Supplementary Material

This supplementary material consists of two sections. The first section provides basic formulae for the MAP estimates under the mixture of Gaussians model; and the second section provides moment calculations for the Laplace and Gaussian mechanisms.

1 MAP estimates under the MoG models

For the maximum a posteriori estimate, we impose the Dirichlet prior on \( \pi \sim \text{Dir}(\alpha) \) and Normal-inverse-Wishart prior on \( p(\mu_k, \Sigma_k) = \text{NIW}(0, \kappa_0, \nu_0, S_0) \), where the MAP estimates are

\[
\begin{align*}
\pi_k^{\text{MAP}} &= \frac{N \pi_k^{\text{MLE}} + \alpha_k - 1}{N + \sum_k \alpha_k - K}, \\
\mu_k^{\text{MAP}} &= \frac{N_k \mu_k^{\text{MLE}}}{N_k + \kappa_0}, \\
\Sigma_k^{\text{MAP}} &= \frac{S_0 + N_k \Sigma_k^{\text{MLE}} + \frac{\kappa_0 N_k}{\kappa_0 + N_k} \mu_k^{\text{MLE}} \mu_k^{\text{MLE}^\top}}{\nu_0 + N_k + d + 2}.
\end{align*}
\]

In this paper we set hyperparameters to conventional values, e.g. \( \alpha = [2, 2, \cdots, 2], \kappa_0 = 1, \nu_0 = d + 2, S_0 = \text{diag}(0.1, \cdots, 0.1) \), rather than optimizing them, cf. [1].

2 \( \lambda \)-th Moment Calculations

2.1 Laplace Mechanism

Univariate Laplace mechanism. Suppose we use the univariate Laplace mechanism where the sensitivity is 1 and we add Laplace noise with parameter \( \frac{1}{\epsilon} \). Then the privacy loss r.v. is:

\[
\begin{align*}
Z &= \epsilon, \quad \text{w.p} \quad \frac{1}{2} \\
&= -\epsilon, \quad \text{w.p} \quad \frac{e^{-\epsilon}}{2} \\
&= \epsilon(1 - 2t), \quad \text{with density} \quad \frac{\epsilon}{2} e^{-\epsilon t}, \quad 0 \leq t \leq 1.
\end{align*}
\]

This leads to the moment generating function:

\[
\alpha_M(\lambda) = \mathbb{E}[e^{\lambda Z}] = \left( \frac{\lambda + 1}{2\lambda + 1} \right) e^{\lambda \epsilon} + \left( \frac{\lambda}{2\lambda + 1} \right) e^{-(\lambda + 1)\epsilon}.
\]
Multivariate Laplace Mechanism with bounded $L_1$ sensitivity Suppose we use the $d$-variate Laplace mechanism where the $L_1$-sensitivity is equal to $\Delta$; in this case, we add Laplace noise with parameter $\Delta/\epsilon$ to each coordinate of the vector. Then the privacy loss random variable may be written as follows:

$$Z = \log \frac{e^{-\epsilon |t|_1}/\Delta}{e^{-\epsilon |\mu - t|_1}/\Delta}, \text{ w.p. } \frac{e^d}{\Delta^d} e^{-\epsilon |t|_1}/\Delta,$$

which is the same as:

$$Z = \frac{\epsilon}{\Delta} (|\mu - t|_1 - |t|_1), \text{ w.p. } \frac{e^d}{(2\Delta)^d} e^{-\epsilon |t|_1}/\Delta.$$ 

Here $\mu$ is the difference between $f(D)$ and $f(D')$ and has the property that $|\mu|_1 \leq \Delta$. Let $\varepsilon = \epsilon/\Delta$. For a given $\mu$, this leads to the moment generating function:

$$\mathbb{E}[e^{\lambda Z}] = \frac{(\varepsilon/2)^d}{\Delta^d} \int_{\mathbb{R}^d} e^{\lambda \varepsilon (|\mu - t|_1 - |t|_1)} \times e^{-\varepsilon |t|_1} \, dt$$

$$= \left( \prod_j \frac{\varepsilon}{2} \int_{-\infty}^{\infty} e^{\lambda \varepsilon |\mu_j - t_j| - (\lambda + 1)\varepsilon |t_j|} \, dt_j \right)$$

$$= \left( \prod_j \left( \frac{\lambda + 1}{2\lambda + 1} e^{\lambda \varepsilon |\mu_j|} + \frac{\lambda}{2\lambda + 1} e^{-(\lambda + 1)\varepsilon |\mu_j|} \right) \right).$$

What we need is an upper bound on the functional $\mathbb{E}[e^{\lambda Z}]$ for any $\lambda$ over all $\mu$ for which $|\mu|_1 \leq \Delta$. For positive $\lambda$, we note that the function $\mathbb{E}[e^{\lambda Z}]$ is convex in $|\mu|_j$, and therefore the maximum value occurs when $\mu = \Delta e_j$ where $e_j$ is some coordinate vector. This maximum value is:

$$\frac{\lambda + 1}{2\lambda + 1} e^{\lambda \varepsilon \Delta} + \frac{\lambda}{2\lambda + 1} e^{-(\lambda + 1)\varepsilon \Delta}.$$

2.2 Multivariate Gaussian Mechanism

Suppose we use the $d$-variate Gaussian Mechanism where the $L_2$-sensitivity is 1 and we add multivariate spherical Gaussian noise $N(0, \sigma^2 I_d)$. Then, the privacy loss random variable can be written as follows:

$$Z = \frac{1 - 2x^T \Delta}{2\sigma^2}, \text{ w.p. } \frac{1}{\sigma^4 (2\pi)^d/2} e^{-||x||^2/2\sigma^2},$$

where $\Delta$ is any $d \times 1$ vector with unit norm. This in turn leads to the moment generating function:

$$\alpha_M(\lambda) = \mathbb{E}[e^{\lambda Z}] = e^{(\lambda + \lambda^2)/2\sigma^2}.$$

References