1 with respect to any of its measures is a cardinal. Jackson’s analysis then gives $\delta_5^1 \geq \aleph_{\omega^{(\omega)}+1}$ so $\delta_5^1 = \aleph_{\omega^{(\omega)}+1}$.

2019 comments. Steve Jackson not only solved Problem #1, but also solved the problem in general for all projective ordinals. He computed δ_{2n+1}^1 to be $\aleph_{\mathbf{e}_{n+1}}$ where $\mathbf{e}_0 := 0$ and $\mathbf{e}_{i+1} := \omega^{(\omega^{\mathbf{e}_i})}$ (i.e., \mathbf{e}_n is an exponential ω -tower of height $2n - 1$).

However, Jackson’s paper [Jac88] where the inequality $\delta_{2n+1}^1 \leq \aleph_{\mathbf{e}_{n+1}}$ is established, is notoriously hard to read, and so in the decades following his solution of the problem, Jackson produced various expositions of the results. As the title “*A computation of δ_5^1* ” suggests, his book [Jac99] focuses on the (complete) computation of δ_5^1 as asked in the original problem, and explains how to proceed to compute all projective ordinals via an inductive analysis. His survey paper [Jac10] also discusses extensions of these results beyond the projective ordinals:

In the early 1980s, Martin [Mar] obtained a result on the ultrapowers of δ_3^1 by the normal measures on δ_3^1 . Building on this and some joint work with Martin, [Jackson] computed δ_5^1 . In the mid-1980s, this was extended to compute all the δ_n^1 , and to develop the combinatorics of the cardinal structure of the cardinals up to that point. The analysis, naturally, proceeded by induction. The complete “first-step” of the induction appears in [Jac99]. The analysis revealed a rich combinatorial structure to these cardinals. [...] A goal, then, is to extend some version of this “very-fine” structure theory to the entire model $\mathbf{L}(\mathbb{R})$. In the late 1980s, [Jackson] extended the analysis further, up to the least inaccessible cardinal in $\mathbf{L}(\mathbb{R})$, although this lengthy analysis has never been written up. It was clear, however, that new, serious problems were being encountered shortly past the least inaccessible. In [Jac91], for example, results were given that show that the theory fell far short of $\kappa^{\mathbb{R}}$, the ordinal of the inductive sets (the Wadge ordinal of the least non-selfdual pointclass closed under real quantification). [Jac10, p. 1755]

Part of the extended results is what is known as *Kechris’s theorem*:

THEOREM 1. *Assume $\mathbf{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$. If κ is an inaccessible Suslin cardinal,³ then κ , κ^+ , and κ^{++} are measurable.*

Kechris’s theorem remained unpublished for many years; a proof and generalisations to polarised partition properties for κ , κ^+ , and κ^{++} can be found in [AJL13].

³E.g., the Kleene ordinal discussed in Problem #7 or $\kappa^{\mathbb{R}}$, the least non-hyperprojective ordinal; cf. [AJL13, Proposition 5].

Further projects to make the proof and its generalisations more accessible are what Jackson calls “description theory” (cf. [JK16]) and the “simple inductive arguments” given in [L ow02, BL07, Bol09]. On the basis of the inductive analysis, Jackson and the second author developed an abstract theory of *canonical measure assignments* that allows blackboxing the proofs of partition properties and deriving consequences (such as the behaviour of the cofinality function or the calculation of the measurable cardinals) directly by induction (cf. [JL13]).

The ZFC context. At the end of the original formulation of Problem #1, the question of calculating bounds for the projective ordinals in $\text{ZFC} + \text{AD}^{\mathbb{L}(\mathbb{R})}$ is mentioned. The original AD-results listed in the original formulation of Problem #1 yield upper bounds in this context: $\delta_1^1 = \aleph_1$, $\delta_2^1 \leq \aleph_2$, $\delta_3^1 \leq \aleph_3$, $\delta_4^1 \leq \aleph_4$, $\delta_5^1 \leq \aleph_7$. In general, Jackson’s analysis shows that under AD, there are exactly $2^n - 1$ regular cardinals below $\delta_{2^{n+1}}^1$; thus $\text{ZFC} + \text{AD}^{\mathbb{L}(\mathbb{R})}$ proves that $\delta_n^1 < \aleph_\omega$ for every natural number n . Martin conjectured that “for all n , $\delta_n^1 = \aleph_n$ ” should follow from $\text{ZFC} + \text{AD}^{\mathbb{L}(\mathbb{R})}$ plus reasonable additional assumptions (cf., e.g., [Woo99, p. 5]).

In [Woo99], Woodin develops a very powerful technique to produce models of ZFC as forcing extensions of models of determinacy, the analysis of which provides a solution to Martin’s conjecture in the case $n = 2$:

THEOREM 2 (Woodin; [Woo99, Theorem 1.1 & §3.1]). *If the nonstationary ideal on ω_1 is ω_2 -saturated and $\wp(\omega_1)^\#$ exists, then $\delta_2^1 = \aleph_2$.*

On the other hand, Woodin points out that current techniques produce models where $\delta_3^1 < \Theta^{\mathbb{L}(\mathbb{R})} = \aleph_3$, and asks—in contrast to Martin’s conjecture—whether it is a theorem of $\text{ZFC} + \text{AD}^{\mathbb{L}(\mathbb{R})}$ that $\Theta^{\mathbb{L}(\mathbb{R})} \leq \aleph_3$ [Woo99, §1.5]. Martin’s conjecture for $n > 2$ and the competing question by Woodin remain open.

#2. The extent of definable scales.

Original problem [KM78A]. A **semiscale** on a set $P \subseteq \mathbb{R}^k$ ($\mathbb{R} = {}^\omega\omega$) is a sequence $\vec{\varphi} = \{\varphi_n : n \in \omega\}$ of **norms** on P , where each $\varphi_n : P \rightarrow \lambda$ maps P into some ordinal λ and the following convergence condition holds: If $x_0, x_1, \dots \in P$ and for each n the sequence $\varphi_n(x_0), \varphi_n(x_1), \varphi_n(x_2), \dots$ is ultimately constant, then $x \in P$. We call $\vec{\varphi}$ a **scale** if, under the same hypotheses, we can infer that

$$\varphi_n(x) \leq \varphi_n(x_i) \text{ for all large } i.$$

A semiscale $\vec{\varphi}$ is in a class of relations Γ if both relations

$$U(n, x, y) \iff x \in P \wedge [y \notin P \vee \varphi_n(x) \leq \varphi_n(y)]$$

$$V(n, x, y) \iff x \in P \wedge [y \notin P \vee \varphi_n(x) < \varphi_n(y)]$$

are in Γ .

It is easy to check that a set P admits a semiscale $\vec{\varphi}$ into λ if and only if P is λ -Suslin, *i.e.*, P is the projection of some tree T on $\omega^k \times \lambda$; moreover, T is definable exactly when $\vec{\varphi}$ is definable. Sets which admit definable scales are well-behaved in many ways, e.g. we can use a scale on $P \subseteq \mathbb{R} \times \mathbb{R}$ to uniformise P .

Granting projective determinacy, we can prove that every projective set admits a projective scale (Moschovakis); on the other hand it is easy to check that $\{(x, y) : x \text{ is not ordinal definable from } y\}$ does not admit a scale which is OD in a real, granting only that for each y there is some x which is not OD in y . Thus not every “definable” set admits a “definable” scale.

The strongest result we can get with current methods is that inductive sets admit inductive scales, granting inductive determinacy; here P is inductive if P is Σ_1 over the smallest admissible set M which contains the reals, $\mathbb{R} \in M$.

PROBLEM # 2. *Assume ZF + DC + AD + V=L(\mathbb{R}); prove or disprove that every coinductive set of reals is λ -Suslin for some λ .*

Progress report [KMM81A]. The problem was solved by Moschovakis who showed (from AD+DC) that every coinductive pointset admits a scale. If we let Σ_0^* be the set of all Boolean combinations of inductive and coinductive sets and then define Σ_n^* by counting quantifiers over \mathbb{R} in front of a Σ_0^* matrix in the usual way, then the proof shows that every coinductive set admits a scale $\{\varphi_n : n \in \omega\}$, where each φ_n is a Σ_{n+1}^* -norm, uniformly in n .

Martin and Steel extended the method used by Moschovakis in this proof and showed that ZF + DC + AD + V=L(\mathbb{R}) implies that every Σ_1^2 set admits a Σ_1^2 -scale; this combines with an earlier result of Kechris and Solovay to

show that $\text{ZF} + \text{DC} + \text{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$ implies that a pointset admits a scale if and only if it is Σ_1^2 .

Martin then combined these ideas with the technique of the Third Periodicity Theorem [Mos80, Theorem 6E.1] and showed that under reasonable hypotheses of determinacy for games on \mathbb{R} , (namely, $\text{AD}_{\mathbb{R}}$), the scale property is preserved by the game quantifier \mathfrak{G}^2 on \mathbb{R} , where

$$(\mathfrak{G}^2\alpha)P(x, \alpha) \iff (\exists\alpha_0)(\forall\alpha_1)(\exists\alpha_2)(\forall\alpha_3)\dots P(x, \langle\alpha_0, \alpha_1, \dots\rangle).$$

This result produces scales for sets that are not Σ_1^2 in $\mathbf{L}(\mathbb{R})$ and leaves open the general question of the extent of scales in the presence of axioms stronger than AD.

2019 comments. Moschovakis's paper appeared as [Mos83]. The result of Martin and Steel appears in [MS83]. Martin's theorem on preservation of scales under the game quantifier \mathfrak{G}^2 is in [Mar83]. Related results by Steel are in [Ste83B] and [Ste83A]. The latter paper introduces the key fine structural analysis of $\mathbf{L}(\mathbb{R})$ via gaps that is now used in the core model induction. These results have been further extended by Steel (under appropriate large cardinals or stronger determinacy assumptions), *cf.*, *e.g.*, [Ste08E, Ste08D], and the introduction [Ste08B]. These extensions are needed for core model inductions whose goal is to reach models of strong determinacy assumptions.

#3. The invariance of $\mathbf{L}[T^3]$.

Original problem [KM78A]. Let n be an *odd* integer. Let P be a complete Π_n^1 set of reals and assuming PD let $\vec{\varphi} = \{\varphi_m : m \in \omega\}$ be a Π_n^1 -scale on P . (It is understood here that each φ_m maps P onto an initial segment of the ordinals.) The tree $T^n = T^n(\vec{\varphi})$ associated with this scale is defined by

$$T^n = \{\langle \alpha(0), \varphi_0(\alpha), \dots, \alpha(k), \varphi_k(\alpha) \rangle : \alpha \in P\}.$$

Let $\text{AD}^{\mathbf{L}(\mathbb{R})}$ be the hypothesis that every set of reals in $\mathbf{L}(\mathbb{R})$ is determined.

PROBLEM # 3. *Assume $\text{ZF} + \text{DC} + \text{AD}^{\mathbf{L}(\mathbb{R})}$. Prove or disprove that $\mathbf{L}[T^3] = \mathbf{L}[T^3(\vec{\varphi})]$ is independent of the choice of the complete Π_3^1 set P and the particular Π_3^1 -scale $\vec{\varphi}$ on P .*

It is known that $\mathbf{L}[T^1] = \mathbf{L}$ (Moschovakis). Also under the above hypothesis it is known that for all odd n and all $T^n = T^n(\vec{\varphi})$, $\mathbf{L}[T^n] \cap \mathbb{R} = C_{n+1}$, where C_{n+1} is the largest countable Σ_{n+1}^1 set of reals (Harrington-Kechris), so that $\mathbb{R} \cap \mathbf{L}[T^n]$ does not depend on the choice of T^n .

In many ways, the model $\mathbf{L}[T^n]$ is an excellent analog of \mathbf{L} for the $(n+1)$ -st level of the analytical hierarchy.

Progress report [KMM81A]. Kechris showed in [Kec81] that if $T^3 = T^3(\vec{\varphi})$ is the tree associated with some Π_3^1 -scale $\vec{\varphi}$ on a Π_3^1 -complete set P and if

$$\tilde{\mathbf{L}}[T^3] = \bigcup_{\alpha \in \mathbb{R}} \mathbf{L}[T^3, \alpha],$$

then $\text{ZF} + \text{AD} + \text{DC} + \delta_3^1 \rightarrow (\delta_3^1)^{\delta_3^1}$ implies that $\tilde{\mathbf{L}}[T^3]$ is independent of the choice of P and $\vec{\varphi}$.

This partial result emphasises the importance of the question of the *strong partition property* for δ_3^1 which is still open.

Progress report [KMM83A]. The problem was solved by Becker and Kechris who showed that $\mathbf{L}[T^3]$ is independent of the choice of T^3 . This is a consequence of the following fact, which is a theorem of $\text{ZF} + \text{DC}$.

THEOREM 3. *Let Γ be an ω -parametrised pointclass closed under \wedge and recursive substitution and containing all recursive sets. Let $P \subset \mathbb{R}$ be a complete Γ set, $\vec{\varphi} = \{\varphi_i : i \in \omega\}$ be an $\exists^{\mathbb{R}}\Gamma$ -scale on P such that all norms φ_i are regular, and $\kappa = \sup\{\varphi_i(x) : i \in \omega, x \in P\}$. Let $T(\vec{\varphi})$ be the tree on $\omega \times \kappa$ associated with $\vec{\varphi}$. For any set $A \subset \kappa$, if A is $\exists^{\mathbb{R}}\Gamma$ -in-the-codes with respect to $\vec{\varphi}$ (that is, if the set $\{\langle i, x \rangle \in \omega \times \mathbb{R} : x \in P \wedge \varphi_i(x) \in A\}$ is $\exists^{\mathbb{R}}\Gamma$), then $A \in \mathbf{L}[T(\vec{\varphi})]$.*

In general, given two such scales $\vec{\varphi}, \vec{\psi}$, it is not known that $T(\vec{\psi})$ is $\exists^{\mathbb{R}}\Gamma$ -in-the-codes with respect to $\vec{\varphi}$, so the invariance of $\mathbf{L}[T(\vec{\varphi})]$ has not been

shown in this generality. However there are special cases where invariance can be proved. Henceforth, assume AD.

In Moschovakis [Mos80, p. 562], a model H_Γ is defined for every pointclass Γ which resembles Π_1^1 ; this includes the pointclasses Π_n^1 for odd n . It follows from Theorem 3, together with known results about the H_Γ 's [Mos80, 8G], that for any Γ , P , $\vec{\varphi}$ such that Γ resembles Π_1^1 and Γ , P , $\vec{\varphi}$ satisfy the assumptions of Theorem 3, $L[T(\vec{\varphi})] = H_\Gamma$, and hence $\mathbf{L}[T(\vec{\varphi})]$ is independent of the choice of P and $\vec{\varphi}$. For $\Gamma = \Pi_3^1$ this solves the third problem.

While the invariance problem for $\mathbf{L}[T^n]$ is thus solved for odd n , for even n the situation is still unclear. Call a Σ_3^1 -scale on a Π_2^1 set **good** if it satisfies the ordinal quantification property of Kechris-Martin [KM78]. It follows from the above theorem that $\mathbf{L}[T^2]$ is independent of the choice of a complete Π_2^1 set and of the choice of a good scale. Whether or not it is independent of the choice of an arbitrary scale is unknown. For even $n > 2$, it is not known whether there exist any good scales.

2019 comments. The result by Becker and Kechris for odd n appears in [BK84]. In [Hjo96B], Hjorth shows that, under $\text{Det}(\underline{\Pi}_2^1)$, the model $\mathbf{L}[T^2]$ is independent of the exact choice of T^2 . His argument uses forcing to analyze Π_3^1 equivalence relations. In [Hjo95], he uses properties of $\mathbf{L}[T^2]$ to draw descriptive set theoretic consequences of the assumption that all reals have sharps, in particular showing that if all reals have sharps and MA_{ω_1} holds, then all Σ_3^1 sets are Lebesgue measurable. Further work on $\mathbf{L}[T^2]$ using fine-structural techniques has been carried out by Hauser [Hau99].

In [Atm], Atmai shows that $\mathbf{L}[T^{2n}]$ is independent of the choice of T^{2n} , assuming $\text{Det}(\underline{\Pi}_{2n}^1)$. His proof involves an appropriate generalisation of the Kechris-Martin theorem to the odd levels of the projective hierarchy. Atmai also shows that the $\mathbf{L}[T^{2n}]$ are not extender models, but satisfy some of their properties, such as GCH.

Meanwhile, developments in inner model theory have provided us both with new methods for analyzing the models $\mathbf{L}[T^n]$, and with the proper analogues of \mathbf{L} for higher levels of the analytic hierarchy, the fine structural models \mathbf{M}_n . Recall that (under appropriate large cardinal assumptions) \mathbf{M}_n is the canonical minimal inner model for the assumption that there are n Woodin cardinals. In [Ste95B], Steel gives a precise definition of \mathbf{M}_n in terms of n -smallness and shows that \mathbf{M}_n is Σ_n^1 -correct, and that $\mathbb{R} \cap \mathbf{M}_n = C_n$ for n even, and $\mathbb{R} \cap \mathbf{M}_n = Q_n$ for odd n . For odd n , it is unknown whether the sets C_n in general have an inner model theoretic characterisation. For $n = 1$, Guaspari, Kechris, and Sacks independently showed that $C_1 = \{x \in \mathbb{R} : x \in \mathbf{L}_{\omega_1^x}\}$ [Gua73, Kec75, Sac76]. For $k > 0$, the analogous statement “ C_{2k+1} is the set of reals Δ_{2k+1}^1 -equivalent to the

first order theory of some level of \mathbf{M}_{2^k} projecting to ω " is open and known as the C_3 **conjecture** [Ste08B, p. 13] (*cf.* also [GH76, Cra85, Zhu17]).

4. The strength of $\text{Sep}(\Sigma_3^1)$ in the presence of sharps.

Original problem [KM78A]. Let $(\#)$ stand for “for all $x \subseteq \omega$, $x^\#$ exists” and let $\text{Sep}(\Sigma_3^1)$ denote “for every $x \subseteq \omega$, every two disjoint $\Sigma_3^1(x)$ sets of reals can be separated by a $\Delta_3^1(x)$ set.”

PROBLEM # 4. *Prove or disprove that*

$$\text{ZFC} + \text{Sep}(\Sigma_3^1) + (\#) \text{ implies } \text{Det}(\Delta_2^1).$$

Harrington has shown that $\text{ZFC} + \text{Sep}(\Sigma_3^1)$ is consistent relative to ZFC . However, using Jensen’s Absoluteness Theorem for the core model \mathbf{K} (which states that if $(\#)$ holds and Σ_3^1 formulas are not absolute for \mathbf{K} , then 0^\dagger exists) one can see that

$$\text{ZF} + \text{DC} + \text{Sep}(\Sigma_3^1) + (\#) \text{ implies that } x^\dagger \text{ exists for all } x \subseteq \omega.$$

2019 comments. Problem # 4 was solved with core model techniques by John Steel, following the approach mentioned in the last paragraph of the original problem. The result appears as [Ste96, Corollary 7.14]. The key result is that if there are no inner models with Woodin cardinals and there exists a measurable cardinal, then \mathbf{K} is Σ_3^1 -correct [Ste96, Theorem 7.9]. In the setting of that book, an additional larger measurable Ω is assumed in the background and a set sized \mathbf{K} is built of height Ω ; this additional assumption is now known not to be necessary; cf. [JS13].

To solve Problem # 4 affirmatively, Steel argues that $\text{Sep}(\Sigma_3^1) + (\#)$ implies that for every real x there is a proper class model M with $x \in M$, and an ordinal δ such that $\mathbf{V}_{\delta+1}^M$ is countable, and δ is Woodin in M . By results of Woodin, this implies $\text{Det}(\Delta_2^1)$ (cf. the **2019 comments** on Problem # 9 and [Nee10, Corollary 6.12]). To see that such a model M exists, one first uses the core model argument mentioned in the original problem: for any real y , the Σ_3^1 -correctness of the Mitchell core model gives a proper class model N with $y \in N$ and two measurable cardinals.⁴

Once we have N , Steel argues that if y is chosen carefully to ensure that $\text{Sep}(\Sigma_3^1)$ relativises down from \mathbf{V} to N , the \mathbf{K}_x construction inside N must fail: Assuming that $(\mathbf{K}_x)^N$ exists, then it is Σ_3^1 -correct inside N . But there is a $\Delta_3^1(x)$ well-ordering of the reals of $(\mathbf{K}_x)^N$, which implies the failure of $\text{Sep}(\Sigma_3^1)$ inside $(\mathbf{K}_x)^N$. But by our choice of y , $\text{Sep}(\Sigma_3^1)$ relativises down from \mathbf{V} to N and the correctness of \mathbf{K}_x inside N implies that it further relativises down from N to $(\mathbf{K}_x)^N$, which is impossible.

Thus, the \mathbf{K}_x construction inside N fails and therefore $(\mathbf{K}_x)^N$ reaches a Woodin cardinal, and an iterate of an appropriate hull of $(\mathbf{K}_x)^N$ is the model M as needed.

⁴If we make use of [JS13], we only need one measurable cardinal and could use the Dodd-Jensen core model here.

It is still open whether there is a Σ_3^1 -correctness theorem for \mathbf{K} (in the absence of Woodin cardinals) without additional assumptions beyond the existence of sharps.

5. A classification of functions on the Turing degrees.

Original problem [KM78A]. We write \mathcal{D} for the set of Turing degrees. A property P of degrees holds almost everywhere (a.e.) if and only if there is a \mathbf{c} such that for all $\mathbf{d} \geq \mathbf{c}$, we have $P(\mathbf{d})$. For $f, g: \mathcal{D} \rightarrow \mathcal{D}$, let $f \leq_m g$ if and only if $f(\mathbf{d}) \leq g(\mathbf{d})$ a.e. A function $f: \mathcal{D} \rightarrow \mathcal{D}$ is **representable** if and only if there is some $F: {}^\omega\omega \rightarrow {}^\omega\omega$ such that for all x , $\deg(F(x)) = f(\deg(x))$.

PROBLEM # 5. Working in $\text{ZF} + \text{AD} + \text{DC}$, settle the following conjectures of D. Martin:

- (a) If $f: \mathcal{D} \rightarrow \mathcal{D}$ is representable and $\mathbf{d} \not\leq f(\mathbf{d})$ a.e., then there is a \mathbf{c} such that $f(\mathbf{d}) = \mathbf{c}$ a.e.
- (b) The relation \leq_m is a prewellorder of $\{f : f \text{ is representable and } \mathbf{d} \leq f(\mathbf{d}) \text{ a.e.}\}$.

Further, if f has rank α in \leq_m , then f' has rank $\alpha+1$, where $f'(\mathbf{d}) = f(\mathbf{d})'$, the Turing jump of $f(\mathbf{d})$.

REMARKS. With regard to (a), it is well known that if $f(\mathbf{d}) \leq \mathbf{d}$ and $\forall \mathbf{c} (\mathbf{c} \leq f(\mathbf{d}) \text{ a.e.})$, then $f(\mathbf{d}) = \mathbf{d}$ a.e. It is known that conjecture (b) is true when restricted to uniformly representable f so that $\mathbf{d} \leq f(\mathbf{d})$ a.e. (A function f is **uniformly representable** if there is an $F: {}^\omega\omega \rightarrow {}^\omega\omega$ such that for all x , we have $\deg(F(x)) = f(\deg(x))$ and, moreover, there is a $t: \omega \rightarrow \omega$ such that for all x and y , if $x \equiv_T y$ via e then $F(x) \equiv_T F(y)$ via $t(e)$.) It is conjectured that every representable $f: \mathcal{D} \rightarrow \mathcal{D}$ is uniformly representable.

A proof of conjecture (b) would yield a strong negative answer to a question of Sacks: is there a degree invariant solution to Post's problem?

Progress report [KMM81A]. It follows from unpublished results of Kechris and Solovay that $\text{ZF} + \text{AD} + \text{DC} + \mathbf{V}=\mathbf{L}(\mathbb{R})$ implies that every function $f: \mathcal{D} \rightarrow \mathcal{D}$ on the degrees is representable. Although this has no direct bearing on a possible solution of the fifth problem, it underscores the generality of the question.

Progress report [KMM83A]. Slaman and Steel have proved two theorems relevant to Problem # 5. The first verifies a special case of conjecture (a):

THEOREM 4. ($\text{ZF} + \text{AD} + \text{DC}$). Let $f: \mathcal{D} \rightarrow \mathcal{D}$ be such that $f(\mathbf{d}) < \mathbf{d}$ a.e.; then for some \mathbf{c} , $f(\mathbf{d}) = \mathbf{c}$ a.e.

The second verifies a special case of conjecture (b). Call $f: \mathcal{D} \rightarrow \mathcal{D}$ **order-preserving a.e.** if and only if there is a \mathbf{c} such that for all $\mathbf{a}, \mathbf{b} \geq \mathbf{c}$, we have that $\mathbf{a} \leq \mathbf{b}$ implies $f(\mathbf{a}) \leq f(\mathbf{b})$.

THEOREM 5. ($\text{ZF} + \text{AD} + \text{DC}$). Let $f: \mathcal{D} \rightarrow \mathcal{D}$ be order-preserving a.e. and such that $\mathbf{d} < f(\mathbf{d})$ a.e. Then either

- (i) $\exists \alpha < \omega_1$ ($f(\mathbf{d}) = \mathbf{d}^\alpha$ a.e.), or
(ii) For a.e. \mathbf{d} , $\forall \alpha < \omega_1^{\mathbf{d}}$ ($f(\mathbf{d}) > \mathbf{d}^\alpha$).

(Here ω_1 is the least uncountable ordinal, and $\omega_1^{\mathbf{d}}$ is the least \mathbf{d} -admissible ordinal greater than ω .)

2019 comments. Problem # 5 is commonly known as **Martin’s Conjecture**. We shall refer to its restriction to uniformly representable functions as the **Uniform Martin’s Conjecture**. Steel had proved part (b) of the Uniform Martin’s Conjecture [Ste82A]; for further work in this direction, cf. [Bec86]. The partial results by Slaman and Steel listed in the 1983 progress report appear in [SS88] and constitute a proof of the Uniform Martin’s Conjecture. Theorem 5 can be improved using results of Woodin [Woo08], so that part (ii) of the conclusion can be strengthened to $f(\mathbf{d}) > \mathcal{O}^{\mathbf{d}}$. Kihara and Montalbán recently refined the Uniform Martin’s Conjecture to functions from the Turing degrees to the many-one degrees [KM18].

A competing conjecture from the theory of Borel equivalence relations is in conflict with Martin’s conjecture: For Polish spaces X and Y and equivalence relations \equiv and \equiv' on X and Y , respectively, we say that \equiv is **Borel reducible** to \equiv' if and only if there is a Borel function $f : X \rightarrow Y$ such that for all $x, x' \in X$ we have

$$x \equiv x' \iff f(x) \equiv' f(x').$$

An equivalence relation on X is **Borel** if and only if it is a Borel subset of $X \times X$, and it is **countable** if and only if all its equivalence classes are countable. A countable Borel equivalence relation is **universal** if and only if all countable Borel equivalence relations are Borel reducible to it. Kechris asked (cf. [Kec92, Problem 17, p. 99]):⁵

QUESTION 6. *Is Turing equivalence $\equiv_{\mathbf{T}}$ universal?*

Slaman and Steel have also shown that *arithmetic* equivalence is universal [MSS16, § 2], but the question remains open for Turing equivalence. A positive answer to Kechris’s question would contradict Martin’s conjecture: if there is a Borel reduction of two disjoint copies of $\equiv_{\mathbf{T}}$ to $\equiv_{\mathbf{T}}$, then the range of one of the copies under the reduction would be a set disjoint from a cone.⁶ A detailed discussion of the current state of knowledge, including a proof of the Slaman-Steel result on arithmetic equivalence, can be found in [MSS16]. Montalbán, Reimann, and Slaman, have shown (in unpublished work) that Turing equivalence is not *uniformly* universal [Sla09].

⁵This question is sometimes stated as a conjecture; cf. [DK00, Conjecture, p. 86].

⁶Details can be found in [DK00, second Fact on p. 86].

6. The extent of definable scales.*Original problem* [KMS88A].

PROBLEM # 6. Assume $\underline{\mathbf{II}}_1^1\text{-AD}^{\Sigma_3}$. Do all $\mathfrak{D}^{\Sigma_2}\underline{\mathbf{II}}_1^1$ sets admit $\mathbf{HOD}(\mathbb{R})$ scales?

The terminology is explained in Steel’s paper [Ste88]. The strongest result in this direction has been Martin’s theorem that for $\lambda < \omega_1$ a limit ordinal, $\underline{\mathbf{II}}_1^1\text{-AD}^\lambda$ implies all $\mathfrak{D}^\lambda\underline{\mathbf{II}}_1^1$ sets admit $\mathfrak{D}^\lambda\underline{\mathbf{II}}_1^1$ scales [Mar83]. Work of Woodin and Steel had shown that a positive answer to #6 implies that some form of definable determinacy (i.e., $\underline{\mathbf{II}}_1^1\text{-AD}^{\Sigma_3}$) yields an inner model of $\text{AD}_{\mathbb{R}}$.

Steel obtained a positive answer to #6 in February 1984; his results in this area are described in [Ste88].

2019 comments. There has been a significant amount of additional work on determinacy of long games and regularity of associated sets. In [Ste08C], Steel extends the work published in [Ste88]:

Say that T is an ω_1 -tree if and only if $T \subseteq \omega^{<\omega_1}$ and T is closed under initial segments.⁷ For an ω_1 -tree T , the game $\mathcal{G}(T)$ is the following (closed) game on ω of length ω_1 : For any countable α , at stage α , player I plays an integer m_α and player II replies an integer n_α . Letting $\langle \cdot, \cdot \rangle$ denote a (natural) pairing function, let $f: \omega_1 \rightarrow \omega$ be the function defined at any α by $f(\alpha) = \langle m_\alpha, n_\alpha \rangle$. Player II wins this run of the game if and only if $f \in [T]$, the set of length- ω_1 branches through T .

We say that ω_1 -**open-projective determinacy** holds if for all ω_1 -trees T definable over $\mathbf{H}(\omega_1)$ from parameters, the game $\mathcal{G}(T)$ is determined. We let \mathfrak{D}^{ω_1} (open-analytical) be the pointclass of all sets of the form $\mathfrak{D}^{\omega_1}(T)$ for such a tree.

THEOREM 7 (Steel). *If ω_1 -open-projective determinacy holds, then the pointclass \mathfrak{D}^{ω_1} (open-analytical) has the scale property.*

The determinacy property is called “ ω_1 -open-projective” because an ω_1 -tree is definable over $\mathbf{H}(\omega_1)$ from parameters if and only if it can be coded by a projective set of reals. In [Nee04], Neeman proved that ω_1 -open-projective determinacy follows from a traditional large cardinal assumption, viz., that for every real x there is a countable, $\omega_1 + 1$ -iterable (coarse) mouse M with $x \in M$ and $M \models \text{ZFC} - \text{P} +$ “there is a measurable Woodin cardinal”, where $\text{ZFC} - \text{P}$ denotes ZFC without the power set axiom. The monograph [Nee04] describes the state of the art in the theory of determinacy of long

⁷The usual definition implies that ω_1 -trees have height ω_1 and that each level be countable. The present form weakens both requirements but keeps that each node has at most countably many immediate successors, while simultaneously providing a uniform way of ensuring the countability of each of these sets of successors.

games around 2004, although a few results of Woodin in the area remain unpublished and the field has further developed since then.

7. The Kleene ordinal.*Original problem* [KMS88A].

PROBLEM # 7. *Let κ be the least ordinal not the order type of a pre-wellordering of \mathbb{R} recursive in Kleene's 3E and a real. Assume $\text{AD}^{\mathbf{L}(\mathbb{R})}$. Is κ the least weakly inaccessible cardinal?*

That the answer is positive is an old conjecture of Moschovakis, who had shown that κ is a regular limit of Suslin cardinals [Mos70, Mos78]. Steel showed in [Ste81A] that κ is the least regular limit of Suslin cardinals. Thus the problem amounted to bounding the growth of the Suslin cardinals below κ . Building on work of Kunen and Martin, Jackson had done this for the first ω Suslin cardinals; this work is described in his long paper [Jac88].

In the fall of 1985, Jackson obtained a positive answer to #7. His new work extends the theory presented in [Jac88]. Because of its length and complexity, as of now no one but Jackson has been through this new work.

2019 comments. Jackson's result remains unpublished; he comments:

Steel has developed a “fine structure theory” for $\mathbf{L}(\mathbb{R})$ assuming $\text{ZF} + \text{AD}$. This suffices to answer certain questions about $\mathbf{L}(\mathbb{R})$, for example, it gives a complete description of the scale property in $\mathbf{L}(\mathbb{R})$. Other problems, however, such as whether every regular cardinal is measurable seem to require a more detailed understanding of $\mathbf{L}(\mathbb{R})$.

Our results provide such a detailed analysis for an initial segment of the $\mathbf{L}_\alpha(\mathbb{R})$ hierarchy. Exactly how far this enables one to go is not clear, and is the subject of current investigation. However, the author has verified that the theory extends through the Kleene ordinal $\kappa = o({}^3E)$, and in fact, considerably beyond. This analysis is quite involved, however, and has not yet been written up. One consequence is the solution to a problem of Moschovakis, who conjectured in $\text{ZF} + \text{AD} + \text{DC}$ that the Kleene ordinal should be the least inaccessible cardinal (this is the seventh Victoria Delfino problem). [Jac89, p. 80]

As already quoted in our comments to Problem # 1, Jackson reports in 2010 that he had

extended the analysis further, up to the least inaccessible cardinal in $\mathbf{L}(\mathbb{R})$, although this lengthy analysis has never been written up. [Jac10, p. 1755]

To the best of our knowledge, no alternative approaches (via the **HOD** analysis or otherwise) have been suggested. Portions of the analysis have appeared in [Jac91, Jac92, Jac10].

8. Regular cardinals in $\mathbf{L}(\mathbb{R})$.

Original problem [KMS88A].

PROBLEM # 8. *Assume $\mathbf{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$. Are all regular cardinals below Θ measurable?*

Moschovakis and Kechris had shown, in $\mathbf{ZFC} + \mathbf{AD}^{\mathbf{L}(\mathbb{R})}$, that if κ is regular (in \mathbf{V} , where \mathbf{AC} holds!) and $\kappa < \Theta^{\mathbf{L}(\mathbb{R})}$, then $\mathbf{L}(\mathbb{R}) \models$ “ κ is measurable”. This led them to conjecture a positive answer to #8. Jackson’s detailed analysis of cardinals and measures had verified the conjecture for κ below the supremum of the first ω Suslin cardinals (cf. [Jac88]).

The only progress on this problem since its addition to the list is Jackson’s new work cited above, which presumably yields a positive answer to #8 for κ below the Kleene ordinal.

2019 comments. Problem # 8 was solved by John Steel using core model techniques, specifically through the beginning of what we now call the **HOD** analysis. The proof is published as [Ste10, Theorem 8.27].

Steel realised, under the assumption of determinacy, the fragment of the model $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ below Θ as a fine structural mouse, specifically as the direct limit of a system whose objects are certain countable mice and whose commuting maps are appropriate iterations. Analysis of this system allows us to conclude (combinatorial or descriptive set theoretic) properties of its direct limit from (fine structural) properties of the mice, and many different results have been established this way. In particular:

THEOREM 8 (Steel; [Ste95A]). *Assume $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$ and work in $\mathbf{L}(\mathbb{R})$. Then for every $x \in \mathbb{R}$ and $\kappa < \delta_1^2$ such that κ is regular in $\mathbf{HOD}(x)$, the following implication holds:*

$$\text{cf}(\kappa) > \omega \text{ implies } \mathbf{HOD}(x) \models \text{“}\kappa \text{ is measurable”}.$$

These measures on κ in $\mathbf{HOD}(x)$ for different x can be amalgamated via the directed system that guides the iterations mentioned above; the result now follows via reflection; cf. also [Ste10, Lemma 8.25]

This analysis of $\mathbf{V}_\Theta \cap \mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ has been extended by Woodin to a full analysis of **HOD** via a longer directed system, while identifying the correct *hybrid* rather than purely fine structural mice that make up **HOD**; cf. [SW16]. A similar analysis of the **HOD** of larger models than $\mathbf{L}(\mathbb{R})$ has become a key tool in recent work in determinacy, in particular, in the proofs of partial versions of the *mouse set conjecture* (cf. the **2019 comments** on Problem # 11).

To illustrate the reach of the **HOD** analysis, we mention some further applications (the list is not exhaustive): Recall that, assuming determinacy, κ_{2n+1}^1 is the cardinal predecessor of the projective ordinal δ_{2n+1}^1 .

In [Sar13B, Theorem 5.2.2], Sargsyan proves Woodin’s theorem that, under $\text{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$, for all $n \in \omega$, κ_{2n+3}^1 is the least cardinal δ of \mathbf{HOD} such that $\mathbf{M}_{2n}(\mathbf{HOD}|\delta) \models$ “ δ is Woodin.” This identifies a purely descriptive set theoretic characterisation of cardinals with a fine structural characterisation, and provides us with precise information of how much large cardinal strength the relevant cardinals retain when passing from \mathbf{V} to nice inner models. In [Sar14], Sargsyan uses the \mathbf{HOD} analysis to prove the strong partition property of $\tilde{\delta}_1^2$, a result first established in [KKMW81]. In [Nee07B], Neeman uses the analysis to provide a characterisation of supercompactness measures for ω_1 in $\mathbf{L}(\mathbb{R})$. In [JKSW14], the authors use the analysis to prove Woodin’s result that, under $\text{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$, every uncountable cardinal below Θ is Jónsson and, if its cofinality is ω , then it is even Rowbottom. This drastically extends previous results of Kleinberg [Kle77] and their generalisation by the second author [Löw02].

9. Large cardinals implying determinacy.

Original problem [KMS88A].

PROBLEM # 9. *Does the existence of a nontrivial, elementary $j: \mathbf{V}_{\lambda+1} \rightarrow \mathbf{V}_{\lambda+1}$ imply $\underline{\Pi}_3^1$ determinacy?*

The world view embodied in the statements of this and the succeeding problem was seriously mistaken. That view was inspired by Martin’s result that the existence of a nontrivial, Σ_1 -elementary $j: \mathbf{V}_{\lambda+1} \rightarrow \mathbf{V}_{\lambda+1}$ implies $\underline{\Pi}_2^1$ determinacy [Mar80], together with work of Mitchell [Mit79] which promised to lead to a proof that nothing much weaker than the existence of such an embedding would imply $\underline{\Pi}_2^1$ determinacy. Martin naturally conjectured that a nontrivial, fully elementary $j: \mathbf{V}_{\lambda+1} \rightarrow \mathbf{V}_{\lambda+1}$ would yield PD; hence the inclusion of # 9 on our list.

Partly because this view was so mistaken, progress in this area since 1984 has been dramatic. From February to April of 1984, Woodin showed that the existence of a nontrivial, elementary $j: \mathbf{L}(\mathbf{V}_{\lambda+1}) \rightarrow \mathbf{L}(\mathbf{V}_{\lambda+1})$ implies PD and in fact $\text{AD}^{\mathbf{L}(\mathbb{R})}$. This was still consistent with the view underlying # 9, and in spirit was a positive answer, although even for $\underline{\Pi}_3^1$ determinacy Woodin’s result required a hypothesis slightly stronger than allowed in # 9. However, at about the same time Foreman, Magidor and Shelah [FMS88] developed a powerful new technique for producing generic elementary embeddings under relatively “weak” large cardinal hypotheses such as the existence of supercompact cardinals. Woodin realised at once the potential in their technique and used it to show, in May 1984, that the existence of a supercompact cardinal implies all projective sets of reals are Lebesgue measurable. Immediately thereafter, Shelah and Woodin improved this to include all sets in $\mathbf{L}(\mathbb{R})$.

If the relationship between large cardinals and determinacy were to exhibit anything like the pattern it had previously, supercompact cardinals had to imply $\text{AD}^{\mathbf{L}(\mathbb{R})}$. In September 1985, Martin and Steel showed that in fact they do (thereby answering # 9 positively). (Their proof of PD is self-contained. Their proof of $\text{AD}^{\mathbf{L}(\mathbb{R})}$ requires work done by Woodin using the generic embedding techniques.) The Martin-Steel theorem required much less than supercompactness; e.g., for $\underline{\Pi}_{n+1}^1$ determinacy it required the existence of n “Woodin cardinals” with a measurable above them all. [The notion of a “Woodin cardinal” had been isolated by Woodin in his work on generic embeddings; it is a refinement of a notion due to Shelah.] In May–July of 1986, Martin and Steel pushed the theory of inner models for large cardinals far enough to show that the hypothesis of their theorem was best possible: the existence of n Woodin cardinals does not imply $\underline{\Pi}_{n+1}^1$ determinacy. More recently, Woodin has obtained relative consistency results in this direction by a different method; cf. Problem # 10 below.

Unfortunately, with the exception of [FMS88], none of this recent work has been published.

2019 comments. The relationship between determinacy and large cardinals is now well documented. Since this relationship is of fundamental importance to the field and the Cabal, we use this opportunity to give a brief exposition of the developments and the current topics of research.

Woodin cardinals. The mentioned Shelah-Woodin results on Lebesgue measurability of all sets of reals in $\mathbf{L}(\mathbb{R})$ in the presence of large cardinals appear in [SW90]. The paper defines the notions now known as Shelah and Woodin cardinals, although the notation it uses is different, cf. [SW90, p. 384 and Definitions 3.5 & 4.1]. Since the paper was not published until after the importance of Woodin cardinals had become apparent, the name *Woodin cardinal* appears in this paper:

We define here two large cardinals: $\text{Pr}_a(\lambda, f)$, $\text{Pr}_a(\lambda)$ by Shelah (Definition 3.5) and $\text{Pr}_b(\lambda)$ by Woodin—now called a Woodin cardinal. [SW90, p. 384]

The mentioned Martin-Steel results appear in [MS88] and [MS89], which also mark the first appearance of the term *Woodin cardinal* in the literature. The definition of Woodin cardinals given in [SW90] is easily seen to be equivalent to the modern definition: suppose that δ is an infinite ordinal and that $A \subseteq \mathbf{V}_\delta$. A cardinal $\lambda < \delta$ is **$<\delta$ - A -strong** if and only if for any $\mu < \delta$ there is a nontrivial elementary embedding $j: \mathbf{V} \rightarrow M$ with critical point λ and such that $j(\lambda) > \mu$, $\mathbf{V}_\mu \subset M$, and $j(A) \cap \mathbf{V}_\mu = A \cap \mathbf{V}_\mu$. The ordinal δ is a **Woodin cardinal** if and only if it is an inaccessible cardinal and for all $A \subseteq \mathbf{V}_\delta$ there is a $<\delta$ - A -strong cardinal.

Woodinness was instantly recognised as a pivotal large cardinal notion, and its properties were immediately studied in detail. The realisation that Woodin cardinals form a key step in the development of the inner model program confirmed their importance for the field: comparison of mice is central to the theory of fine structural models; comparisons at the level of cardinals that could be reached by the techniques of the early 1980s were linear and this imposed serious limitations on the nature of the corresponding models, *e.g.*, all of them admitted Δ_3^1 well-orderings of their set of reals. As a consequence, none of them could be models of projective determinacy. Thus, if inner model theory had any hope of reaching supercompact cardinals, essential changes were needed.

The crucial change connected to Woodin cardinals was the increase in the complexity of the comparison process from linear iterations to what are now called *iteration trees*. The development of the appropriate fine structure followed shortly thereafter [MaS94, MiS94] and led to the precise determination of the effect of Woodin cardinals on the complexity of the

reals present in canonical inner models, on the amount of determinacy outright provable or provably consistent, and on the amount of correctness that a model would satisfy or that could be forced of an iterate of the model. As a consequence, the set theoretic landscape transformed significantly thanks to the introduction of Woodin cardinals.

Determinacy from large cardinals. The Martin-Steel theorem mentioned in the original problem, “ n Woodin cardinals and a measurable above imply $\text{Det}(\mathbb{R}^1_{n+1})$ ”, is published in [MS89]. The optimal result is that if for every real x there is a suitable model M that is iterable and contains x and n Woodin cardinals, then $\text{Det}(\mathbb{R}^1_{n+1})$ holds [Nee95].

That assuming just n Woodin cardinals does not suffice follows from inner model theory: from the existence of n Woodin cardinals, a fine structural model with n Woodin cardinals can be obtained, in which the reals admit a Δ^1_{n+2} well-ordering [MaS94, Ste95B], and therefore $\text{Det}(\mathbb{R}^1_{n+1})$ fails in the model by [Kan94, Exercise 27.14].

$\text{AD}^{\mathbf{L}(\mathbb{R})}$ from infinitely many Woodin cardinals and a measurable cardinal above them is due to Woodin, proved using the methods of stationary tower forcing and an appeal to the main theorem, Theorem 5.11, in Martin-Steel [MS89]. A proof using Woodin’s genericity iterations and fine structure instead of stationary tower forcing is due to Steel, and the proof reached in this chapter (using a second form of genericity iterations and no fine structure) is due to Neeman. [Nee10, p. 1880]

These arguments can be pushed much further, and the determinacy of stronger pointclasses than $\wp(\mathbb{R}) \cap \mathbf{L}(\mathbb{R})$ is provable by similar methods from large cardinals still in the region of Woodin cardinals (in particular, well before reaching the level of rank-to-rank embeddings or even supercompactness).

The consistency strength of the Axiom of Determinacy. Woodin’s derived model theorem shows that infinitely many Woodin cardinals without a measurable above suffice to establish the consistency of determinacy in $\mathbf{L}(\mathbb{R})$:⁸

THEOREM 9. *If λ is a limit of Woodin cardinals, G is $\text{Col}(\omega, < \lambda)$ -generic over \mathbf{V} , and $\mathbb{R}^* = \bigcup_{\alpha < \lambda} \mathbb{R} \cap \mathbf{V}[G|\alpha]$, then $\mathbf{L}(\mathbb{R}^*)$ is a model of determinacy.*

In fact, we have that $\mathbb{R}^ = \mathbb{R} \cap \mathbf{V}(\mathbb{R}^*)$ and, letting Γ denote the collection of all sets of reals $A \subseteq \mathbb{R}^*$ in $\mathbf{V}(\mathbb{R}^*)$ such that $\mathbf{L}(A, \mathbb{R}^*) \models \text{AD}^+$, then we have that $\mathbf{L}(\Gamma, \mathbb{R}^*)$ is also a model of determinacy.*

Here, AD^+ denotes Woodin’s strengthening of the axiom of determinacy; cf. the **2019 comments** on Problem # 14 below.

⁸Appropriate weakenings hold for finitely many Woodin cardinals; e.g., if δ is Woodin and G is $\text{Col}(\omega, \delta)$ -generic over \mathbf{V} , then $\text{Det}(\Delta^1_2)$ holds in $\mathbf{V}[G]$ [Nee95, Corollary 2.3].

Conversely, if $\text{AD}^{\mathbf{L}(\mathbb{R})}$ holds, then in a forcing extension there is a model of choice with ω Woodin cardinals (*cf.* [Ste09, KW10, ST10, Zhu10, Zhu15]).

One of the important early results concerning proofs of the existence of large cardinals in inner models from determinacy is the following theorem (*cf.* [KW10, Theorem 5.1]):

THEOREM 10 (Woodin). *Assume $\text{AD} + \mathbf{V}=\mathbf{L}(\mathbb{R})$. Then Θ is a Woodin cardinal in \mathbf{HOD} .*

Lightface determinacy. Harrington's results on getting sharps from analytic determinacy are lightface: if x is a real and $\text{Det}(\Pi_1^1(x))$ holds, then $x^\#$ exists [Har78].

Moving up to Π_2^1 , we get that if both pointclasses $\mathbf{\Pi}_1^1$ and Π_2^1 are determined, then $\mathbf{M}_1^\#$ exists and is ω_1 -iterable [SW16, Corollary 4.17]. In 1995, Woodin claimed the following boldface generalisations of this result:⁹

THEOREM 11. *If $\text{Det}(\mathbf{\Pi}_{n+1}^1)$ holds, then $\mathbf{M}_n^\#(x)$ exists and is ω_1 -iterable for all reals x .*

The result remained unpublished until [MSW16], where the following strengthening is established:

THEOREM 12 (Woodin). *Assume $\text{Det}(\Pi_{n+1}^1)$ and $\text{Det}(\mathbf{\Pi}_n^1)$. If there is no Σ_{n+2}^1 sequence of length ω_1 of distinct reals, then $\mathbf{M}_n^\#$ exists and is ω_1 -iterable.*

The proof uses inner model theory and relativizes to give Theorem 11.

In [MSW16, § 4.2], the authors further conjecture the strengthening of Theorem 12 where the assumption about the existence of uncountable sequences of reals is removed. For $n = 1$, this is [SW16, Corollary 4.17] mentioned above. The conjecture remains open in general, but was settled affirmatively by Zhu for odd numbers in [Zhu16].

The Solovay sequence. Theorem 10 has been significantly generalised and is part of the \mathbf{HOD} analysis mentioned in connection with Problem # 8. In [Sol78B], Solovay introduced the **Solovay sequence** $\langle \Theta_\alpha \mid \alpha \leq \Omega \rangle$ as a way of measuring the strength of determinacy models: We assume determinacy and let Θ_0 is the supremum of all ordinals α for which there is an ordinal definable pre-wellordering of a subset of \mathbb{R} of length α . If Θ_α is defined for all $\alpha < \beta$, and β is limit, then Θ_β is defined as their supremum. Finally, if Θ_α is defined and is less than Θ , then $\Theta_{\alpha+1}$ is the supremum of the lengths of all pre-wellorderings of subsets of \mathbb{R} that are definable from ordinals and a set of reals of Wadge rank Θ_α . The sequence ends once an ordinal Ω is reached such that $\Theta_\Omega = \Theta$.

⁹*Cf.* [Nee95, p. 328] and [Nee04, p. 9].

In $\mathbf{L}(\mathbb{R})$, we have that $\Theta = \Theta_0$, but longer sequences are possible and correspond to models of stronger versions of determinacy. It turns out that all $\Theta_{\alpha+1}$ are Woodin cardinals in **HOD**. The situation at limit ordinals is more delicate and still being explored; *cf.* [Sar15]. Conversely, starting with models with many Woodin cardinals, the derived model construction provides us with models of strong versions of determinacy; *cf., e.g.*, [Ste08A].

Very large cardinals. Although no longer relevant to the goal of deriving determinacy from large cardinals, Woodin's original approach led to the development of the theory of large cardinals past the level of rank-to-rank embeddings. The motivation was the realisation that there was a strong analogy between the theory of $\mathbf{L}(\mathbb{R})$ in the presence of determinacy, and the theory of $\mathbf{L}(\mathbf{V}_{\lambda+1})$ in the presence of nontrivial embeddings $j: \mathbf{L}(\mathbf{V}_{\lambda+1}) \rightarrow \mathbf{L}(\mathbf{V}_{\lambda+1})$ with λ being the supremum of the associated critical sequence. Some results illustrating this can be found in [Kaf04], where versions of the coding lemma are established. For more recent developments, *cf., e.g.*, [Dim11, BKW17].

10. Supercompacts in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$.*Original problem* [KMS88A].

PROBLEM # 10. Assume $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$. Does $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfy “there is a κ such that κ is 2^κ -supercompact”?

Becker and Moschovakis [BM81] had shown that $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ “there is a κ such that $\mathfrak{o}(\kappa) = \kappa^+$ ”. Martin (unpublished) then showed $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ “there is a κ such that κ is μ -measurable”. Steel (unpublished) then showed $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ “there is a κ such that κ is λ -strong, where $\lambda > \kappa$ is measurable”. Inspired by these results, the Cabal conjectured that the model $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfies all large cardinal hypotheses weaker than that which implies $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$ (which is false in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$). Problem #10 resulted from our mistaken guess as to what these hypotheses are.

The Woodin-Shelah Theorem that the existence of supercompacts implies all sets in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable settles #10 negatively, since, assuming $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ “there is a wellorder of \mathbb{R} in $\mathbf{L}(\mathbb{R})$ ”. However, except for the mistake about the cardinals involved, the answer to #10 is positive. Woodin has recently (February 1987) shown that, assuming $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models$ “there is a κ such that κ is a Woodin cardinal”, and under the same assumption found a natural submodel of $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfying “there are ω Woodin cardinals”. The work of Martin, Steel and Woodin referred to in the discussion of #9, together with further work of Woodin reducing its large cardinal hypothesis, shows that $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$ follows from the existence of ω Woodin cardinals with a measurable above them all, so that Woodin’s recent work is in spirit a positive answer to #10.

2019 comments. The remarks we gave on Problem # 9 apply here as well. The paper [KW10] shows how to find Woodin cardinals in \mathbf{HOD} . Assuming strong forms of determinacy, the question of precisely which large cardinals can be present in \mathbf{HOD} remains open, with modern research in descriptive inner model theory motivated by the expectation that at least a very large initial segment of the large cardinal hierarchy should be realised within the \mathbf{HOD} models of strong models of determinacy [Sar13A].

11. The GCH in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$.

Original problem [KMS88A].

PROBLEM # 11. Assume $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$. Does $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfy the GCH?

Becker [Bec80] has shown that, assuming $\mathbf{AD}^{\mathbf{L}(\mathbb{R})}$, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models 2^\kappa = \kappa^+$ for many cardinals κ . There has been little progress on this question since January 1984. Woodin’s recent work on large cardinals in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ does show that

$$\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} \models “(\delta_1^2)^{\mathbf{L}(\mathbb{R})} \text{ is } \Theta^{\mathbf{L}(\mathbb{R})}\text{-strong}”.$$

It follows by an easy reflection argument that if $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ satisfies the GCH below $(\delta_1^2)^{\mathbf{L}(\mathbb{R})}$, then it satisfies the GCH.

2019 comments. Steel’s analysis of \mathbf{HOD} below δ_1^2 , mentioned in the solution to Problem # 8, also solves # 11, *cf.* [Ste10, Corollary 8.22]. Beyond the fine structural analysis, Steel’s argument uses the result mentioned in the original wording of the problem, that under \mathbf{AD} , δ_1^2 is strong up to Θ in $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$; *cf.* [KW10]. It also uses that there is a set $P \subseteq \Theta$ in $\mathbf{L}(\mathbb{R})$ such that $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})} = \mathbf{L}(P)$. Both these results are due to Woodin. The second follows from the analysis of the Vopěnka algebra; *cf.* [SW16].

The argument generalises to the \mathbf{HOD} of larger models of determinacy, as long as the models allow a version of the \mathbf{HOD} analysis. At the moment, this falls within the region below a Woodin limit of Woodin cardinals or, in terms of determinacy assumptions, somewhere in the neighbourhood of $\mathbf{AD}_{\mathbb{R}} + “\Theta$ is regular” [Sar15, Tra14, AS19, ST16]. The expectation is that the result should hold in general.

12. Projective uniformisation, measure, and category.*Original problem* [KMS88A].

PROBLEM # 12. *Does the theory ZFC+ “Every projective relation can be uniformised by a projective function” + “Every projective set is Lebesgue measurable and has the property of Baire” prove PD?*

Woodin [Woo82] showed that the theory in question proves $\forall x \subseteq \omega (x^\dagger \text{ exists})$ and more in this direction, together with some other consequences of PD, and conjectured a positive answer to #12.

There has been no direct progress on this problem since 1984.

2019 comments. Although the expectation was a positive answer, Problem # 12 was solved negatively by Steel in 1997. The precise strength of the theory in question is that of ZFC together with the existence of a cardinal δ with countable cofinality that is the limit of cardinals that are δ -strong.

Details can be found as handwritten notes by Schindler [Sch99], and in Philipp Doebler’s Master’s thesis [Doe06]. Steel showed that the large cardinal mentioned above suffices to produce a model of the theory under consideration, and Schindler proved that this is indeed an equiconsistency, cf. [Sch02, Theorem 9.1].

We sketch Steel’s argument. If there is a cardinal δ as required, then there is a minimal, fully iterable, fine structural inner model $\mathbf{L}[E]$ witnessing that there is such a cardinal δ ; this model admits a Σ_3^1 well-ordering of its reals and this means that $\text{Det}(\mathbf{\Delta}_2^1)$ must fail.

Steel argues by forcing with $\text{Col}(\omega, \delta)$ over $\mathbf{L}[E]$. In the resulting model, all projective sets are Lebesgue measurable, and have the Baire property and we have projective uniformisation. Furthermore, $\mathbf{L}[E]$ is the core model of any of its forcing extensions, and thus $\mathbf{L}[E] \prec_{\Sigma_3^1} \mathbf{L}[E][G]$. Since $\text{Det}(\mathbf{\Delta}_2^1)$ is a Σ_3^1 -statement, we obtain that $\text{Det}(\mathbf{\Delta}_2^1)$ must fail in $\mathbf{L}[E][G]$.

Using the additional assumption that $\mathbb{R}^\#$ exists (in order to implement the core model theory of [Ste96]) and results of Schindler on the complexity of $K \cap \mathbf{H}(\omega_1)$,¹⁰ Hauser and Schindler showed that the theory in Problem # 12 gives us an inner model with a cardinal δ and an ω -sequence of cardinals cofinal in δ and δ -strong [HS00]. Finally, in [Sch02], Schindler shows that, at the level of the theories under consideration, core model theory works without this additional assumption and therefore provides us with a genuine equiconsistency.

From further results in [HS00] and the same argument from [Sch02], we also have that the theory ZF+ “Every projective relation can be uniformised by a projective function” + “Every projective set is Lebesgue measurable

¹⁰Cf. [HS00, Theorems 3.4 & 3.6] which in turn relied on earlier work by Hauser and Hjorth [HH97].

and has the property of Baire” (*i.e.*, the theory considered in Problem # 12 without the Axiom of Choice) gives us an inner model with a cardinal δ and an ω -sequence of cardinals cofinal in δ and that are λ -strong for all $\lambda < \delta$. This is also an equiconsistency, as can be verified by starting with the corresponding minimal $\mathbf{L}[E]$ model for this large cardinal assumption, and forcing now with the symmetric collapse of δ .

Two variations of Problem # 12 remain open:¹¹

In the first variation, we strengthen the theory by changing the assumption of projective uniformisation with its level-by-level version, namely, that for each n , any $\mathbf{\Pi}_{2n+1}^1$ subset of \mathbb{R}^2 can be uniformised by a function with a $\mathbf{\Pi}_{2n+1}^1$ graph. Steel has shown that this version implies $\text{Det}(\mathbf{\Delta}_{\frac{1}{2}}^1)$; *cf.* [Ste96, Corollary 7.14].

In the second variation, we replace the assumption with its lightface version, *i.e.*, that all lightface projective subsets of \mathbb{R}^2 can be uniformised by a function with a lightface projective graph.

¹¹*Cf.* [Hau00] for more information on both of them.

13. The cofinal branches hypothesis.

The **cofinal branches hypothesis**, introduced by Martin and Steel [MaS94, pp. 50–53], is the statement that every countable iteration tree on \mathbf{V} has at least one cofinal well-founded branch; we write CBH for this statement.

PROBLEM # 13. *Does CBH hold?*

The **unique branches hypothesis**, UBH, also introduced by Martin-Steel [MaS94], is the statement that every countable iteration tree on \mathbf{V} has at most one cofinal well-founded branch. As long as the iteration tree \mathcal{T} under consideration is sufficiently closed, UBH for \mathcal{T} implies CBH for \mathcal{T} .

2019 comments. A few years after the problem was formulated, Woodin refuted UBH using large cardinals at the level of embeddings $j : \mathbf{V}_\lambda \rightarrow \mathbf{V}_\lambda$. Later, in 1999, he also refuted CBH, from the existence of a supercompact with a Woodin above, showing from these assumptions that there is an iteration tree of length ω^2 with no cofinal well-founded branch. The tree is formed by an ultrapower by an extender, followed by an ω -sequence of alternating chains on the ultrapower model.

The argument also refutes UBH from the same assumptions, the counterexample being a single ultrapower, now followed by an alternating chain on the ultrapower model, both of whose branches are well-founded.

Details for the case of UBH were presented by Woodin at a meeting at the *American Institute of Mathematics* (AIM) in December 2004. Later, Neeman and Steel significantly lowered the large cardinal assumption needed for both results, to something weaker than the existence of a cardinal strong past a Woodin. More precisely, Neeman and Steel obtained their counterexamples (using the same tree structure as in Woodin’s results) from the assumption that there exists a cardinal δ and an extender F such that F has critical point below δ , support δ , and is δ -strong, and δ is Woodin in the smallest admissible set containing $\mathbf{V}_\delta \cup \{F\}$.

Details, including a discussion of revised versions of both hypotheses that remain open, together with partial positive results, can be found in [NS06].

14. ∞ -Borel sets.

Informally, a set is ∞ -Borel if it can be generated from open sets by closing under the operations of complementation and well-ordered union. Since we are in a choiceless context, we need to give the formal definition in terms of ∞ -Borel codes. In analogy to standard Borel codes, we define the class of ∞ -Borel codes by recursion as follows: a tree T is an ∞ -**Borel code** if and only if

- (i) either $T = \{\langle n \rangle\}$ for some $n \in \omega$,
- (ii) or $T = \bigvee_{\alpha} T_{\alpha} := \{\langle \bigvee, \alpha \rangle \frown t : \alpha < \tau \text{ and } t \in T_{\alpha}\}$, where τ is an ordinal, and each $T_{\alpha} \in \text{BC}$,
- (iii) or $T = \neg T' := \{\langle \neg \rangle \frown t : t \in T'\}$, where $T' \in \text{BC}$.

Now fix a bijection $\ulcorner \cdot, \cdot \urcorner : \omega^2 \rightarrow \omega$; given $T \in \text{BC}$, we define its interpretation by recursion via

- (i) $A_T = \{x \in {}^{\omega}\omega : x(k) = \ell\}$ if $T = \{\langle \ulcorner k, \ell \urcorner \rangle\}$,
- (ii) $A_T = \bigcup_{\alpha < \tau} A_{T_{\alpha}}$ if $T = \bigvee_{\alpha} T_{\alpha}$, and
- (iii) $A_T = {}^{\omega}\omega \setminus A_{T'}$ if $T = \neg T'$.

Then we say that a set A is ∞ -**Borel** if and only if there is an ∞ -Borel code T such that $A = A_T$.

PROBLEM # 14. *Does AD imply that all sets of reals are ∞ -Borel.*

A possibly weaker version of the problem is: Does $\text{AD} + \text{DC}_{\mathbb{R}}$ imply that all sets of reals are ∞ -Borel?¹²

2019 comments. Both versions of Problem #14 are open. The problem is now considered part of the question whether Woodin's AD^+ is equivalent to AD.

In order to define AD^+ , we first need to formulate the concept of *ordinal determinacy*: if $\lambda < \Theta$, we endow λ with the discrete topology, and consider the product topology on ${}^{\omega}\lambda$. Given a set $A \subseteq {}^{\omega}\omega$ and a function $f : {}^{\omega}\lambda \rightarrow {}^{\omega}\omega$, we consider the game $G(f, A)$ to be the game of length ω on λ with payoff set $f^{-1}[A]$. We say that **ordinal determinacy** holds if for any $\lambda < \Theta$, any continuous $f : {}^{\omega}\lambda \rightarrow {}^{\omega}\omega$, and any set of reals A , the game $G(f, A)$ is determined. Now AD^+ is the conjunction of “All sets are ∞ -Borel”, $\text{DC}_{\mathbb{R}}$, and ordinal determinacy.¹³ It is not known whether any of the three components of AD^+ follows from AD.

¹²The axiom $\text{DC}_{\mathbb{R}}$, or (more precisely) $\text{DC}_{\omega}(\mathbb{R})$, is the statement that whenever $R \subseteq \mathbb{R}^2$ satisfies that for any real x there is a y with $x R y$, then there is a function $f : \omega \rightarrow \mathbb{R}$ such that $f(n) R f(n+1)$ for all n . Equivalently, any tree T on a subset of \mathbb{R} with no end nodes has an infinite branch.

¹³Cf. [CK11, §2] for an introduction to AD^+ .

It is known that AD^+ holds in natural models of determinacy, such as models of the form $\mathbf{L}(\wp(\mathbb{R}))$ obtained through the derived model construction. Woodin has shown that $\text{AD}_{\mathbb{R}}$ (in fact, $\text{AD} + \text{Uniformisation}$) implies that all sets of reals are ∞ -Borel (*cf.*, *e.g.*, [IW09, Theorem 4.10]).

The problem is closely connected to a number of other famous open problems in the area:

If every set of reals is ∞ -Borel and there is no uncountable sequence of distinct reals, then all sets of reals are Ramsey, Lebesgue measurable, have the Baire property, and the perfect set property (*cf.* [CK11]); therefore a positive answer for Problem #14 would imply that AD implies that every set of reals is Ramsey (*cf.* [Kan94, Question 27.18]).

In unpublished work, Woodin has shown that from the consistency of $\text{ZF} + \text{DC} + \text{AD} +$ “not every set of reals is ∞ -Borel” one can prove the consistency of $\text{ZF} + \text{DC} + \text{AD} +$ “there exists $\kappa > \Theta$ with the strong partition property”. This connects the problem with the open problem whether it is consistent to have a strong partition cardinal above Θ .

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