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Topological Semantics for Conditionals

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Abstract: In this paper we explore the topological semantics for conditional logic that arises from the Alexandroff equivalence between preorders and topological spaces. This clarifies the relation between the standard order semantics and premise semantics for conditionals. As an application we provide a construction of relative similarity orders between possible worlds from topologies of relevant propositions. The conditional logic over topologies is intertranslatable with the modal logic S4u.

Keywords: Conditionals, Counterfactuals, Similarity Orders, Premise Semantics, Evidence Models, Topological Semantics, S4u

1 Introduction

Preorders are the standard semantics for conditional logic. The Alexandroff correspondence associates to every preorder a unique Alexandroff topological space, which is a topological space closed under arbitrary intersections. This suggests using Alexandroff topological spaces as a semantics for conditional logic. We show that this topological semantics captures all that is relevant in premise semantics for the evaluation of conditionals.

We apply this topological approach to the logic of counterfactual conditional. Concretely, we provide a construction of order frames, which are relative similarity orders between worlds, from an Alexandroff topology of relevant propositions. This yields a well-motivated way to obtain a relative similarity order from information which is more basic than similarity among worlds. We completely characterize the class of order frames which results from this construction. This yields strong constraints on order frames which imply the commonly assumed centring condition. We completely axiomatize the validities of this restricted semantics via an intertranslatability result between conditional logic and S4u, which is the usual S4 modal logic on topological spaces, augmented with a universal modality.

2 Alexandroff Correspondence for Conditionals

2.1 Conditional Logic

Various minimization procedures over preorders have naturally arisen in contexts such as the formal semantics of modal notions, belief revision theory or default reasoning (Baltag & Smets, 2006; Imielinski, 1987; Kratzer, 1977; Lewis, 1973). In this paper, we restrict our attention to a setting where a binary modal operator is used to express such a minimization condition. This means we are working with a modal language built according to the following grammar:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi \rightsquigarrow \varphi$$

where $p \in \mathsf{Atom}$ is any atomic sentence from a given infinite set Atom. We will call formulas of the form $\varphi \rightsquigarrow \psi$ conditionals. Depending on the application the conditional $\varphi \rightsquigarrow \psi$ can be read in different ways, for instance as "If φ had been the case then ψ would have been the case", "An agent believes ψ conditional on believing φ", or as "If φ then you must ψ".

The semantics for this modal language based on preorders, variously worked out in (Lewis, 1973; Pollock, 1976; Veltman, 1985), has been influential. This approach associates a preorder to every possible world to evaluate the conditional at that world. This yields the following definition:

Definition 1 (Order frame) *An order frame over a set of possible worlds W is a family $(\leq_x)_{x \in W}$, which associates to every world $x \in W$ a preorder (i.e. reflexive, transitive relation) $\leq_x \subseteq W \times W$ over the set W.*

Depending on the setting the preorder associated with a world in a order frame can have different interpretations. In the logic of counterfactual conditionals, the preorders represents metaphysical similarity of the worlds. Thus, $y \leq_x z$ means that the world y is more similar to the world x than the world z. On an doxastic interpretation of the conditional, the preorder associated to a world represents the plausibility of worlds according to the mental state a given agent at that world. Thus, $y \leq_x z$ means that at x the agent considers it more plausible that y is the actual world than that z is the actual world. We will mostly focus on these two interpretations of the preorder as metaphysical similarity or as plausibility for an agent. It is important to notice that similarity is an ontological notion, while plausibility is an agent relative epistemic notion. For instance for similarity orders, but not for plausibility orders, it is common to require some form of centring, meaning that the world x is in some sense minimal in $\leq_x$. On the other hand for plausibility orders, but not for similarity orders, it is common to require doxastic introspection, meaning that the order associated to any world that the agent considers possible is the same as the preorder in the actual world.

To evaluate formulas on an order frame $(\leq_x)_{x \in W}$ we also need to provide an interpretation for the atomic sentence in Atom. As usual this is done by considering an evaluation function $V : \mathsf{Atom} \to \mathcal{P}W$ which maps every atomic sentence to the set of worlds where it is true. The semantic clause for the atomic sentences make use of the valuation V, the Boolean connectives are defined as usual, while the semantics of the conditional is as follows:

$$x \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad \text{for all } y \in [\![\varphi]\!] \text{ there is a } z \in [\![\varphi]\!] \text{ such that } z \leq_x y \text{ and } u \models \varphi \to \psi \text{ for all } u \leq_x z$$

Given a frame $(\leq_x)_{x \in W}$ and a valuation V, we use $[\![\cdot]\!]$ as a shorthand for the set of all worlds which satisfy the formula φ, i.e. $[\![\varphi]\!] = \{x \in W \mid x \models \varphi\}$.

It is possible to simplify the semantic clause for the conditional by assuming an additional condition on the frame $(\leq_x)_{x \in W}$ which is called the limit assumption (Warmbrod, 1982). It requires that for any $x \in W$ and set of worlds $X \subseteq W$, X has $\leq_x$-minimal elements. In this case one can check that $x \in [\![\varphi \rightsquigarrow \psi]\!]$ iff for all the $\leq_x$-minimal elements y in $[\![\varphi]\!]$ we have that $y \in [\![\psi]\!]$. It is this fact which explains the notion of minimisation on a preorder we mention above.

The set of validities for the above semantics can be completely axiomatized, see (Veltman, 1985) for details and a completeness proof.

2.2 Alexandroff Correspondence

We make use of a correspondence between preorders and topological spaces, first noticed by Alexandroff in (Alexandroff, 1937), to derive a topological semantics for the conditional from its order semantics. A topology over a set of points W is family $\tau \subseteq \mathcal{P}W$ of sets of points such that $\emptyset, W \in \tau$ and τ is closed under arbitrary unions and finite intersections of sets. The family τ is called the topology of the topological space (W, τ) and its elements are called open sets. We shall just write τ for the topological space (W, τ). The topological spaces that stand in correspondence to preorders are Alexandroff topological spaces, which are additionally closed under arbitrary, not necessarily finite, intersections. We denote an Alexandroff topology by the symbol $\mathcal{A}$ instead of τ to emphasize that the topology is Alexandroff.

We shall now explicitly describe how the Alexandroff correspondence. Let $\leq$ be a preorder over over a set W, then the corresponding Alexandroff topology $\mathsf{Do}(\leq) \subseteq \mathcal{P}W$ is defined as the set of all downsets of the preorder:

$$\mathsf{Do}(\leq) = \{U \subseteq W \mid \text{if } x \in U \text{ and } y \leq x \text{ then } y \in U\}$$

In the other direction, for an Alexandroff topology $\mathcal{A} \subseteq \mathcal{P}W$ one can define the specialization preorder $\leq^{\mathcal{A}} = \mathsf{Sp}(\mathcal{A})$ on W by:

$$x \leq^{\mathcal{A}} y \quad \text{iff} \quad \text{for all } U \in \mathcal{A} \text{ if } y \in U \text{ then } x \in U$$

The classic result by Alexandroff is that these constructions establish a bijective correspondence between preorders and Alexandroff spaces.

Theorem 2 *For all preorders $\leq$ on a set W it holds that* $\mathsf{Sp}(\mathsf{Do}(\leq)) = \leq$ *and for all Alexandroff topologies $\mathcal{A}$ it holds that* $\mathsf{Do}(\mathsf{Sp}(\mathcal{A})) = \mathcal{A}$.

2.3 Topological Semantics

The Alexandroff correspondence associates to every preorder an Alexandroff topology; conversely, every Alexandroff topology arises from a unique preorder. We use this fact to obtain a topological semantics for the conditional. We need just replace every preorder $\leq_x$ associated to a world x in a order frame $(\leq_x)_{x\in W}$ with its corresponding Alexandroff topology $\mathsf{Do}(\leq_x)$.

Definition 3 (Neighbourhood space) *A neighbourhood space over a set of possible worlds W is a structure $(\mathcal{A}_x)_{x\in W}$ which associates to each world $x \in W$ an Alexandroff topology $\mathcal{A}_x \subseteq \mathcal{P}W$ over the set W.*

By pointwise application of $\mathsf{Do}(\cdot)$ we obtain a neighbourhood space $\mathsf{LDo}((\leq_x)_{x\in W})$ for every order frame $(\leq_x)_{x\in W}$. Analogously, for every neighbourhood space $(\mathcal{A}_x)_{x\in W}$ we obtain an order frame $\mathsf{LSp}((\mathcal{A}_x)_{x\in W})$. This is a bijective correspondence meaning that $\mathsf{LSp}(\mathsf{LDo}((\leq_x)_{x\in W})) = (\leq_x)_{x\in W}$ and $\mathsf{LDo}(\mathsf{LSp}((\mathcal{A}_x)_{x\in W})) = (\mathcal{A}_x)_{x\in W}$.

To define the topological semantics of conditional logic on a neighbourhood space $(\leq_x)_{x\in W}$ we again need a valuation function $V : \mathsf{Atom} \to \mathcal{P}W$ to fix the truth values of atomic sentences. The semantics for atomic and Boolean formulas is as usual, while for the conditional we have:

$$x \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad \text{for all } \varphi\text{-consistent } U \in \mathcal{A}_x \text{ there is a } \varphi\text{-consistent } V \in \mathcal{A}_x \text{ with } V \subseteq U \text{ and } V \cap [\![\varphi]\!] \subseteq [\![\psi]\!]$$

Again $[\![\varphi]\!] \subseteq W$ is the set of worlds which make the formula φ true and we call a set $U \subseteq W$ φ-consistent if $U \cap [\![\varphi]\!] \neq \emptyset$. The reader can verify that this semantic clause for the conditional on a neighbourhood space corresponds to its semantic clause on the corresponding order frame.

Our topological semantics is akin to the premise semantics for counterfactuals developed in (Kratzer, 1981; Lewis, 1981) and to the evidence models of (van Benthem & Pacuit, 2011). In premise semantics every world in a frame is associated to an arbitrary subset of the powerset of the set of worlds which is not required to be a topology. The elements of this set of sets of worlds associated to a world are called the premises and are thought of as being the relevant background information against which a conditional is evaluated. Lewis (Lewis, 1981) studies the relation between this premise semantics and the order semantics and gives a construction which associates to any order frame an logically equivalent premise frame and viceversa. Lewis does not mention the fact that the construction he uses is the same as the Alexandroff correspondence. Since Lewis does not require the premise sets to be Alexandroff topologies he does not obtain a 1-to-1 correspondence between order frames and premise frames. However, closing an premise set under arbitrary intersections and unions does preserve the truth of all counterfactuals. Therefore one does not collapse any substantial semantic distinctions by requiring premise sets to be Alexandroff topologies, but one gains the 1-to-1 correspondence with order frames.

In the context of epistemic logic, premise frames reappear in the guise of evidence models (van Benthem & Pacuit, 2011). Here the elements of the premise set associated to a world represent a piece of evidence that some agent has about the actual world. Again, Pacuit and van Benthem establish the same relation to order frames but do not mention the underlying Alexandroff correspondence.

3 Coherent Neighbourhood Spaces and Order Frames

In this and the following section we discuss an application of the Alexandroff correspondence to the semantics of counterfactual conditional logic. We observed in the previous section that one can think of the open sets in the local topology $\mathcal{A}_x$ associated to a world x as the facts obtaining at x which are relevant to determine similarity of other worlds to x. The definition of the similarity order $\leq_x$ from the topology $\mathcal{A}_x$ was such that another world y is the more similar to x the more of the relevant propositions in $\mathcal{A}_x$

are true at y. We present a plausible procedure to construct a neighbourhood frame, and hence an order frame, from an independently given Alexandroff topology of relevant propositions. A proposition is deemed to be relevant if its truth or falsity at worlds matters for the relative similarity of worlds. Additionally, we characterize the neighbourhood spaces and order frames that arise in this way, which we name "coherent".

Assume that we are given a set $\mathcal{A}$ of propositions which are relevant to determine the similarity among worlds. For instance one might in general consider the physical laws holding at a world as relevant whereas accidental facts are not relevant. We also assume that $\mathcal{A}$ is an Alexandroff topology, meaning that that relevant propositions are closed under arbitrary conjunctions and disjunctions. We mention possible weakenings of this assumption in section 5. The propositions relevant to determine relative similarity to a particular world x are now all the relevant propositions in $\mathcal{A}$ which are actually true at x. This motivates the following definition.

Definition 4 *For a topological space $\mathcal{A}$ define the corresponding neighbourhood space $\mathsf{Loc}(\mathcal{A}) = (\mathcal{A}_x)_{x \in W}$ by:*

$$\mathcal{A}_x = \{U \in \mathcal{A} \mid x \in U\} \cup \{\emptyset\}$$

Once we have a neighbourhood space one can apply the Alexandroff correspondence to obtain the corresponding order frame. In this sense Definition 4 provides a construction of relative similarity orders $\mathsf{LSp}(\mathsf{Loc}(\mathcal{A}))$ from any set of relevant propositions $\mathcal{A} \subseteq \mathcal{P}W$.

In the rest of this section we characterize the neighbourhood spaces and order frames which arise from an Alexandroff space via this construction.

3.1 Coherent Neighbourhood Spaces

The neighbourhood spaces arising from the construction above satisfy the following properties:

(C) If $U \in \mathcal{A}_x$, $U \neq \emptyset$ then $x \in U$

(Ov) If $U \in \mathcal{A}_x$ and $y \in U$, then $U \in \mathcal{A}_y$

(CU) If $\mathcal{U} \subseteq \bigcup_{x \in W} \mathcal{A}_x$ then $\bigcup \mathcal{U} \in \mathcal{A}_x$ for some x

We call a neighbourhood space satisfying the above three conditions a coherent neighbourhood space. The first condition (C) is a version of centring for premise frames. The second condition (Ov) captures our assumption that relevance of propositions is not world relative. If a proposition is relevant somewhere then it is relevant wherever it is true.

Theorem 5 *Coherent neighbourhood spaces are in bijective correspondence with Alexandroff topological spaces.*

Proof. The reader can check that the construction from Definition 4 always yields a coherent neighbourhood space.

For the other direction assume we are given any coherent neighbourhood space $(\mathcal{A}_x)_{x\in W}$. We define the following Alexandroff topology on W:

$$\mathsf{Col}((\mathcal{A}_x)_{x\in W}) = \bigcup_{x\in W} \mathcal{A}_x$$

Using the conditions on coherent neighbourhood spaces one can check that $\bigcup_{x\in W} \mathcal{A}_x$ is indeed an Alexandroff topology.

It is easy to check that $\mathsf{Col}(\mathsf{Loc}(\mathcal{A})) = \mathcal{A}$ for any Alexandroff space $\mathcal{A}$. For the other direction, we show that $(\mathcal{A}'_x)_{x\in W} = \mathsf{Loc}(\mathsf{Col}((\mathcal{A}_x)_{x\in W}))$ is equal to $(\mathcal{A}_x)_{x\in W}$ for any neighbourhood space $(\mathcal{A}_x)_{x\in W}$. We need that $\mathcal{A}_x = \mathcal{A}'_x$ for all $x \in W$. So take any $U \in \mathcal{A}_x$. Then $U \in \mathsf{Col}((\mathcal{A}_x)_{x\in W})$. Hence it is also in $\mathcal{A}'_x = \{U \in \mathsf{Col}((\mathcal{A}_x)_{x\in W}) \mid x \in U\} \cup \{\emptyset\}$ because by (C) it follows from $U \in \mathcal{A}_x$ that $x \in U$. If on the other hand $U \in \mathcal{A}'_x = \{U \in \bigcup_{x\in W} \mathcal{A}_x \mid x \in U\} \cup \{\emptyset\}$ then U is either empty hence clearly $U \in \mathcal{A}_x$ or there is some $y \in W$ such that $U \in \mathcal{A}_y$ and $x \in U$. By (Ov) it follows that $U \in \mathcal{A}_x$. □

3.2 Coherent Order Frames

Corresponding to the notion of a coherent neighbourhood space we define coherent order frames as the class of order frames $(\mathcal{A}_x)_{x\in W}$ which satisfy the following two axioms:

(Ex) For all $x, y, z \in W$: If $x \leq_y z$ then $x \leq_z y$

(St) For all $x, y, z \in W$: If $x \leq_y z$ then $x \leq_y y$ or $x \leq_z z$

A version of centring is implied by (Ex): Because by reflexivity $x \leq_y x$ it follows that $x \leq_x y$ for all $x, y \in W$. This version of centring is slightly weaker than the strong centring which requires $x \leq_x y$ and not $y \leq_x x$ for all $x, y \in W$. It is, however, stronger than weak centring which requires only hat x is a minimal element in the order $\leq_x$. For discussion of different centring conditions see (Veltman, 1985).

The two conditions (Ex) and (St) are quite strong. One can check that neither satisfied by relative closeness of points in a metric space, as considered e.g. in (Lehmann, Magidor, & Schlechta, 2001). However, the two conditions corresponds exactly to the coherence conditions on neighbourhood spaces, which are justified by the intuitive construction of a neighbourhood space from the set of relevant propositions.

Theorem 6 *Coherent order frames are in bijective correspondence with coherent neighbourhood spaces.*

We first need some lemmas which require an additional notion. Define the minimal neighbourhood of y in the topology $\mathcal{A}_x$ of a neighbourhood space $(\mathcal{A}_x)_{x \in W}$ by:

$$N_x(y) = \bigcap \{U \in \mathcal{A}_x \mid y \in U\}$$

Since $\mathcal{A}_x$ is an Alexandroff topology $N_x(y)$ is again open and it is in fact the minimal open set in $\mathcal{A}_x$ containing y.

Lemma 7 *Let $(\leq_x)_{x \in W}$ be an order frame. Then $(\leq_x)_{x \in W}$ satisfies (Ex) if and only if $\mathsf{LDo}((\leq_x)_{x \in W})$ satisfies for all $y, z \in W$*

(Ex') $N_y(z) = N_z(y)$

and it satisfies (St) if and only if $\mathsf{LDo}((\leq_x)_{x \in W})$ satisfies for all $y, z \in W$

(St') $N_y(z) \subseteq N_y(y) \cup N_z(z)$

Proof. Omitted. □

Lemma 8 *For every neighbourhood space $(\mathcal{A}_x)_{x \in W}$ we have:*

1. *(C) and (Ov) imply (Ex')*
2. *(Ov) and (CU) imply (St')*
3. *(Ex') implies (C)*
4. *(Ex') and (St') imply (Ov)*
5. *(St') and (Ov) imply (CU)*

Proof. We leave the points 1 and 3 to the reader.

For point 2: By (CU), $N_y(y) \cup N_z(z) \in \mathcal{A}_w$ for some w. But then because $y \in N_y(y)$ we have by (Ov) that $N_y(y) \cup N_z(z)$ is in $\mathcal{A}_y$. Because $z \in N_z(z) \subseteq N_y(y) \cup N_z(z)$ and $N_y(z)$ is the minimal open set in $\mathcal{A}_y$ containing z it follows that $N_y(z) \subseteq N_y(y) \cup N_z(z)$.

For point 4: consider $U \in \mathcal{A}_x$ and $y \in U$. To show that $U \in \mathcal{A}_y$, we show that for any $z \in U$, there exist an open set $O \in \mathcal{A}_y$ such that $z \in O$ and $O \subseteq U$. Let $z \in U$, and consider $N_y(z)$: it is open in $\mathcal{A}_y$, and z belongs to it. We show that $N_y(z) \subseteq U$. By (St'), $N_y(z) \subseteq N_y(y) \cup N_z(z)$. Hence it is w.l.o.g. enough to show that $N_y(y) \subseteq U$:

$$\begin{aligned} N_y(y) &\subseteq N_y(x) && y \in N_y(x) \text{ by (C)} \\ &= N_x(y) && \text{(Ex')} \\ &\subseteq U && y \in U \in \mathcal{A}_x \end{aligned}$$

For point 5: consider a family of sets $\mathcal{U} \subseteq \bigcup_{x \in W} \mathcal{A}_x$. If $\bigcup \mathcal{U} = \emptyset$, then certainly $\bigcup \mathcal{U} \in \mathcal{A}_x$ for any x. Otherwise, pick $B \in \mathcal{U}$ non-empty, and point $b \in B$. We prove that $\bigcup \mathcal{U} \in \mathcal{A}_b$. It suffices to show that for any $x \in \bigcup \mathcal{U}$ the set $N_b(x)$ is open in $\mathcal{A}_b$, and $x \in N_b(x) \subseteq \bigcup \mathcal{U}$. By definition $N_b(x)$ is open in $\mathcal{A}_b$ and $x \in N_b(x)$. To show that $N_b(x) \subseteq \bigcup \mathcal{U}$ it is enough to prove that $N_b(b) \subseteq \bigcup \mathcal{U}$ and that $N_x(x) \subseteq \bigcup \mathcal{U}$, because by (St'), $N_b(x) \subseteq N_b(b) \cup N_x(x)$. The former holds because $N_b(b) \subseteq B$, since $b \in B \in \mathcal{A}_b$, and $B \subseteq \bigcup \mathcal{U}$, since $B \in \mathcal{U}$. For the latter take a $U \in \mathcal{U}$ such that $x \in U$. By (Ov) it follows that $U \in \mathcal{A}_x$ and hence $N_x(x) \subseteq U \subseteq \bigcup \mathcal{U}$. □

We can now prove Theorem 6.

Proof. (of Theorem 6) Let $(\mathcal{A}_x)_{x \in W}$ be a coherent neighbourhood space. By 1. and 2. of Lemma 8 it must satisfy conditions (Ex') and (St') of Lemma 7. Hence, the corresponding order frame $\mathsf{LSp}((\mathcal{A}_x)_{x \in W})$ is coherent.

Let $(\leq_x)_{x \in W}$ be a coherent order frame. By Lemma 7 the corresponding neighbourhood space $\mathsf{LDo}((\leq_x)_{x \in W})$ satisfies (Ex') and (St'). By points 3. 4. and 5. of Lemma 8 it follows that $\mathsf{LDo}((\leq_x)_{x \in W})$ is a coherent neighbourhood space.

The correspondence between order frames and neighbourhood spaces remains bijective when restricted to coherent order frames and coherent neighbourhood spaces. □

4 S4u and Completeness

The purpose of the present section is to show that the conditional logic over coherent neighbourhood spaces and S4u over topological spaces are intertranslatable. This gives us an easy route to the completeness of conditional logic on coherent neighbourhood spaces. We obtain the complete axiomatization by translating the already know axiomatization of S4u into the language of the conditional and by adding axioms that guarantee the provability of the the translation rules.

The results of this section also hold for coherent neighbourhood spaces in which the local topologies associated to a world are not required to be Alexandroff. Such a space can be obtained from an arbitrary topological space of relevant propositions by means of the localisation procedure described in Definition 4. In the following we however stick to the notation $\mathcal{A}$ and $\mathcal{A}_x$ for the topology of a space and for the local topology at a world x.

We give a brief overview of the aspects of S4u that we need here. For further details consult (Aiello, van Benthem, & Bezhanishvili, 2003). The language of S4u is a bimodal language with two unary modalities $\Box$ and $\forall$. On a topological space the semantics of $\Box$ is the interior modality whereas $\forall$ is a universal modality. We formulate the semantics on the level of the coherent neighbourhood space generated from a topological space. By Theorem 5 this is the same as working with just topological spaces. The most convenient formulation of the semantics for our purposes is:

$$\begin{aligned} x \models \forall\varphi \quad &\text{iff} \quad U \subseteq [\![\varphi]\!] \text{ for all } U \in \mathcal{A}_x \\ x \models \Box\varphi \quad &\text{iff} \quad U \subseteq [\![\varphi]\!] \text{ for some non-empty } U \in \mathcal{A}_x \end{aligned}$$

Note that this induces the following clauses for the duals:

$$\begin{aligned} x \models \exists\varphi \quad &\text{iff} \quad \text{some } U \in \mathcal{A}_x \text{ is } \varphi\text{-consistent} \\ x \models \Diamond\varphi \quad &\text{iff} \quad \text{all non-empty } U \in \mathcal{A}_x \text{ are } \varphi\text{-consistent} \end{aligned}$$

The validities of this semantics are axiomatized by the modal logic S4u in which $\Box$ is an S4 modality, $\forall$ an S5 modality and the interaction axiom $\forall\varphi \to \Box\varphi$ holds.

The intuitive reading of the universal $\forall$ modality is metaphysical necessity. This also becomes clear in the first translation rule of the following theorem, which amounts to the same embedding of metaphysical necessity into conditional logic as suggested in appendix 1 of (Williamson, 2007):

Theorem 9 *These equivalences hold on coherent neighbourhood spaces:*

$$\forall\varphi \equiv \neg\varphi \rightsquigarrow \bot$$
$$\Box\varphi \equiv \top \rightsquigarrow \varphi$$
$$\varphi \rightsquigarrow \psi \equiv (\Diamond\varphi \to \Box(\varphi \to \psi)) \land (\neg\Diamond\varphi \to \forall(\varphi \to \Diamond(\varphi \land \Box(\varphi \to \psi))))$$

Proof. The proof of the first two equivalences is omitted.

For the third equivalence we start with the left to right direction. Let $(\mathcal{A}_x)_{x\in W}$ be a coherent neighbourhood space and $x \in W$ such that $x \models \varphi \rightsquigarrow \psi$. We distinguish two cases. Either all non empty opens in $\mathcal{A}_x$ are φ-consistent or there is a non empty $U \in \mathcal{A}_x$ which is not φ-consistent.

In the first case where they are all φ-consistent it follows that $x \models \Diamond\varphi$. To show $x \models \Box(\varphi \to \psi)$ pick any non empty $U \in \mathcal{A}_x$. We can assume that U is φ-consistent, because otherwise U witnesses the truth of $\Box(\varphi \to \psi)$. It follows by $x \models \varphi \rightsquigarrow \psi$ that there is a $V \in \mathcal{A}_x$ such that $V \cap [\![\varphi]\!] \subseteq [\![\psi]\!]$. So V witnesses the truth of $\Box(\varphi \to \psi)$.

In the second case, where there is a non empty $U \in \mathcal{A}_x$ that is not φ-consistent, it follows that $x \models \neg\Diamond\varphi$. We need to show that $x \models \forall(\varphi \to \Diamond(\varphi \land \Box(\varphi \to \psi)))$. To prove this it is sufficient to take an arbitrary $v \in [\![\varphi]\!]$ and to show that $v \models \Diamond(\varphi \land \Box(\varphi \to \psi))$. Pick any non empty $V \in \mathcal{A}_v$. We need the existence of a $z \in V$ such that $z \models \varphi \land \Box(\varphi \to \psi)$.

Consider $U \cup V$. Since $U \in \mathcal{A}_x$ and $V \in \mathcal{A}_v$, then it follows by (CU) that $U \cup V \in \mathcal{A}_w$ for some $w \in W$. Because $U \in \mathcal{A}_x$ it follows by (C) that $x \in U \subseteq U \cup V$. So it follows by (Ov) that $U \cup V \in \mathcal{A}_x$. Also because $V \in \mathcal{A}_v$ it follows again by (C) that $v \in V \subseteq U \cup V$. So $U \cup V$ is φ-consistent because $v \in [\![\varphi]\!]$. Now we use the assumption that $x \models \varphi \rightsquigarrow \psi$ to obtain a φ-consistent $Z \in \mathcal{A}_x$ with $Z \subseteq U \cup V$ and $Z \cap [\![\varphi]\!] \subseteq [\![\psi]\!]$. Pick a $z \in Z$ which satisfies φ. Then $z \in V$ because $Z \subseteq U \cup V$ and U is assumed to be not φ-consistent. Now we can see that $z \models \Box(\varphi \to \psi)$ since $Z \subseteq [\![\varphi \to \psi]\!]$, Z is not empty, and $Z \in \mathcal{A}_z$ by (Ov).

For the right to left direction suppose that the formula on the right is true at some world x. We need to show that then $x \models \varphi \rightsquigarrow \psi$. Again, we distinguish the cases where all non empty elements of $\mathcal{A}_x$ are φ-consistent and where there is an element in $\mathcal{A}_x$ that is not φ-consistent.

If all non empty $U \in \mathcal{A}_x$ are φ-consistent then $x \models \Diamond\varphi$ and by assumption we also have that $x \models \Box(\varphi \to \psi)$. Hence there is a not empty $Z \in \mathcal{A}_x$ with $Z \subseteq [\![\varphi \to \psi]\!]$. To prove that $x \models \varphi \rightsquigarrow \psi$ pick any φ-consistent $U \in \mathcal{A}_x$. Now consider the intersection $U \cap Z \subseteq U$. It is in $\mathcal{A}_x$ since

$\mathcal{A}_x$ is a topology. By (C) we have that $x \in U \cap Z$ hence it is a non empty element of $\mathcal{A}_x$. Since we are in the case where all non empty elements of $\mathcal{A}_x$ are φ-consistent it follows that $U \cap Z$ is also φ-consistent. It holds that $U \cap Z \cap [\![\varphi]\!] \subseteq [\![\psi]\!]$ because $U \cap Z \subseteq Z \subseteq [\![\varphi \to \psi]\!]$.

In the other case there is a non empty $Z \in \mathcal{A}_x$ which is not φ-consistent. Then $x \models \neg\Diamond\varphi$ and so $x \models \forall(\varphi \to \Diamond(\varphi \wedge \Box(\varphi \to \psi)))$. To show that $x \models \varphi \rightsquigarrow \psi$ take any φ-consistent $U \in \mathcal{A}_x$. Because it is φ-consistent there is a $u \in U$ with $u \models \varphi$. So U is a non-empty element of $\mathcal{A}_x$ and hence by $x \models \forall(\varphi \to \Diamond(\varphi \wedge \Box(\varphi \to \psi)))$ we have that $U \subseteq [\![\varphi \to \Diamond(\varphi \wedge \Box(\varphi \to \psi))]\!]$. So $u \models \Diamond(\varphi \wedge \Box(\varphi \to \psi))$. Because of (Ov) we have $U \in \mathcal{A}_u$ and so U is $\varphi \wedge \Box(\varphi \to \psi)$-consistent. Hence there is a $v \in U$ with $v \models \varphi$ and $v \models \Box(\varphi \to \psi)$. By the latter it follows that there is a non-empty $V \in \mathcal{A}_v$ such that $V \subseteq [\![\varphi \to \psi]\!]$. By (CU) $V \cup Z \in \mathcal{A}_w$ for some world W. Because $x \in Z$ it follows that $V \cup Z \in \mathcal{A}_x$ by (Ov). Since $\mathcal{A}_x$ is closed under intersections we then obtain $(V \cup Z) \cap U \in \mathcal{A}_x$. Clearly $(V \cup Z) \cap U \subseteq U$ and $(V \cup Z) \cap U \subseteq V \cup Z \subseteq [\![\varphi \to \psi]\!] \cup [\![\neg\varphi]\!] \subseteq [\![\varphi \to \psi]\!]$ because Z is not φ-consistent. $(V \cup Z) \cap U$ is φ-consistent because $v \in V$ by (C) and hence $v \in (V \cup Z) \cap U$. □

Corollary 10 *The validities of conditional logic over coherent neighbourhood spaces are completely axiomatizable.*

Proof. One can obtain an axiomatization by translating the S4u rules and axioms into the language of the conditional using the translation clauses in Theorem 9 and adding the translation of the third equivalence from Theorem 9 as an axiom. □

5 Conclusions and Further Work

In this paper we have done the following. We have used the Alexandroff correspondence between preorders and Alexandroff topological spaces to:

- clarify the formal relation between the order semantics for conditionals and its semantics based on sets of sets of worlds, as in the case of premise frames and evidence models.
- present a topological semantics for conditional logic.
- provide a construction to generate a similarity order among worlds starting from a set of relevant propositions. We also characterized the order frames that arise from this construction.

Additionally, we established the intertranslatability of the logic of counterfactual over topological spaces and S4u, by means of which we obtained a completeness result.

Some possible directions for further work are:

- One might try to weaken the somewhat implausible assumption that the set of relevant propositions is closed under disjunctions. This might lead to a weaker notion of coherence for neighbourhood spaces and order frames.
- One can try to abstract away from possible worlds and construct them and the relative similarity order from an algebra of propositions.
- The axiomatization of conditional logic on coherent order frames obtained in Corollary 10 is not aesthetically pleasing. It remains an open question whether there are more natural axioms.
- Our construction of a relative similarity order from a set of relevant propositions treats all relevant propositions as equally relevant. However, it seems natural to rank propositions according to their importance, for instance see (Lewis, 1979, p. 472). This ranking might be implemented by an ordering over the set of relevant propositions.

We hope to address some of these points in a future paper.

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