Extended operators in quantum field theory

Light-ray operators and generalized symmetries

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The defense will take place on Friday 22nd of October 2021 at 11:00 in the Kik Laboratory, Singel 411 in Amsterdam. The defense will be followed by a lunch with colleagues.

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EXTENDED OPERATORS IN QUANTUM FIELD THEORY

LIGHT-RAY OPERATORS AND GENERALIZED SYMMETRIES

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aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K.I.J. Maex
ten overstaan van een door het College voor Promoties
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in het openbaar te verdedigen in de Aula der Universiteit
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[1] The author contributed to all conceptual discussions. In addition, the author performed all the two-dimensional computations of section 2 and wrote the first draft of this section. The author also computed all the necessary OPEs that were needed to derive the differential operators in higher-dimensional CFTs. The derivation of the differential operators themselves was a joint effort. The author also wrote all the appendices.

[2] The author contributed to all conceptual discussions. The author performed the free field theory computations as well as the two-point function computations necessary for sections 4, 5 and 6. Section 5 was a joint effort. He also did the OPE computation of section 6. The author wrote the first draft for all these sections as well as the appendices A to F.

[3] The author contributed to all conceptual discussions. The author did the hydrodynamics expansion of section 3 and 4. He wrote the draft of these sections as well as the appendices A, B, and C.
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1

Introduction

1.1 Motivation

In the last centuries, the theoretical physics community has made impressive progress. This has radically improved our current understanding of the world we live in. That joint effort culminated in the twentieth century with the discovery of quantum mechanics and general relativity. These two theories, which are nowadays the two pillars of theoretical physics, have been put through rigorous tests, and are to this day the most successful frameworks that we have to describe the phenomena we encounter. These two theories are valid at different scales. Quantum mechanics describes physics at very small distances, at the scale of atoms and subatomic particles. It can be used to account for a wide variety of physical systems, ranging from the structure of the hydrogen atom to the current search for a quantum computer. Once you combine quantum mechanics with special relativity, you obtain quantum field theory (QFT), which describes, for example, the realm of elementary particles and accounts for their interactions. This is the framework that is used to understand the scattering experiments that happen in the Large Hadron Collider at CERN. It is also used to describe many interesting condensed matter systems. Quantum field theory successfully accounts for three of the four fundamental forces of nature, which are known as the electromagnetic, weak, and strong interactions. The fourth, gravity, is negligible at the scale where quantum field theory applies due to its weakness. However, at very big distances, the effects of gravity need to be taken into account, and its properties are described using general relativity, the famous theory developed by Einstein. General relativity is the geometrical theory that describes gravitation as a manifestation of curved space and time. It has also been thoroughly verified through countless observations, describing the perihelion precession of Mercury with great accuracy for example. In addition, gravitational waves, that have been recently experimentally measured [7, 8], are an integral part of general relativity.

Despite the massive success of these two frameworks, there are cases, such as black holes, where quantum mechanics and general relativity need to be combined into a single theory that is necessary to be able to account for their very rich physics. For example, general relativity breaks down at the center of a black hole due to the extreme value of the
curvature, and hence we need another theory to describe the physics happening in this case. A theory that would successfully unify gravity and quantum mechanics would be a theory of *quantum gravity*. The search for such a theory has been one of the major endeavours in theoretical high-energy physics since the time of Einstein himself. Sadly, such a theory that could yield predictions for our own Universe and unify physics at small and large scales has not been found yet. The most promising candidate for such a theory is *string theory*.

The main idea of string theory, which is an attempt at describing quantum gravitational physics, is to replace the usual concept of point particles with extended objects, such as strings or particular membranes that are called *D-branes*. The excitations of these extended objects can account for both the usual degrees of freedom, such as the *photon* and for the quantum mechanical excitation mediating gravity, the *graviton*. This seems to bring us closer to a possible resolution, as string theory is precisely one instance of such a theory of quantum gravity. Nevertheless, a theory is only useful provided it can be used to make rigorous predictions about the world we daily experience, or explain the data gathered through experiments. This has been one of the main challenges that string theorists have faced.

When we want to use string theory to compute physical observables, all sorts of complications appear along the way. One of the most striking ones is the fact that string theory is not sensible in only four spacetime dimensions, which is what we daily experience as human beings. String theory requires extra dimensions to be a mathematically well-defined theory, and we thus need to compactify these extra dimensions one way or another. Besides, it is not clear whether string theory can produce the vacuum solution of general relativity that approximates well the accelerated expansion of our universe. This is by all means not an exhaustive list of complications, but merely an illustration that there is still a lot of work needed to be able to find a theory of quantum gravity, and that this is still a very active field of research.

Moreover, many of physics’ most important phenomena can be described using quantum field theory, and do not require quantum gravity. Within the realm of quantum field theory, we can separate physical systems into two big categories: the so-called *weakly-coupled* and *strongly-coupled* systems. One of the big success of twentieth-century physics has been to develop extremely powerful perturbative techniques, which work very well for weakly-coupled systems. The general idea is that the *coupling constant*, which is the parameter that accounts for the strength of the interaction, is sufficiently small for weakly-coupled systems such that we can use it as a parameter to perform a perturbative expansion around a solvable non-interacting theory. With this technique, we can compute observables order by order in perturbation theory to higher and higher accuracy. Nevertheless, for strongly-coupled systems, this method breaks down due to the lack of a small parameter to perform an expansion. This is problematic since many of the most interesting phenomena admit a description in terms of strongly-coupled quantum field theories. For example, QCD, which is the theory of strong interactions, becomes strongly-coupled at the low energies that are accessible through experiments. Another
famous example is high-temperature superconductivity. For strongly-coupled QFTs, the
naive way to picture the problem is the following. Interactions become so intense that
the usual methods that are available to physicists break down.

Developing new non-perturbative tools is thus both important and pressing to be able to
get a better understanding of strongly-coupled QFTs. Two very promising examples of
such new ideas and methods are holography and the conformal bootstrap. In addition,
generalized symmetries provide a new important organizing principle, suited for extended
objects in QFT such as strings, and can help decipher strongly-coupled QFTs. Developing
new insights on strongly-coupled systems as well as a better understanding of quantum
gravity using these ingredients is the main motivation for the work presented in this
thesis, and we will explain many important concepts along the way.

1.2 Holography

1.2.1 The general idea

As we described in section 1.1, string theory is still facing many challenges to be able
to account for the plethora of physics phenomena that we observe. Nevertheless, it has
been an extremely fruitful ground, providing many novel ideas, sometimes with deep
connections to pure mathematics. A particularly innovative and prolific idea that has
yielded impressive results since its proposal is the holographic principle.

The first hint that a proposal like the holographic principle could be true originated in
the study of the interplay between black hole physics and thermodynamics. If black holes
carried no entropy, one could violate the second law of thermodynamics by throwing mass
into a black hole. To forbid this from happening, black holes must have an entropy that
increases by a greater amount compared to the entropy of the mass thrown in. With this
idea in mind, Jacob Bekenstein conjectured that the entropy of a black hole $S_{BH}$ should
be \[ S_{BH} = \frac{A}{4G}, \]  (1.1)
with $G$ the Newton constant and $A$ the area of the event horizon of the black hole. In a
subsequent paper, Stephen Hawking [10] showed that this entropy is a thermodynamic
entropy, and as such, the degrees of freedom that are necessary to describe a black hole
appear to live not in the volume encircled by the black hole but on the boundary of it, i.e.
in one dimension less than the black hole itself. This is a hologram. This reorganization
of degrees of freedom is a property of quantum gravity itself.

Typically, the thermodynamic entropy scales with the volume in quantum field theory,
and not with the area as in (1.1). The naive picture is that the constituents tend to
spread evenly and cover all of the available volume. In this regard, quantum gravity
seems to behave very differently, and the degrees of freedom repackage themselves on a
lower-dimensional surface, which is the boundary of the objects of interest. In the case of a black hole, this is the horizon.

Although quite vague, and surprising to say the least, this idea was materialized quite explicitly later on in what is known as the AdS/CFT correspondence.

1.2.2 The AdS/CFT correspondence

A specific realization of the holographic idea is the AdS/CFT correspondence. This is a conjecture that originated from string theory and was put forward a bit over twenty years ago [11]. The original conjecture states that providing one takes the appropriate decoupling limit, the same physical system can be described by two seemingly very different theories. In particular, it relates type IIB supergravity living on AdS$_5 \times$ S$^5$ to $\mathcal{N} = 4$ Super Yang-Mills (SYM), which is a supersymmetric gauge theory living on the conformal boundary of the AdS$_5$ space.

More generally, the AdS/CFT correspondence is a conjecture that relates some quantum theories of gravity in the bulk of a $(d+1)$-dimensional Anti-de Sitter (AdS) spacetime (which is a solution to Einstein equation with negative cosmological constant) to lower-dimensional conformal field theories (CFTs) living on the $d$-dimensional boundary of AdS$_{d+1}$. Conformal field theories are quantum field theories with an additional symmetry structure: Conformal symmetry. The conformal group is the group of space transformations that preserve angles, and they locally look like a rotation and a rescaling. We will give more details in section 1.3.3.

Since the original proposal, the amount of evidence accumulated in support of this duality is overwhelming, strongly suggesting that this duality holds. In addition, even though the original conjecture was fairly precise and related two specified theories, many more examples of holographic dualities have now been discovered. They are part of a more general family that is called gauge/gravity dualities.

The correspondence introduces a duality between both formulations such that they describe the same physical system in a different language. This duality has proven particularly powerful as it is what is called a weak/strong duality, which relates the weakly-coupled regime of one description to the strongly-coupled regime of the other. We will see an explicit example of this in section 1.2.5. It is thus a powerful tool to study strongly-coupled problems in QFT through studying weakly-interacting gravitational theories. But it also works the other way around and we can learn about quantum gravity by studying the space of consistent and allowed CFTs. Since the original proposal, a whole dictionary has been developed that relates quantities on both sides of the duality. If the proposal holds, we should be able to map the symmetries, the degrees of freedom as well as the dynamics on both sides of the duality. Let us review some of the most basic maps between the two descriptions.
1.2. Holography

1.2.3 The holographic dictionary

The AdS/CFT correspondence relates two theories that live in spacetimes of different dimensions. To be able to define the correspondence, we need to have a map between quantities in the two different formulations. This is the idea of the holographic dictionary.

Matching the symmetries

First and foremost, let us explain what happens with the symmetries in the two formulations. Gauge symmetries of the bulk theory become global symmetries in the boundary theory. In particular, the AdS bulk isometries (which are the symmetries that leave the AdS spacetime unchanged) are mapped to the global symmetry group of the boundary theory. The isometry group of the \((d + 1)\)-dimensional anti-de Sitter space \(\text{AdS}_{d+1}\) is \(SO(d, 2)\) that contains the Lorentz group in \(d\) dimensions together with a dilatation and an inversion symmetry. It happens that \(SO(d, 2)\) is also the conformal group in \(d\) dimensions as we will explain in section 1.3.3. We thus see that the symmetries can be mapped on both sides of the duality.

Scale/radius correspondence

The extra dimension of the bulk, which we call the holographic (radial) direction is related to the energy scale of the field theory. To illustrate this connection, we can consider \(\text{AdS}_{d+1}\) in Poincaré coordinates. The metric is

\[
ds^2 = \frac{\ell_{\text{AdS}}}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right),
\]

where \(z\) is the holographic direction, \(\ell_{\text{AdS}}\) the AdS radius and \(\eta_{\mu\nu}\) is the Minkowski metric with \(\mu = 0, \ldots, d - 1\). This metric covers only the Poincaré patch of AdS, and the boundary of AdS is located at \(z = 0\) in these coordinates.

The field theory is conformal, and so is invariant under rigid scale transformations \(x^\mu \rightarrow \lambda x^\mu\). This transformation rescales the energy of particles as \(E \rightarrow E/\lambda\). In the bulk, this symmetry has to be dual to the diffeomorphism \(x^\mu \rightarrow \lambda x^\mu, z \rightarrow \lambda z\), which is an isometry of the metric (1.2). This implies that we can identify the extra bulk direction \((z)\) with an energy scale in the gauge theory, i.e. \(z \sim 1/E\) [12]. The boundary region of AdS, which is at \(z \ll 1\), is thus associated with the UV regime in the CFT, and vice-versa. This gives rise to a UV/IR duality, where high energies (short distances) in the CFT correspond to large radii in the bulk, which are closer to the boundary. This weak/strong nature of the AdS/CFT correspondence is crucial, and we will return to it shortly.

Bulk fields and boundary operators

To make sense of the AdS/CFT proposal, we must be able to provide a map between bulk fields and boundary operators. The general picture is the following: Consider a field
\[ \Phi(x, z) \] that propagates in the bulk. This field is related to a local gauge invariant operator \( \mathcal{O}(x) \) in the field theory. The boundary operator \( \mathcal{O}(x) \) couples to the restriction of the bulk field on the boundary \( \phi_{(0)}(x) \) through a boundary term of the form \( \int_{\mathcal{B}_d} \phi_{(0)} \mathcal{O}(x) \) where \( \mathcal{B}_d \) is the boundary manifold.

The bulk fields and boundary operators need to have the same Lorentz structure and quantum numbers for this boundary coupling to make sense. For example, a scalar bulk field is dual to a scalar boundary operator, while a boundary conserved current associated with a global symmetry is dual to a bulk dynamical gauge field. Moreover, the conformal dimensions of the boundary operators are related to the mass of the bulk fields. For a scalar field, we have

\[ m^2 \ell_{\text{AdS}}^2 = \Delta(\Delta - d), \]

where \( m \) is the mass of the bulk field \( \Phi(x, z) \) and \( \Delta \) the conformal dimension of the operator \( \mathcal{O}(x) \), which characterizes its transformation under conformal transformations. We will define it precisely in section 1.3.3.

### Equivalence of partition functions

One of the most important entry of the AdS/CFT dictionary followed rapidly after the original conjecture, and is necessary to be able to use the correspondence for actual computations [13, 14]. The observation is that the partition functions are equal in both theories, i.e.

\[ Z_{\text{string}} = Z_{\text{CFT}}. \]

We want to make this idea more precise as well as review the basic mechanism. For this, we will restrict ourselves to the weak form of the correspondence, where instead of considering the full string partition function (which is in general not known), we will use the fact that when gravity is weakly coupled, it is well-approximated by semiclassical supergravity. We want to discuss the Euclidean case.

In the gauge theory, this weak limit corresponds to taking the index of the gauge group \( N \) to infinity together with the coupling constant \( g_{\text{YM}}^2 \) to zero while keeping the product \( g_{\text{YM}}^2 N \) fixed. In this case, the \( n \)-point functions are dominated by planar contributions [15], and we can rewrite (1.4) in a slightly more precise manner

\[ Z_{\text{string}} \sim e^{-S_{\text{Supergravity}}} = Z_{\text{CFT}} = e^W, \]

where \( W \) is the generating functional for connected Green’s functions that we will define shortly, and \( S_{\text{Supergravity}} \) is the on-shell supergravity action.

Consider a Lagrangian \( \mathcal{L}_{\text{CFT}} \) in the CFT. Then, every operator of conformal dimension \( \Delta \) that we call \( \mathcal{O}(x) \) can be associated to a source (background gauge field) \( \phi_{(0)}(x) \) as

\[ \mathcal{L}_{\text{CFT}} \rightarrow \mathcal{L}_{\text{CFT}} - \int d^d x \phi_{(0)}(x) \mathcal{O}(x). \]

We can then define \( W[\mathcal{O}, \phi_{(0)}] \) as the generating functional for connected correlation
functions of $\mathcal{O}$. As usual, we have

$$Z_{\text{CFT}} = e^{W[\mathcal{O}, \phi(0)]} = e^{\int d^4x \, \phi(0)(x) \mathcal{O}}.$$  \hspace{1cm} (1.7)

We can obtain the connected correlators in the usual manner

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_c = \left. \frac{\delta^n W[\mathcal{O}, \phi(0)]}{\delta \phi(0)(x_1) \ldots \delta \phi(0)(x_n)} \right|_{\phi(0) = 0}. \hspace{1cm} (1.8)$$

In the bulk, the theory is described by an effective action, which is the on-shell supergravity action. In addition, $\langle \mathcal{O}(x) \rangle$ and $\phi(0)(x)$ are related to the bulk field $\Phi(x, z)$. In particular, solving the bulk equation of motion for the field $\Phi(x, z)$ and looking at the solution close to the boundary ($z \to 0$), we have

$$\Phi(x, z) \sim z^{d - \Delta} \phi(0)(x) + z^{\Delta} \phi(2\Delta - d)(x) + \ldots,$$ \hspace{1cm} (1.9)

where the dots are subleading terms in the expansion. We can thus impose the following Dirichlet boundary condition

$$\phi(0)(x) = \lim_{z \to 0} z^{-(d - \Delta)} \Phi(x, z), \hspace{1cm} (1.10)$$

with $\phi(0)(x)$ the boundary value of the field that needs to be identified with the source of the dual operator $\mathcal{O}(x)$ using (1.7). We can now write the partition function in the bulk as

$$Z_{\text{bulk}} = \exp \left( - S_{\text{on-shell}}[\Phi(x, z)] \big|_{\lim_{z \to 0} (z^{\Delta - d}\Phi(x, z)) = \phi(0)(x)} \right). \hspace{1cm} (1.11)$$

We thus have

$$W[\mathcal{O}, \phi(0)] = -S_{\text{on-shell}}[\Phi(x, z) \sim \phi(0)(x)], \hspace{1cm} (1.12)$$

where we evaluate the supergravity action on solutions of the bulk equation that are compatible with the boundary condition (1.10). Using (1.7) it is then clear that we can compute the correlation functions of $\mathcal{O}(x)$ in the CFT by taking functional derivatives of the on-shell action $S_{\text{on-shell}}$.

Nowadays, the dictionary contains many more entries, but we will not provide more details in this dissertation. There are many excellent reviews, for example [16–21].

### 1.2.4 Strong version of the AdS/CFT

Since the original conjecture that related two very precise theories in a particular limit, the perspective on AdS/CFT has evolved. The current view, which is widely agreed on, is called the strong version of AdS/CFT. AdS/CFT strongly suggests that spacetime emerges, as an approximate low-energy description, from microscopic degrees of freedom living on the boundary. We can thus expect that the holographic duality extends to a very large class of CFTs that all have very different microscopic details. However, they all produce emergent spacetime at low energies. This universality in the space of QFT
at strong coupling is remarkable, but the general mechanism responsible for it is not yet fully understood.

As we already explained, AdS/CFT is a weak/strong duality, and this implies that all the results that can be derived in the weakly-coupled gravity theory in AdS have to hold on the CFT side. Moreover, an independent derivation should be possible using only CFT techniques. A particular example is the universality of Einstein’s gravity. Any consistent theory of quantum gravity should reduce, in the low energy limit, to Einstein gravity (or general relativity). The strong version of the AdS/CFT correspondence suggests that every CFT gives rise to a quantum gravity theory in the bulk of AdS, but for most of these CFTs, the bulk theory will be highly curved and quantum. It is thus a natural question to wonder: What are the necessary and sufficient conditions for a CFT to admit a weakly-coupled Einstein gravity dual in the bulk? A CFT with such a gravity dual is called a holographic CFT.

We want to discuss an example, which is the case of AdS$_3$/CFT$_2$. This case is better understood thanks to the presence of Virasoro symmetry and modular invariance of the partition function on the torus, and will provide a useful toy example. We then want to comment on the higher-dimensional analog, which will ultimately be the focus of chapter 2.

### 1.2.5 Universality in AdS$_3$/CFT$_2$

Let us start with Einstein gravity in three bulk dimensions in the presence of a cosmological constant $\Lambda$. The Einstein-Hilbert action is given as

$$S_{EH} = \frac{1}{16\pi G} \int d^3 x \sqrt{-g} (R - 2\Lambda) + S_B ,$$

(1.13)

with $G$ the three-dimensional Newton constant, $g_{\mu\nu}$ the metric and $g$ its determinant and $R$ the Ricci scalar. The action (1.13) is defined up to a boundary term $S_B$ which is necessary to ensure that the complete action has a well-defined action principle.

Three-dimensional gravity is peculiar as it has no propagating degree of freedom, and is therefore topological. In addition, solutions must be maximally symmetric spaces of constant curvature. Here, we are interested in the case $\Lambda < 0$, which is Anti-de Sitter space. Despite looking trivial, this theory has a lot of interesting features. For example, it can be written as a Chern-Simons theory [22, 23]. Moreover, the naive guess would be that AdS$_3$ has $SO(2, 2)$ asymptotic symmetry algebra, but it has an infinite asymptotic symmetry algebra: The Virasoro algebra [24]. In addition, three-dimensional gravity also admits black hole solutions. We want to discuss these points in more detail.

The metric of AdS$_3$

In any dimension $d$, AdS$_d$ space can be defined through an hyperboloid equation in an embedding space of dimension $d + 1$. For the case of AdS$_3$, we can start with $\mathbb{R}^{(2,2)}$ with
coordinates \( \{X^0, X^1, X^2, X^3\} \) and the metric
\[
ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + (dX^3)^2,
\]
and obtain AdS\(_3\) as the hypersurface that solves the constraint
\[
-(X^0)^2 - (X^1)^2 + (X^2)^2 + (X^3)^2 = -\ell_{AdS}^2,
\]
where \( \ell_{AdS} \) is the AdS length. This has the isometry group \( SO(2, 2) \).

The metric of (the universal cover of) AdS\(_3\) can be written in global coordinates as
\[
ds^2 = -\left(1 + \frac{r^2}{\ell_{AdS}^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell_{AdS}^2}\right)^{-1} dr^2 + r^2 d\phi^2,
\]
with \( 0 \leq r < \infty, \ -\infty < t < \infty \) and \( 0 \leq \phi < 2\pi \).

**BTZ black hole**

It came as a surprise when in 1992, Bañados, Teitelboim, and Zanelli (BTZ) showed that three-dimensional gravity admits a black hole solution [25] (reviews can be found in [26, 27]). The BTZ black hole of mass \( M \) and angular momentum \( J \) is described by the metric
\[
ds^2 = -N(r) dt^2 + N(r)^{-1} dr^2 + r^2 (d\phi + N^\phi(r) dt)^2,
\]
with
\[
N(r) = \left(-8GM + \frac{r^2}{\ell_{AdS}^2} + \frac{16G^2J^2}{r^2}\right), \quad N^\phi(r) = -\frac{4GJ}{r^2}.
\]
The metric (1.17) is a solution of the Einstein equation with cosmological constant \( \Lambda = -1/\ell_{AdS}^2 \). It has horizons at
\[
r_\pm = \ell_{AdS} \left[4GM \left(1 \pm \sqrt{1 - \frac{J^2}{M^2\ell_{AdS}^2}}\right)\right]^{1/2},
\]
and the mass and angular momentum can be rewritten in terms of the two radii \( r_\pm \) as
\[
M = \frac{r_+^2 + r_-^2}{8G\ell_{AdS}^2}, \quad J = \frac{r_+ - r_-}{4G\ell_{AdS}}.
\]
We can compute its Hawking temperature, and it is given as
\[
T_{BTZ} = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell_{AdS}^2},
\]
while its Bekenstein entropy is now one quarter of its perimeter, as we would have predicted using (1.1). We obtain
\[
S_{BTZ} = \frac{2\pi r_+}{4G} = 2\pi \left(\sqrt{\frac{\ell_{AdS}}{8G} (M\ell_{AdS} + J)} + \sqrt{\frac{\ell_{AdS}}{8G} (M\ell_{AdS} - J)}\right).
\]
The Brown-Henneaux central charge

Before AdS/CFT was even proposed, Brown and Henneaux [24] derived an important result relating AdS$_3$ and CFT$_2$ in a non-trivial way. By carefully analyzing the asymptotic symmetries of AdS$_3$, they realized that the generators $L_m$ and $\bar{L}_m$ of the diffeomorphisms that preserve the appropriate boundary conditions satisfy two commuting copies of the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{n+m,0},$$  

with the central charge

$$c = \frac{3\ell_{AdS}}{2G}.\quad (1.24)$$

This result is remarkable. It shows that the algebra of charges that is associated with the symmetry transformations preserving the form of the asymptotic AdS$_3$ consists of two copies of the Virasoro algebra. This is precisely the (infinite-dimensional) symmetry structure of local conformal transformations in two dimensions as we will describe in more detail in section 1.3.3. In the CFT, the existence of a central charge is a consequence of a conformal anomaly in the quantum theory. In other words, the theory is conformal classically, but quantum effects spoil its conformal nature. However, the computation of Brown and Henneaux is purely classical. This is quite surprising, and a reason why this result is often seen as a precursor of the AdS/CFT correspondence. With this point of view, this is interpreted as a classical AdS$_3$ bulk computation describing a property of the effective action of the quantum boundary theory, which in this case is a CFT$_2$ living on the boundary of the AdS$_3$ space.

Moreover, the result (1.24) is a concrete example of the strong/weak nature of the duality that we described in section 1.2.4. The naive way to understand the central charge is as a number counting the number of degrees of freedom in a given CFT. Ultimately, we see that in (1.24), the two quantities $c$ and $G$ are inversely related. This implies that a big $G$ (gravity is strongly-coupled) corresponds to a small $c$, and thus the number of degrees of freedom in the CFT$_2$ at asymptotic infinity is small. Conversely, a small $G$ implies a big $c$. Because classical gravity is weakly-coupled, which is equivalent to considering small $G$, this last case is the most relevant. In this case, the CFT has a large central charge, which in practice suppresses quantum gravitational effects. We just discovered one of the necessary ingredients to have a holographic CFT. We need a large number of degrees of freedom, or large $c$. In the extreme case, where $c \rightarrow \infty$, gravity is effectively turned off in the bulk.

Another way to understand the requirement of a large central charge $c$ is to study the stress-tensor three-point function suitably normalized. This is dual to the three graviton vertex in the bulk and directly proportional to $\sqrt{G_N}$. In two dimensions, this three-point function has a single tensor structure, and it scales as $1/\sqrt{c}$. This implies that two-dimensional CFTs with a weakly-coupled gravity dual should have a large central charge. All in all, holographic CFTs need to have at least both conformal symmetry and a large central charge to describe semi-classical bulk gravitational physics in asymptotically AdS spacetimes.
BTZ black hole entropy from the CFT

Soon after the original paper on AdS/CFT, Strominger provided one of the most beautiful hints on the holographic principle \[28\]. In particular, he managed to reproduce the entropy of the BTZ black hole (1.22) in the CFT, and we want to review the main idea of this derivation here.

In a CFT\(_2\) in the limit of large conformal dimension \(\Delta\), the degeneracy of states is given by the Cardy formula \[29\], which is ultimately a consequence of modular invariance of the partition function on the torus. More precisely, for a CFT\(_2\) with central charge \(c\), the universal result for the entropy for states of large conformal dimension \(\Delta\) is given by

\[
S(\Delta, \bar{\Delta}) = 2\pi \sqrt{\frac{c}{6} \left( \Delta - \frac{c}{24} \right)} + 2\pi \sqrt{\frac{c}{6} \left( \bar{\Delta} - \frac{c}{24} \right)},
\]

(1.25)

where we can also see \((\Delta, \bar{\Delta})\) as the eigenvalues of the energy operators \((L_0, \bar{L}_0)\) of the right- and left-moving sectors respectively. We want to use the Cardy formula (1.25) to compute the number of microstates that are associated to a BTZ black hole of mass \(M\) and angular momentum \(J\) in the semi-classical limit, which is the large \(M\) limit. In the CFT, the large \(M\) limit implies that \(\Delta + \bar{\Delta} \gg c\). Then, using the expressions for \(\Delta\) and \(\bar{\Delta}\) in terms of \(M\) and \(J\) as well as the Brown-Henneaux central charge (1.24) within (1.25), we obtain

\[
S \approx 2\pi \left( \sqrt{\frac{c\Delta}{6}} + \sqrt{\frac{c\bar{\Delta}}{6}} \right) = 2\pi \left( \sqrt{\frac{\ell_{AdS}}{8G}} (M\ell_{AdS} + J) + \sqrt{\frac{\ell_{AdS}}{8G}} (M\ell_{AdS} - J) \right),
\]

(1.26)

which is exactly the entropy of the BTZ black hole that we derived in (1.22). This is again a remarkable result! It asserts that, using the Cardy formula in the boundary CFT\(_2\), we can reproduce the entropy of the AdS\(_3\) black hole.

Universality in CFT\(_2\)

Finally, we want to comment on the actual CFT\(_2\) that lives at the asymptotic boundary of AdS\(_3\). To carry out the computation we just described, we never needed to specify a particular two-dimensional CFT. Generally, we have a complete family of two-dimensional CFTs that are labeled by their central charge \(c\). In addition, using the strong version of the AdS/CFT that we introduced in section 1.2.4 we can see every two-dimensional CFT as describing quantum gravity in AdS\(_3\). However, not every CFT\(_2\) has the appropriate properties to describe semi-classical gravity in the bulk. We already explained why a large central charge is necessary, but extra conditions might be decisive, and this question was tackled in \[30\]. We want to comment on their findings.

There is an important difference in the regime of validity of the Cardy formula and the black hole entropy formula. The Cardy formula (1.25) holds for unitary modular invariant CFT\(_2\) in the Cardy limit

\[
c \text{ fixed, } \quad (\Delta, \bar{\Delta}) \to \infty,
\]

(1.27)
whereas the Bekenstein-Hawking entropy formula holds in the semi-classical limit
\[ c \to \infty, \quad (\Delta, \bar{\Delta}) \sim c. \] (1.28)

This implies that a decisive feature of holographic CFTs is that they must have an extended range of validity for the Cardy formula. Specifically, it is clear that any CFT\(_2\) will match the BTZ entropy for infinite energy (or temperature), just as a consequence of the Cardy formula. But holographic CFTs have to satisfy the Cardy formula all the way down to the lowest physical temperature, which is the temperature of the Hawking-Page transition. In terms of the CFT, this implies that the Cardy formula should be valid down to energies \( \Delta \sim c \), where the black hole starts dominating in the bulk. This is a necessary condition for a two-dimensional CFT to be holographic. It can be shown that this is the case, i.e. that \([30]\)
\[ Z_{\text{CFT}}(\beta) = Z_{\text{grav}}(\beta), \] (1.29)
where \( \beta \) is the inverse temperature if and only if the two following conditions are met
\[ c \gg 1, \] (1.30)
\[ \rho(\Delta) \leq \exp(2\pi \Delta)) \, , \quad \text{for} \, \Delta < \frac{c}{12}, \] (1.31)
where \( \rho(\Delta) \) is the density of states at conformal dimension \( \Delta \). This last condition is now referred to as **sparseness condition** on the spectrum. Ultimately, our current understanding of the necessary conditions for a CFT to be holographic relies heavily on the fact that all stress-tensor interactions in two-dimensional CFT are captured by the Virasoro block of the identity. This is not the case in higher dimensions, and this is why answering this question is more involved in these setups.

### 1.2.6 Universality in higher dimensions

The question of what are the necessary and sufficient conditions for a CFT to be holographic is also extremely relevant in higher dimensions. We want to explain the main idea in this section.

One constraint that emerges from the AdS/CFT dictionary is the fact that for a CFT to admit a semi-classical bulk dual, the CFT must have a large number of degrees of freedom, as we explained for the two-dimensional case. For higher-dimensional analogs, this large number of degrees of freedom is usually parametrized by the index of the gauge group \( N \). A large \( N \) is necessary to have a semi-classical bulk. This is because the inverse of \( N \) scales as a positive power of the Planck length, once written in units of AdS.

Finding the necessary ingredients for a CFT to be holographic has triggered a whole lot of attention recently. In particular, it was implemented in the conformal bootstrap framework, which is the focus of section 1.3, in the seminal paper \([31]\). They conjectured that

\[
\text{Large } N \, + \, \text{Large higher spin gap} \Rightarrow \text{Weakly coupled, local gravity dual}.
\]
1.2. Holography

More precisely, the large higher spin gap condition is encoded in the following quantity: Define $\Delta_{\text{gap}}$ as the conformal dimension of the lightest single-trace operator of spin greater than two. Then, higher-dimensional holographic CFTs have large $N$ and large $\Delta_{\text{gap}}$.

From the bulk perspective, the universality of Einstein gravity is simply an effective field theory statement. As we already mentioned, any consistent theory of quantum gravity should, in the low-energy limit, be described by the Einstein-Hilbert action together with small corrections that are suppressed by a mass scale $M$ characteristic of new physics. In particular, a generic bulk effective Lagrangian can be written as

$$S_{\text{bulk}} = \frac{1}{16\pi G_N} \int \sqrt{|g|} d^{d+1}x \left( -2\Lambda + R + \frac{c_2}{M^2} R^2 + \frac{c_4}{M^4} R^3 \right), \quad (1.32)$$

where $M$ is the string mass and $c_2, c_4 \sim O(1)$ [32]. This implies that for a CFT to admit a weakly-coupled Einstein gravity dual, we need to show that it corresponds to a bulk theory where higher derivative terms, i.e. $R^2$ and $R^3$, are suppressed by the string scale. This is the indication of local bulk physics.

It was shown in [32] by studying micro-causality in high energy scattering processes in the bulk of AdS that the two coefficients $c_2, c_4$ must effectively vanish. The precise statement is

$$c_n \lesssim \frac{1}{\Delta_{\text{gap}}}, \quad (1.33)$$

which implies that $c_2$ and $c_4$ are effectively suppressed in a large gap theory. This was derived by studying the graviton three-point coupling, and the universality of the graviton three-point coupling is a fundamental property of quantum gravity that should be reproducible directly in CFT.

To probe this universality on the CFT side, we need to consider the quantity that is dual to the graviton three-point coupling. Using the usual AdS/CFT dictionary, this is the stress-energy tensor three-point function $\langle TTT \rangle$. In $d \geq 3$, this correlator is fixed by symmetry up to numbers and contains three independent tensor structures

$$\langle TTT \rangle = \langle TTT \rangle_{\text{Einstein}} + c_2 \langle TTT \rangle_{R^2} + c_4 \langle TTT \rangle_{R^3}, \quad (1.34)$$

where we have parametrized each tensor structure in terms of the gravity theories that active them. It is then clear that Einstein’s gravity is very special, in the sense that only the first tensor structure of (1.34) survives. In particular, in four dimensions, the constraint (1.33) on $c_2$ becomes a parametric bound involving the conformal anomalies $a$ and $c$, which we define in (1.103). Einstein gravity universally predicts

$$\frac{|a-c|}{c} \lesssim \frac{1}{\Delta_{\text{gap}}^2}, \quad (1.35)$$

or in other words

$$a = c, \quad (1.36)$$

1. Note that this parametrization is not unique.
in a holographic CFT with large $N$ and large $\Delta_{\text{gap}}$.

In chapter 2, we will explore this question using only CFT techniques. We will show that for a holographic CFT, we can reproduce (1.36) using the conformal bootstrap and light-ray operators that we introduce in sections 1.3 and 1.4 respectively.

1.3 The Conformal bootstrap

1.3.1 The bootstrap philosophy

There exist multiple reasons to study conformal field theories. As we already explained in section 1.2.2, we can use them to decipher the behavior of quantum gravity through the AdS/CFT correspondence. In addition, they are also very important in the study of quantum field theories themselves. The reason is that, in general, quantum field theories become scale-invariant at long distances, and very different microscopic systems admit a description in terms of the same CFT at long distances. This notion is called critical universality. This then raises the hope of being able to solve the long-distance dynamic of any QFT by understanding the complete set of well-defined CFTs.

In addition, a modern view on a large class of UV-complete QFTs is that they can be viewed as deformed CFTs. The starting point is a UV CFT at short distance, which is the UV fixed point. This CFT can be perturbed by a relevant operator to flow either to another CFT at long distance, which is the IR fixed point, or to a massive phase at long distance. Hence, we can see CFTs as signposts in the space of QFTs. Classifying and solving them would be a huge step forward to understand the space of QFTs and one particularly promising approach to tackle this problem systematically is the conformal bootstrap.

As we already mentioned, the conformal group is the group of angle-preserving transformations of space. It consists of rotations, translations, rescalings as well as special conformal transformations. In general dimension, this group is finite-dimensional but in two dimensions, the conformal group becomes infinite-dimensional (we will explain this later). This infinite-dimensional symmetry structure and its consequences are powerful enough to be able to completely solve and classify two-dimensional minimal models, such as the Ising model [33]. The solution to such models can be found using the constraints that originate from the two-dimensional conformal symmetry. To do so, we can use the conformal bootstrap philosophy that boils down to [34]:

1. Focus on a specific CFT itself in terms of its data, and do not rely on a specific realization,
2. Use the full power of conformal symmetry and determine all its consequences,
3. Impose consistency conditions that follow from the existence of an associative operator product expansion (OPE).
By combining the two last points above, it is possible to at least constraint but sometimes solve the theory. Note that many of these two-dimensional models can be exactly solved using different methods. An example is the exact solution to the two-dimensional Ising model found by Onsager [35], which does not rely on the conformal bootstrap.

The bootstrap philosophy is actually an old idea that dates back to the original work of Ferrara, Gatto, Grillo, and Polyakov [36, 37]. Despite its success in two dimensions, where the infinite-dimensional symmetry structure provides powerful constraints, the results following from using the conformal bootstrap in higher dimensions have been slower mostly due to technical reasons. Nevertheless, an important feature of the bootstrap method is that it only uses non-perturbative structures. This implies that this is a very powerful tool to uncover mysteries of strongly-coupled quantum field theories, without using explicit Lagrangians.

In the rest of this section, we first want to review the main ingredients of the conformal bootstrap without too many technical details in section 1.3.2, before discussing the relevant technicalities in section 1.3.3.

### 1.3.2 The conformal bootstrap: A quick overview

In this section, we want to give a quick overview of the main ingredients that are at the heart of the bootstrap techniques. We will also present examples where the conformal bootstrap was used with great success. The technical details are presented in the next section 1.3.3. We will broadly follow the presentation of [38].

**Correlation functions**

To characterize a conformal field theory, we need an infinite set of local operators $O_1(x), O_2(x), \ldots$. These local operators represent measurements of some given interesting physical quantities of the system at a point $x$. They can be scalar or spinning operators, but we will mostly use scalars for concreteness. The objects of interest in a CFT are the *correlation functions* (or *correlators*) of multiple local operators evaluated at different positions $x_i$,

$$\langle O_1(x_1) \cdots O_n(x_n) \rangle . \quad (1.37)$$

We are considering a correlation function in a conformally invariant theory. This implies that the correlation function (1.37) has to be covariant under any conformal transformation. In particular, we can consider an arbitrary conformal transformation that acts as $x \rightarrow x'$ on our coordinates. Then, (1.37) has to be related to the correlation function of the same operators evaluated at positions $x'_1, \ldots, x'_n$. This requirement imposes strong constraints on the possible form of correlation functions of the form (1.37).

Let us consider a scalar primary operator $O(x)$ that transforms under an arbitrary con-
formal transformation as

\[ \mathcal{O}(x') = \Omega(x)^{-\Delta_\mathcal{O}} \mathcal{O}(x), \quad (1.38) \]

with \( \Omega(x) = |\partial x'/\partial x|^{1/d} \) the (position dependant) scale factor, \( d \) the dimension of the space and \( \Delta_\mathcal{O} \) the scaling or conformal dimension of the operator \( \mathcal{O}(x) \). Imposing the co-variance of CFT correlation functions under the transformation (1.38) severely constrains the two- and three-point functions [39]. The two-point function of scalar operators \( \mathcal{O}_i(x) \) of conformal dimension \( \Delta_i \) can be diagonalized such that it vanishes if the operators are different. It is given as

\[ \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}}, \quad (1.39) \]

The three-point function takes the form

\[ \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle = \frac{\lambda_{ijk}}{|x_{12}|^{h_{ijk}} |x_{13}|^{h_{ikj}} |x_{23}|^{h_{jki}}}, \quad (1.40) \]

with \( x_{ij} \equiv x_i - x_j \) and \( h_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k \). The numbers \( \lambda_{ijk} \) are called the three-point function coefficients and are crucial in CFT. Note that these are the results for scalar operators, but some operators transform under rotations and thus carry non-trivial spin \( \ell_\mathcal{O} \). For such operators, the numerator of equations (1.39) and (1.40) gets modified in a straightforward manner [40].

**The operator product expansion**

The set of parameters \( \{ \Delta_i, \lambda_{ijk}, \ell_i \} \) is customarily denoted as CFT data. Astonishingly, knowing the CFT data is sufficient, not only to determine two and three-point functions but to compute all correlation functions (including higher-point correlators) of local operators in CFT. This is a consequence of the existence of an operator product expansion (OPE) in CFT, which implies that inside a correlation function of local operators, two nearby operators can be replaced by a (possibly infinite) sum over local operators as

\[ \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) = \sum_k f_{ijk}(x_1, x_2, x_3, \partial x_3) \mathcal{O}_k(x_3), \quad (1.41) \]

where \( k \) runs over primary operators, which we will describe in more detail in section 1.3.3. The coefficients \( f_{ijk} \) are operators packaging together the primary and its descendants in the conformal multiplet of the \( k \)-th primary operator.

Moreover, the coefficients \( f_{ijk} \) depend on the scaling dimensions and the spins of the operators appearing in the OPE. In addition, there is some freedom in choosing the position \( x_3 \), as long as it is within the radius of convergence of the OPE. Different choices are related by a Taylor expansion of the operators \( \mathcal{O}_k \).

The OPE is very powerful in CFT for two main reasons. First, conformal invariance imposes some constraints on the coefficients \( f_{ijk} \) appearing in (1.41). In particular, they are completely determined up to a number, which happens to be the three-point function coefficient \( \lambda_{ijk} \) appearing in (1.40). We can thus write

\[ f_{ijk} = \lambda_{ijk} \hat{f}_{ijk}(x_1, x_2, x_3, \partial x_3), \quad (1.42) \]
where $\hat{f}_{ijk}$ depend on the scaling dimensions as well as the spins of the operators, together with the space dimension $d$.

Second, the radius of convergence of the OPE is finite in CFT. It is determined by the distance to the closest operator. Ultimately, an OPE will converge provided there exists a sphere centered at $x_3$ separating the position of the two operators $x_1$ and $x_2$ from any other operator.

**Conformal block decomposition**

We are now ready to explain why the CFT data are enough to compute arbitrary higher-point functions of local operator. We will use the example of a $n$–point correlator of scalar local operators, which is given as

$$\langle O_1(x_1)O_2(x_2)\ldots O_n(x_n) \rangle .$$

(1.43)

Then, we can use the OPE (1.41) to replace, for example $O_1(x_1)O_2(x_2)$ as

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\ldots O_n(x_n) \rangle = \sum_k f_{12k} \langle O_k(x_k)O_3(x_3)\ldots O_n(x_n) \rangle ,$$

(1.44)

which is now a sum of $(n-1)$–correlators. It is then clear that we can perform this operation recursively and obtain a sum over two-point functions, which conformal symmetry fixes to be of the form given in (1.39). This implies that the full higher-point correlators in CFT are completely fixed in terms of the position of the operators, the CFT data and the dimensionality of space.

Let us consider the simpler case of a four-point function of scalar operators and introduce the conformal block decomposition [36, 39], which is a way of gluing three-point functions into higher-point functions. Concretely, the four-point function of four identical operators can be obtain by gluing the three-point functions $\langle OOOO_k \rangle$ and $\langle OOOk \rangle$ and summing over $O_k$. This is the statement that

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = \sum_k \lambda^2_{OOGO_k} G_{O_k}^{\Delta O} (x_1, x_2, x_3, x_4) .$$

(1.45)

The factor $\lambda^2_{OOGO_k}$ is due to the multiplication of two identical three-point functions while the function $G_{O_k}^{\Delta O}$ $(x_1, x_2, x_3, x_4)$ is called a *conformal block* and comes from gluing the three-point functions. We will give more details in section 1.3.3, but conformal blocks encode the contribution from a complete conformal family to the four-point function.

To fully solve a CFT, we then need to obtain the CFT data in this theory as they would grant us access to any correlator using the aforementioned strategy. This is sadly a really
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hard task to perform in practice. A simpler approach is thus to impose consistency conditions that carve out the space of allowed CFT data such that we obtain a well-defined theory. Two main consistency conditions need to be imposed.

Crossing equation

The first one is the requirement that the three-point function coefficients $\lambda_{ijk}$ are real for real operators. This is ultimately a consequence of unitarity and the requirement that the probabilities are real and add up to one. This implies that $\lambda_{ijk}^2$ in (1.46) is positive. Other bounds on the CFT data of the operators that are allowed in the spectrum of a conformal field theory follow from unitarity and are known as unitarity bounds. We will not describe these in this dissertation.

The second consistency condition follows from using the OPE decomposition (1.41) inside a four-point function in two different ways. In practice, we can fuse the two operators $\mathcal{O}(x_1)\mathcal{O}(x_2)$ together with $\mathcal{O}(x_3)\mathcal{O}(x_4)$ which would result in an infinite sum over two-point functions. But we can also do the OPE between $\mathcal{O}(x_1)$ and $\mathcal{O}(x_3)$ together with $\mathcal{O}(x_2)$ and $\mathcal{O}(x_4)$. Ultimately, these two different decompositions need to agree in their overlapping region of convergence. This results in the crossing equation, that needs to be true in any unitary CFT and take the schematic form

$$\sum_k \mathcal{O}_k^{1234} = \sum_k \mathcal{O}_k^{1324} \tag{1.47}$$

The crossing equation (1.47) relates the CFT data in non-trivial ways and needs to be satisfied for all configurations of local operators. In practice, if we were able to impose the crossing equation (1.47) for every four-point function of the theory, we would be able to solve for all the CFT data. Solving the crossing equation exactly is an extremely difficult task to carry out in practice, especially for higher-dimensional CFTs, and this is the reason why progress in using the bootstrap in $d > 2$ has been slower. Instead of solving the crossing equation (1.47) exactly, we can see it as a geometric equation to derive universal bounds on the CFT data.

Geometric interpretation of the crossing equation

We can rewrite the crossing equation (1.47) using (1.46) and we arrive to

$$0 = \sum_k \lambda^2 \mathcal{O}_k \mathcal{O}_k^{1234} \left[ G^{\Delta_{\mathcal{O}_k}}_{\mathcal{O}_k, \mathcal{O}_k} (x_1, x_2, x_3, x_4) - G^{\Delta_{\mathcal{O}_k}}_{\mathcal{O}_k, \mathcal{O}_k} (x_4, x_2, x_1, x_3) \right]. \tag{1.48}$$

The idea is then to derive universal bounds (bounds that hold regardless of the other unknown parameters of the theory) on CFT data by studying equation (1.48) geometrically
1.3. The Conformal bootstrap

[41]. We think of the equation (1.48) as a geometric equation that takes the schematic form

\[ 0 = \sum_{\Delta, \ell} \lambda^2_{O(x)O(y)} F_{\Delta, \ell}^{\Delta, \ell}, \]  

(1.49)

where \( F_{\Delta, \ell}^{\Delta, \ell} \) is the function in the parentheses of (1.48) and \( \Delta, \ell \) run over the dimensions and spins of the operators that appear in the \( \mathcal{O}(x)\mathcal{O}(y) \) OPE. The idea is now to think about this quantity as a vector in the (infinite-dimensional) vector space of functions of four positions. This is extremely powerful because (1.49) is a sum with positive coefficients (because \( \lambda_{O(x)O(y)} \) is squared) that needs to vanish. It implies that if all vectors \( F_{\Delta, \ell}^{\Delta, \ell} \) lie in the same direction, it is impossible for them to add up to zero. We are now solving a geometric problem where we are trying to find which configurations of vectors have a chance of solving (1.49) and which configurations must be discarded because they will never solve (1.49). In practice, this task is achieved by looking for a surface, called a separating plane \( \delta \). This surface passes through the origin and we would like all vectors to lie on only one side of the surface. If such a surface exists, we are in the situation where there is no chance to solve (1.49), and we can thus determine that these CFT data cannot be CFT data of a consistent unitary CFT. This can be represented pictorially, as shown in figure 1.1. Note that this still sounds like a hard technical problem, because the vectors \( F_{\Delta, \ell}^{\Delta, \ell} \) live in an infinite-dimensional space. Nevertheless, finding a separating plane in any finite-dimensional subspace is sufficient to be able to use the aforementioned strategy. This yields inequalities that all need to be verified at the same time. This method has yield impressive results by providing exclusion plots that separate the allowed region of parameter space from the disallowed region. In particular, we can start with well-chosen values of scaling dimensions, and search for a separating plane. If it exists, it means that the starting parameters are forbidden, and we have to restart the same search with different values.

Examples

To conclude this section, we would like to provide one example where this strategy was used with massive success. Let us consider three-dimensional CFTs with a \( \mathbb{Z}_2 \) symmetry. Then, the operators \( \sigma \) and \( \epsilon \) are the two leading operators in terms of scaling dimension. \( \sigma \) is the lowest \( \mathbb{Z}_2 \)-odd operator, and it is the continuum version of the Ising spin while \( \epsilon \) is the leading \( \mathbb{Z}_2 \)-even operator, which is the continuum version of the product of two neighboring spins. Then, running the aforementioned strategy yields an exclusion plot for the two conformal dimensions \( (\Delta_\sigma, \Delta_\epsilon) \), which separates the allowed and disallowed regions for these parameters. This plot exhibits a kink that sits very close to the values of the three-dimensional Ising model [43]. This is displayed on figure 1.2.

Moreover, it is possible to add extra ingredients to this computation, which were not present in the previous analysis. In particular, we can add crossing symmetry of the four-point functions of the two operators \( \sigma \) and \( \epsilon \), and impose that they are the only relevant operators in the theory (relevant operators have scaling dimensions \( \Delta_\mathcal{O} < d \)). This results in the most precise determination of the two scaling dimensions, yielding a small
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Figure 1.1 – This is the geometric problem that follows from the crossing equation. On the left-hand side, the vectors have a chance to add up to zero even with strictly positive coefficients, and this can yield a well-defined CFT. On the right, such a configuration will never add up to zero, and we have found a separating plane $\delta$ that has all vectors on one side. This picture is heavily inspired by the one in [42].

island in $(\Delta_\sigma, \Delta_r)$–space [44–46]. This conformal bootstrap computation outperformed dramatically other methods. This is displayed on figure 1.3.

We can now present the relevant technicalities that are relevant to be able to use the conformal bootstrap in higher-dimensional CFTs.

1.3.3 Basics of $d$–dimensional CFT

In this section, we want to review the basic necessary material of $d$–dimensional CFT that is needed for the rest of this dissertation. This is standard material, and this is by no means an exhaustive review. We will follow standard presentations, amongst which we can cite [34, 38, 47, 48].

Conformal transformations

Consider a set of $d$-dimensional coordinates $x^\mu$ with $\mu = 1, \ldots, d$ together with a metric tensor $\eta^{\mu\nu}$. Then, a conformal transformation is a diffeomorphism

$$x^\mu \to (x')^\mu,$$  (1.50)
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**Figure 1.2** – Exclusion plot of the leading $\mathbb{Z}_2$–even scalar scaling dimension $\Delta_{\epsilon}$ in terms of the leading $\mathbb{Z}_2$–odd scalar scaling dimension $\Delta_{\sigma}$ in a three-dimensional CFT using conformal invariance and unitarity. This exhibits a kink close to the 3d Ising values. The blue region is allowed while the white is not allowed [43].

**Figure 1.3** – Most precise determination of the allowed value for the leading $\mathbb{Z}_2$–even scalar scaling dimension $\Delta_{\epsilon}$ and the leading $\mathbb{Z}_2$–odd scalar scaling dimension $\Delta_{\sigma}$ in the three-dimensional Ising model using the conformal bootstrap. This result is compared to the best previous available Monte Carlo simulation [46].

that leaves the metric invariant up to a position-dependant rescaling

$$\eta_{\mu\nu} \rightarrow \Omega^2(x)\eta_{\mu\nu}. \quad (1.51)$$

This is equivalent to saying that a conformal transformation is a diffeomorphism that locally looks like a rotation $\Lambda^\mu_{\nu}(x)$ composed with a rescaling $\Omega(x)$, which we usually call dilatation. This implies that the coordinate diffeomorphism (1.50) obeys

$$\frac{\partial (x')^\mu}{\partial x^\nu} = \Omega(x)\Lambda^\mu_{\nu}(x), \quad \eta_{\rho\sigma}\Lambda^\rho_{\mu}(x)\Lambda^\sigma_{\nu}(x) = \eta_{\mu\nu}. \quad (1.52)$$

In $d > 2$, any conformal transformation (1.50) is a composition of four more basic transformations: translations, rotations, dilatations and inversions $(x')^\mu = x^\mu / x^2$. Note that the inversions are disconnected from the identity.

This results in the conformal group, which is the largest subgroup of diffeomorphism of $\mathbb{R}^d$, and is a Lie group of dimension $(d+1)(d+2)/2$. The finite conformal transformations are

- **translation**: $x^\mu \rightarrow x^\mu + a^\mu$, \hspace{1cm} (1.53)
- **rotation**: $x^\mu \rightarrow \Delta^\mu_{\nu} x^\nu$, \hspace{1cm} (1.54)
- **dilatation**: $x^\mu \rightarrow \lambda x^\mu$, \hspace{1cm} (1.55)
- **SCT**: $x^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$, \hspace{1cm} (1.56)

where the special conformal transformation (SCT) are obtained by composing an inversion, a translation and a second inversion, and $a^\mu, b^\mu$ are arbitrary vectors.
By considering the infinitesimal version of these four transformations, we can obtain the generators of the conformal algebra. The rotations are generated by $M_{\mu\nu}$, the translations by $P_\mu$, the dilatations by $D$, and the special conformal transformations by $K_\mu$. Their non-vanishing commutation relations are [49]

\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho},
\]

\[
[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu,
\]

\[
[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu,
\]

\[
[D, P_\mu] = P_\mu,
\]

\[
[D, K_\mu] = -K_\mu,
\]

\[
[K_\mu, P_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}.
\]

Moreover, there is an isomorphism between the conformal algebra in Euclidean signature and the $SO(\text{d}+1,1)$ algebra. The mapping is the following: Write the generators of $SO(d+1,1)$ as $J_{AB}$ with $A = 1, \ldots, d+2$ and define

\[
M_{\mu\nu} = J_{\mu\nu}, \quad P_\mu = -J_{d+1\mu} + J_{d+2\mu}, \quad (1.58)
\]

\[
K_\mu = -(J_{d+1\mu} + J_{d+2\mu}), \quad D = J_{d+1d+2}. \quad (1.59)
\]

With these definitions, the commutators (1.57) satisfy the usual $SO(d+1,1)$ commutation relations

\[
[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{BD} J_{CA} - \eta_{AD} J_{CD},
\]

with $\eta_{AB} = \text{diag}(1, \ldots, 1, -1)$. This map is the root of the embedding formalism of CFT, where we think of the action of the conformal group in terms of $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$ [36, 50–54]. We will not present this here, but there are many excellent reviews in the literature, such as [47–49] for example.

### Conformal transformations in two dimensions

Let us rapidly make a small detour to the two-dimensional case. It is well-known that this case is special because the group of conformal transformations is infinite-dimensional.

Consider an infinitesimal coordinate transformation with a small parameter $\epsilon(x) \ll 1$. Up to first order, it reads

\[
(x')^\mu = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2).
\]

Then, this coordinate transformation is responsible for the following transformation of the metric

\[
\eta_{\mu\nu} \rightarrow \eta'_{\mu\nu} = \frac{\partial x^\rho}{\partial (x')^\mu} \frac{\partial x^\sigma}{\partial (x')^\nu} \eta_{\rho\sigma} = \eta_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2).
\]
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For this transformation to be conformal, it needs to be compatible with (1.51), which implies that the transformation parameter needs to solve

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial^\rho \epsilon_\rho) \eta_{\mu\nu}.$$  \hspace{1cm} (1.63)

In higher dimensions, the most generic solution to this equation corresponds to the four transformations that we introduced in the previous section, but in two dimensions, the condition (1.63) reduces to the Cauchy-Riemann equations

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \quad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0.$$  \hspace{1cm} (1.64)

These equations are solved by any holomorphic function $$f(z) = z + \epsilon(z)$$ with $$z = x^0 + ix^1$$. It gives rise to an infinitesimal conformal transformation with $$z \to z' = f(z)$$ and equivalently for $$\bar{z}$$.

Let us now describe Virasoro symmetry in two-dimensional CFTs. Since any holomorphic function is a conformal transformation, it implies that a general infinitesimal conformal transformation can be written as

$$z' = f(z) = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}),$$  \hspace{1cm} (1.65)

with a similar relation for $$\bar{z}'$$. Focusing on the holomorphic sector, the infinitesimal parameters $$\epsilon_n$$ in (1.65) are constant. For a particular $$n$$, this transformation is generated by

$$l_n \equiv -z^{n+1} \partial_z,$$  \hspace{1cm} (1.66)

and the number of independent infinitesimal conformal transformations is infinite as $$n \in \mathbb{Z}$$ in (1.66). The generators (1.66) have commutation relations

$$[l_m, l_n] = (m - n) l_{m+n},$$  \hspace{1cm} (1.67)

which is one copy of the Witt algebra. There is a second copy for the anti-holomorphic sector and the two copies commute.

It is important to note that the generators $$l_n$$ are not defined everywhere. In particular, there is an ambiguity at $$z = 0$$. We should thus work on the Riemann sphere $$\mathbb{C} \cup \infty$$ instead of $$\mathbb{C}$$. The globally defined conformal transformations on the Riemann sphere are generated by $$\{l_{-1}, l_0, l_1\}$$. These three generators generate a subalgebra called the algebra of infinitesimal global conformal transformations that is isomorphic to $$sl(2, \mathbb{C})$$. The Lie group that corresponds to these is the group of global conformal transformations on the Riemann sphere

$$z \to \frac{az + b}{cz + d}, \quad \text{with} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$  \hspace{1cm} (1.68)

However, this is not the end of the story. The Witt algebra admits a central extension, and central extensions allow projective representations to become honest representations. In particular, in quantum theory, we know that these symmetry transformations must act projectively on states because the state $$|\phi\rangle$$ is physically indistinguishable from any
multiple $\kappa |\phi\rangle$. It is a fact that projective representations of an algebra are equivalent to the representations of the centrally extended algebra. It thus makes sense to look for the central extension of the Witt algebra, which is called the Virasoro algebra. It is generated by $L_n$ with $n \in \mathbb{Z}$. The Virasoro algebra is the unique central extension of the Witt algebra and its commutation relations are

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m-1)\delta_{m+n,0},$$

where the number $c$ is the central charge. Note that we already encountered this algebra when describing the Brown-Henneaux result in equation (1.23).

The fact that the group of conformal transformations in two dimensions is infinite-dimensional is the reason why we have a lot of control over these theories. Moreover, this powerful structure has other consequences. In two dimensions, we can compute higher-point correlation functions involving the stress-energy tensor easily thanks to the Ward identities. They are the quantum manifestation of symmetries in QFT.

For an infinitesimal conformal transformation, the current is $T(z) \epsilon(z)$, where $T(z)$ is the stress-energy tensor. The charge is then given by the integral of the current, which is (for the holomorphic sector)

$$Q_\epsilon = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z),$$

where the contour integral is a consequence of the usual radial quantization in CFT. The charge operator generates the following conformal transformation on a scalar field

$$\phi(w, \bar{w}) \rightarrow \phi'(w, \bar{w}) = \left( \frac{\partial f(w)}{\partial w} \right)^h \phi(f(w), \bar{w}),$$

with $f(w) = w + \epsilon(w)$, $w$ a coordinate on the complex plane and $h$ the conformal dimension of $\phi(w, \bar{w})$. The infinitesimal form of such a transformation is

$$\delta_\epsilon \phi(w, \bar{w}) = h \partial_w \epsilon(w) \phi(w, \bar{w}) + \epsilon(w) \partial_w \phi(w, \bar{w}).$$

Let us insert this general infinitesimal transformation $\oint dz \epsilon(z) T(z)$ within a correlation function. We want to evaluate the following correlation function

$$\langle 0| \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi(w_1) \ldots \phi(w_n) |0 \rangle,$$

where the contour encircles all points $w_i$. This contour can be deformed, such that it encircles all $w_i$ separately. This implies that we obtain

$$\sum_i \langle 0| \phi(w_1) \ldots \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi(w_i) \ldots \phi(w_n) |0 \rangle = \sum_i \langle 0| \phi(w_1) \ldots \delta_\epsilon \phi(w_i) \ldots \phi(w_n) |0 \rangle.$$

Using (1.72), and the fact that this must hold for any $\epsilon(z)$, this implies that

$$\langle 0| T(z) \phi(w_1) \ldots \phi(w_n) |0 \rangle = \sum_i \left[ \frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i} \frac{\partial}{\partial w_i} \right] \langle 0| \phi(w_1) \ldots \phi(w_n) |0 \rangle.$$
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This gives a recipe to compute arbitrary higher-point functions with the insertion of $T(z)$ in terms of lower-point functions. There is a more complicated, but completely analogous formula for computing correlation functions with multiple insertions of the stress-energy tensor $T(z)$. This formula is displayed and used in appendix A.4.

In chapter 2, we will see that an analogous statement holds for the ANEC operator in holographic CFTs and that we can compute arbitrary higher-point functions using a more complicated differential operator on lower-point functions as in (1.75).

Two-dimensional CFTs have been the ground of many impressive results over the last fifty years, and we will not go into more details in the present dissertation. Many nice references cover this in detail, for example [55–58].

Primary and descendant operators

The primary objects of study in CFT are the correlation functions of local operators. Moreover, conformal symmetry strictly constrain the possible form of such correlation functions, as we have seen in (1.39) and (1.40). These constraints follow from requiring covariance properties of the correlators when acting with conformal transformations. Besides, the correlators should also be compatible with the representation theory of the conformal group.

To discuss this representation theory, we can focus on operators inserted at the origin because we can obtain the transformation properties at other positions by using a translation and the different commutation relations in (1.57) [51].

If the theory is invariant under rotation, local operators at the origin transform in an irreducible representation of the rotation group $SO(d)$. It implies that if $R_{\mu\nu}$ are generators of this irreducible representation, we have

$$[M_{\mu\nu}, \mathcal{O}^i(0)] = (R_{\mu\nu})_{ji} \mathcal{O}^j(0),$$

(1.76)

where $i, j, ...$ are indices for the $SO(d)$ representation of $\mathcal{O}(x)$. Nevertheless, because CFTs are scale-invariant theories, it is also sensible to diagonalize the dilatation operator. At the origin, we have

$$[D, \mathcal{O}^i(0)] = \Delta_\mathcal{O} \mathcal{O}^i(0),$$

(1.77)

where the eigenvalue $\Delta_\mathcal{O}$ is the scaling dimension of the operator $\mathcal{O}(x)$. Using the commutation relations (1.57), we can show that $K_\mu$ acts as a lowering operator while $P_\mu$ acts a raising operator. Namely, we have

$$[D, K_\mu \mathcal{O}^i(0)] = (\Delta_\mathcal{O} - 1)K_\mu \mathcal{O}^i(0),$$

(1.78)

$$[D, P_\mu \mathcal{O}^i(0)] = (\Delta_\mathcal{O} + 1)P_\mu \mathcal{O}^i(0).$$

(1.79)

Hence, $P_\mu$ and $K_\mu$ generate the so-called conformal multiplet of the operator $\mathcal{O}$. Acting many times with $K_\mu$, we can generate operators of arbitrary low dimensions, but in unitary CFTs, the spectrum of the dilatation operator $D$ needs to be real and bounded.
from below. This implies that it exists, in the conformal multiplet, an operator of lowest dimension. This operator is called primary operator, and is characterized by
\[ [K_\mu, O^i(0)] = 0. \] (1.80)

Starting from this primary operator, we can construct operators with higher dimensions, that are called descendant operators. They are obtained by acting repeatedly with \( P_\mu \).

Note that these definitions are sufficient to derive the general transformation rules of a given operator \( O^i(x) \) under conformal transformations at arbitrary positions [47].

**Correlation functions in CFT**

Conformal invariance fixes the two- and three-point functions of local operators. We will comment on how to extend these results for spinning operators later on, but we first want to describe four-point functions in CFT in more details. Conformal invariance alone is not sufficient to fully fix them, but four-point functions play a crucial role in the bootstrap program, as they are the main ingredient of the crossing equation (1.48). We want to present the result for four primary scalar operators.

The starting point is to consider the two conformal cross-ratios, which are commonly denoted \( u \) and \( v \) and defined as
\[
\begin{align*}
    u &= \frac{x_{12}^2 x_{34}^2}{x_{13} x_{24}^2}, \\
    v &= \frac{x_{23}^2 x_{14}^2}{x_{13} x_{24}^2},
\end{align*}
\] (1.81)

with \( x_{ij} = x_i - x_j \). These conformal cross-ratios are invariant under all conformal transformations.

There are exactly two independent cross-ratios. Understanding why this is the case will allow us to find out why conformal symmetry is powerful enough to fix two- and three-point functions, but not four-point functions.

Consider a set of four positions. We can use conformal transformations to arrange these four positions in a convenient configuration. First, using SCT, we can move \( x_4 \) to infinity. Using translations, we can then move \( x_1 \) to zero. Using rotations and dilatations, we can move \( x_3 \) to a unit vector \((1, 0, \ldots, 0)\). Using the rotation that fix \( x_3, x_2 \) is then moved to \((x, y, 0, \ldots, 0)\) [59]. This configuration is displayed on figure 1.4.

For two and three points, it is clear that this procedure leaves no unknown quantity, which aligns with the intuition that conformal symmetry is powerful enough to fix these correlation functions. For four points, there are two free parameters, which are \( x \) and \( y \). This is the reason why there are exactly two independent cross-ratios \( u \) and \( v \). In the special configuration of figure 1.4, these cross-ratios evaluate to
\[
\begin{align*}
    u &= z \bar{z}, \\
    v &= (1 - z)(1 - \bar{z}),
\end{align*}
\] (1.82)

with \( z = x + iy \).

It is now clear that four-point functions in CFT can depend arbitrarily on the two cross-ratios. For four scalar primary operators \( O_i \) of dimension \( \Delta_i \), the four-point function is
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Figure 1.4 – Example of a configuration of four points that can be reached using conformal transformations. The figure is taken from [59]

\[ (O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)) = \frac{g(u,v)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left( \frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_{12}}{2}} \left( \frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_{34}}{2}}, \]

(1.83)

\[ \equiv g(u,v)K_4(\Delta_i, x_i), \]

(1.84)

with \( \Delta_{ij} \equiv \Delta_i - \Delta_j \). The second line is a definition for the factor \( K_4(\Delta_i, x_i) \). This factor transforms appropriately under conformal transformations and hence (1.84) is the CFT four-point function for any function \( g(u,v) \), which is left unfixed. However, \( g(u,v) \) can be obtained in terms of CFT data using the conformal block decomposition. We will return to this shortly.

Let us briefly comment on correlation functions for spinning operators. The story is similar, but slightly more involved. They can be obtained by different methods, out of which we mention the embedding formalism [54], or the conformal frame approach [40, 60]. The two-point functions are again fixed by conformal symmetry, and are non-vanishing for operators with same dimensions and spins. We present, for example, the two-point function of spin-1 operators of dimension \( \Delta \), which is given by

\[ \langle J^\mu(x)J'^\nu(y) \rangle = C_J \frac{1}{(x - y)^{2\Delta}} \left( \delta^{\mu\nu} - \frac{2x^\mu x'^\nu}{x^2} \right), \]

(1.85)

where \( C_J \) is a constant [40].

**The operator product expansion**

We already briefly explained the operator product expansion in section 1.3.2, but we want to give a bit more detail.

In a CFT, primary operators form an algebra under the OPE. This implies that, within correlation functions, we can replace the product of two primary operators at nearby points by a sum over other local operators times some functions as

\[ \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k f_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2), \]

(1.86)
provided a sphere that separates the two operators from all other operators present in the correlation function exists. It is also important to note that it is also possible for any of the operators involved to have spin, in which case the OPE is a bit more involved, but the philosophy is completely analogous.

Obviously, the OPE expansion needs to be compatible with conformal invariance, and this severely constrains the possible form of the differential operator $f_{ijk}$. The easiest way to see this is to insert the OPE (1.86) within a correlation function with a third scalar operator $O_k(x_k)$ while assuming $|x_{jk}| \geq |x_{ij}|$ such that the OPE is valid. In particular, we obtain

$$\langle O_i(x_1)O_j(x_2)O_k(x_3) \rangle = \sum_p f_{ijp}(x_{12}, \partial_2) \langle O_p(x_2)O_k(x_3) \rangle .$$  \hspace{1cm} (1.87)

Conformal invariance fixes the three-point function that appears on the left-hand side of (1.87) to be of the form given in (1.40). In addition, the two-point function is orthonormal, as shown in (1.39), such that we can rewrite (1.87) as

$$\lambda_{ijk} = f_{ijk}(x_{12}, \partial_2) \frac{1}{|x_{23}|^{2\Delta_k}} .$$  \hspace{1cm} (1.88)

We thus see that, as we already explained near equation (1.42), $f_{ijk}$ is proportional to $\lambda_{ijk}$ times a differential operator that can be obtained by matching the small $|x_{12}|/|x_{23}|$ expansion of (1.88). We will use this method on several occasions throughout this dissertation to compute OPE expansions in four-dimensional CFTs in chapter 2 and 3.

### Conformal block decomposition

Conformal blocks are a central ingredient of the conformal bootstrap and they were studied since the early days of the conformal bootstrap [37, 61–63]. There was a renewed interest later on, with new significant results coming from [64–66]. Using the embedding formalism, new results became available as well for correlators involving spinning operators [54, 67]. Let us review some basic facts about conformal blocks in this section.

For concreteness, we will use a four-point function of scalar primary operators, and elude the case of spinning operators. We already explained, around equation (1.44) how we can compute higher-point functions in CFT by using the OPE multiple times to reduce them to a sum of two-point functions. Let us consider the following correlator $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle$ and perform the OPE between $\phi_1(x_1)\phi_2(x_2)$ and $\phi_3(x_3)\phi_4(x_4)$. We refer to this as the (12) − (34) OPE channel, and this is the channel displayed in equation (1.45). Then, using (1.42), we can write this as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = \sum_\mathcal{O} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} W_\mathcal{O}(x_i) ,$$  \hspace{1cm} (1.89)

where $W_\mathcal{O}(x_i)$ are called **conformal partial waves** (CPWs). They are given as

$$W_\mathcal{O}(x_i) = \hat{f}_{12\mathcal{O}}(x_1, x_2, y, \partial_y)\hat{f}_{34\mathcal{O}}(x_3, x_4, z, \partial_z) \langle \mathcal{O}(y)\mathcal{O}(z) \rangle ,$$  \hspace{1cm} (1.90)
where the summation is over the same operator $O$ in the two OPEs as the two-point function (1.39) is diagonal. Using the conformal invariance of the OPE, it can be shown that each conformal partial wave transforms as the four-point function under conformal transformations [67]. We can thus strip off a factor of $K_4(\Delta_i, x_i)$ defined in equation (1.84) from the CPW to obtain the conformal block $g_{\Delta_O, \ell O}^{\Delta_1, \Delta_2}$ as

$$W_O(x_i) = g_{\Delta_O, \ell O}^{\Delta_1, \Delta_2}(u,v)K_4(\Delta_i, x_i).$$

(1.91)

Each conformal block encodes the contribution of the operator $O$ as well as all its descendant to the four-point function. Comparing (1.84) and (1.91), we can obtain a series representation for the function $g(u,v)$ in terms of conformal blocks as

$$g(u,v) = \sum_O \lambda_{12O} \lambda_{34O} g_{\Delta_O, \ell O}^{\Delta_1, \Delta_2}(u,v).$$

(1.92)

The domain of convergence of the conformal block decomposition (1.92) is guaranteed for $|z| < 1$ in Euclidean signature, which is sufficient for many applications [68]. Also, note that despite the fact that the formal definition (1.92) is correct, this is not the most useful representation when it comes to practical computations, because it requires to know the OPE explicitly. There are different methods available to compute explicitly these conformal blocks. We will not present them here, but a review can be found in [38] for example. We just want to point out that in even dimensions, closed-form expressions are available [61, 64–66]. They are given in terms of hypergeometric functions and read, in $d = 2$

$$g_{\Delta_O, \ell O}^{\Delta_1, \Delta_2}(z, \bar{z}) = \frac{1}{(-2)^\ell(1 + \delta_{\ell 0})} (k_{\Delta+\ell}(z)k_{\Delta-\ell}(\bar{z}) + k_{\Delta+\ell}(\bar{z})k_{\Delta-\ell}(z)),

(1.93)$$

and in $d = 4$

$$g_{\Delta_O, \ell O}^{\Delta_1, \Delta_2}(z, \bar{z}) = \frac{1}{(-2)^\ell} \frac{z\bar{z}}{z - \bar{z}} (k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) + k_{\Delta+\ell}(\bar{z})k_{\Delta-\ell-2}(z)),

(1.94)$$

where we used

$$k_{\beta}(z) = z^{\beta/2} \left( \frac{\beta - \Delta_{12}}{2}, \frac{\beta + \Delta_{34}}{2}, \beta; z \right),

(1.95)$$

with $z, \bar{z}$ defined in (1.82). For odd dimensions, closed form expressions are currently unavailable, but recursion relations, for example, are available. However, this is out of the scope of this dissertation, and we will not discuss this further.

After this lighting review of $d$–dimensional CFT, we would like to describe a specific set of non-local operators in CFT which are the main objects of interest of chapter 2 and 3.

### 1.4 Light-ray operators and the bootstrap

#### 1.4.1 Energy conditions and the ANEC

In section 1.3, we have presented the main ingredients of the conformal bootstrap as well as state-of-the-art results that were obtained using the numerical bootstrap and linear
programming. To derive these results, we have focused solely on local operators, without ever needing to consider more exotic operators. This is because much of the work done on the conformal bootstrap has focused on constraints that arise from purely local physics. In general, however, non-local operators may probe physics that is invisible to these local probes. There are multiple non-local operators in quantum field theory, but most of this dissertation focuses on a special class of such operators, called light-ray operators.

The usual intuition that we have from our daily experiences is that energy is a positive definite quantity. In classical general relativity, different energy conditions have been discussed that are trying to capture this fact. In particular, any smooth Lorentzian manifold can be a spacetime. Solving the Einstein equation with this given spacetime would yield a stress-energy tensor $T^{\mu\nu}$ that is necessary to produce such a solution. The energy-momentum tensor $T^{\mu\nu}$ captures the distribution of the mass, momentum, and stress due to matter and non-gravitational fields. To restrict exotic phenomena and non-physical configurations, we thus need to restrict the possible stress-energy tensors. This is precisely the goal of energy conditions. In the context of classical general relativity, many such local energy conditions should supposedly be true at any given point of spacetime. These energy conditions play a key role in the basic understanding of our universe, for example through theorems [69, 70], or by forbidding wormholes and time-machines [71–73].

These pointwise energy conditions can nevertheless all be violated in some state of the theory [74], and in particular, they are violated in quantum field theory [75]. This is a consequence of quantum fluctuations that forbid any local operator from having a positive expectation value in every state. In quantum field theory though, non-local energy conditions can be satisfied. A prototypical example is the Averaged Null Energy Condition (ANEC), which in general relativity states that

$$\int_{-\infty}^{+\infty} d\lambda T_{\alpha\beta} u^\alpha u^\beta \geq 0,$$

where the integral is over a null geodesics $\gamma$ with affine parameter $\lambda$ and $u^\mu$ is the tangent null vector.

The inequality (1.96) has been central to prove many of the classic theorems of general relativity [76–79]. It would be possible to build time machines provided matter violating the ANEC existed [80, 81]. In addition, matter that does not obey the ANEC would also violate the second law of thermodynamics [82]. Ultimately, the power of the ANEC resides in the fact that it lies at the interface of gravity and QFT, where it yields nontrivial constraints even in Minkowski space without gravity.

In quantum field theory, the ANEC takes the following form: Define the ANEC operator as follows

$$\mathcal{E}(x^+, \bar{x}_\perp) = \int dx^- T_{--}(x^+, x^-, \bar{x}_\perp),$$

where $x^\pm = t \pm x$ are the lightcone coordinates and the integral is over a null ray. This operator is a light-ray operator. Then, the ANEC in QFT is the statement that the
1.4. Light-ray operators and the bootstrap

The expectation value of the ANEC operator (1.97) is positive for any state $|\psi\rangle$ in the Hilbert space, i.e.

$$\langle \psi | E | \psi \rangle \geq 0.$$ \hfill (1.98)

This is a property of quantum field theory.

For a long time, the ANEC (1.98) was not proven, but no counterexample was found in quantum field theory provided the null geodesic is achronal (it does not contain any points that are connected by a timelike path) [83]. In addition, it was shown to hold explicitly in specific examples, such as free theories or two dimensions [84–88]. The first general proof of the ANEC in QFT originated from quantum information theory [89], and relies on the monotonicity of relative entropy. A second proof followed [90], which relies on causality of four-point functions in the lightcone limit and uses the conformal bootstrap. Ultimately, both proofs show that the ANEC is a consequence of unitarity.

1.4.2 The conformal collider

The positivity of the ANEC operator was already assumed in CFT before the ANEC (1.98) was rigorously proven. The motivation for this assumption is the fact that the energy is a positive-definite quantity, and hence the ANEC should hold. In particular, we want to explain the setup of [91], called the conformal collider which is an efficient way to derive the consequences of the ANEC condition (1.98) in CFT.

The idea is to consider a gedanken experiment where we create a localized excitation in conformal field theory and measure the energy deposited at given angles at future null infinity using calorimeters. This is displayed in figure 1.5 and 1.6.

The energy is measured in states created by acting with local operators $\mathcal{O}(x)$ on the Lorentzian vacuum $|0\rangle$. We want our local operator $\mathcal{O}(x)$ to have energy $q$ and zero momentum. To localize these states, we define them as

$$|\mathcal{O}(q,\epsilon)\rangle \equiv Q \int d^4x \ e^{-iq\cdot x} e^{-\frac{x^2}{\sigma^2}} \epsilon \cdot \mathcal{O}(x) |0\rangle,$$ \hfill (1.99)

with $q\sigma \gg 1$ such that the operator is localized, has finite norm and has four momentum $q^\mu = (q,\vec{0}) + \mathcal{O}(1/\sigma)$. Moreover, $\epsilon$ is a polarization tensor that allows to discuss spinning operators $\mathcal{O}(x)$, and $Q$ is fixed by requiring the states (1.99) to have unit norms.

In $d$ dimensions, null infinity is a sphere $S^{d-2}$ of radius $r$, and we need a unit vector $n^i$ that points to the location where the calorimeter is inserted on $S^{d-2}$. The energy at null infinity in a given state (1.99) is then measured as

$$\langle \mathcal{E}(n) \rangle_{\epsilon, \mathcal{O}} = \lim_{r \to \infty} r^{d-2} \int_{-\infty}^{+\infty} dx^- \langle \mathcal{O}(q,\epsilon) | T^-_- (x^-, rn) | \mathcal{O}(q,\epsilon) \rangle.$$ \hfill (1.100)

If we evaluate the energy in a scalar state, the symmetry of the problem implies that the energy density should be uniform on the sphere $S^{d-2}$. In addition, integrating the
1. Introduction

Figure 1.5 — We measure the energy created by the localized excitation (blue) in the CFT using calorimeter far away (red). Figure adapted from [91].

Figure 1.6 — For a CFT, this is equivalent to measuring the energy at future null infinity $J^+$ in the limit of infinite radius. Figure adapted from [91].

energy density over the angle should give the total energy, which implies that we have

$$\langle O(q) | \mathcal{E}(n) | O(q) \rangle = \frac{q}{\Omega_{d-2}}. \quad (1.101)$$

While this result can be guessed by the fact we just explained, it can also be verified by direct computation.

The ANEC implies that the energy one-point function defined in (1.100) is non-negative, and we can use this to derive bounds in four-dimensional CFTs. In particular, the integrand of (1.100) is just the CFT three-point function $\langle OTC \rangle$ in Lorentzian signature once we order the operators appropriately. It is then clear that the quantity $\langle \mathcal{E}(n) \rangle_{c,O}$ is explicitly depending on OPE coefficients. Combining this with the ANEC (1.98), we can derive bounds on OPE coefficients using this method, which is the insight of [91].

We want to quickly review the main steps of this computation in four dimensions, using stress-energy tensors as external operator while evaluating (1.100). In this case, the CFT three-point function of interest is $\langle TTT \rangle$ which takes the schematic form

$$\langle TTT \rangle = t_0 \left[ \langle TTT \rangle_0 + t_2 \langle TTT \rangle_2 + t_4 \langle TTT \rangle_4 \right], \quad (1.102)$$

where the labels $\langle TTT \rangle_i$ labels the irreducible representations in terms of the spin.\(^3\)

These coefficients are related to the two conformal anomalies $a$ and $c$ that parametrize the trace of the energy-momentum tensor in a generic metric background as

$$\langle T_{\mu}^{\mu} \rangle = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E, \quad (1.103)$$

where $W$ is the Weyl tensor and $E$ the Euler density $E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$.

3. Note that this is just a different parametrization compared to (1.34) which is more practical for the discussion at hand. The coefficients $t_2$ and $t_4$ are related to the anomaly coefficients $a$ and $c$ [40].
Then, we can use equation (1.100) with stress-tensor external states. Using the symmetries, which are rotations on the null $S^2$, we obtain the most general expression

$$\langle \mathcal{E}(n) \rangle_{\epsilon \cdot T} = \frac{q}{4\pi} \left[ 1 + t_2 \left( \frac{\epsilon^i_{ij} \epsilon_{ik} n^j n^k}{|\epsilon|^2} - \frac{1}{3} \right) + t_4 \left( \frac{\epsilon^i_{ij} \epsilon_{ik} n^j n^k n^l}{|\epsilon|^2} - \frac{2}{15} \right) \right],$$  \hspace{1cm} (1.104)$$

where $t_2$ and $t_4$ are just a convenient way to parametrize the anomaly coefficients $a$ and $c$, and the constants in the parentheses are fixed such that the total energy of the state is $q$.

By imposing positivity of (1.104), which is a consequence of the ANEC (1.98), we can obtain constraints on $t_2$ and $t_4$ which translate, once written in terms of the two anomaly coefficients $a$ and $c$ to the bounds [91, 92]

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}.$$  \hspace{1cm} (1.105)$$

These bounds are known as the conformal collider (or Hofman-Maldacena) bounds. They were rigorously proven in [93] using Lorentzian conformal bootstrap techniques (see also [94, 95]). When these bounds are saturated, it implies that the theory is free [96].

Subsequently, many more bounds of the form (1.105) were derived. For example, positive sum rules for integrated higher spin operators of even spin were obtained in [95, 97]. Bounds on transport coefficients were also derived in [98]. Besides, instead of considering the three-point function of the ANEC operator in states created by a single local operator, such as the stress tensor, we can consider three-point functions evaluated in states created by superposing the stress-energy tensor and a scalar operator. This provide constraints for the off-diagonal OPE coefficients $\lambda_{TTO}$ in terms of the diagonal ones $\lambda_{TTT}$ and $\lambda_{OTO}$ [99].

In addition, the ANEC operator belongs to a larger class of non-local operators in CFT, that are called light-ray operators. In recent years, there has been a lot of activity involving these operators, including higher spin generalizations [100, 101]. Light-ray operators have been used to provide an alternative proof of the Lorentzian inversion formula [102]. Moreover, in a non-perturbative QFT, the existence of an OPE of local operators is easy to prove [103], but this reasoning cannot be easily extended to the case of non-local operators. Nevertheless, it was shown that there exists a convergent OPE for light-ray operators [104].

Note that as we discussed in section 1.2.4 for holographic CFTs, we know that $a$ and $c$ obey much stronger bounds than (1.105). In particular, the Einstein gravity result (1.36) is

$$1 \leq \frac{a}{c} \leq 1,$$  \hspace{1cm} (1.106)$$

which holds at infinite $N$ and gets corrected as a $1/N$ expansion. This is a statement about the universality of the stress-tensor three-point function for holographic CFTs. This also implies that the ANEC gets replaced by a stronger condition in holographic CFTs. This is the subject of chapter 2.


1. Introduction

1.4.3 Generalized ANEC operators

It is also possible to consider more general light-ray operators. In particular, we can build more general null-integrals of the stress-energy tensor, that we define as

\[ E_f(x^+, \vec{x}^\perp) \equiv \int dx^- f(x^-) T_{--}(x^+, x^-, \vec{x}^\perp). \]  

(1.107)

When \( f(x^-) = 1 \), then this operator is the ANEC operator (1.98). When two such operators live at the same \( \vec{x}^\perp \) position, they are embedded in a two-dimensional plane. This offers a promising framework to look for an algebra amongst these operators that would be similar to the two-dimensional Virasoro algebra (1.69). In a different context, and by considering properties of modular Hamiltonians on deformed half-spaces, it was proposed that the operators

\[ L_n(x^+, \vec{x}^\perp) \equiv E(x^-)^{n+2}(x^+, \vec{x}^\perp), \]  

(1.108)

satisfy, when inserted on the same light-sheet at \( x^+ = 0 \), the following algebra in four dimensions [105]

\[ [L_m(\vec{x}^\perp), L_n(\vec{y}^\perp)] = -i\delta^{(2)}(\vec{x}^\perp - \vec{y}^\perp)(m - n)\delta^{(2)}(\vec{x}^\perp + \vec{y}^\perp). \]  

(1.109)

The situation is illustrated on figure 1.7, where we depicted two light-ray operators on the same null-sheet. When they are separated in the transverse direction, i.e. \( \vec{x}_{ij}^\perp \neq 0 \), they are spacelike separated, and we thus expect them to commute. When they are on top of each other, we expect some non-trivial commutation relations, which are delta function supported as in (1.109).

We now want to review the origin of the proposed algebra in more detail.

Origin of the proposed algebra

We want to review the origin of this proposal, following the original work of [105] and the presentation of [106]. To do this, we will consider the two operators \( L_{-2}(x^+ = 0, \vec{x}^\perp) \) and \( L_{-1}(x^+ = 0, \vec{x}^\perp) \). These two operators are the local versions of some of the charges of the Poincaré algebra. We can thus integrate them over the transverse directions to obtain generators of the Poincaré algebra as

\[ P_+ = \int d^2 \vec{x}^\perp L_{-2}(x^+ = 0, \vec{x}^\perp), \quad J_{01} = \int d^2 \vec{x}^\perp L_{-1}(x^+ = 0, \vec{x}^\perp), \]  

(1.110)

where \( P_+ \) is the generator of translations in the \( x^- \) direction and \( J_{01} \) generates boost in the \( (x^+, x^-) \) plane. Their commutator is \( [J_{01}, P_+] = -iP_+ \).

To compute the commutator of the operators (1.110) on the same light-sheet at \( x^+ = 0 \) but separated in transverse directions, we will use the following assumptions [105, 106]:

1. Microcausality: Operators that are spacelike separated commute.
1.4. Light-ray operators and the bootstrap

Figure 1.7 – Two light-ray operators (blue and yellow lines) inserted on the same null-sheet at \( x^+ = 0 \). In the case where these two operators are separated in the transverse directions \( x_{ij}^\perp \neq 0 \), they are spacelike separated. We expect them to commute. When they are on top of each other, the commutator is non-trivial.

2. Unitarity: Local operators transform in unitary representations of the conformal group \( SO(d, 2) \).

3. Ward identities: The global charges \( J_{01} \) and \( P_− \) implement the Poincaré transformations on operators.

4. Closure: The commutator of two parallel light-ray operators on the same light-sheet at \( x^+ = 0 \) that are integral of local operators (that are conserved current) can be expressed as the integral of a local operator.

Before presenting the actual computations, let us comment on the closure assumption. The commutators that we want to compute are supported only on a null-line, which is given by the null-line we integrate over to obtain light-ray operators. They thus have to commute with all fields that are spacelike separated from them. The closure assumption is the statement that such an operator can be written as a light-ray integral of local operators [107].

With these four assumptions, we are now ready to compute the commutators of interest, and we will start with the commutator of two ANEC operators

\[
[L_{-2}(\vec{x}_1^\perp), L_{-2}(\vec{x}_2^\perp)] = \left[ \int dx_1^+ T_{--}(x_1^+ = 0, x_1^-, \vec{x}_1^\perp), \int dx_2^+ T_{--}(x_2^+ = 0, x_2^-, \vec{x}_2^\perp) \right].
\]

These two operators are spacelike separated provided \( \vec{x}_{12}^\perp = \vec{x}_1^\perp - \vec{x}_2^\perp \neq 0 \), which implies that using our microcausality assumption, the commutator can only have support at
\[ \vec{x}_1^+ = \vec{x}_2^+ . \] We can thus write the Ansatz
\[ [L_{-2}(\vec{x}_1^+), L_{-2}(\vec{x}_2^+)] = \delta^{(2)}(\vec{x}_{12}^+) L(\vec{x}_2^+) + \partial A \delta^{(2)}(\vec{x}_{12}^+) L^A(\vec{x}_2^+) + \ldots , \] (1.112)
where \( A = 2, 3 \) runs over the transverse directions, and \( \ldots \) are terms with more derivatives acting on \( \delta^{(2)}(\vec{x}_{12}^+) \). \( L \) and \( L^A \) are some non-local operators that are integrals of local operators.

The next step is then to integrate (1.112) over \( \vec{x}_1^+ \) to obtain, using (1.110)
\[ [P_-, L_{-2}(\vec{x}_2^+)] = L(\vec{x}_2^+) = 0 , \] (1.113)
where the last equality follows from the translation invariance of the operator \( L_{-2}(\vec{x}_2^+) \). This implies that \( L = 0 \).

Now, by computing the twist, which is the conformal dimension minus the spin on both sides of equation (1.112), and assuming that \( L^A \) is the integral of a local operator \( \phi^A \) (this is the closure assumption), we conclude that \( \phi^A \) needs to have twist one. In a generic CFT, no such operator exists, and it implies that \( \phi^A \) must vanish. This argument excludes also the terms hidden in \( \ldots \) of equation (1.112), which are more and more violating unitarity bounds. We thus conclude that two ANEC operators commute
\[ [L_{-2}(\vec{x}_1^+), L_{-2}(\vec{x}_2^+)] = 0 , \] (1.114)
which was already put on firm ground by [108].

For the next commutator of interest, we can write the Ansatz
\[ [L_{-1}(\vec{x}_1^+), L_{-2}(\vec{x}_2^+)] = \delta^{(2)}(\vec{x}_{12}^+) \tilde{L}(\vec{x}_2^+), \] (1.115)
where we have directly excluded the other terms by the same aforementioned twist argument. We can then integrate this over \( \vec{x}_1^+ \) and \( \vec{x}_2^+ \), and use (1.110) and the commutator \([J_{01}, P_-]\) to obtain
\[ [L_{-1}(\vec{x}_1^+), L_{-2}(\vec{x}_2^+)] = -i\delta^{(2)}(\vec{x}_{12}^+) L_{-2}(\vec{x}_2^+) . \] (1.116)
This reasoning can be generalized to obtain the proposed algebra (1.109). In chapter 3, we will take this proposal seriously and investigate whether it holds in four-dimensional CFTs by doing computations in free theories and holographic CFTs. In addition, we will check the calculations in holographic CFTs by reproducing the result in AdS. To be able to perform these computations in AdS, we needed to derive new exact solutions to the Einstein equation. These solutions are the (shockwave) geometries dual to the insertions of generalized light-ray operators in the CFT.

If such an algebra would hold in higher-dimensional CFT, it would have far-reaching consequences. In particular, we have already explained how the Virasoro symmetry strongly constrains two-dimensional CFTs, and how this has allowed for groundbreaking discoveries. If an algebra of the form we just discussed holds, we could generate an infinite set of constraints, that all need to be true simultaneously in higher-dimensional CFTs.
1.5 Generalized symmetries

In this section, we want to introduce another important concept for this dissertation, which is generalized global symmetries. In general, symmetries are one of the most powerful guiding principles to understand a physical system, and they tightly constrain most of the world we experience. The recently formalized so-called generalized symmetries are a generalization of regular symmetries. Instead of acting on pointlike objects as the usual symmetries, they act on extended objects and are ubiquitous in quantum field theory.

Most of the known results for regular symmetries have analogs for generalized symmetries. They have also proven to be extremely powerful to probe strongly-coupled QFTs, where they provide a new organization principle suited for extended objects.

1.5.1 Generalized global symmetries

Symmetry is one of the great unifying principles in physics. Symmetries determine the form, interactions, and evolution that can occur in physical systems. In light of Noether’s theorem, we understand how every continuous symmetry is responsible for a conserved quantity. In some cases, the breaking of a symmetry can be as powerful as the symmetry itself. It often happens that the solutions of a theory do not have a symmetry that the equations of the theory do have, and this phenomenon goes under the name spontaneous symmetry breaking (SSB). Further, when a continuous symmetry is spontaneously broken, massless scalar particles called Nambu-Goldstone bosons appear in the spectrum. This is Goldstone’s theorem [109–111].

Typically, the greater the symmetry of a system, the easier it is to solve. Recently, a new type of symmetries has been formalized in the literature [112], which provides a new way to organize our thinking about symmetries. Whereas regular symmetries act on point-like objects, these so-called generalized (or higher-form) symmetries act on and help classify extended objects such as strings and flux tubes in QCD, or membranes. The conserved charges associated with regular symmetries count the number of particles. For higher-form symmetries, the conserved quantity is the density of higher-dimensional objects, such as strings. A $p$-form symmetry acts on $p$-dimensional objects. Let us formalize this more precisely.

0-form symmetry

For a regular symmetry, which in this language is a 0-form symmetry, the charged operators are point-like, and we call them $\theta(t,x)$. They create charged excitations that propagate through spacetime with a 1–dimensional worldline. Noether’s theorem states that for any generator of a continuous symmetry, we have a conserved current $J$, which is a $(p + 1)$-form for a $p$–form symmetry. We can build charge operators $Q$, that count
the number of such point-like objects, by integrating the Hodge star of the conserved current $J$ over a co-dimension one surface $M_{d-1}$

$$Q(M_{d-1}) = \int_{M_{d-1}} \star J.$$  \hfill (1.117)

This is illustrated in figure 1.8. Charge operators can be non-Abelian, which is the statement that two such charge operators might have non-trivial commutation relations. Naively, this is because if we want to commute two such charge operators, they have no choice but to touch one another. When they get superposed and nothing happens, we have an Abelian symmetry. In the case where something non-trivial happens when they touch, we obtain a non-Abelian symmetry.

The symmetry operators, which are the operators that produce the symmetry transformations, are associated with a $d$-dimensional manifold $M_d$ and a group element $g$ of the global symmetry. We denote them as $U_g(M_d)$. In the case of connected Lie groups, the symmetry operators are just the exponentiated version of the charges and are given as

$$U_\alpha(M_d) = e^{i\alpha^a Q_a}.$$ \hfill (1.118)

In general, for continuous symmetry groups, $U_g(M_d)$ can be obtained by exponentiating the charge $Q(M_d)$. These symmetry group elements should obey a fusion rule

$$U_g(M_d) U_{g'}(M_d) = U_{g''}(M_d),$$ \hfill (1.119)

with $g'' = gg'$ in the group. Moreover, the dependence of $U_g$ on $M_d$ is only topological, and the value of $U_g$ changes under deformations of the manifold $M_d$ only when $M_d$ crosses a charged object. This discussion is just a slightly more involved version of Gauss’s law. If we have a conserved current, we can build a charge associated to it. If we then want to compute the charge contained within some closed boundary, we only need to measure how much goes in and out of the boundary.

**Higher $p$-form symmetry**

We can now use this general presentation of 0-form symmetries to generalize this to higher-form symmetries. A $p$–form symmetry acts on operators that are supported on a $p$–dimensional manifold. In particular, we have extended charged operators $\theta(M_p)$ that create conserved excitations. In this case, the Noether current is a $(p + 1)$–form, and the charge operators $Q$ that count excitations are obtained by integrating this over a $d - 1 - p$ surface as

$$Q(M_{d-p-1}) = \int_{M_{d-p-1}} \star J.$$ \hfill (1.120)

We can think of these charges as counting the number of charged objects that go through the surface $M_{d-p-1}$.

A crucial difference between regular and higher-form symmetries is that because any charge operator is supported on a manifold that has codimension greater than one, we
1.5. Generalized symmetries

Figure 1.8 – For a 0-form symmetry, we have a point-like charged operator that creates excitations. They propagate through spacetime and are counted by the charge operator $Q$. This is a three-dimensional picture for a 0-form symmetry.

can always smoothly deform the charge operators such that they do not touch each other. This implies that, on a space of trivial topology, the charge operators commute for higher-form symmetries, and thus they are always Abelian.

As for the usual symmetries, the symmetry operators are obtained by exponentiating the charges, and in the $U(1)$ case, we have

$$U_{g} = e^{i\alpha(M_{d-1-p})} = e^{i\alpha Q(M_{d-1-p})}.\quad (1.121)$$

We measure the charge of the operator $\theta(M_{p})$ by wrapping it with the symmetry operator

$$U_{g}(M_{d-1-p})\theta(M_{p}) = g^{a}\theta(M_{p}),\quad (1.122)$$

where $q$ is the charge of the operator. This is displayed on figure 1.9.

Let us finally mention the following fact. The charged objects that are counted by the charge operator are always $(p + 1)$-dimensional for a $p$-form symmetry. This implies that we can couple a theory with a $p$-form symmetry to a background $(p + 1)$-form connection. This is analogous to how we can couple a 0-form theory to a 1-form vector potential $A_{\mu}$. For the 0-form symmetry, the charged objects have a 1-dimensional world-volume such that particles are the natural charged objects of 0-form symmetries.

We are now ready to discuss the common features that higher-form symmetries share with usual zero-form symmetries.

**Common features**

Many known results for ordinary symmetries have generalized symmetry analogs. For example, higher-form symmetries can be gauged or can have anomalies, and this will
1. Introduction

Figure 1.9 – This is a picture depicting a 1-form symmetry in 3 spacetime dimensions. We have a 1–dimensional charged operator $\theta(M_p)$ that create excitations that are counted by the charge operator $Q$. To measure the charge of the operator $\theta(M_p)$, we wrap it with $U_g(M_{d-p-1})$ as displayed on the right.

be relevant in chapter 4 [113, 114]. Crucially, there is also a version of Goldstone’s theorem, which proves that spontaneously broken continuous higher-form symmetries are responsible for massless Goldstone modes. The proof follows from integrating Ward identities for charged operators [115, 116]. It is thus proven that the Goldstone modes are the fields that shift by the addition of a ”constant” factor when we act with the symmetry group. For usual symmetries, when $p = 0$, this is the usual shift $\phi \to \phi + \varphi$, where $\varphi$ is a constant, while for $p \geq 1$, we have $A \to A + \alpha$ with $\alpha$ a flat connection ($d\alpha = 0$). There is also a version of the Coleman-Mermin-Wagner theorem for higher-form symmetries. It states that continuous $p$–form symmetries in $d$ spacetime dimensions are never broken if $p \geq d - 2$ [112].

Understanding higher-form symmetries can give new ways of thinking about well-known problems. For example, magnetohydrodynamics can be rephrased in this language, where the only constraints come from symmetries [117]. The same can be done with elasticity theory [118, 119] and superfluid hydrodynamics [3]. Higher-form symmetries were also studied in the context of holographic QFTs, thus extending the holographic dictionary. For example, discrete global symmetries on the boundary are dual to discrete gauge theories in the bulk [120, 121]. These symmetries are ubiquitous in well-known physical systems and a prototypical example is free electromagnetism in four dimensions, which possess two global 1-form symmetries. The objects charged under these symmetries are respectively electric and magnetic flux lines, and we want to describe this in more detail.
1.5. Generalized symmetries

1.5.2 Example: Maxwell free electrodynamics

In this section, we want to give an explicit example of a relatively simple system that exhibits higher-form symmetries. We will describe how one-form symmetries appear in the context of free electromagnetism. This discussion is much more elegant in form language, and this is the description that we have chosen here. We will follow the presentation of [112, 120].

The theory we want to describe is four-dimensional free electromagnetism without matter. This theory has a gauge coupling \( g \), which is marginal. In this theory, we have two \( U(1) \) one-form symmetries, which are usually referred to as electric and magnetic symmetry. We introduce the so-called electric one-form gauge field \( A_e \) whose associated field strength is, as usual, \( F = dA_e \).

For concreteness, we can start with a pure gauge theory action, that is

\[
S_{EM} = -\frac{1}{2g^2} \int_M F \wedge \star F,
\]

where \( M \) is just a four-manifold. Then, the equation of motion and the Bianchi identity are respectively

\[
d \star F = 0, \quad dF = 0.
\]

This theory has two 2–form conserved currents, each associated to a 1–form symmetry, that are given as

\[
J_e = \frac{1}{g^2} F, \quad J_m = g^2 G,
\]

where

\[
G = \frac{1}{g^2} \star F.
\]

These two currents are conserved, in the sense that \( d \star J_e = d \star J_m = 0 \). Each of these two currents is associated with a \( U(1) \) 1-form global symmetry. In particular, \( J_e \) is conserved thanks to the source-free Maxwell equation and counts electric flux lines. The electric 1-form symmetry shifts the gauge field by a flat connection as \( A \to A + \alpha \), where \( \alpha \) is flat (\( d\alpha = 0 \)). \( J_m \) is conserved thanks to the Bianchi identity and counts magnetic flux line. The objects charged under the electric 1–form symmetry are Wilson lines and the objects charged under the magnetic 1–form symmetry are ’t Hooft lines, which are both supported on 1-manifolds. The conserved charges are obtained as \( Q_E = \oint_{S^2} \star F \) and \( Q_M = \oint_{S^2} F \), where \( S^2 \) is just a two-sphere. These charges measure the electric and magnetic charges of the particle whose wordline they surround.

The Bianchi identity \( dF = 0 \) implies that \( J_e \sim dA_e \), and that the 1-form symmetry whose associated conserved current is \( J_e \) is non-linearly realized and that this symmetry is spontaneously broken. In particular, we can think of the electric photon \( A_e \) as the Goldstone mode whose gaplessness is protected by the fact that the electric \( U(1)_E \) is spontaneously broken [112]. This provides a new perspective on the use of higher-form symmetries in well-known physical systems.
In general, when a symmetry is spontaneously broken, it can be realized nonlinearly and the current takes the general form $J \sim d\theta$ where $\theta$ is the Goldstone mode. However, this implies that the current $\star J$ is also automatically conserved as $d \star (\star J) \sim d^2 \theta = 0$ in this phase. We thus see that spontaneous symmetry breaking and the appearance of a second conserved current are closely related. We will return to this in chapter 4.

**Introducing background gauge fields**

We can now gauge these two 1-form global symmetries by introducing background 2–sources for the symmetry currents. It is clear, looking at (1.125) that both currents are related through a Hodge star operation $\star$ such that a single background 2–form gauge field $b_e$ for the electric current is sufficient to gauge this theory. The source for the magnetic current is then constructed as

$$b_m = \frac{1}{g^2} \star b_e.$$  \hfill (1.127)

The currents (1.125) are not gauge invariant anymore once we introduce the background gauge field $b_e$. This is a consequence of the fact that in a symmetry broken phase, the covariant derivative is not linear on the Goldstone fields $A_{e,m}$. We thus need to define gauge invariant currents as

$$J_e \rightarrow \frac{1}{g^2} (F - b_e), \quad J_m \rightarrow g^2 (G - b_m).$$  \hfill (1.128)

while the gauge transformations are given as

$$A_e \rightarrow A_e + \phi_e, \quad b_e \rightarrow b_e + d\phi_e,$$

$$A_m \rightarrow A_m + \phi_m, \quad b_m \rightarrow b_m + d\phi_m.$$  \hfill (1.129)

We can now write the action (1.123) with the background gauge fields turned on. This action can be written in the two formulations (electric and magnetic). The action in the electric formulation is

$$S_{EM} = -\frac{1}{2g^2} \int (F - b_e) \wedge \star (F - b_e),$$  \hfill (1.130)

which is the universal low energy effective action for the Goldstones when the symmetries are spontaneously broken. This effective action holds irrespective of any UV completion.

It is clear that we can obtain the currents (1.128) by taking functional derivatives with respect to the sources $b_{e,m}$ appropriately. We now want to investigate the equation of motion, because this is where we will discover an anomaly that breaks the conservation laws when background gauge fields are turned on. We want to focus on the electric formulation, and thus take $A_e$ as our fundamental field. Then, the equation of motion is obtained by varying the action with respect to $A_e$, and we obtain

$$d \star \frac{1}{g^2} (F - b_e) = 0.$$  \hfill (1.131)
1.5. Generalized symmetries

The Bianchi identity becomes anomalous, and in particular, we can write it as

$$d \ast g^2 (G - b_m) = H_e,$$

(1.132)

where $H_e = db_e$. We thus see that the improved electric 2-current is still conserved while the magnetic 2-current is not conserved anymore. In particular, it is broken by the curvature of the electric background gauge field $b_e$. This is the manifestation of the fact that both symmetries cannot be gauged simultaneously, which is the reason for the appearance of the anomaly in equation (1.132). Another perspective on this anomaly is the following. If we require current conservation, this introduces some contact terms in the field-strength two-point function. These contact terms break the other symmetry such that when you gauge this symmetry, the action is not invariant. Note that this whole discussion could have been performed using the magnetic formulation, in which case $A_m$ is the fundamental field. An interested reader could refer to [120] for more details.

To conclude, we note that the non-conservation for the magnetic 2-current is given by a source (in this case $H_e$) and not by a dynamical operator: this is an anomaly, preventing the simultaneous gauging of both electric and magnetic 1-form symmetries. In chapter 4, we will see an analogous structure emerge in the case of superfluids.

We have explained how higher-form symmetries can protect massless phases as a consequence of spontaneous symmetry breaking and the Goldstone theorem. We want to explain how this can also be framed as a consequence of an anomaly, as in equation (1.132).

### 1.5.3 How anomalies protect massless phases: A toy example

Consider the following two-dimensional effective field theory for a $U(1)$ compact boson $\phi$ of radius $R \sim 1/g$, where $g$ is the gauge coupling. The action is then given as

$$S_{EFT} = \frac{1}{g^2} \int d^2 x (\partial \phi)^2.$$  

(1.133)

This action has a shift symmetry $\phi \rightarrow \phi + \varphi$, where $\varphi$ is just a constant. The conserved current associated with this symmetry is

$$J = d\phi, \quad d \ast J = 0,$$

(1.134)

where the conservation is due to the equation of motion. However, there is also a dual current, which is topological (always conserved). It is given as

$$\tilde{J} = d\tilde{\phi} = \ast d\phi, \quad d \ast \tilde{J} \propto d^2 \phi = 0,$$

(1.135)

which is conserved because the operator $d$ is nilpotent, i.e. $d^2 = 0$. We see that the symmetry structure of this effective theory is thus really $U(1) \times \tilde{U}(1)$ and not only
the naive $U(1)$. The Coleman-Mermin-Wagner theorem forbids spontaneous symmetry breaking in two dimensions, but the theory (1.133) is gapless. It is thus natural to wonder what protects this massless phase if we cannot see $\phi$ as a Goldstone boson?

Imagine that the $U(1)$ symmetry associated with the conserved current $J$ is accidental at low energy, and write an interaction that preserves the topological symmetry $\tilde{U}(1)$. The prototypical example that is consistent with the compactness of the boson $\phi$ is

$$S_{\text{EFT}} = \int d^2x \left( \frac{1}{g^2} (\partial \phi)^2 + \gamma \cos(\phi) + \ldots \right).$$  \hspace{1cm} (1.136)

The parameter $\gamma$ needs some mass dimension, which makes the theory gapped. In particular, the mass dimension for $\gamma$ is $[\gamma] = 2 - [\cos(\phi)] = 2 - g^2$. We thus see that for $g = \sqrt{2}$, $\gamma$ is marginal while for $g \geq g_c$, $\gamma$ is irrelevant. We can thus integrate it out and the infra-red becomes gapless again. We can write a phase diagram, which is displayed on figure 1.10.

We see that in this case, the global symmetry structure is different, and this happens without spontaneous symmetry breaking. The gapless phase is protected by the fact that the $U(1)$ symmetry is in fact anomalous. We will explain this, in the case of superfluid, in chapter 4, where we will prove Goldstone’s theorem using anomalous higher-form symmetries.

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1.6 Hydrodynamics

In this section, we will introduce the theory of hydrodynamic fluctuations in QFT. This is relevant for chapter 4 and we will roughly follow the presentation of [122].

Hydrodynamics is an effective field theory that describes the long-distance evolution of a system at non-zero temperature. For relativistic systems, the microscopic constituents of the system are constrained by Lorentz symmetry. Hydrodynamics is useful for many physical phenomena that we encounter in our universe. In particular, hydrodynamic fluctuations can be thought of as gravitational fluctuations of black holes, and...

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Figure 1.10 – Phase diagram of our effective field theory. At $g_c$, the deformation is irrelevant. For $g \leq g_c$, the deformation is relevant and the theory is gapped with symmetry structure $\tilde{U}(1)$ while for $g \geq g_c$, the deformation is irrelevant and can be integrated out. This restore the $U(1)$ symmetry such that the symmetry structure is just $U(1) \times \tilde{U}(1)$. In this case, the theory is gapless again.

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1.6 Hydrodynamics

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Hydrodynamics is an effective field theory that describes the long-distance evolution of a system at non-zero temperature. For relativistic systems, the microscopic constituents of the system are constrained by Lorentz symmetry. Hydrodynamics is useful for many physical phenomena that we encounter in our universe. In particular, hydrodynamic fluctuations can be thought of as gravitational fluctuations of black holes, and...
this equivalence works in both directions [123, 124]. The connection between gravity and hydrodynamics works even for the non-linear response of the system, and the Einstein equation encodes the full non-linear relativistic Navier-stokes equations, which are the equations describing the evolution of a relativistic (super-)fluid. This relation is often called the fluid-gravity correspondence, and is a specific instance of the gauge/gravity correspondence described in section 1.2.2 [125].

This point of view has been very successful recently, especially due to the strong/weak nature of the correspondence. In particular, it allows to derive transport coefficients of strongly-coupled systems, where the usual perturbative methods are not adequate, by performing computations using black hole physics. With this technique, it is possible to compute the ratio of the shear viscosity $\eta$ to the volume density of entropy $s$, which characterizes how close a given fluid is to being perfect. This ratio admits a universal value for a large class of strongly-interacting QFT and takes the value

$$\frac{\eta}{s} = \frac{1}{4\pi k_B},$$  \hspace{1cm} (1.137)

where $k_B$ is the Boltzmann constant [126].

The hydrodynamic fluctuations that we want to focus on are small and long-wavelength fluctuations near thermal equilibrium that are long-lived, which means that they can propagate long distances. They are described by hydrodynamic equations and we want to describe the relativistic case in more detail.

### 1.6.1 Relativistic hydrodynamics

In this section, we want to describe the relativistic hydrodynamics of a normal fluid. The case of superfluids will be the subject of chapter 4. In addition, we will exclude dynamic electromagnetic fields, which is the theory of magnetohydrodynamics. The speed of light is set to $c = 1$.

**Hydrodynamic variables**

Noether theorem states that for every continuous global symmetry, there is a conserved quantity. To be able to write hydrodynamic equations, we need to identify the quantities that are conserved and write conservation equations for them. For generic relativistic systems, the spacetime symmetries are the one of the Lorentz group, given by translations, rotations, and boosts. In addition, if there is an extra symmetry in our system, we also need to write a conservation equation for the current associated with it. That could be, for example, a $U(1)$ associated with baryon number symmetry.

Let us start by describing the conserved current associated with spacetime symmetries. The current associated with translations is the usual stress-energy tensor $T^{\mu\nu}$, which is symmetric, and defined using the variation of the action with respect to the spacetime
metric $g_{\mu\nu}$. The conservation law of the stress-energy tensor is the usual conservation equation

$$\partial_{\mu}T^{\mu\nu} = 0.$$  \hfill (1.138)

For rotations and boosts, the conserved currents are defined as $M^{\mu\nu\lambda} = 2\varepsilon^{[\mu}T^{\nu]\lambda}$, which are identically conserved $\partial_{\lambda}M^{\mu\nu\lambda} = 0$. This implies that the conservation equation (1.138) accounts for the complete spacetime symmetries. In addition, we want to consider a (parity-symmetric) system with a global $U(1)$ symmetry. This symmetry will imply a conserved current $J^\mu$ whose conservation equation reads

$$\partial_{\mu}J^\mu = 0.$$  \hfill (1.139)

In terms of counting, $T^{\mu\nu}$ has $(d + 1)(d + 2)/2$ free components in $d$ spatial dimensions while $J^\mu$ has $d + 1$ components. Equation (1.138) is a vector equation, and accounts then for $d + 1$ equations while the current conservation equation (1.139) is a single equation. This implies that, as of now, there are more unknown components than there are equations to fix them. We thus need some simplifying assumptions relating components between each other.

In hydrodynamics, the simplifying assumption is the fact that we can express the conserved quantities $T^{\mu\nu}$ and $J^\mu$ in terms of $d + 2$ local fields, which are commonly taken to be the temperature $T(x)$, the fluid velocity $\vec{v}(x)$ and the chemical potential $\mu(x)$. With this, the number of unknowns matches the number of equations, and the system can be solved.

To understand this choice, we can consider the equilibrium state, which is characterized by the density operator $\hat{\rho}$. This operator is proportional to the exponential of the conserved charges \[\hat{\rho} = e^{\beta_{\mu}P^\mu + \gamma N}/\text{Tr}(e^{\beta_{\mu}P^\mu + \gamma N}),\]  \hfill (1.140)

where $P^\mu$ is the momentum operator and $N$ the operator of the conserved charge. In equation (1.140), we introduced a timelike vector $\beta^\mu$ and a scalar $\gamma$. They parametrize the manifold of thermal equilibrium states and are needed to characterize the density operator $\hat{\rho}$. These parameters are actually defined as follows: Consider a unit timelike vector $u^\mu$ that is normalized as $u^2 = -1$ and is the four-velocity of the fluid, then $\beta^\mu = \beta u^\mu$ with $\beta = 1/T$ the inverse temperature and $\gamma = \beta \mu$. The fluid four-velocity is related to the usual fluid spatial velocity $\vec{v}$ as usual: $u^\mu = (1 - \vec{v}^2)^{-1/2}(1, \vec{v})$. It is a fact that this equilibrium state breaks boost invariance because we used a preferred timelike vector $u^\mu$ in these definitions.

Ultimately, when doing hydrodynamic perturbation, we are interested in states that deviate slightly from the thermal equilibrium state (1.140) which is defined with a constant $\beta^\mu$ and $\gamma$. To be able to describe these states that are close to equilibrium, we use the slowly varying functions $u^\mu(x)$, $T(x)$, $\mu(x)$. The hydrodynamic equations must be Lorentz covariant, and the hydrodynamic effective action must be Lorentz invariant since the microscopic theory that we want to describe is.
We are now ready to write the relativistic hydrodynamic equations in terms of the usual hydrodynamic variables, that we take to be $u^\mu$, $T$, and $\mu$.

**Constitutive relations**

The goal is to write the conserved quantities $T^{\mu\nu}$ and $J^\mu$ in terms of our chosen timelike vector $u^\mu$. In particular, once you have picked the timelike vector $u^\mu$, you can define a projector that projects on directions transverse to $u^\mu$ as $\Gamma^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$ where $\eta^{\mu\nu}$ is the mostly plus flat-space metric. Then, we can write the constitutive equations for the conserved quantities as

$$T^{\mu\nu} = (E + P)u^\mu u^\nu + P\eta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu},$$

$$J^\mu = N u^\mu + j^\mu,$$

where the coefficients $E$, $P$ and $N$ are scalars while the vectors $q^\mu$ and $j^\mu$ are transverse to the four-velocity $u^\mu q_\mu = u^\mu j_\mu = 0$ and the symmetric two-tensor $t^{\mu\nu}$ is transverse and traceless. It is easy to extract these functions by contracting appropriately with the four-velocity and the projector. For example, we have

$$E = u_\mu u_\nu T^{\mu\nu}, \quad P = \frac{1}{d} \Delta_{\mu\nu} T^{\mu\nu}, \quad N = -\mu_\nu J^\mu,$$

and more complicated expressions exist for the rest [122]. The definitions (1.141) are only identities that are correct locally for any symmetric tensor $T^{\mu\nu}(x)$ and $J^\mu(x)$ once you have specified a vector $u^\mu(x)$. The simplifying assumption of hydrodynamics implies that we can express the unknown quantities, such as $E$, $P$, $N$, $q^\mu$, $j^\mu$ and $t^{\mu\nu}$ in terms of the chosen hydrodynamic variables and derivatives thereof. Scalar coefficients such as $P$, $E$, $N$ will be functions of $T$, $\mu$, $\partial_\mu u^\nu$, $u^\mu \partial_\mu T$, ... while transverse vectors will depend on transverse vectors such as $\Gamma_{\mu\nu} \partial^\nu T$, ... Ultimately, we will be able to write expressions for $T^{\mu\nu}$ and $J^\mu$ in terms of the three hydrodynamic variables and these are called constitutive equations or constitutive relations.

As of now, this strategy appears a bit cumbersome, because they are infinitely many terms that you can write in the constitutive equations with the correct properties. We need an organizing principle to guide the construction of the constitutive equations, and we will use a derivative expansion. This implies that we organize the constitutive equations in terms of how many derivatives act on the hydrodynamic variables. This is a good idea because we are considering slowly varying hydrodynamic variables whose variations from thermal equilibrium are small. Hence, a term with one derivative acting on the hydrodynamic variables will be smaller than a term without derivative and so on.

When we consider only terms without derivatives in the constitutive equations, this is called ideal hydrodynamics, because there is no dissipation. To be able to treat dissipative hydrodynamics, we need to consider at least non-derivative and one-derivative terms in the constitutive equations for the conserved quantities $T^{\mu\nu}$ and $J^\mu$. We first want to describe ideal (or zeroth-order) hydrodynamics.
1. Introduction

Zeroth-order (or ideal) hydrodynamics

We want to write constitutive equations for our conserved quantities without derivatives acting on the hydrodynamic variables. First, we note that $q^\mu$, $t^\mu$, and $t^{\mu\nu}$ can only be written using derivatives. They are thus not present at this order in the hydrodynamic expansion. The remaining coefficients have pretty simple physical interpretation and in particular, the stress-energy tensor in static equilibrium is given as $T^{\mu\nu} = \text{diag}(\epsilon, p, \ldots, p)$ where $\epsilon$ is the equilibrium energy density and $p$ the equilibrium pressure. Also, the current is given as $J^\mu = (n, \vec{0})$ with $n$ the charge density in equilibrium. If the fluid moves at constant four-velocity $u^\mu$, the conserved quantities are given as

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu},$$
$$J^\mu = nu^\mu.$$  \hfill (1.143)

In this case, the slowly-varying fields are $\epsilon$, $p$, $n$ and $u^\mu$, and the equilibrium result (1.143) corresponds to the ideal hydrodynamic constitutive equations. In addition, we can use the equation of state in equilibrium. It provides the pressure as a function of the temperature and the chemical potential $p(T, \mu)$, which allows us to extract the energy density $\epsilon$, the entropy density $s$ and the charge density $n$ using

$$s = \frac{\partial p}{\partial T}, \quad n = \frac{\partial p}{\partial \mu}, \quad \epsilon = -p + Ts + \mu n.$$  \hfill (1.144)

We can then project the conservation equation $\partial_\mu T^{\mu\nu}$ and pick up only the longitudinal component as $u_\nu \partial_\mu T^{\mu\nu} = 0$ and also consider the current conservation $\partial_\mu J^\mu = 0$ to write

$$\partial_\mu ((\epsilon + p)u^\mu) = u^\mu \partial_\mu p,$$
$$\partial_\mu (nu^\mu) = 0.$$  \hfill (1.145)

We can now use the thermodynamic relations $\epsilon + p = Ts + \mu n$ and $dp = sdT + nd\mu$ to combine the two conservation equations (1.145) into

$$\partial_\mu (su^\mu) = 0,$$  \hfill (1.146)

which is the statement that the entropy current is conserved. It is interpreted as the fact that ideal hydrodynamic is non-dissipative, and that locally the entropy does not increase.

Toward first-order hydrodynamics: the choice of frame

If we want to consider the next order in the perturbative expansion, we need to address a new concept that was not required for ideal hydrodynamics. We need to choose a frame (frame choice). This is because there is no unique definition of local temperature, fluid velocity and chemical potential away from equilibrium. For example, if we define a different local temperature away from equilibrium using gradients of the hydrodynamic variables, the two different definitions must agree at equilibrium, where the gradient vanishes. This implies that we can define a different coefficient $E$ in (1.141) as

$$E = c(T, \mu) + f_E(\partial T, \partial \mu, \partial u),$$  \hfill (1.147)
where $\epsilon$ is determined by the equilibrium equation of state while the gradient corrections $f_E$ that are necessary out of equilibrium depend on the definition of local hydrodynamic variables. In hydrodynamics, we commonly refer to the redefinition of the hydrodynamic variables $T(x), \mu(x)$ and $u^\mu$ as choice of frame. In particular, while the conserved quantities $T^{\mu\nu}$ and $J^\mu$ have a microscopic definition in equilibrium, this is not the case for the hydrodynamic variables, and they can be redefined at will provided they do not change the quantities that are conserved [129].

We can use this freedom to remove some coefficients in (1.141). The liberty to redefine $u^\mu$ can be used to set $j^\mu = 0$ (Eckart frame [128]) or to set $q^\mu = 0$ (Landau frame [130]). Both have different physical implications as the first implies that there is no charge flow in the local rest frame of the fluid while the second implies no energy flow. In addition, the possible redefinitions of $\mu$ and $T$ allow us to set the corrections to two of the three scalar quantities to zero. It is customary to choose the off-equilibrium definition of $T$ and $\mu$ such that $E = \epsilon$ and $N = n$. For more details, see [122].

**First-order hydrodynamics**

We are now ready to write the constitutive equations at first order in derivatives, and we have chosen to do so in the Landau frame, where we pick the four-velocity $u^\mu$ such that we can set $q^\mu = 0$ and the scalar functions to $E = \epsilon$ and $N = n$. We are thus left with $P, T^{\mu\nu}$ and $j^\mu$ in (1.141) that need to be written in terms of derivatives of the hydrodynamic variables.

Let us list the allowed building blocks. We have five scalars at our disposal up to one derivative: $T$, $\mu$, $u^\lambda \partial_\lambda T$, $u^\lambda \partial_\lambda \mu$ and $\partial_\lambda u^\lambda$. In addition, we can list three transverse vectors that are $\Gamma^{\mu\nu} \partial_\nu T$, $\Gamma^{\mu\nu} \partial_\nu \mu$ and $\Gamma^{\mu\nu} u^\lambda \partial_\lambda u_\nu$, and a single traceless, symmetric and transverse tensor

$$\sigma^{\mu\nu} \equiv \Gamma^{\mu\alpha} \Gamma^{\nu\beta} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d \eta_{\alpha\beta}} \partial_\mu u^\mu \right) .$$

We can use these quantities to write the first order hydrodynamics constitutive equations and we start with the only remaining scalar function, which is $P$. The most general expression that we can write is then naturally

$$P = p + c_1 u^\lambda \partial_\lambda T + c_2 u^\lambda \partial_\lambda \mu + c_3 \partial_\lambda u^\lambda + \ldots ,$$

(1.149)

where the terms in $\ldots$ contain at least two derivatives, $p$ is the thermodynamic pressure in the local rest frame, and $c_{1,2,3}$ are some unknown coefficients. Now, we can make use of our knowledge of ideal hydodynamics to simplify equation (1.149) quite a bit. We can use the two scalar equations $\partial_\mu J^\mu = u_\nu \partial_\mu T^{\mu\nu} = 0$ to fix two one-derivative scalars in terms of the third, and the error we are making by doing so contains at least two derivatives and is thus not relevant for first-order hydrodynamics. The usual choice is to keep only $\partial_\lambda u^\lambda$ and eliminate the other two one-derivative scalars such that we can rewrite (1.149) as

$$P = p - \zeta \partial_\lambda u^\lambda + \ldots ,$$

(1.150)
where \( \zeta \) is the bulk viscosity and it needs to be determined using the microscopic theory. Playing the same game for the transverse vector \( j^\mu \), we have one transverse vector equation at our disposal, which is \( \Gamma_{\lambda\nu} \partial_\mu T^{\mu\nu} = 0 \), which allows us to remove one of the three transverse vectors. We obtain

\[
j^\mu = -\sigma \Gamma^{\mu\nu} \partial_\nu \left( \frac{\mu}{T} \right) + \chi T \Gamma^{\mu\nu} \partial_\nu T + \ldots ,
\]

where \( \sigma \) is the charge conductivity and \( \chi_T \) is a free coefficient. They both need to be determined using the microscopic theory. Because \( \sigma^{\mu\nu} \) is the only tensor with the appropriate properties, we obtain

\[
i^{\mu\nu} = -\eta \sigma^{\mu\nu} + \ldots ,
\]

where \( \eta \) is the shear viscosity. We can thus summarize the first-order hydrodynamic constitutive equations in Landau frame as

\[
T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + n \eta \Gamma^{\mu\nu} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\mu u^\mu \right) - \zeta \Gamma^{\mu\nu} \partial_\lambda u^\lambda + \ldots ,
\]

\[
J^\mu = nu^\mu - \sigma T \Gamma^{\mu\nu} \partial_\nu \left( \frac{\mu}{T} \right) + \chi T \Gamma^{\mu\nu} \partial_\nu T + \ldots .
\]

We have used Lorentz covariance to write constitute equations for the quantities that are conserved. This fixes the form of these up to four free coefficients, which are commonly referred to as transport coefficients, and are \( \eta, \zeta, \sigma \) and \( \chi_T \) in our example. Using the equation of state, we can get \( \epsilon(T, \mu) \) and \( n(T, \mu) \) from \( p(T, \mu) \), but the way on which the transport coefficients depends on the hydrodynamic variables in dictated by the microscopic theory.

In the end, the transport coefficients are just parameters that are necessary to write an effective theory such as hydrodynamics, but they need to be compared with the microscopic theory itself. One way to do this is to use linear response theory, which gives explicit expressions for the transport coefficients in terms of correlation functions of the conserved quantities. It is nevertheless important to note that the transport coefficients do not depend on our frame choice. A different frame can only change the place where they appear in the constitutive relations but not their value. Linear response theory is beyond the scope of this dissertation, and we refer the interested reader to [122] for example.

**Entropy dissipation**

At ideal order, the entropy current in an equilibrium state at constant four-velocity \( u^\mu \) is given as \( S^\mu = su^\mu \). We already explained that at ideal order this current is conserved as \( \partial_\mu S^\mu = 0 \). We can assume that this current gets corrected at first order in hydrodynamics by some gradient that needs to vanish in equilibrium. We thus write

\[
S^\mu = su^\mu + \text{(gradient corrections)} ,
\]
1.6. Hydrodynamics

that needs to satisfy $\partial_{\mu} S^\mu \geq 0$. This requirement puts constraints on the different transport coefficients [130]. To write an entropy current, we can use a “covariant” version of the usual thermodynamic relation $Ts = p + \epsilon - \mu n$ which reads [127]

$$TS^\mu = pu^\mu - T^\mu_\nu u^\nu - \mu J^\mu.$$  \hspace{1cm} (1.156)

This implies when using (1.141) that

$$S^\mu = \left[ s + \frac{E - \epsilon - \mu(N - n)}{T} \right] u^\mu + \frac{q^\mu}{T} - \frac{\mu j^\mu}{T}.  \hspace{1cm} (1.157)$$

It is easy to check that $S^\mu$ is frame invariant. Imposing its positivity as $\partial_{\mu} S^\mu \geq 0$ implies for the transport coefficients that

$$\eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0, \quad \chi_T = 0.  \hspace{1cm} (1.158)$$

We thus determined that one transport coefficient is forced to vanish to not violate $\partial_{\mu} S^\mu \geq 0$. In addition, note that this hydrodynamics expansion can be continued to higher orders, see for example [131, 132].

Now that we have described the basics of relativistic hydrodynamics in its modern presentation, we will refer the reader to chapter 4. There, we use the interplay between higher-form symmetries and their anomalies to revisit the usual superfluid hydrodynamics. In particular, we will show that this is the simplest example of how anomalies enter the hydrodynamic expansion. It was understood a while ago, using the fluid/gravity correspondence how anomalies enter the hydrodynamic expansion. In the previous example, they entered and fixed coefficients at first order in the hydrodynamic expansion [133]. In the case of relativistic superfluid, the anomalous higher-form symmetry fixes coefficients already at ideal order in the hydrodynamic expansion, which is a novel feature. In this case, we write conservation equations for the higher-form conserved current and write constitutive relations for them. This is done systematically in chapter 4, where we provide a new understanding of the usual superfluid hydrodynamics.
1. Introduction

Outline of this thesis

The rest of this thesis is organized as follows. In chapter 2, we recast the action of the ANEC operator as a differential operator in four-dimensional holographic CFTs with large \( N \) and large gap to higher-spin operators. This repackaging of the ANEC operator holds within correlation functions. In holographic CFTs, this other representation of the ANEC operator is exact, and it allows us to compute arbitrary higher-point functions of the ANEC operator in external local operators. In particular, subtracting appropriately four-point functions of two ANEC operators in external local states, we can obtain the commutator of two ANEC operators in local external states. This ANEC commutator must vanish, and we can use this requirement to derive new constraints on the OPE (or equivalently the anomaly) coefficients in these theories. In particular, we show that this forces the anomaly coefficients to reproduce the Einstein gravity result. This implies that holographic CFTs must have a holographic dual with Einstein gravity minimally coupled to matter.

In chapter 3, we investigate a possible algebra amongst light-ray operators in higher-dimensional CFTs. This algebra, which was suggested in previous literature, involves operators that are generalizations of the ANEC operator. This algebra resembles the two-dimensional Virasoro algebra, and is supposedly holding in four-dimensional CFTs. To analyze this, we investigate such an algebra in free field theories as well as holographic CFTs and find disagreement. In addition, we derive new solutions to Einstein’s equation which are dual to the insertion of (exponentiated) generalized ANEC on the boundary. With these solutions, we compute commutators in the bulk and find perfect agreement with the holographic CFT computations. This suggests that this naive algebra does not hold in these cases.

In chapter 4, we investigate the interplay between higher-form symmetries, their anomalies, and hydrodynamics. First, we show that we can provide an alternative proof of Goldstone’s theorem where the starting point is a system with a regular symmetry and a higher-form symmetry that are connected through a mixed anomaly. With that, we can prove that there is a massless mode in the spectrum that transforms non-linearly. This is more general than the usual spontaneous symmetry breaking. We then recast superfluid hydrodynamics at the hydrodynamics theory of a system with a regular \( U(1) \) and a higher-form \( U(1)^{(d-2)} \) symmetries that are connected through a mixed anomaly. This streamlines the hydrodynamics expansion, and we do not need to add a Josephson condition by hand, as is standard in the usual treatment of the subject.

In chapter 4, we conclude and provide a summary of the thesis. Moreover, we present an outlook with interesting directions in which we could expand and improve the work presented in this thesis.

Several appendices contain technical details that had been omitted in the main text. In appendix A, we present technical details for the two-dimensional case. We then present the OPEs that are necessary to derive the differential operators that replace the action.
of the ANEC within correlation functions. In addition, we give the most general form of these differential operators for the case of a scalar operator, a conserved current as well as the stress-energy tensor.

In appendix B, we provide many details on the computations that are necessary for chapter 3. In particular, we list all the necessary integrals and derive the most general form of the commutator for arbitrary global operators in free field theory. We subsequently derive many commutators explicitly and check the algebra for multiple cases in free field theory. We then explain why the finite separation contribution that we find is non-integrable. We also work out the OPE between two stress-energy tensors in free field theory and use it to identify the non-local operator that contributes to the finite separation contribution in the \([L_1, L_1]\) commutator. We conclude with a discussion of conformal blocks in position space and confirm that \([L_{-1}, L_{-2}]\) is non-vanishing at finite transverse separation for holographic CFTs.

In appendix C, we provide a thermodynamic argument to fix the tension in the superfluid hydrodynamics. We then provide the map between our results and results that were already available in the literature. Finally, we construct the building blocks that are necessary to work out the first order hydrodynamics expansion.
1. Introduction
In this chapter, which is based on [1], we study correlation functions with multiple averaged null energy (ANEC) operators in conformal field theories. For large $N$ CFTs with a large gap to higher spin operators, we show that the OPE between a local operator and the ANEC can be repackaged as a particularly simple differential operator acting on the local operator. This operator is simple enough that we can resum it and obtain the finite distance OPE. Under the large N, large gap assumptions, the vanishing of the commutator of ANEC operators tightly constrains the OPE coefficients of the theory. In particular, we reach the conclusion that $a = c$ in four dimensions. This implies that the bulk dual of such a CFT is semi-classical Einstein-gravity with minimally coupled matter.

2.1 Introduction

The AdS/CFT correspondence [11] relates conformal field theories (CFTs) in $d$ dimensions to gravitational theories in Anti de Sitter spacetimes in $d + 1$ dimensions ($\text{AdS}_{d+1}$) as we described in section 1.2.2. This duality was originally discovered in the context of string theory by considering D-brane constructions, which relate particular CFTs that arise in the low energy limit of brane theories to string theory in AdS. When the gravitational dual is described by weakly coupled Einstein gravity, the CFTs are pushed into some extreme corners of theory space, with a large number of degrees of freedom and a strong coupling.

A more modern approach to the subject is to consider the strongest form of the AdS/CFT correspondence which states that every conformal field theory can be viewed as giving rise to a theory of quantum gravity in AdS. Typically, however, the bulk dual will be highly curved and quantum. One then tries to answer the following questions: what makes a CFT holographic? What are necessary and sufficient conditions that a CFT must satisfy in order to have a weakly coupled Einstein gravity dual? How does the bulk emerge from the CFT? These questions have mostly been tackled following two main frameworks. The first is the conformal bootstrap, originally conceived in [33, 36, 37] and
revived and modernized in [41]. We reviewed this in section 1.3. Here the idea is to use the analytic properties of CFT correlation functions to determine the physics of the dual gravitational theory. The second is that of quantum information theory where one might try to derive Einstein’s equations from CFT entanglement, see e.g. [134–138]. The current work discussed in this chapter will follow the first approach to the problem, even though the full power of the bootstrap will not be exploited.

The implementation of the conformal bootstrap in this context was originally introduced by Heemskerk, Penedones, Polchinski and Sully [31]. They showed that there was an equivalence between the number of solutions to the crossing equations and bulk effective Lagrangians in AdS. Furthermore, they conjectured that the locality of the bulk theory was encoded in the dimension of the lowest single-trace operator with spin greater than two. Since then, much evidence has been gathered in favour of this conjecture, as well as a more quantitative understanding of the effect of the gap, see e.g. [32, 139–143]. There has also been a great amount of progress by bootstrapping partition functions in the large $N$ limit [6, 30, 144–159].

The statement that the bulk theory should be weakly coupled is really a statement about the three-point function of the CFT stress-tensor, suitably normalized.

\[
\frac{\langle TTT \rangle}{\langle TT \rangle^{3/2}} \sim \sqrt{G_N} \quad \tag{2.1}
\]

In two dimensions, there is a single tensor structure for the stress-tensor three-point function, and the normalized three-point function scales as $c^{-1/2}$, where $c$ is the central charge. Two-dimensional CFTs with a weakly-coupled gravity dual should therefore have a large central charge. In higher dimensions, there are multiple tensor structures for the stress-tensor three-point function. Concretely, in $d \geq 4$ dimensions, we have three independent structures:

\[
\langle TTT \rangle = t_0 \left[ \langle TTT \rangle_0 + t_2 \langle TTT \rangle_2 + t_4 \langle TTT \rangle_4 \right] \quad \tag{2.2}
\]

where the $\langle TTT \rangle_i$ are known tensor structures given in [40]. The index $i$ labels the irreducible representations appearing in the three-point function in terms of its spin. The $t_i$’s are theory-dependent structure constants and one would expect them all to be large for the gravitational theory to be weakly coupled.

On the other hand, a generic bulk effective Lagrangian will be

\[
S_{\text{bulk}} = \frac{1}{16\pi G_N} \int \sqrt{|g|} d^{d+1}x \left( -2\Lambda + R + \frac{c_2}{\Lambda} R^2 + \frac{c_4}{\Lambda^2} R^3 \right) + \cdots \quad \tag{2.3}
\]
2.1. Introduction

where $\Lambda$ is dimensionful and $c_2$ and $c_4$ are dimensionless constants. The \ldots represent terms suppressed by powers of $G_N$. The statement made above about the largeness of the $t_i$’s translates to the fact that we expect a hierarchy of scales that separates $\Lambda$ from $G_N$ while keeping $\alpha_1$ and $\alpha_2$ order one. The reason we could isolate just these terms in the effective action is that we can use field redefinitions to codify the information of the three-point functions discussed above in just these terms.

An immediate concern that arises from these expressions is that they seem to violate the logic of effective field theory where we expect all higher derivative terms to be suppressed by the cutoff scale $G_N$ and not the IR scale $\Lambda$. This amounts to a large amount of fine tuning. But since holography allows this as a consequence of the large $N$ limit there is no obvious contradiction.

Eventually, causality considerations in the gravitational theory showed that only a finite, order one, range for the dimensionless constants $\alpha_1$ and $\alpha_2$ was allowed. This was first observed in [160, 161] and complete bounds were obtained in [92, 162]. It turns out these bounds follow directly from an exact calculation in CFT at finite $N$ performed in [91] where bounds on the central charges were obtained by assuming a form of the averaged null energy condition in the context of a gedanken collider experiment. We review these results briefly in the following subsection.

Still, after these results were understood, it remained surprising that the effective field theory results could be violated by the possibility of finite values for $\alpha_1$ and $\alpha_2$. This puzzle was beautifully resolved in [32] where a careful study of micro-causality in high energy scattering processes in the bulk of AdS showed that in a theory with a large gap $\alpha_1$ and $\alpha_2$ must effectively vanish. In this chapter, an argument was presented that deviations from this result are constrained by the dimension ($\Delta_{gap}$) of the lightest operator with spin $J > 2$ in the dual CFT. In particular,

$$|c_2| \lesssim \frac{1}{\Delta_{gap}}.$$  \hfill (2.4)

For example, one cannot have a CFT that is dual to Gauss-Bonnet gravity with a large correction to the Einstein-Hilbert action where all other higher derivative corrections are small. Through (2.2), this implies the vanishing of $t_2$ and $t_4$. It is known that these coefficients are related to values of central charges. The result is that holographic CFTs (i.e. large $N$, large gap) must satisfy in $d = 4$:

$$a = c,$$  \hfill (2.5)

where $a$ and $c$ are the anomaly coefficients. This result was later proven using CFT techniques in [140]. The argument is technically quite involved, it requires an understanding of the Regge limit and rests ultimately on the chaos bound [163].

The goal of this paper is to derive this result by simpler arguments in a large $N$ CFT by assuming that there is a large gap to the higher spin operators. We will make heavy use of the averaged null energy operator, discussed in the next subsection, and its commutation
relations, see [106, 107]. We hope this approach will open simpler ways to access the question of possible deviations from $a = c$ in terms of $\Delta_{gap}$ and sheds light on the sub-algebra of light-ray operators in general and their universal properties.

The averaged null energy operator

The averaged null energy operator (which we will call the ANEC operator) is defined to be

$$\mathcal{E}(x^+, \vec{x}_\perp) = \int dx^- T_{--}(x^+, x^-, \vec{x}_\perp),$$

(2.6)

where $x^\pm$ are null directions. It is an example of a larger class of non-local CFT operators known as light-ray operators, see [164, 165] and [101, 106] for more recent developments. This operator has remarkable properties, in particular its expectation value is positive for any state in the Hilbert space

$$\langle \psi | \mathcal{E} | \psi \rangle \geq 0.$$

(2.7)

This inequality is known as the averaged null energy condition (ANEC) and is an astonishing property of quantum field theory. It has recently been proven in [89, 90] by different methods. Interestingly, the two proofs originate again from either quantum information theory or the bootstrap. Positivity of the ANEC operator was assumed to derive bounds on the anomaly coefficients in any 4d conformal field theory in [91]. One simply evaluates the one-point function of an ANEC operator in a state created by the stress-tensor. The bounds, that we reviewed in section 1.4.2 read

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}.$$

(2.8)

These bounds are usually referred to as the conformal collider bounds as they were first suggested in [91] in the context of the gedanken collider experiment. They were first proven in [93] by Lorentzian bootstrap techniques [94, 95]. The saturation of these bounds implies that the CFT is actually free, see [96].

Now that the ANEC has been proven to be true the bounds (2.8) follow trivially as a particular instance of a much more general statement. One could generate many more inequalities using these type of techniques. For example, bounds on the OPE coefficients of other operators, e.g $\langle TTO \rangle$, were obtained in [99].

Once defined, the ANEC operator can be used to compute higher point correlation functions of the type

$$\langle \psi | \mathcal{E}_1 \cdots \mathcal{E}_k | \psi \rangle,$$

(2.9)

which is a $k + 2$-point correlation function in the CFT. Notice that, if inserted at the same light-like coordinate $x^+$ but at different transverse positions $\vec{x}_\perp$, these operators are space-like separated and, therefore, should commute [106, 107]. However, this argument is in fact too quick since the two ANEC operators still touch at infinity, as can be seen on the Penrose diagram of Minkowski space. One could even perform a conformal transformation to map the point where they touch to a finite distance, making the concern more manifest.
2.1. Introduction

Nevertheless, while the null integral of generic operators may be problematic, one can show that the commutator still vanishes for ANEC operators (see [108] for an in-depth discussion of the issue).

Quite trivially, the product of commuting positive operators is also positive. This signals that once the ANEC is satisfied no further information in terms of bounds should be accessible from the higher point correlation functions.

These operators were related in [91] by a conformal transformation to light-ray operators inserted at the conformal boundary of Minkowski space. In this context their interpretation corresponds to the insertion of a detector that measures the integrated energy deposited in a calorimeter in the celestial sphere in a sort of collider experiment. Because of the properties of the ANEC operators, these energy operators are also known to be positive and commuting.\(^1\)

The energy operators can be obtained by sending the ANEC operator to the boundary of space-time as:

\[
E(n^i) \sim \lim_{r \to \infty} r^2 \int dx^- T_{--}(x^+_n = 2r, x^-, 0),
\]

where \(n^i\) is a unit vector in space transforming under \(SO(3)\). We have picked here a coordinate system \(x^\pm_n = t \pm n^i x^i\) in order to take the limit in a simple way. Given this connection we will be a bit careless in the rest of this chapter and refer to these operators as energy or ANEC operators indistinctively.

Higher point correlators of these operators can also be studied. In this chapter, we will study such correlation functions for generic CFTs at large \(N\), and pay particular attention to the situation where there is a large gap for the single-trace higher spin operators. To do so, we will develop the OPE between an ANEC operator and a local operator. When there is a large gap, the only single-trace operator that will appear in the OPE between \(E\) and \(O\) is the local operator \(O\) itself. In such a case, the ANEC operator acts as a differential operator which takes a relatively simple form.

We will show that the OPE expansion can be resummed to obtain an exact expression at finite distance between the ANEC operator and the local operator insertion. This will allow us to compute the conformal collider higher point correlation functions for a large \(N\) CFT. We will see that consistency of the commutator of energy operators \(E\) or, equivalently, demanding that their product remains positive singles out the dual AdS bulk theory to be Einstein Gravity.

From the point of view of the large \(N\) CFT what we observe is that the range of allowed central charges (2.8) is drastically reduced. By looking at commutators or higher point

\(^1\) In this context, the concern about commutation arises from a possible contribution coming from the point of future infinity in the Penrose diagram.
functions of energy correlators, we deduce
\[ 1 \geq \frac{a}{c} \geq 1. \]  \hspace{1cm} (2.11)

Concretely we show:

**Result 1:** \( \langle T \left[ E_1, E_2 \right] T \rangle = 0 \implies a = c \)

**Result 2:** \( \langle T \left[ E_1 \ldots E_n \right] T \rangle \geq 0 \implies |a - c| \leq \delta_n \),  \hspace{1cm} (2.12)

where \( \delta_n \) are strictly decreasing. We will also derive equivalent properties for the coupling to currents. From a bulk point of view, we are deriving minimal Einstein-gravity couplings by assuming the large gap condition.

The result (2.11) was obtained, as mentioned above, first in [140] by different methods. We expect our discussion to be somewhat simpler and also allow for applications of this techniques to other problems. A direct application, which we do not exploit in this chapter, is the possibility, within our formalism, to compute arbitrary higher point correlators of ANEC operators at finite distances in large \( N \) CFTs.

Ultimately, what allows these computations is that, as a consequence of the large \( N \) limit and the large gap, higher dimensional CFTs acquire, as far as the ANEC operators go, a structure reminiscent of that found in two-dimensional CFTs, where the Virasoro algebra fixes all correlation functions. Interesting examples where similar structures have been uncovered in higher dimensional CFTs include: infinite dimensional algebras in supersymmetric theories [166, 167], BMS algebras for light-ray operators [106] and affine Kac-Moody algebras for theories with higher form symmetries [116]. We hope our formalism will provide a tool to further study these results.

This chapter is organized as follows: in section 2.2, we start by performing a warm up calculation in two-dimensional CFTs, from which we draw analogies to higher dimensions. In section 2.3, we compute the OPE between local operators and the ANEC operator and use it to obtain an expression for the energy operator in terms of a differential operator. In section 2.4, we compute higher-point functions of the ANEC operators and derive constraints on the central charges of the CFT. We show the holographic dual theory must be Einstein gravity minimally coupled to other light fields. We conclude in section 2.5 with a discussion.

## 2.2 A 2d warmup

We start by considering the ANEC operator in two dimensions. As we will see this example is somewhat trivial, but it will nevertheless illustrate some important properties
2.2. A 2d warmup

that will carry over to higher dimensions. In two dimensions, the ANEC operator is simply

\[ \mathcal{E} = \int dz T(z), \]

where \( T(z) \) is the holomorphic stress-tensor. Let us start by considering the three-point function

\[ \langle T(z_1)\mathcal{E}T(z_3) \rangle. \]

The local three-point function is given by

\[ \langle T(z_1)T(z_2)T(z_3) \rangle = \frac{c}{z_{12}^2 z_{13}^2 z_{23}^2}, \]

where \( z_{ij} = z_i - z_j \). We can easily integrate this expression to obtain

\[ \langle T(z_1)\mathcal{E}T(z_3) \rangle = 2\pi i \frac{2c}{z_{13}^5}. \]

However, there is a quicker way to arrive at this answer. In two dimensions, the ANEC operator is simply a differential operator

\[ \mathcal{E} = \int dz T(z) \sim \partial_z, \]

which follows from the Laurent expansion of the stress-tensor. Thus, we have

\[ \langle T(z_1)\mathcal{E}T(z_3) \rangle = -2\pi i \partial_{z_1} \langle T(z_1)T(z_3) \rangle = 2\pi i \frac{2c}{z_{13}^5}, \]

in agreement with the previous calculation. Having understood this, one can easily compute a \( k \)-point function of ANEC operators, by applying the differential operator \( k \) times

\[ \langle T(z_1)\mathcal{E}_1..\mathcal{E}_kT(z_2) \rangle \sim \partial^k \langle T(z_1)T(z_2) \rangle, \]

which means, in particular, that it is fixed by symmetry. While this is natural in light of Virasoro symmetry, it is quite surprising from a global conformal group point of view, where for example a four-point function is given by an infinite sum over the exchange of quasi-primaries. Let us illustrate this fact by considering the four-point function

\[ \langle T(z_1)\mathcal{E}_2\mathcal{E}_3T(z_4) \rangle. \]

This correlator, much like the local four-point function, is completely fixed by symmetry in two dimensions due to the Ward identities [33]. Let us start from the canonically normalized stress-tensor two-point function

\[ \langle T(z)T(w) \rangle = \frac{c/2}{(z - w)^4}. \]

2. We use an \( i\epsilon \) prescription such that we pick up the poles that lie in the lower half plane and we close the contour in that direction, see more details about this in section 3.
By using the Ward identity (see appendix A.4) twice, we get the local four-point function
\[
\langle T(z_1)T(z_2)T(z_3)T(z_4) \rangle = \frac{c^2/4}{z_{12}z_{34}} + \frac{c^2/4}{z_{23}z_{14}} + \frac{c^2/4}{z_{13}z_{24}} + \frac{2c}{z_{12}^2z_{23}^2z_{14}^2z_{24}^2z_{34}^2}.
\] (2.22)

We can now integrate this four-point function twice. The terms on the first line of (2.22), which are crucially proportional to \(c^2\), do not have simple poles and therefore vanish. The answer comes solely from the terms on the second line and reads
\[
\langle T(z_1)E_2E_3T(z_4) \rangle = 2 \frac{1}{z_{12}z_{23}z_{14}z_{24}} F_{TTTT}(\eta),
\] (2.23)

We would now like to rederive this result from the point of view of a global conformal block expansion, where one sums over an infinite set of quasi-primaries.

Using conformal blocks

We want to explicitly check the computation of the double integral above using the conformal block decomposition. Note that it corresponds to the exchange of an infinite number of quasi-primaries. These operators are the stress tensor itself, as well as composites of the stress-tensor of the schematic form: \(T \partial^k T\): which are also quasi-primaries. We will now show that the only global block relevant for computing the integrated four-point function is the stress-tensor one. All the composites will drop out once we integrate. We will use the results of [168]. The four-point function can be written as
\[
\langle T(z_1)T(z_2)T(z_3)T(z_4) \rangle = \frac{1}{4} \frac{c^2}{z_{12}z_{34}} F_{TTTT}(\eta),
\] (2.24)

where the cross-ratio is defined as
\[
\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}.
\] (2.25)

The result (2.22) can be repackaged in terms of the cross-ratio as
\[
F_{TTTT}(\eta) = \frac{1}{4} c^2 \left( 1 + \eta^4 + \frac{\eta^4}{(1-\eta)^4} \right) + 2c\eta^2 \frac{1-\eta+\eta^2}{(1-\eta)^2},
\] (2.26)

which can be written as a sum of global conformal blocks as
\[
F_{TTTT}(\eta) = \frac{1}{4} c^2 + \sum_{p=0}^{\infty} a_{2p} \eta^{2p+2} F(2p+2,2p+2;4p+4;\eta),
\] (2.27)

where \(F(a,b;c;z)\) is the hypergeometric function and the coefficients \(a_{2p}\) are
\[
a_{2p} = \left( \frac{1}{144} c^2 (2p - 1)! + 2c(1 + 2p(2p + 3)) \right) \frac{(2p)!(2p+1)!}{(4p+1)!}.
\] (2.28)
For our purposes, we can drop the constant piece coming from the identity operator. It is the first term in (2.22) and we saw it vanishes when we integrate. We can thus consider
\[ \tilde{F}_{TTTT}(\eta) = F_{TTTT}(\eta) - \frac{1}{4} c^2 = \sum_{p=0}^{\infty} a_{2p} \eta^{2p+2} F(2p + 2, 2p + 2; 4p + 4; \eta). \] (2.29)

Let us look at the expansion of \( \tilde{F}_{TTTT}(\eta) \) for small \( \eta \). Taking into account the factor of \( \frac{1}{z_1 z_3} \), the series will contain terms, for a given \( m \), of the form
\[ \frac{1}{z_1 z_3} \eta^m = \frac{1}{(z_1 z_3)^{1-m}} \frac{1}{(z_3 z_4)^m}. \] (2.30)

It is clear that when taking the integral as \( z_1 \to z_2 \), only terms where \( 0 \leq m \leq 3 \) can contribute since otherwise there is no pole at \( z_1 = z_2 \). By inspecting (2.29), it is clear that the only block for which this happens is \( p = 0 \) which is the exchange of the stress-tensor. The \( p = 0 \) block gives a contribution
\[ \int \int \frac{\tilde{F}_{TTTT}(\eta)}{z_1 z_3} \bigg|_{p=0} dz_2 dz_3 = -\left(\frac{4\pi^2}{(z_1 - z_4)}\right)^6, \] (2.31)
in agreement with the full answer. All other blocks corresponding to the composites vanish once we integrate. This is quite remarkable: an infinite number of blocks are needed to reproduce the local four-point function but a single one survives the integrals.

In a large \( c \) theory, the composites \( T\partial^k T \) can be thought of as double-trace operators. Even though Virasoro symmetry does not prevail in higher dimensions, we will see that the double-trace operators also drop out of correlators with the ANEC. This is the first lesson to draw from this simple two-dimensional case. Secondly, the fact that the ANEC operator is a differential operator in two dimensions (2.17) is no longer true in higher dimensions. However, we will see that in a large \( N \) theory with a large gap, it becomes approximately true again. The differential operator is slightly more complicated, but it involves a finite number of minus derivatives. With this newly gained insight, we are ready to discuss the structure of ANEC operators in higher dimensions.

### 2.3 Action of \( E \) on local operators

We now study correlation functions of the ANEC operator in higher dimensions. While our technology should apply to any dimension \( d > 2 \), we will work in four dimensions for the rest of the paper. The goal of this section is to develop the OPE between the ANEC operator \( E \) and any local operator \( O_{\mu_1...\mu_s} \) of spin \( s \). In the spirit of the previous section, we will recast the ANEC operator as a differential operator. This will be an exact statement at the level of CFT three-point functions, which are fixed by symmetry. While most results in this section are well known, the advantage of this formalism is that under certain assumptions on the CFT (large \( N \), large gap), the differential operator we find will enable us to compute higher-point correlation functions with multiple ANEC operators,
which will be the subject of the following section. For now, we restrict to three-point functions. We will start by explaining the general structure of this differential operator and then we move on to concrete calculations for operators of different spin.

### 2.3.1 ANEC operator as a differential operator

The goal of this section is to show that the ANEC operator can be recast as a differential operator. At the level of three-point functions with two identical local operators, this expression is exact. We will obtain

\[
\langle O \mathcal{E} O \rangle = D \langle OO \rangle ,
\]

for some differential operator \( D \). In the above expression \( O \) is a local operator of arbitrary spin. This differential operator follows directly from considering the expansion of the OPE between \( \mathcal{E} \) and \( O \) as we now discuss. Throughout this section, we only care about the terms in the OPE where \( O \) itself appears as that is the only relevant information for this three point function. In the next section we will argue why in large \( N \) theories with a gap this is all we need even for higher point-functions.

We start by discussing the building blocks of the differential operator. Consider a spacelike vector \( n^i \) of unit norm. We define two null coordinates as

\[
x^{\pm} = t \pm n^i x^i ,
\]

and define two associated null vectors \( \xi^{\pm} \). The two spacelike coordinates will be denoted \( \vec{x}_\perp \). This gives a natural decomposition of the Lorentz symmetry as

\[
SO(1,3) \to SO(1,1) \times SO(2) ,
\]

where the \( SO(2) \) leaves \( n^i \) invariant. This splitting corresponds to separating the two null directions and the two space-like directions. To proceed, pick a null vector ( say \( \xi^\mu_+ \) ) that will point in the direction along which we will integrate the stress tensor component \( \xi^-_\mu \xi^\nu T_{\mu\nu} \). Then, we need the spin information of the local operator. If we have an operator of spin \( s \), we will consider its contraction with a polarization tensor

\[
\mathcal{O} = \epsilon^{\mu_1 \cdots \mu_s} O_{\mu_1 \cdots \mu_s} .
\]

We can now build the most general differential operator that respects the symmetries at hand. The rules for the OPE \( \mathcal{E}(x_1)\mathcal{O}(x_2) \) are the following:

1. It is a scalar, so all indices must be contracted.
2. It is built from the constituents: \( \xi^\mu_+ , \xi^-_\mu , x^\mu_{12} = x^\mu_1 - x^\mu_2 , \epsilon^{\mu_1 \cdots \mu_s} , O^{\mu_1 \cdots \mu_s} , g^{\mu\nu} , \partial^\mu \).
3. It is linear both in the operator and in the polarization tensor.
4. The position vector \( x^\mu_{12} \) can only be contracted with \( \xi^-_3 \).

3. We are going to look at the OPE when the operators are separated only in + direction. The more general result can be obtained by \( SO(1,3) \) transformations.
5. The differential operator must have weight 3.

6. It must carry a $SO(1,1)$ index.

7. If the operator is a conserved current, the derivative operator cannot be contracted with $O^{\mu_1 \cdots \mu_s}$. If it is traceless, the operator cannot be contracted with the metric either.

For example, consider the case of a scalar operator. The most general operator that we can write down under the conditions above is

$$\mathcal{E}O = \sum_{q,r,s,t} c_{q,r,s,t} (x_{12} \cdot \xi)^q (\partial \cdot \xi_+)^r (\partial \cdot \xi_-)^s (\partial \cdot \partial)^t O,$$

(2.36)

with

$$- q + r + s + 2t = 3, \quad q - r + s = 1,$$

(2.37)

and where $c_{q,r,s,t}$ are some coefficients that are unfixed for now. The constraints on $q, r, s, t$ follow directly from the rules above. Furthermore, this expansion must be local at short distances. This means that $r, s, t$ are positive integer powers. We can then rewrite the differential operator as

$$\mathcal{D} = \frac{1}{(x_{12} \cdot \xi_\perp)} \sum_{k=0}^{\infty} \left( a_k (\partial \cdot \xi_-)^2 + b_k (x_{12} \cdot \xi_-) (\partial \cdot \partial) + c_k (x_{12} \cdot \xi_-)^2 (\partial \cdot \partial)^2 \right) (x_{12} \cdot \xi_- \cdot \partial \cdot \xi_+)^k.$$

(2.38)

The coefficients $a_k, b_k, c_k$ can be computed explicitly by comparing with the three-point function. We will do this in detail in the next subsection. Similar expressions exist for operators with spin and we write down the explicit expression for $U(1)$ currents and the stress tensor, see Appendix A.3.

Before deriving the precise differential operator (2.38) for scalars, let us first discuss the properties of the differential operator once we send it infinitely far away to the celestial sphere, following (2.10). This object will be relevant when we compute correlation function of energy detectors in scattering experiments. By symmetry considerations, we are able to restrict the form of the differential operator, up to a few coefficients that may be extracted once (2.38) is known. This is just a rephrasing of the well known analysis of [91] that was reviewed in section 1.4.2.

### The large distance limit and matrix elements

We would like to view these correlation functions as scattering experiments with the ANEC operators being energy detectors [91]. To do so, it is important to send the ANEC operators to spatial infinity, inserted at a given angle on the celestial sphere. This

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4. In later sections, it will sometimes be more convenient to use a slightly different basis: we will use $\Box_\perp$ instead of $\partial^\mu \partial_\mu$ which is simply a reshuffling of the basis elements written here.
is illustrated in Fig. 2.1. From this point of view, it is therefore more useful to split the Lorentz symmetry as

$$SO(1, 3) \rightarrow \mathbb{R} \times SO(3).$$  (2.39)

After the limit, and an appropriate rescaling by $r^2$, our ANEC operator has undergone the limit (2.10) and we now refer to it as an energy operator $E(n_i)$. We are now interested in the matrix elements

$$\langle \mathcal{O}| E(n_i) | \mathcal{O} \rangle \langle \mathcal{O}| \mathcal{O} \rangle. $$  (2.40)

It is useful to go to momentum space so we consider the Fourier transform of the two-point function

$$F(q) \equiv \int e^{i q x_{12}} \langle \mathcal{O}(x_2)| \mathcal{O}(x_1) \rangle,$$  (2.41)

and we will be particularly interested in the four-momentum

$$q = (q^0, 0, 0, 0),$$  (2.42)

namely $\vec{q} = 0$ momentum eigenstates. For such states, the matrix elements become extremely simple. By dimensional analysis, the differential operator will necessarily be proportional to the energy $q^0$. We will give explicit expressions in the following subsection. For now, consider the general form of the energy density one-point function [91].
2.3. Action of $\mathcal{E}$ on local operators

We have

$$\langle \mathcal{O} | E(n^i) | \mathcal{O} \rangle = \frac{q^0}{4\pi \epsilon \epsilon} \epsilon^* M(g^{ij}, n^i, C_{OOT}) \epsilon,$$

(2.43)

where $C_{OOT}$ represents the relevant OPE coefficients and our notation is

$$\epsilon^* \epsilon \equiv \epsilon_{i_1 \ldots i_s} \epsilon^{i_1 \ldots i_s},$$

$$\epsilon^* M \epsilon \equiv \epsilon_{i_1 \ldots i_s} M^{i_1 \ldots i_s} \epsilon_{j_1 \ldots j_s} \epsilon^{j_1 \ldots j_s}.$$ 

(2.44)

From this point of view, one should really view the ANEC operator as a transfer matrix $M$ between the polarizations. The only important detail is that this must be considered in an in-in formalism. As we made manifest above, it depends on the OPE coefficients between the stress tensor and the operator creating the state and its indices are built from the unit vector and the metric. For scalar operators, we have no polarization and the energy is uniformly spread over the sphere, namely $M = 1$. For operators with spin, the situation is more interesting. For example, the transfer matrices for $U(1)$ currents and stress-tensors read [91]

$$\epsilon^* M_J \epsilon = \epsilon^* \left[ g^{ij} + a_2 \left( n^i n^j - \frac{1}{3} g^{ij} \right) \right] \epsilon_j,$$

(2.45)

$$\epsilon^* M_T \epsilon = \epsilon^* \left[ g^{ik} g^{jl} + t_2 g^{ik} \left( n^i n^j - \frac{g^{ij}}{3} \right) + t_4 \left( n^i n^k n^l - \frac{2 g^{ik} g^{jl}}{15} \right) \right] \epsilon_{kl}.$$ 

(2.46)

where the constants $a_2, t_2, t_4$ depend on the OPE coefficients and the coefficient of the two-point function. We work them out in detail below by comparing with the OPE. For operators of spin $s$, the transfer matrix carries $2s$ indices and its explicit dependence on the OPE coefficients can be worked out in a similar fashion.

These expressions, including the values of the coefficients $a_2, t_2, t_4$ can be obtained explicitly by resumming the expression for the differential operator of the type (2.38) which has an infinite radius of convergence as we show below.

At this point, it is worthwhile to pause and ask why we should go through the trouble of computing this differential operator, since we could have obtained (2.45) directly from the three-point functions. It turns out, that because of the infinite radius of convergence of the OPE expansion within this subsector, we can use these expressions to compute higher order correlation functions of the form

$$\langle \mathcal{O} | E(n^i_1) \ldots E(n^i_k) | \mathcal{O} \rangle.$$ 

(2.47)

These could even be generalized to finite distance in flat space, away from the celestial sphere where the symmetry considerations are no longer directly applicable.

We will see in the following section that under the large gap assumption, this higher point matrix element is completely determined by the transfer matrix we have just described, once we take the appropriate limit. To see this, note that when the ANEC can be recast as a differential operator in higher point functions, then the product of ANECs is simply

5. For operators with spin, note that there is more than one OPE coefficient.
the product of the differential operators. Upon taking the appropriate limit, equation (2.47) becomes
\[
\langle \mathcal{O} | E(n_1) \ldots E(n_k) | \mathcal{O} \rangle = \left( \frac{q^0}{4\pi} \right)^k \frac{1}{\epsilon^* \cdot \epsilon} M_1 \ldots M_k \epsilon.
\] (2.48)
This will provide an efficient tool to obtain bounds on the OPE coefficients. We will now discuss the details of extracting the differential operator for scalar operators as well as the way to take the large distance limit.

\subsection{2.3.2 Scalar operators}

In this section, we derive the differential operator when it acts on scalar local operators, namely (2.38). The derivation for operators with spin follows but is more tedious. We give some of the details of the computation for a current and the stress-tensor in Appendix A.3. We will see that we can give the form of the operator not only as a series expansion, but also in a compact resummed expression. From this, it becomes very simple to extract the large distance limit and derive the transfer matrix described above, as we explain below.

\section*{Exact result}

We start with the general expression for the three-point function of the stress-tensor with two scalar operators \[40\]
\[
\langle T_{\mu\nu}(x_1)O(x_2)O(x_3)\rangle = \frac{C_{TOO}}{x_{12}^d x_{13}^d x_{23}^{2\Delta-d}} t_{\mu\nu}(X),
\] (2.49)
where in 4 dimensions
\[
C_{TOO} = \frac{2\Delta}{3\pi^2},
\] (2.50)
and
\[
t_{\mu\nu}(X) = \frac{X_{\mu}X_{\nu}}{X^2} - \frac{1}{d} \delta_{\mu\nu}, \quad X_{\mu} = \frac{(x_{12})_{\mu}}{x_{12}^2} - \frac{(x_{13})_{\mu}}{x_{13}^2}.
\] (2.51)
It will be more convenient for us to define
\[
s^{\mu} = x_{12}^{\mu}, \quad v^{\mu} = x_{23}^{\mu},
\] (2.52)
which also gives \(x_{13}^{\mu} = s^{\mu} + v^{\mu}\). The three-point function is only a function of these two vectors by translational invariance. We will always think of the vector \(s^{\mu}\) as being the vector between the ANEC operator \(E\) and \(O\) and \(v^{\mu}\) being the vector between the two scalars. We now wish to compute
\[
\langle \mathcal{E}(x_1)O(x_2)O(x_3) \rangle.
\] (2.53)
To do this, we need to specify time orderings for the operators, which is done by giving the appropriate \(i\epsilon\) prescription. We thus perform the shifts
\[
x_i^\pm \to x_i^\pm + i\epsilon_i.
\] (2.54)
2.3. Action of $E$ on local operators

We refer the reader to Appendix A.1 for our conventions for the $\pm$ notation.

We will perform the $x^-$ integral by means of a contour integral in the complex $x^-$ plane. From that point of view, the role of the $i\epsilon$ prescription will be to determine which poles lie inside our contour. To obtain a non-vanishing answer, we need to have solely the singularity due to $O(x_2)$ in our contour and not that of $O(x_3)$ (or the other way around). This can be achieved by picking

$$\epsilon_{12} > 0, \quad \epsilon_{13} < 0,$$

and closing the contour through the bottom. This corresponds to a time ordering where we first create the state with $O(x_2)$, then insert $E$ and then we go back in time to $O(x_3)$ to create another in state.

We can now compute the correlator with the ANEC operator, obtained by integrating (2.49). We have

$$\langle O\mathcal{E}O \rangle = \int ds^{-} \frac{-2\Delta}{8\pi^2} \frac{\delta^2}{(s + v)^{2\Delta - d}} d^{-}(X).$$

(2.56)

The integrand has a pole at

$$s^{-} = \frac{s^2}{s^+} - i\epsilon_{12} \frac{s^2 + (s^+)^2}{(s^+)^2} + O(\epsilon_{12}^2),$$

(2.57)

and the integral can be computed using Cauchy’s theorem by considering the residue at the pole. Upon further taking the limit $\epsilon_{12}, \epsilon_{13} \to 0$, we obtain

$$\langle O(x_3)\mathcal{E}(x_1)O(x_2) \rangle = (-2\pi i)^{\Delta/2} \frac{(v^+)^2 (1 + v^+)^2}{s_{\perp}^+ (s_{\perp}^+)^3}.$$  

(2.58)

We will now show how to reproduce this answer using an OPE expansion.

The OPE expansion

We will now consider the OPE expansion of the ANEC operator with a scalar operator. The OPE between a scalar operator and the stress tensor is given by

$$T_{\mu\nu}(x_1)O(x_2) = A_{\mu\nu}O(x_2) + B_{\mu\nu}^\alpha \partial_\alpha O(x_2) + C_{\mu\nu}^{\alpha\beta} \partial_\alpha \partial_\beta O(x_2) + ...$$

(2.59)

We give the explicit expression for the tensors in appendix A.2, but they are basically the most general tensors built out of the vector $s_\mu$ and the metric $g_{\mu\nu}$ that satisfy the basic symmetry properties of the indices, namely symmetric traceless in $\mu, \nu$ and symmetric in $\alpha, \beta, ...$. The coefficient in front of every tensor structure can be extracted by matching to the expansion of the three-point function (2.49).

Note that only the tensors $A, B, C, ...$ carry an $s^\mu$ dependence, so we can perform the $s^-$ integral term by term to extract the OPE between $\mathcal{E}$ and $O$.\footnote{There are many simplifications that occur once we take the integral. For example, it is easy to see that only the tensor structures with at most two metric factors can contribute once we take the $s^-$ integral. Having three or more metric factors would be accompanied by enough powers of $s$ to cancel the pole at $s^2 = 0.$} We will now compare...
the expressions that we get at each order in the OPE expansion to the exact answer (2.58). It is easy to see that the contributions from $A_{\mu\nu}$ and $B^{\alpha}_{\mu\nu}$ drop out and the first contribution to the three-point function comes from $C^{\alpha\beta}_{\mu\nu}$ and yields

$$(-2\pi i) \frac{\Delta}{\pi^2} \frac{(v^+)^2}{s^+ v^2 \Delta + 4}.$$  

(2.60)

At the next order, the contribution from $D^{\alpha\beta\gamma}_{\mu\nu}$ is

$$(-2\pi i) \frac{\Delta}{\pi^2} \frac{v^+}{(s^+)^2 v^2 \Delta + 6} \left( -6s^+ v^+ \vec{s}_\perp \cdot \vec{v}_\perp + 3(v^+)^2 s^2_\perp + (s^+)^2 (2v^2_\perp + v^+ v^-) \right).$$  

(2.61)

At this point, it is already easy to understand where this expression comes from by looking at the exact result (2.58). Rewriting the full answer as

$$\langle O(x_3) E(x_1) O(x_2) \rangle = (-2\pi i) \frac{\Delta}{\pi^2} \frac{(v^+)^2 (1 + \beta_1)^2}{s^+ v^2 \Delta + 4 (1 - \beta_2)^3}.$$  

(2.62)

with

$$\beta_1 = \frac{s^+}{v^+},$$

$$\beta_2 = \frac{(s^+)^2 v^- - 2s^+ v^+ \vec{s}_\perp \cdot \vec{v}_\perp + v^+ s^2_\perp}{s^+ v^2}.$$  

(2.63)

The k-th order in the OPE gives the homogeneous polynomial of order $k - 2$ in the $\beta_{1,2}$ expansion of the exact answer. This shows that the ANEC operator can be recast as a differential operator acting on the scalar operator. This is an exact statement at the level of three-point functions. We will now proceed to write this operator explicitly. We will do so for $\vec{s}_\perp = 0$, which will be enough for our purposes. The full expression for $\vec{s}_\perp \neq 0$ can be recovered by using $SO(1,3)$ transformations to change coordinates. For states created by operators with spin, we will no longer be able to fix the polarization vectors since most of the rotational symmetry has been used to align the ANEC operator in the $s^+$ direction. This will render the expressions slightly more complicated but the concept remains the same.

**The explicit form of the differential operator**

It is straightforward to work out the differential operator by integrating the OPE between the scalar and the stress-tensor. Writing the operator in the form (2.38) we find

$$D = (-2\pi i) \frac{\Delta}{\pi^2} \frac{1}{s^+} \sum_{k \geq 0} \left( \frac{a_k}{\Delta_{k+2}} \partial^2 + \frac{b_k}{\Delta_{k+3}} \partial_\perp \Box_\perp + (s^+)^2 \frac{c_k}{\Delta_{k+4}} \partial_\perp \right) (s^+ \partial_+)^k.$$  

(2.64)

with

$$a_k = 1,$$

$$b_k = \frac{k + 1}{2},$$

$$c_k = \frac{k^2 + 3k + 2}{32}.$$  

(2.65)

(2.66)

7. It is more convenient to use $\Box_\perp$ rather than $\partial^2$, which is a change of basis.
2.3. Action of $E$ on local operators

and where we have used the Pochhammer symbol

$$\Delta_x = \frac{\Gamma(x + \Delta)}{\Gamma(\Delta)}. \quad (2.67)$$

Fortunately, we can explicitly resum the operator. We find

$$\mathcal{D} = (-2\pi i) \frac{\Delta}{\pi^2} \left[ \frac{\partial^2}{s^+} \frac{e^{s^+ \partial_+}}{(s^+ \partial_+)^{\Delta+1}} \frac{\Gamma(\Delta + 1) - \Gamma(\Delta + 1, s^+ \partial_+)}{\Delta} \right]$$

$$+ \frac{s^+ \Box_\perp}{2\Delta_2} \left( 1 + \frac{e^{s^+ \partial_+}(s^+ \partial_+ - \Delta - 1)(\Gamma(\Delta + 2) - \Gamma(\Delta + 2, s^+ \partial_+))}{(s^+ \partial_+)^{\Delta+2}} \right)$$

$$+ \frac{s^+ \Box_\perp}{32\Delta_5} \left( \frac{\Delta_5}{\Delta_3} (s^+ \partial_+ - \Delta) \right)$$

$$+ \frac{e^{s^+ \partial_+}((s^+ \partial_+)^2 - 2\Delta_2 s^+ \partial_+ + \frac{\Delta_2}{\Delta})(\Gamma(\Delta + 5) - \frac{\Delta_5}{\Delta_3} \Gamma(\Delta + 3, s^+ \partial_+))}{(s^+ \partial_+)^{\Delta+3}} \right], \quad (2.68)$$

which is a relatively simple operator. Here $\Gamma(s, x)$ is the incomplete Gamma function. At this point, it is worth comparing this answer to the one we found in two dimensions. First, we see that the differential operator involves only a finite number of minus derivatives, as advertised. In two dimensions, the operator truncated to a single minus derivative. Here it is more complicated but the number of derivatives remains bounded. For operators with spin, it is also bounded. Second, we notice the appearance of an exponentiation of the plus derivative. This is a new feature compared to two dimensions.

There is also another phenomenon happening. The OPE expansion had a finite radius of convergence, given essentially by $\beta_2 = 1$ in (2.63). Now, in terms of the differential operator, the series can be resummed with infinite radius of convergence, as is manifest by the exponential factor in (2.68). This fact is reminiscent of Borrel resummation and is of importance in taking the large distance limit, over which we now have control. We are now ready to send the ANEC operator to the celestial sphere and consider energy correlators. The differential operator will simplify even further.

The large distance limit

We are now completely set up to study energy correlators. The kinematic setup we are interested in consists of states that are created by inserting local operators near the center of Minkowski space. Furthermore, we wish to send the ANEC operator(s) far away in the radial direction (in these coordinates, in the $x^+$ direction). The limit corresponds to taking $s^+ \to \infty$ and the leading term in the exact three-point function (2.58) becomes

$$\langle O(x_3)E(x_1)O(x_2) \rangle_{s^+ \to \infty} \sim (2\pi i) \frac{\Delta}{\pi^2} \frac{1}{(s^+)^2 \frac{v}{2\Delta} \frac{v^2}{(v^-)^3}}. \quad (2.69)$$

One can study the differential operator $\mathcal{D}$ given in (2.68) in this limit. We find that the operator becomes particularly simple

$$\mathcal{D} \sim (-2\pi i) \frac{\Delta}{\pi^2} \frac{1}{(s^+)^2} \left( -\frac{1}{\Delta} \frac{\partial^2}{\partial_+} + \frac{1}{2\Delta} \frac{\Box_\perp}{\partial_+} - \frac{1}{16\Delta} \frac{\Box_\perp^2}{\partial_+} \right). \quad (2.70)$$
2. Einstein gravity from ANEC correlators

It is worthwhile to mention that there are two asymptotic behaviours for the incomplete regularized Gamma function and one of them contains an exponential. In the regime of real momenta that we are interested in, this exponential is a pure phase and it does not dominate the long distance limit. This is particularly clear when the differential operator acts on momentum eigenstates. Asymptotically, for states satisfying (2.42):

\[
D \sim \frac{2i}{\pi (s^+)^2} \frac{\partial^2}{\partial_+} \left( 1 - \Gamma(\Delta + 1) \frac{e^{s^+ \partial_+}}{(s^+ \partial_+)^\Delta} \right) = \frac{q^0}{\pi (s^+)^2} \left( 1 - \Gamma(\Delta + 1) \frac{e^{-is^+ q^0/2}}{(-is^+ q^0/2)^\Delta} \right).
\]

From this is obvious that the second term is much smaller than the piece we have kept for any \( \Delta > 0 \).

Now consider the Fourier transform of the two-point function (2.41) where we use (2.42)

\[
F(q) = \int e^{i q \cdot x} v^{2\Delta}.
\]

The action of the operator then becomes extremely simple and we find

\[
D F(q) \sim (2\pi i) \frac{1}{\pi^2} \frac{1}{(s^+)^2} \frac{\partial^2}{\partial_+} F(q) = \frac{q^0}{4\pi r^2} F(q).
\]

The energy operator (2.10) is extremely simple when acting on the momentum space two-point function. We then have (in momentum space)

\[
E(n^i)O = \frac{q^0}{4\pi} O.
\]

Note that this is not an approximate formula, it is exact (provided we do not care about other operators appearing in the OPE). At the level of expectation values, we find

\[
\langle E(n^i) \rangle \equiv \frac{\langle OEO \rangle}{\langle OO \rangle} = \frac{q^0}{4\pi},
\]

namely a uniform energy distribution on the celestial sphere as expected for scalar states. This will drastically change once we consider states built out of operators with spin, which we now discuss.

### 2.3.3 Operators with spin

For operators with spin, one repeats the same procedure in a straightforward fashion. The most general expression for the differential operator of the form (2.38) is given explicitly for conserved currents and the stress-tensor in (A.36) and (A.40), respectively. One then compares the general expression with the direct computation of the integrated three-point function and extracts the values of the expansion coefficients. Note that there is some gauge freedom in the OPE because of the conservation/tracelessness of the operators but it can be dealt with reasonably painlessly. In practice, we pick a gauge that makes the resummation easier, which we can do without loss of generality.
Having the differential operator in the form (2.64), we can simply perform the sum and obtain the resummed version which is a (much) lengthier version of (2.68). We omit the exact expression from this draft for environmental reasons. Once again, taking the large distance limit and acting on momentum eigenstates drastically simplifies the operators and we obtain the equivalent of (2.74) for operators with spin. Namely,

\[ E\epsilon^\mu J_\mu = \frac{\pi q^0}{4} \left( \frac{3(\tilde{c} - 2\tilde{e})}{2c_v} \epsilon \cdot J - 3\tilde{c} - 8\tilde{e} \right) (\xi_+ \cdot J) (\xi_+ - \xi_-) \cdot \epsilon ) , \]  
\[ E\epsilon^\mu \epsilon^\nu T^\mu_\nu = \frac{\pi q^0}{4} \left( \frac{5}{3} \frac{7\tilde{a} + 2\tilde{b} - \tilde{c}}{c_T} \epsilon^{\mu\nu} T^{\mu\nu} + 10 \frac{13\tilde{a} + 4\tilde{b} - 3\tilde{c}}{c_T} \xi_+^{\mu} T^{\mu\nu} \epsilon^{\nu\rho} (\xi_\rho^- - \xi_\rho^+) + \frac{15}{6} \frac{81\tilde{a} + 32\tilde{b} - 20\tilde{c}}{c_T} \xi_+^{\mu} \xi_+^{\nu} T^{\mu\nu} \epsilon^{\nu\sigma} (\xi_\sigma^- - \xi_\sigma^+) (\xi_\rho^- - \xi_\rho^+) \right) . \]

From this, we can extract the transfer matrices (2.45) and we find well known expressions for the coefficients

\[ a_2 = \frac{3(8\tilde{e} - \tilde{c})}{2(\tilde{c} + \tilde{e})} , \]  
\[ t_2 = \frac{30(13\tilde{a} + 4\tilde{b} - 3\tilde{c})}{14\tilde{a} - 2\tilde{b} - 5\tilde{c}} , \]  
\[ t_4 = -\frac{15(81\tilde{a} + 32\tilde{b} - 20\tilde{c})}{2(14\tilde{a} - 2\tilde{b} - 5\tilde{c})} . \]

The tilde and hat coefficients correspond to the OPE coefficients appearing respectively in (3.13) and (3.19) of [40]. One can relate the OPE coefficients to the anomaly coefficients \( a \) and \( c \) given by

\[ T^\mu_\mu = \frac{c}{16\pi^2} W^2 - \frac{a}{16\pi^2} E , \]

where \( W \) is the Weyl tensor and \( E \) is the Euler density. The relation to the OPE coefficients is

\[ \frac{a}{c} = \frac{9\tilde{a} - 2\tilde{b} - 10\tilde{c}}{3(14\tilde{a} - 2\tilde{b} - 5\tilde{c})} . \]

The values of the OPE coefficients (2.78) naturally agree with the results in [91], obtained directly from the integrated three-point function without going through the OPE.

As explained in [91], the positivity of the ANEC operator for arbitrary polarizations yields the conformal collider bounds

\[ -\frac{3}{2} \leq a_2 \leq 3 , \]  
\[ 0 \leq 1 - \frac{t_2}{3} - \frac{2t_4}{15} , \]  
\[ 0 \leq t_2 + 2\left( 1 - \frac{t_2}{3} - \frac{2t_4}{15} \right) , \]  
\[ 0 \leq t_2 + t_4 + \frac{3}{2} \left( 1 - \frac{t_2}{3} - \frac{2t_4}{15} \right) . \]
The last three-inequalities can be recast to constrain the anomaly coefficients $a$ and $c$ as

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}.$$  \hspace{1cm} (2.84)

Now, the point is that in a consistent finite $N$ CFT this is the end of the story. No other information can be obtained by looking at higher point functions, as the energy operators at different positions commute and, therefore, their product is automatically positive.

It turns out that large $N$/large gap CFTs are somewhat sick, unless stronger constraints than (2.84) are imposed. We will see below that either by studying the commutator of energy operators (effectively a four point function in the CFT) or by looking at higher point functions we will obtain that the bounds above need to be strengthened to

$$a_2 = t_2 = t_4 = 0, \quad \Rightarrow \quad \frac{a}{c} = 1.$$  \hspace{1cm} (2.85)

Therefore, we now turn our attention to higher-point functions.

### 2.4 Correlation functions of the ANEC operator

We have seen that we can rewrite the ANEC operator as a differential operator, which is an exact statement at the level of three-point functions within the subspace of operators involved. We are now interested in computing correlation functions with multiple ANEC operators. For four-point functions and higher, this is a complicated task since all operators can run in the exchange channel and the four-point function therefore knows about the entire spectrum of the theory. We will focus on large $N$ theories, where large-$N$ factorization will give us a lot of mileage. We start by reviewing the properties of correlators at large $N$. Our counting will be adapted to theories that have order $N^2$ degrees of freedom like $\mathcal{N} = 4$ SYM or adjoint theories in general. It is straightforward to adapt it to other types of large $N$ theories if needed.

The upshot of this section is that, for holographic CFTs, the computations from the previous section are enough to compute all higher point functions of ANEC operators. This, in turn, results in strong constraints for the OPE coefficients in the theory.

#### 2.4.1 Review of large $N$ factorization

In large $N$ theories, it is useful to separate the operators into light and heavy operators. Light operators have a conformal dimension $\Delta$ that is fixed as $N \to \infty$. From a gravitational point of view, these operators correspond to fields from the bulk effective field theory living in AdS. In particular, this excludes all operators that create black holes, as their conformal dimensions scales with $N$. We further separate these operators into two classes: single-trace and multi-trace operators. Single-trace operators correspond to bulk
2.4. Correlation functions of the ANEC operator

fields following the usual AdS/CFT dictionary whereas multi-trace operators correspond to multi-particle states of the bulk fields.

What distinguishes single-trace and multi-trace operators is the way correlation functions scale. This is easiest to see in the normalization where the single-trace operator is normalized such that

\[ \langle OO \rangle \sim \mathcal{O}(1) \]  

\[ (2.86) \]

It is important to note that this is not the canonical normalization for the stress-tensor or conserved currents, which typically have a two-point function that scales like \( N^2 \). We will review the case of the stress-tensor separately. The higher point functions are then given by

\[ \langle OOO \rangle \sim \mathcal{O}(N^{-1}) \], \quad \langle OOOO \rangle_c \sim \mathcal{O}(N^{-2}) \]  

\[ (2.87) \]

where \( \langle \ldots \rangle_c \) is the connected correlation function. On the other hand, the multi-trace operators satisfy

\[ \langle :OO:OO: \rangle \sim \mathcal{O}(1) \], \quad \langle :OO:O :OO: \rangle \sim \mathcal{O}(1) \]  

\[ (2.88) \]

In this paper, we will be interested in computing the connected four-point function, which is of order \( 1/N^2 \). There are two types of contributions at this order:
— The exchange of all single-trace operators. We will denote this contribution schematically by ST.
— The contribution coming from the anomalous dimensions and the correction to the OPE coefficients of the double-trace operators. We will denote these contributions schematically as DT\(^{(1)}\).

The correction of the double-trace data has two separate origins. Part of it corresponds to quartic couplings in the bulk, and this part can be added freely in a crossing symmetric way \([31]\). On the other hand, any single-trace operator that runs in one channel will induce its own correction to the double-trace data, as required by crossing \([169]\). The corrections can be systematically extracted using Caron-Huot’s inversion formula \([102]\).

To summarize this section, we write schematically a local four-point function as

\[
\langle OOOO \rangle = 1 + DT^{(0)} + \frac{1}{N^2} (ST + DT^{(1)}) + O(1/N^4).
\]

(2.93)

We will shortly see that when two of the operators are ANEC operators instead of local operators, the only contribution that survives the integral is the single-trace contribution. This will be a key point in what follows. Before we derive this fact, we start by reviewing the \(N\)-counting for operators whose two-point functions is not \(O(1)\) and we focus on the stress-tensor.

**N-scaling for the stress-tensor**

The \(N\)-scaling for the stress-tensor is slightly different since

\[
\langle TT \rangle \sim N^2, \quad \langle TTT \rangle \sim N^2, \quad \langle T^2 \ T^2 \rangle \sim N^4.
\]

(2.94)

We therefore have the following scaling of the four-point function

\[
\langle TTTT \rangle = N^4(1 + DT^{(0)}) + N^2 (ST + DT^{(1)}) + O(1).
\]

(2.95)

The connected piece is still subleading compared to the disconnected piece, although the general scaling of the correlation function is different.

### 2.4.2 Four-point functions

We would now like to compute four-point functions in large \(N\) theories to first nontrivial order in the 1/\(N\) expansion. We can decompose a four-point function of two local operators and two ANECs using the OPE. We will always use the OPE channel where the local operator and the ANEC fuse, as illustrated in Fig. 2.2

Following the discussion in section 2.4.1, the operators we need to take into account are all single-trace operators, as well as the double-trace operators and their corrected data at order 1/\(N^2\). We will start by showing that the contribution of all double-trace operators vanishes, both at leading level and when one takes into account their corrected data at order 1/\(N^2\).
2.4. Correlation functions of the ANEC operator

Figure 2.2 – The OPE expansion of the four-point function in the channel we have picked. To compute the correlator to the first two orders in $1/N^2$, the sum over $O'$ is over all single-trace and double-trace operators.

The fate of the double-trace operators

In this section, we will show that the double-trace operators vanish in the four-point function of two ANECs and two local operators. To see this, we will consider the OPE of local operators

$$T_{--} \times O \sim T_{--} O : ,$$

(2.96)

and integrate on both sides. At leading order in $N$, the three-point function (from which one could extract the OPE) is given by

$$\langle T_{--}(x_1)O(x_2) : T_{--} : (x_3) \rangle \sim \langle T_{--}(x_1)T_{--}(x_3) \rangle \langle O(x_2)O(x_3) \rangle .$$

(2.97)

We now integrate on both sides. One can directly check that the integral vanishes

$$\int \langle T_{--}(x_1)T_{--}(x_3) \rangle \, dx_1^2 \sim \int dx_1^2 \frac{(x_{12}^4)}{(x_{12}^2 - x_{12}^4)^6} = 0 .$$

(2.98)

We thus conclude that double-trace operators $DT^{(0)}$ do not contribute at leading order in the $1/N$ expansion. Also, the identity is trivially projected out by the same argument.

Therefore, we just need to discuss the corrections that appear at order $1/N^2$. Let us consider double trace operators in (2.93) denoted by $DT^{(1)}$. We will now argue that these corrections vanish in the four-point function with two ANECs. To see this, first note that a change in the OPE coefficient would simply change the prefactor in (2.97), but the fact that it vanishes comes from the integral and a change in the overall coefficient is therefore irrelevant. The correction from the OPE coefficient thus doesn’t contribute to the four-point function with two ANECs at this order.

The case of the anomalous dimensions is more subtle. If the operator acquires an anomalous dimension, the equations (2.97) and (2.98) no longer hold, and the integral no longer

---

8. If $O$ is the stress-tensor, there are other Wick contraction but they will vanish as well.
vanishes. Instead, one can check that it picks up a piece proportional to $\gamma_{n,l}/N^2$, which comes from the discontinuity of the integrand in (2.97) once one includes the anomalous dimension. However, to compute the four-point function, one needs to do the OPE with both local operators, or in other words, to integrate twice. This means that the correction from the anomalous dimension to the double-trace operator will give a contribution of the order
\[
\frac{1}{N^4} \gamma_{n,l}^2 .
\]  
(2.99)

This is a direct consequence of the leading term $\text{DT}^{(0)}$ vanishing. If that was not the case we would indeed have corrections of order $1/N^2$.

The upshot is that $\text{DT}^{(1)}$ does not contribute at the order we are considering. It is worth mentioning that we are performing a computation at an order that corresponds to tree-level bulk physics. From the AdS point of view, the ANEC operators can be viewed as shockwaves and our computation should be thought of as propagating a particle through a shockwave [91, 92]. At tree-level, particle number must be conserved through the shockwave and all that happens to the particle is that it gets displaced. This is particularly clear for high energy particles that can only follow bulk geodesics. This is illustrated in Fig. 2.3. To consider the effect of particle creation, one would need to go one order higher in the $1/N$ expansion, which corresponds to loops in AdS. This is plotted in Fig. 2.4. This is precisely the order $1/N^4$ and it is therefore not surprising that the effect of the double-trace operators can only be seen at that order.

We have now shown that the double-trace operators do not contribute at all to the order we care about, as advertised before. In two dimensions, it was true as an exact statement due to a symmetry. Here, it is valid thanks to the large $N$ limit, and only at this order. At higher orders, the double-trace operators would become important. Overall, we have shown that a four-point function will be given solely by the sum over single-trace operators ST. We now discuss their contribution.

### Single-trace operators and the effect of large gap

We have shown that we only need to keep single-trace operators in the four-point function. This means that our calculation schematically reduces to Fig. 2.5.

There is an operator that stands out in this sum. It is the operator that was used to create the state in the first place. When this particular operator is exchanged, the ANEC simply acts as a differential operator. To see this, note that at each three-point vertex, we have precisely the three-point function we computed exactly in section 3 using the differential operator. We therefore have
\[
\langle O \partial \partial O \rangle_{O \text{-block}} = \mathcal{D}_1 \mathcal{D}_2 \langle OO \rangle .
\]  
(2.100)

This turns out to be particularly simple for ANEC operators on the celestial sphere. In that setting, we showed that the differential operator becomes a transfer matrix between
2.4. Correlation functions of the ANEC operator

Figure 2.3 – A bulk picture of a graviton scattering through a shock-wave. At tree-level the particle simply gets shifted when it passes through the shockwave and particle number is conserved.

Figure 2.4 – A bulk picture of graviton production when passing through a shockwave. One can see that it is necessarily a loop effect.

Figure 2.5 – The sum over operators has reduced to a sum over only the single-trace operators.

polarizations. We therefore have

$$\langle O| E(n_1)E(n_2) |O\rangle_{\text{O-block}} = \left( \frac{q^0}{4\pi} \right)^2 \frac{1}{\epsilon^* \cdot \epsilon} \epsilon^* M_1 M_2 \epsilon. \quad (2.101)$$

Thus, we simply multiply the transfer matrices. This is the whole advantage of thinking about the ANEC operator as a differential operator. Once we understand how it acts, if we have multiple ANECs we can simply apply one after the other even at finite distances. On the celestial sphere we just multiply the transfer matrices. It would also be straightforward to compute the all O block to the k-point function of ANECs as well. It would be given by

$$\langle O| E(n_1)^k |O\rangle_{\text{All O}} = \left( \frac{q^0}{4\pi} \right)^k \frac{1}{\epsilon^* \cdot \epsilon} \epsilon^* M_1 ... M_k \epsilon. \quad (2.102)$$

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2. Einstein gravity from ANEC correlators

In a large $N$ CFT, this block would however not be enough. One would need to add to this the contribution of all other light single-trace operators. At this point, we will focus on theories with a large gap in the dimension of single-trace operators with spin $s > 2$. In a large gap scenario, the higher spin single-trace operators are heavy and they do not contribute (or rather give small corrections suppressed by $1/\Delta_{\text{gap}}$). One could worry about other light operators of lower spin, but it was shown in [142] that in a large gap scenario, couplings of these form are suppressed by the gap as well. We can, therefore, also neglect them. In a generic holographic CFT with no supersymmetry or other accidental symmetries we do not expect to have other low spin single trace operators, in any case.

We have thus arrived at the following conclusion:

**CFT with large N, large gap $\implies$ Only $O$ appears in the $\mathcal{E} O$ OPE.**

The remainder of this paper will focus on drawing consequences or constraints from this statement. For example, we will see that the conformal collider bounds get squeezed in to give a definite value of $a/c = 1$. In general, the OPE coefficients will be “minimal” in that they match what Einstein gravity minimally coupled to matter would predict in the CFT.

### 2.4.3 Einstein gravity from commutators

In order to derive Einstein gravity and minimal couplings, we will study the commutator of two ANECs in the state created by a local operator. This amounts to computing a CFT four point function of the type described above. The commutator can be computed quite easily now that we know the transfer matrices. It is simply given by the commutator of the transfer matrices. As explained in the introduction, two ANEC operators must commute:

$$[\mathcal{E}_1, \mathcal{E}_2] = 0.$$ (2.103)

For two ANEC operators on the same null-sheet (that means they are separated in the transverse direction) all points of the two null rays are space-like separated. Therefore, we expect the operators to commute, see [106, 107]. When taken to the celestial sphere, one might worry about contributions to the commutator coming from the point at infinity. Given that in a CFT these observables can be related to the commutator above by a conformal transformation [91], we take this to be true even in this case.

We start by studying the commutator in states created by currents.

**$U(1)$ current states**

The commutator of the transfer matrices can be worked out from (2.45) and reads

$$[M_1, M_2]^{i k} = \frac{3(8\tilde{e} - \tilde{c})}{2(\tilde{c} + \tilde{e})} (n_1 \cdot n_2) \left( n_1^i n_2^k - n_2^i n_1^k \right).$$ (2.104)

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2.4. Correlation functions of the ANEC operator

Once again the tilde coefficients appearing above are related to the \( JJT \) OPE and are defined in [40]. For this matrix to vanish for arbitrary polarization states, we must have

\[
\tilde{c} = 8\tilde{e} \quad \Rightarrow \quad a_2 = 0.
\]  

(2.105)

This constraint on the OPE coefficients has a natural interpretation in the AdS dual. It corresponds to a minimal coupling between the bulk gauge field and the graviton. The effective action would be given by [91]

\[
S \sim \int d^5x \sqrt{g} F_{\mu\nu} F^{\mu\nu},
\]  

(2.106)

namely a Maxwell term. From a bulk effective field theory point of view, one could have written down non-minimal couplings involving curvature tensors, for example a coupling with the Weyl tensor \( \int d^5x \sqrt{g} W_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \), but they would have modified the value of \( \tilde{c} - 8\tilde{e} \). We have therefore shown that the large gap assumption implies minimal couplings between gauge fields and the graviton.

Stress-tensor states

The transfer matrix for stress-tensor states is given from (2.45), upon taking a suitable symmetrization and removing the traces. We find

\[
M_{ij}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{jk} g^{il}) - \frac{1}{3} g^{ij} g^{kl} + t_2 \left[ \frac{1}{4}(g^{ik} n^j n^l + g^{jk} n^i n^l + g^{il} n^j n^k + g^{jl} n^i n^k) - \frac{1}{3}(g^{ij} n^k n^l + n^i n^j g^{kl}) - \frac{1}{6}(g^{ik} g^{jl} + g^{jk} g^{il}) + \frac{2}{9} g^{ij} g^{kl} \right] 
\]  

(2.107)

\[
\]  

We now compute the commutator which reads

\[
[M_1, M_2]^{ijkl} = t_2 \left[ \frac{n_1 \cdot n_2}{8} (g^{ik} n_1^i n_2^i + g^{jk} n_1^j n_2^j + g^{il} n_1^i n_2^k + g^{jl} n_1^j n_2^k) - \frac{1}{3} \left( n_1^i n_2^j k^l n_2^l - \frac{1}{3} (g^{ij} n_2^k n_2^l + n_1^i n_1^j g^{kl}) \right) \right] 
\]  

(2.108)

\[
+ t_4 \left[ \left( n_1 \cdot n_2 \right)^2 - \frac{1}{3} \left( n^i n_1^j k^l n_2^l - \frac{1}{3} (g^{ij} n_2^k n_2^l + n_1^i n_1^j g^{kl}) \right) \right] - t_2 t_4 \left[ \frac{2}{3} \left( n_1^i n_1^j n_2^k n_2^l - \frac{1}{3} (n_1^i n_1^j g^{kl} + g^{ij} n_2^k n_2^l) \right) + \frac{(n_1 \cdot n_2)^2}{3} \left( n_1^i n_1^j g^{kl} + g^{ij} n_2^k n_2^l \right) \right] - 1 \leftrightarrow 2.
\]  

(2.109)

Demanding that this vanishes when inserted in states of arbitrary polarizations yields

\[
t_2 = t_4 = 0.
\]  

(2.109)
In terms of the anomaly coefficients, we have from (2.82)
\[ \frac{a}{c} = 1. \] (2.110)
This corresponds to a bulk effective theory given by general relativity, without higher derivative corrections. More precisely, we have shown that all higher derivative corrections in (2.3) are suppressed by a UV scale much larger than the IR scale \( \Lambda \). In short, in order for the commutator of the ANEC operators to vanish in a theory with a large gap, the bulk dual must be given by Einstein gravity.

2.4.4 Strengthening of bounds from higher point correlators

There is an alternative route to these results. It is also possible to derive Einstein gravity with minimal couplings by considering higher-point functions of the ANEC operator.\(^9\) The product of positive commuting operators must also be a positive operator. This implies
\[ \langle E_1...E_k \rangle \geq 0. \] (2.111)
We will now show that positivity of such correlators will strengthen the conformal collider bounds in this large \( N \) scenario. Therefore, the window for non-minimal couplings will close in from both sides. To do so, we need to define a correlator that is “blind” to the commuting properties of the ANEC operators. The most natural object to consider is the symmetrized correlator
\[ \langle E_1...E_k \rangle_{\text{SYM}} \equiv \frac{1}{k!} \sum_{g \in S_k} \langle E_{g(1)}...E_{g(k)} \rangle. \] (2.112)
One way to convince yourself that this is a reasonable observable is to think a bit about its holographic computation. This is discussed in [91] for scalar states. The way to perform this computation is to push an incoming particle through a gravitational shockwave with insertions associated to each \( E \) operator. When one performs the expansion of this solution the result is naturally symmetric under the reshuffling of all \( E \)'s as they all exist in the same light-like plane and have no natural ordering associated to them.

Consider, as an example, the observable above in a state created by a local current operator. We can now solve for the eigenvalues of such a matrix as a function of the parameter \( a_2 \) defined in (2.78) using (2.45). When one of the eigenvalues becomes zero, we are in danger of finding negative expectation values. The edges of the \( a_2 \) parameter space where the expectation values are positive are therefore given by the values of \( a_2 \) such that an eigenvalue vanishes for some angle on the celestial sphere.

\(^9\) D.H. would like to thank Sasha Zhiboedov for early discussions concerning this point. In particular for bringing up that the holographic computations of these quantities in the \( AdS \) bulk show a similar phenomenon.
2.4. Correlation functions of the ANEC operator

We compute this numerically below. We distribute $k$ ANEC operators randomly over the celestial sphere and iterate the procedure many times to find the strongest possible constraint for a given $k$. We plot the results for current states in Fig. 2.6 for $k = 1, \ldots, 8$. We clearly see that the bounds on $a_2$ close in on zero as we increase $k$. Demanding positivity of an arbitrary number of operator insertions will therefore close the allowed range down to $a_2 = 0$ which is again minimally coupled Maxwell theory in the bulk.

![Figure 2.6](image)

**Figure 2.6** – A plot of the allowed parameter space for the coupling $a_2$, as demanded by the positivity of the $k$-point function of ANEC operators. For $k = 1$, we have the conformal collider bounds. As we increase the number of operators, the region gets more and more constrained and is slowly closing in on $a_2 = 0$. We can fit the bounds by a power law and we find $\delta_{\text{max}} \sim 2.98k^{-0.66}$ and $\delta_{\text{min}} \sim -1.53k^{-0.46}$. 
2.5 Discussion

In this paper, we have studied correlation functions of ANEC operators in states created by a local operator. We developed an OPE between the local operators and the ANEC operator and recast it as a differential operator. This statement is exact at the level of three-point functions and becomes approximately true for higher point functions in a CFT at large \( N \) and with a large gap. The form of this differential operator is given as a series expansion which we were able to resum. In the limit where the ANEC operator is sent to the celestial sphere infinitely far away, the differential operator becomes particularly simple.

This formalism is particularly useful to compute correlation functions with multiple ANEC operators. In a large \( N \) CFT with a large gap to higher spin operators, we showed that the contribution of double-trace operators completely drops out from the correlator at the order that we care about and the correlation function of multiple ANEC operators is simply given by acting with a sequence of differential operators on the two-point function. The emerging structure is reminiscent of \( d = 2 \) physics, as it was previewed in section 2.

We used this property to compute the commutator of two ANEC operators and demanded it must vanish. In a CFT with a large gap, we showed that this constrains the OPE coefficients of the theory to be “minimal”, which, in particular, forces the anomaly coefficients to satisfy

\[
a = c. \tag{2.113}
\]

The bulk version of this statement is that any large \( N \) theory with a large gap must have a holographic dual with Einstein gravity minimally coupled to matter. We have also computed the \( k \)-point function of ANEC operators and demanded it to be positive. This implies a strengthening of the conformal collider bounds. In the large \( k \) limit, the bounds close in again on the minimal couplings. The two approaches turn out to be equivalent.

The most important direction in which this discussion could be improved concerns relaxing the assumption of an infinite gap to higher spin operators. If one kept a large but finite value of \( \Delta_{\text{gap}} \), it would be possible to perform a systematic expansion in terms of this quantity in order to study how equalities like \( a = c \) can be corrected by powers of \( \Delta_{\text{gap}}^{-1} \). This way one could obtain precise expressions including numerical factors that would build on the results in [32, 142]. The obstacle is that this computation cannot be done reliably entirely in the conformal channel used in this paper where light-ray operators act on local operators creating a state. The reason is that it is actually easy to see that the addition of a finite number of operators to the computations described in this work cannot change the strong contraints coming from the vanishing of commutators. What we find in this case, instead, is the requirement that further non-minimal couplings to this new heavy operators must vanish as well. In order to have an effect on the constraints an infinite number of heavy operators need to be included. But this is equivalent to considering a finite number of them in the cross-channel. Therefore, our
2.5. Discussion

techniques don’t apply directly to this case and they need important improvements to account for this physics.

If one could somehow improve the analysis, a new kind of results would become available. If many single trace operators can appear in the intermediate channel of the calculation of the ANEC commutator one would expect interesting sum rules to arise of the form\(^\text{10}\)

\[
\left(\frac{a - c}{c}\right)^2 \sim \sum_{\Delta^\Delta \neq \mu \nu} |C_{TTO}|^2 f(\Delta, s_O).
\] (2.114)

One direct application that is readily available from the results presented here is the computation of arbitrary high-point correlation functions of ANEC operators, even when they are separated by a finite distance from each other in the large \(N\) limit. While we have not looked at these observables in detail, it seems their structure is universal and can be thought of as a generalization of the well known structure present in two dimensional CFTs. The fact that we have written the operators in differential form gives us direct access to study the Ward identities for these theories, in the spirit of [33]. We hope this approach will help provide a better understanding of the appearance of infinite dimensional algebras in some contexts in \(d = 4\) CFTs [106, 116, 166, 167].

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10. A similar type of structure was observed in [99].
2. Einstein gravity from ANEC correlators
In this chapter, which is based on [2], we study correlation functions involving generalized ANEC operators of the form \( L_n = \int dx^- (x^-)^{n+2} T_-(\vec{x}) \) in four dimensions. We compute two, three, and four-point functions involving external scalar states in both free and holographic conformal field theories. From this information, we extract the algebra of these light-ray operators. We find a global subalgebra spanned by \( n = \{-2, -1, 0, 1, 2\} \) which annihilate the conformally invariant vacuum and transform among themselves under the action of the collinear conformal group that preserves the light-ray. Operators outside this range give rise to an infinite central term, in agreement with previous suggestions in the literature. In free theories, even some of the operators inside the global subalgebra fail to commute when placed at spacelike separation on the same null-plane. This lack of commutativity is not integrable, presenting an obstruction to the construction of a well defined light-ray algebra at coincident \( \vec{x} \) coordinates. For holographic CFTs the behavior worsens and operators with \( n \neq -2 \) fail to commute at spacelike separation. We reproduce this result in the bulk of AdS where we present new exact shockwave solutions dual to the insertions of these (exponentiated) operators on the boundary.

### 3.1 Introduction and summary of results

Two-dimensional conformal field theories occupy a special place in the landscape of all Conformal Field Theories (CFTs). In two dimensions, conformal invariance of a field theory implies the existence of an infinite-dimensional symmetry – the Virasoro symmetry [33]. The presence of this symmetry has far-reaching consequences, going from the existence of CFTs with a finite number of (Virasoro) primary operators (rational CFTs) to the actual solvability of the conformal bootstrap in such cases [37].

Virasoro symmetry also plays an important role for quantum gravity in AdS\(_3\) through the AdS/CFT correspondence. Gravitational dynamics are much simpler in three-dimensions due to their topological nature, in fact so simple that they are completely universal. The CFT analog of this statement is that the stress-tensor sector of a 2d CFT should be completely universal, which is indeed the case as enforced by Virasoro symmetry.
3. On the Stress Tensor Light-ray Operator Algebra

These observations have enabled a powerful machinery to derive gravitational dynamics in holographic two-dimensional CFTs by assuming the dominance of the Virasoro identity block [170–178].

Given the successes of Virasoro symmetry in \( d = 2 \), it is natural to ask whether such a symmetry can exist in higher-dimensional CFTs. An intuitive way to think about the Virasoro symmetry is that in two dimensions, only the stress-tensor \( T \) and its composites can appear in the \( T \times T \) OPE, with coefficients uniquely determined by the central charge. This immediately presents serious challenges for higher-dimensional CFTs since all (neutral) operators of the theory can in principle appear in the stress-tensor OPE, making any form of universality seem hopeless. Potential ways around this obstruction have been suggested. Let us consider two such proposals. \(^2\)

The first is to consider a class of non-local CFT observables known as light-ray operators \([1, 101, 104, 106–108, 179]\). The operators we will be particularly interested in are built from null-integrals of the stress-tensor as follows

\[
E_f(x^+, \bar{x}^+) \equiv \int_{-\infty}^{\infty} dx^- f(x^-) T_{--}(x) .
\]

For \( f(x^-) = 1 \), this operator satisfies the averaged null energy condition (ANEC) \([89, 90]\) meaning that the operator is positive in any quantum field theory (QFT). This fact has far reaching consequences and it encodes important constraints on QFTs consistent with causality. In particular, it imposes bounds on the range of central charges of unitary CFTs. This was first suggested in [91] and finally proven in [93] (see [99, 100, 180] for other bounds).

A family of operators of the form (3.1) all living at the same \( \bar{x}^\perp \) are embedded in a two-dimensional plane. This offers a promising framework to look for a Virasoro algebra. In fact, by considering properties of modular Hamiltonians on deformed half-spaces, [107] proposed that the operators

\[
L_n(x^+, \bar{x}^+) \equiv E_{x+n}(x^+, \bar{x}^+) ,
\]

satisfy the Virasoro algebra (in \( d = 4 \))

\[
[L_m(\bar{x}^\perp), L_n(\bar{y}^\perp)] = -i\delta^{(2)}(\bar{x}^\perp - \bar{y}^\perp)(m-n)L_{m+n+1}(\bar{y}^\perp) .
\]

This algebra is really a Witt algebra rather than Virasoro since one does not obtain a finite central charge (see [181] for a proposal to do so by balancing UV and IR divergences). A version of this algebra was proposed to hold for arbitrary CFTs in \( d > 2 \). A part of the “global” version of this algebra was also argued for in [106] (see also [182]), with explicit

1. Note that not all observables are necessarily reproduced by the identity block even in holographic CFTs, see for example [150].

2. A third proposal that we will not discuss in the context of this chapter relates to the algebra of chiral operators placed on a 2d plane in four and six dimensional Superconformal Theories [166, 167].
3.1. Introduction and summary of results

checks in free field theory. A careful consideration of this proposal for a Virasoro algebra in \( d = 4 \) will be presented in this work.

The second possible way to find a Virasoro algebra is to consider very special CFTs: holographic large \( N \) CFTs in \( d \) dimensions which capture the dynamics of Einstein gravity in \( \text{AdS}_{d+1} \). While a generic CFT will have all kinds of operators appearing in the stress-tensor OPE, holographic CFTs have the special property that

\[
T \times T \sim T + \text{composites} + \mathcal{O}(1/N^2) + \mathcal{O}(1/\Delta_{\text{gap}}),
\]

where \( \Delta_{\text{gap}} \) controls the corrections to Einstein gravity in the bulk. This suggests that while the stress-tensor sector is highly non-universal in a generic CFT, it becomes universal at large \( N \) and large \( \Delta_{\text{gap}} \). Some evidence has been gathered in this direction [32, 140–143, 183–190]. One may hope that this universality is controlled by a Virasoro symmetry, emergent at large \( N \) and large \( \Delta_{\text{gap}} \), which can recast gravitational dynamics of Einstein gravity in terms of a symmetry.

It is with this overarching goal in mind that we will study the algebra of light-ray operators (3.1). In the CFT context we will study mostly free theories and will comment on how to use the conformal block decomposition to extrapolate some of these results to holographic CFTs. We then turn to computations in AdS gravity where we explicitly obtain shockwave solutions that allow us to explore the algebra of these operators directly. As we will see, the algebra (3.3) as advocated for in [107], does not seem to hold, neither in free field theory nor in holographic CFTs where one would expect the most universality 3.

3.1.1 Summary of results

In this chapter, we present various results for the expectation values and commutators of operators (3.1) in certain states. We provide calculations in free field theories and holographic theories in \( d = 4 \), for which we discuss both the gravitational and CFT sides of the computations. While we give numerous explicit computations throughout the chapter, we would like to highlight the following three results.

Collinear transformations and a family of five light-ray operators

There is a subset of the conformal group known as the collinear subgroup [165]. The group action maps five light-ray operators into one another. These operators are the \( L_n \) operators of (3.2) for \(-2 \leq n \leq 2\), the simplest of which is the ANEC operator \( (L_{-2}) \). These operators combine into a five-dimensional representation of the collinear algebra and it is thus natural to discuss them together. It is interesting to note that from the point

3. Possible obstructions to these type of constructions were already put forward in [104, 108] by pointing out issues with the convergence properties of (3.1). We will comment on this as we encounter these issues in our computations.
of view of the Virasoro algebra (3.3), one may want to call the operators \(-2 \leq n \leq 0\) the “global” part, but we will see that the action of these operators on the vacuum make it more natural to refer to this whole family of five operators as global. In general \(d\) dimensional CFTs this global algebra has dimension \(d + 1\), consistent with the well known three-dimensional global subalgebra of the Virasoro generators in CFTs in \(d = 2\).

**A breakdown of the algebra**

We will explicitly see that the operators (3.1) do not in general commute when inserted on the same null-plane even at finite spacelike separation. For example, in free field theory we have

\[
\langle \phi(x_1)[L_1(x_2), L_0(x_3)]\phi(x_4) \rangle = -\left( x_1^- - \frac{|\vec{x}_{12}|^2}{x_{12}} \right) - \left( x_4^- + \frac{|\vec{x}_{24}|^2}{x_{24}} \right) \frac{3\pi^2 x_1^+ x_2^+ x_2^+ x_{23}^+ |\vec{x}_{23}|^2}{x_{12} x_{24}}. \quad (3.5)
\]

For a holographic CFT, the problem actually worsens and lower modes of the \(L_n\)’s fail to commute. A simple way to visualize the result in this case is to quote the expression for operators inserted on the celestial sphere at different angles as is familiar in collider experiments [91]. For example, taking spherically symmetric scalar states of definite timelike momentum \(p^0\), we find

\[
\frac{\langle O(p^0)[[L_{-1}, L_{-2}]]O(p^0) \rangle}{\langle O(p^0)O(p^0) \rangle} = -i \frac{p^0}{16\pi^2} (1 + 3 \cos \theta_{12}), \quad (3.6)
\]

where \(\theta_{12}\) is the angle between the two operators on the celestial sphere.

The fact that the operators do not commute at spacelike separation makes it extremely challenging to define an algebra. The result (3.5) is not integrable in the \(x^+\) direction rendering the short distance singularity ambiguous. We will expand on this issue throughout the chapter and in the discussion section.

Similar observations were made in [108], where it was shown that four-point functions involving two light-ray operators \(L_n\) and \(L_m\) are only unambiguously defined (i.e. that the integrals of the Wightman functions are absolutely convergent) provided \(n + m\) satisfies a bound that depends on the Regge intercept of the CFT. We will discuss how our results connect to this statement.

**Generalized shockwave geometries in AdS**

It is long known that the gravity dual of ANEC operator insertions are shockwaves [91]. In this chapter, we present new exact solutions to Einstein’s equations which are generalized shockwaves in AdS, with a source given by one of the global \(L_n\)’s at the boundary. We were able to find exact solutions for all operator insertions but \(L_0\), for reasons that we detail in the main text (see section 3.8.3 for explicit metrics corresponding to \(L_{-1}\) and \(L_2\)). By scattering waves through these shocks, we can compute correlators in the bulk.

4. See section 3.2.1 for conventions.
3.2 Generalized ANEC operators

and compare to the computation in a holographic CFT. We find perfect agreement with the CFT answer, and find that these shocks do not commute.

This chapter is organized as follows: In section 2, we present our conventions and define carefully the operators we want to investigate. In section 3, we study the action of the collinear subgroup, which is the subgroup of conformal transformations that maps the light-ray onto itself, and identify a family of five light-ray operators that map into one another under the group action. In section 3, we evaluate two- and three-point functions involving light-ray operators and compute the would-be central charge of the algebra we are investigating. In section 4, we compute the four-point functions as well as the commutator involving two global light-ray operators in free field theory, and investigate the algebra of the five global operators. In section 6, we explain the finite transverse separation contribution in the commutator of two light-ray operators by studying a subset of the OPE of two light-ray operators, with the specific example of $[L_1, L_1]$ in mind. In section 7, we perform a conformal block decomposition relevant for holographic CFTs. In section 8, we describe the gravitational shockwaves dual to the generalized ANEC operators. We conclude in section 9. Many details are provided in the appendices.

3.2 Generalized ANEC operators

In this section, we introduce the conventions that we use throughout this work, as well as precisely define the light-ray operators that we consider.

3.2.1 Conventions

We will be working in $d = 3 + 1$ spacetime dimensions, with the coordinates

$$x^\pm \equiv x^0 \pm x^3, \quad \vec{x}^\perp \equiv (x^1, x^2),$$

(3.7)

and the associated metric

$$ds^2 = -dx^+ dx^- + d\vec{x}^{\perp 2}.$$  

(3.8)

Coordinates with lowered indices are

$$x_\pm \equiv \frac{1}{2}(x_0 \pm x_3) = -\frac{1}{2}x^\mp, \quad \vec{x}_\perp \equiv (x_1, x_2) = \vec{x}^\perp,$$

(3.9)

and the invariant distance is

$$x^2 = -x^+ x^- + |\vec{x}^\perp|^2 = -4x_+ x_- + |\vec{x}_\perp|^2.$$  

(3.10)

Throughout this work, we will be studying Wightman functions of the general form

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle.$$  

(3.11)
3. On the Stress Tensor Light-ray Operator Algebra

We can ensure this particular fixed ordering in Lorentzian signature (with $O_i$ to the left of $O_{i+1}$) by using the following $i\epsilon$ prescription \[94\]

\[x_i^\pm \rightarrow x_i^\pm - i\epsilon_1,\] (3.12)

with $\epsilon_1 > \cdots > \epsilon_n$. In practice, one can do this by using the following prescription: when evaluating an $n$-point Wightman function of the form (3.11), choose

\[x_1^\pm \rightarrow x_1^\pm - ni\epsilon,\quad x_2^\pm \rightarrow x_2^\pm - (n-1)i\epsilon,\quad \ldots,\quad x_n^\pm \rightarrow x_n^\pm - i\epsilon.\] (3.13)

When evaluating integrals over $x^-$, we will often encounter poles from the OPE singularity where the distance between two operators goes to zero, i.e. $x_{ij}^2 = 0$. The locations of these poles take the general form

\[x_j^- = x_i^- - \frac{|x_{ij}^\perp|^2}{x_{ij}^+}.\] (3.14)

It will be convenient to represent this combination of coordinates with the shorthand notation

\[x_{i,j}^- \equiv x_i^- - \frac{|x_{ij}^\perp|^2}{x_{ij}^+}.\] (3.15)

Note that the first index indicates the location of the pole (in $x^-$), while the second index indicates which variable was integrated over. The coordinate $x_{i,j}^-$ is thus specifically the location of a pole in the $x_j^-$ plane.

3.2.2 Definition of generalized ANEC operators

We study a set of light-ray operators that have been considered previously in [106–108, 181]. They are generalizations of the ANEC operator where we integrate $T_{--}(x)$ along the null direction $x^-$, weighted by an arbitrary function $f(x^-)$,

\[E_f(x^+, \vec{x}^\perp) \equiv \int_{-\infty}^\infty dx^- f(x^-) T_{--}(x).\] (3.16)

In particular, for $f(x^-) = 1$ we recover the ANEC operator. One must be careful when inserting these operators in Wightman functions, since the resulting integral over $x^-$ may not converge. This of course depends on the behaviour of the function $f$ near infinity. We will detail below the precise function class from which we draw $f$.

As explained in the introduction, we will consider the functions $f(x^-) = (x^-)^{n+2}$, in analogy with Virasoro generators in $d = 2$. We will denote the associated operators by $L_n$, thus defined as

\[L_n(x^+, \vec{x}^\perp) \equiv E_{x^{n+2}}(x^+, \vec{x}^\perp) = \int_{-\infty}^\infty dx^- (x^-)^{n+2} T_{--}(x^+, x^-, \vec{x}^\perp).\] (3.17)

Our convention (3.17) is slightly different than that used in [107, 108, 181] (the label $n$ is shifted by 1). It will become clear in section 3.3 why we find our convention more
3.3. Conformal transformations of light-ray operators

convenient. Note that throughout this work, we will use a slight abuse of notation and write \( L_n(x) \equiv L_n(x^+, \vec{x}^\perp) \).

Let us now return to the function class from which we would like to draw the functions \( f(x^-) \). For the integral to converge inside arbitrary correlation functions, it is manifest that the function \( f \) should have nice boundedness properties near infinity. In particular, choosing \( f(x^-) = (x^-)^{n+2} \) will be ill-behaved for sufficiently high \( n \). Rather than directly working with bounded functions, we will define the operators \( L_n \) through a limiting procedure. We define

\[
L_n(x^+, \vec{x}^\perp) = \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} dx^- e^{i\delta x^-} (x^-)^{n+2} T_-(x^+, x^-, \vec{x}^\perp)
\]

where we close the integration contour in the upper half-plane before taking the \( \delta \to 0^+ \) limit. This contour integral is now well-behaved inside arbitrary correlation functions. To avoid cluttering the equations, we will omit the limiting procedure and not explicitly write out the contour integral in the rest of the chapter, but it should be understood that we implement this procedure on all operators. This procedure is harmless when considering convergent real integrals (like the ones we will study) and amounts to a particular regularization when they don’t. A physical way of thinking about this is to restrict the matrix elements of the operators involved to states with support confined to a localized enough region in \( x^- \).

That being said, we will mostly be focused on \( f(x^-) = (x^-)^{n+2} \) for \(-2 \leq n \leq 2\). We will show that in all correlation functions we consider, the integrand for this set of functions is bounded at infinity and we can close the contour without any need for a regularization procedure. This observation will be useful in practice when evaluating integrals in the following sections.

3.3 Conformal transformations of light-ray operators

Before studying the structure of correlation functions involving the light-ray operators \( L_n \), it will be useful to first understand their behavior under conformal transformations. In particular, we shall focus on the so-called “collinear” subgroup of conformal transformations, which preserve the null line \( x^+ = \vec{x}^\perp = 0 \). Under these transformations, the operators \( L_n \) in general dimensions behave similarly to their \( d = 2 \) inspiration, with a set of “global” operators forming a finite-dimensional representation of the collinear subgroup, and the remaining \( L_n \) grouped into two infinite towers of “Virasoro” operators.
3.3.1 Collinear subgroup

In $d$ dimensions, the conformal group $SO(d,2)$ is built from translations $P_\mu$, Lorentzian boosts and rotations $M_{\mu\nu}$, dilatations $D$, and the special conformal transformations $K_\mu$. We will follow the conventions of [165] for the commutation relations of these conformal generators.

We are particularly interested in the set of conformal transformations which map the light-ray along the $x^-$ direction,

$$x^\mu = \alpha n^\mu, \quad n^\mu \equiv (n^+, n^-, \vec{n}^\perp) = (0, 1, \vec{0}),$$  \hspace{1cm} (3.19)

to itself. These transformations are generated by the four generators $D$, $P_-$, $M_{+-}$, and $K_+$, which form the *collinear subalgebra* of the full conformal algebra. If we arrange these generators into the useful form

$$J_{-1} \equiv i P_-, \quad J_0 \equiv \frac{i}{2} (D - 2 M_{+-}), \quad J_1 \equiv -i K_+,$$  \hspace{1cm} (3.20)

we see that they satisfy the familiar algebra of $SL(2,\mathbb{R})$,

$$[J_0, J_{\pm 1}] = \mp J_{\pm 1}, \quad [J_1, J_{-1}] = 2 J_0,$$  \hspace{1cm} (3.21)

similar to the “global” conformal algebra in $d = 2$. We also have the remaining combination

$$\tilde{J}_0 \equiv \frac{i}{2} (D + 2 M_{+-}),$$  \hspace{1cm} (3.22)

which commutes with all $J_i$ and measures the collinear “twist” $\tilde{h} \equiv \frac{1}{2} (\Delta - m)$, where $m$ is the spin component in the $x^\pm$ plane.

Under a general collinear transformation, the coordinate $x^-$ transforms as

$$x^- \to \frac{ax^- + b}{cx^- + d},$$  \hspace{1cm} (3.23)

with the constraint $ad - bc = 1$. The remaining coordinates transform as

$$x^+ \to x^+ - \frac{c|\vec{x}^\perp|^2}{cx^- + d}, \quad \vec{x}^\perp \to \frac{\vec{x}^\perp}{cx^- + d}.$$  \hspace{1cm} (3.24)

We therefore clearly see that these transformations preserve the null line $x^+ = \vec{x}^\perp = 0$.

The action of the collinear generators on a general primary operator $\mathcal{O}(x^-)$ located on this null line is

$$[J_{-1}, \mathcal{O}(x^-)] = \partial_+ \mathcal{O}(x^-),$$
$$[J_0, \mathcal{O}(x^-)] = \left( \tilde{h} + x^- \partial_+ \right) \mathcal{O}(x^-),$$  \hspace{1cm} (3.25)
$$[J_1, \mathcal{O}(x^-)] = \left( 2hx^- + (x^-)^2 \partial_+ \right) \mathcal{O}(x^-),$$
3.3. Conformal transformations of light-ray operators

where $h \equiv \frac{1}{2} (\Delta + m)$. We are specifically interested in the operator $T_{-}$, which at arbitrary $x$ transforms as

$$[J_{-1}, T_{-}(x)] = \partial_- T_{-}(x),$$

$$[J_{0}, T_{-}(x)] = \left(h_{T} + x^\perp \partial_+ + \frac{1}{2} x^\perp \cdot \vec{\partial}_\perp \right) T_{-}(x),$$

$$[J_{1}, T_{-}(x)] = \left(2h_{T} x^+ + (x^-)^2 \partial_- + x^- x^\perp \cdot \vec{\partial}_\perp + |x^\perp|^2 \partial_+ \right) T_{-}(x) + 2 x^\perp \cdot \vec{T}_\perp(x),$$

with $h_{T} = 3$ (and $\bar{h}_{T} = 1$) for $d = 4$. Away from $x^\perp = 0$, $T_{-}$ therefore mixes with other components of the stress tensor under collinear transformations.

3.3.2 Transformations of generalized ANEC operators

Let’s now consider the behavior of the light-ray operators $L_n$ under general collinear transformations, which can be derived from that of $T_{-}$ in eq. (3.26). These transformations are simplest for $x^\perp = 0$, in which case we find\(^5\)

$$[J_{-1}, L_n(x^+)] = -(n + 2) L_{n-1}(x^+),$$

$$[J_{0}, L_n(x^+)] = -n L_n(x^+),$$

$$[J_{1}, L_n(x^+)] = -(n - 2) L_{n+1}(x^+).$$

The operator $L_n$ thus has collinear weight $-n$. Unsurprisingly, we see that $J_{\pm 1}$ act as raising and lowering operators, moving us between the different $L_n$.

However, if we specifically look at the ANEC operator $L_{-2}$, we see that it is annihilated by $J_{-1}$,

$$[J_{-1}, L_{-2}(x^+)] = 0,$$

which simply follows from the fact that $L_{-2}$ is translation-invariant along $x^-$. If we now repeatedly act with $J_1$ on $L_{-2}$, we move through the higher $L_n$ until we reach $L_2$, which is annihilated by $J_1$,

$$[J_{1}, L_2(x^+)] = 0.$$

The central five operators

$$\{L_{-2}, L_{-1}, L_0, L_1, L_2\},$$

thus form a finite-dimensional representation of $SL(2, \mathbb{R})$ when acting at $x^\perp = 0$. We shall refer to these five operators as “global” light-ray operators, in analogy with two dimensions. None of the remaining generalized ANEC operators $L_{\pm n}$ with $n \geq 3$ are annihilated by the $J_i$, and instead form two infinite towers with respect to the collinear subgroup, as shown schematically in figure 3.1.

---

\(^5\) In deriving these expressions, we have used the fact that all support for $T_{-}(x)$ vanishes at $x^- \to \infty$. This follows from the limiting procedure (3.18).
Figure 3.1 – Schematic representation of the action of the collinear generators $J_{\pm 1}$ on the light-ray operators $L_n$. The “global” operators with $n = \{-2, -1, 0, 1, 2\}$ form a five-dimensional representation of $SL(2, \mathbb{R})$, with the remaining operators in two infinite towers.

We can also consider the behavior of these light-ray operators under the finite $SL(2, \mathbb{R})$ transformation (3.23). At $\vec{x}^\perp = 0$, the stress tensor simply transforms as

$$T_{-+}(x^+, x^-) \rightarrow (c x^- + d)^{2h_T} T_{-+}(x^+, x^-),$$

which we can use to derive the transformation of the general light-ray operator

$$L_n(x^+) \rightarrow \int dx^- (ax^- + b)^{n+2} (cx^- + d)^{n-2} T_{-+}(x^+, x^-).$$

Expanding this expression as a series in $x^-$, we see that for $|n| > 2$ the operator $L_n$ mixes with an infinite number of other light-ray operators. However, the five “global” operators only mix with each other, as expected. The transformation of the five global operators under a general collinear transformation are

$$L_{-2} \rightarrow d^4 L_{-2} + 4cd^3 L_{-1} + 6c^2 d^2 L_0 + 4c^3 d L_1 + c^4 L_2,$$

$$L_{-1} \rightarrow bd^3 L_{-1} + d^2 (ad + 3bc)L_{-1} + 3cd(ad + bc)L_0 + c^2 (3ad + bc)L_1 + ac^3 L_2,$$

$$L_0 \rightarrow b^2 d^2 L_{-2} + (ad + 3bc) L_{-1} + (a^2 d^2 + 4abcd + b^2 c^2)L_0 + 2ac(ad + bc)L_1 + a^2 c^2 L_2,$$

$$L_1 \rightarrow b^3 d L_{-2} + b^2 (ad + 3bc)L_{-1} + 3ab(ad + bc)L_0 + a^2 (ad + 3bc)L_1 + a^3 c L_2,$$

$$L_2 \rightarrow b^4 L_{-2} + 4ab^3 L_{-1} + 6a^2 b^2 L_0 + 4a^3 b L_1 + a^4 L_2,$$

where all operators are functions of $x^+$ with $\vec{x}^\perp = 0$.

Things become more complicated if we move the light-ray operators to general $\vec{x}^\perp$, because the collinear transformations were specifically chosen to preserve light-rays along $\vec{x}^+ = 0$. At finite $\vec{x}^\perp$, the direction of the null line changes under general collinear transformations, such that $T_{-+}$ mixes with other components of the stress tensor, as we saw in eq. (3.26). At arbitrary $x$, we therefore obtain the general transformations

$$[J_{-1}, L_n(x)] = -(n + 2)L_{n-1}(x),$$

$$[J_0, L_n(x)] = -\left(n - \frac{1}{2} \vec{x}^\perp \cdot \vec{\partial} \right) L_n(x),$$

$$[J_1, L_n(x)] = -\left(n - 2 - \vec{x}^\perp \cdot \vec{\partial} \right) L_{n+1}(x) + |\vec{x}^\perp|^2 \partial_+ L_n(x)$$

$$+ 2 \int dx^- (x^-)^{n+2} \vec{x}^\perp \cdot \vec{T}_{-+}(x).$$
Finally, we can consider the action of the generator $\bar{J}_0$, obtaining

$$[\bar{J}_0, L_n(x)] = \left( \bar{h}_T + x^+ \partial_+ + \frac{1}{2} \vec{x}^\perp \cdot \vec{\partial}_\perp \right) L_n(x).$$

Unsurprisingly, every light-ray operator has the same collinear twist $\bar{h}_T = 1$, since the raising and lowering operators commute with $\bar{J}_0$.

### 3.4 Two- and three-point functions and the tale of integration

In this section, we would like to gain some intuition for the behaviour of these light-ray operators. In addition, we want to explain with the simplest examples how to perform the light-ray integral that converts $(x^-)^{n+2} T_{- -}$ into a light-ray operator $L_n$. The easiest playgrounds to probe this question are the two- and three-point functions:

$$\langle T_{- -}(x_1) T_{- -}(x_2) \rangle, \quad \langle O(x_1) T_{- -}(x_2) O(x_3) \rangle,$$

whose structure is fixed by conformal symmetry. In this fairly simple setup, we can understand how light-ray operators act on the vacuum and on scalar operators, as well as gather information about the candidate central charge that a possible algebra could have.

#### 3.4.1 Two-point functions of light-ray operators

The stress-tensor two-point function in $d$ dimensions takes the following form [40]

$$\langle T_{\mu \nu}(x) T_{\rho \sigma}(0) \rangle = \frac{c_T}{x^{2d}} \left[ \frac{1}{2} I_{\mu \rho}(x) I_{\nu \sigma}(x) + I_{\mu \sigma}(x) I_{\nu \rho}(x) \right] - \frac{1}{d} \eta_{\mu \rho} \eta_{\nu \sigma},$$

where we have defined

$$I_{\mu \nu}(x) \equiv \eta_{\mu \nu} - 2 \frac{x_{\mu} x_{\nu}}{x^2}.$$  

To compute correlators involving light-ray operators, we are specifically interested in the component $T_{- -}$ in $d = 4$, which has the simple correlator

$$\langle T_{- -}(x) T_{- -}(0) \rangle = 4 c_T \frac{(x^-)^4}{x^{12}} = \frac{c_T}{4} \frac{(x^+)^4}{x^{12}}.$$  

### Two-point function involving global operators

Using this two-point function, we can integrate over one of the positions, weighted by a function $f(x^-)$ which we first take to be holomorphic up to possible poles at infinity, which are outside the contour of integration due to the regularization procedure (3.18).
3. On the Stress Tensor Light-ray Operator Algebra

To ensure that the operators are properly ordered, we use the $i\epsilon$ prescription given in (3.13), namely

$$x_{12}^\pm \rightarrow x_{12}^\pm - i\epsilon.$$  \hspace{1cm} (3.40)

We now want to compute

$$\langle \mathcal{E}_f(x_1) T_{\ldots}(x_2) \rangle \equiv \int_{-\infty}^{\infty} dx^-_1 f(x^-_1) \langle T_{\ldots}(x_1) T_{\ldots}(x_2) \rangle$$

$$= \frac{c_T}{4} \int_{-\infty}^{\infty} dx^-_1 f(x^-_1) \frac{(x_{12}^+)^4}{-\left(x_{12}^- - i\epsilon\right)\left(x_{12}^- - i\epsilon\right) + |\vec{x}_{12}|^2}.$$  \hspace{1cm} (3.41)

The integral over $x^-_1$ can be evaluated using Cauchy’s theorem (recall that the integral has been transformed into a contour integral by the limiting procedure described in section 3.2). There is a sixth-order pole for $x^-_1$, located at

$$x^-_1 = x^-_2 + \frac{|\vec{x}_{12}|^2}{x^-_{12}} + i\epsilon \equiv x^-_{2,1} + i\epsilon,$$  \hspace{1cm} (3.42)

from which we obtain

$$\langle \mathcal{E}_f(x_1) T_{\ldots}(x_2) \rangle = \frac{(2\pi i)}{20} \frac{f^{(5)}(x^-_{2,1})}{(x_{12}^+)^2}.$$  \hspace{1cm} (3.43)

If we instead integrate over the position $x^-_2$ of the second operator, there is no pole in the contour and we obtain

$$\langle T_{\ldots}(x_1) \mathcal{E}_f(x_2) \rangle = 0.$$  \hspace{1cm} (3.44)

The appearance of a fifth derivative in (3.43) is rather important. From this, we immediately see that the five “global” operators that form a finite-dimensional representation of the collinear subalgebra annihilate the vacuum when acting both to the left and to the right

$$\langle L_n(x_1) T_{\ldots}(x_2) \rangle = \langle T_{\ldots}(x_1) L_n(x_2) \rangle = 0, \quad (|n| \leq 2).$$  \hspace{1cm} (3.45)

This observation follows directly from the fact that the functions

$$f(x^-) \in \{(x^-)^0, (x^-)^1, (x^-)^2, (x^-)^3, (x^-)^4\},$$  \hspace{1cm} (3.46)

have a vanishing fifth derivative. For these five functions, there is no pole at infinity in the two-point function, such that we could have closed the integration contour in either direction, without the need for our regularization procedure.

**Two-point function involving non-global $L_n$**

We now turn to the remaining $L_n$ operators with $|n| > 2$. Let’s start with the operators with higher powers of $x^-$, i.e. $n \geq 3$. Since the only poles in this choice of $f(x^-)$ are at infinity, the procedure is exactly the same as for the global operators, and we find

$$\langle T_{\ldots}(x_1) L_n(x_2) \rangle = 0,$$  \hspace{1cm} (3.47)

$$\langle L_n(x_1) T_{\ldots}(x_2) \rangle = (n + 2)(n + 1)n(n - 1)(n - 2) \frac{i\pi c_T(x^-_{2,1})^{n-3}}{240(x_{12}^+)^2}.$$  \hspace{1cm} (3.48)
3.4. Two- and three-point functions and the tale of integration

These $L_n$ therefore act like annihilation operators on the vacuum. Note that because the integrands contained poles at infinity, unlike the case for the global operators, it was necessary to implement our regularization procedure in evaluating these expressions.

For the operators $L_{-n}$ with $n \geq 3$, the story is slightly more involved. The integral that we need to perform is given by

$$\langle T_{-n}(x_1)L_{-n}(x_2) \rangle = \frac{c_T}{4} \int_{-\infty}^{\infty} dx_2 \frac{(x_{12}^+)^4}{(x_2 - i\epsilon)^{n-2}[-(x_{12}^+ - i\epsilon)(x_{12}^- - i\epsilon) + |x_{12}^\perp|^2]^6}. \quad (3.49)$$

The new feature is that the integrand in (3.49) now has two poles, one of which is introduced by the function $f$ itself. They are located at

$$x_2^- \rightarrow i\epsilon, \quad x_2^+ \rightarrow x_{1,2}^- - i\epsilon. \quad (3.50)$$

One is in the upper half-plane while the other is in the lower half-plane, so only a single pole contributes to the contour integral. We obtain

$$\langle T_{-n}(x_1)L_{-n}(x_2) \rangle = (n+2)(n+1)n(n-1)(n-2) \frac{i\pi c_T}{240(x_{12}^2)^2(x_{12}^-)^{n+3}} \quad (3.51)$$

which implies that $L_{-n}$ does not annihilate the vacuum when acting to the right.

For the opposite ordering, both poles are now in the upper half-plane and the integral thus vanishes

$$\langle L_{-n}(x_1)T_{-n}(x_2) \rangle = 0. \quad (3.52)$$

The operators $L_{-n}$ for $n \geq 3$ therefore act as creation operators on the vacuum.

3.4.2 What about the central charge?

We now consider the two-point function between two light-ray operators. From the discussion above, it is clear that the only non-vanishing two-point function is

$$\langle L_m(x_1)L_n(x_2) \rangle, \quad (3.53)$$

with $m > 2$ and $n < -2$. For notational simplicity, we can relabel $n \rightarrow -n$ such that both $m > 2$ and $n > 2$. The integral to perform is then

$$\langle L_m(x_1)L_{-n}(x_2) \rangle = \frac{c_T}{4} \int_{-\infty}^{\infty} dx_1 dx_2 \frac{(x_1^+ - 2i\epsilon)^{m+2}}{(x_2 - i\epsilon)^{n-2}[-(x_1^+ - i\epsilon)(x_2^- - i\epsilon) + |x_{12}^\perp|^2]^6}. \quad (3.54)$$

The easiest approach is to evaluate the $x_1^+$ integral first to obtain (3.48) and then perform the $x_2^-$ integral. We find

$$\langle L_m(x_1)L_{-n}(x_2) \rangle = (2\pi i)^2 \frac{c_T \Gamma(n-m)}{480 \Gamma(n-2)\Gamma(-m-2)} \frac{(-1)^n |x_{12}^\perp|^{2(m-n)}}{(x_2^- - i\epsilon)^{m-n+2}}. \quad (3.55)$$
3. On the Stress Tensor Light-ray Operator Algebra

which is symmetric under the simultaneous exchange \( m \leftrightarrow -n \) and \( x_1 \leftrightarrow x_2 \), and vanishes when \( m \leq 2 \) or \( n \leq 2 \). Notice that this implies that these operators are not an orthogonal basis for the algebra.

This correlator is also the commutator, because the reversed ordering vanishes due to (3.47). We can thus look at the case where \( m = n \) to probe a possible central charge in the algebra,

\[
\langle [L_m(x_1), L_{-m}(x_2)] \rangle = -\frac{(m + 2)(m + 1)m(m - 1)(m - 2)\pi^2}{120(x_1^+ - i\epsilon)^2} c_T. \tag{3.56}
\]

To probe the algebra, one would specifically want to consider the case \( x_1^+ = x_2^+ \). However, (3.56) diverges once we further take the \( \epsilon \to 0 \) limit. At face value, this would suggest that the central charge of a putative algebra is infinity, which matches the observations made in [107, 191]. An important remark is that this divergent central term is not strictly speaking the one that one would guess by the form of the proposed Witt algebra discussed in [107]. That term is forbidden by the collinear conformal group. Note however that a term of the form (3.56) has appeared before in [181].

One may try to regularize the operators to obtain a finite answer, but we will not attempt do so here (see [181] for a discussion of such an approach). We come back to this issue briefly in the discussion, see section 3.9.

3.4.3 Three-point functions

Three-point functions are also fixed by conformal symmetry (up to OPE coefficients), so we can compute three-point functions involving light-ray operators without specifying a CFT (unlike four-point functions, which will be the focus of the subsequent sections).

In four dimensions, the three-point function between the stress-energy tensor \( T_{-} \) and a general scalar operator \( O \) takes the form [40]

\[
\langle O(x_1)T_{-}(x_2)O(x_3) \rangle = -\frac{\Delta}{6\pi^2} \frac{1}{x_1^{2\Delta-2}} \left( \frac{(x_{12}^+)^2}{x_{12}^6x_{23}^2} + 2x_{12}^+x_{23}^+ + \frac{(x_{23}^+)^2}{x_{12}^6x_{23}^6} \right), \tag{3.57}
\]

where we have normalized the operator \( O \) such that

\[
\langle O(x)O(0) \rangle = \frac{1}{x^{2\Delta}}. \tag{3.58}
\]

We can now integrate (3.57) to obtain three-point functions involving light-ray operators.

\[
\langle O\mathcal{E}_f\mathcal{O} \rangle
\]

Let’s first consider a general function \( f(x^-) \) to obtain the generalized ANEC correlator

\[
\langle O(x_1)\mathcal{E}_f(x_2)O(x_3) \rangle \equiv \int_{-\infty}^{\infty} dx_2^- f(x_2^-) \langle O(x_1)T_{-}(x_2)O(x_3) \rangle, \tag{3.59}
\]

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where we assume for now that the inclusion of the function \( f \) does not introduce new singularities in the integrand. There are therefore two poles in the \( x^- \) plane:

\[
x^-_1 = x^-_{1,2} - i\epsilon, \quad x^-_2 = x^-_{3,2} + i\epsilon.
\] (3.60)

Using the integrals from appendix B.1, we can then evaluate this expression, obtaining either the residue of the pole in the lower half-plane,

\[
\langle \mathcal{O}(x_1) \mathcal{E}_f(x_2) \mathcal{O}(x_3) \rangle = -\frac{i\Delta}{6\pi} \frac{1}{x^-_{12}x^+_{23}x^-_{13}} \left[ \frac{f''(x^-_{1,2})}{(x^-_{1,2} - x^-_{3,2})} - 6 \frac{f'(x^-_{1,2})}{(x^-_{1,2} - x^-_{3,2})^2} + 12 \frac{f(x^-_{1,2})}{(x^-_{1,2} - x^-_{3,2})^3} \right].
\] (3.61)

or the residue of the pole in the upper half-plane,

\[
\langle \mathcal{O}(x_1) \mathcal{E}_f(x_2) \mathcal{O}(x_3) \rangle = -\frac{i\Delta}{6\pi} \frac{1}{x^-_{12}x^+_{23}x^-_{13}} \left[ \frac{f''(x^-_{3,2})}{(x^-_{1,2} - x^-_{3,2})} + 6 \frac{f'(x^-_{3,2})}{(x^-_{1,2} - x^-_{3,2})^2} + 12 \frac{f(x^-_{3,2})}{(x^-_{1,2} - x^-_{3,2})^3} \right].
\] (3.62)

Note that these two expressions are equivalent *only* for functions \( f \) which introduce no additional singularities. As we will now discuss, for our set of functions \( f(x^-) = (x^-)^{n+2} \) this specifically corresponds to the case \(-2 \leq n \leq 2\), which are the five global operators.

\[ \langle \mathcal{O} L_n \mathcal{O} \rangle \]

Let’s now focus on the operators \( L_n \) with \( n \geq 3 \). These functions introduce a pole at infinity, in which case we must follow our regularization procedure (3.18) and close in the upper half-plane. We can then use eq. (3.62) for the case \( f(x^-) = (x^-)^{n+2} \), obtaining

\[
\langle \mathcal{O}(x_1) L_n(x_2) \mathcal{O}(x_3) \rangle = -\frac{i\Delta}{6\pi} \frac{(x^-_{3,2})^n}{x^-_{12}x^+_{23}(x^-_{1,2} - x^-_{3,2})^3x^-_{13}} \left[ (n+1)(n+2)(x^-_{1,2})^2 - 2(n+2)(n-2)x^-_{1,2}x^-_{3,2} + (n-1)(n-2)(x^-_{3,2})^2 \right].
\] (3.63)

Next, we can consider \( L_{-n} \) with \( n \geq 3 \). With our \( i\epsilon \) prescription, these functions introduce a new pole at \( x^- = i\epsilon \). We therefore must include the contribution of this pole when closing in the upper half-plane. Equivalently, we can evaluate the contour in the lower half-plane, as this function introduces no pole at infinity. Either way, we obtain the resulting expression

\[
\langle \mathcal{O}(x_1) L_{-n}(x_2) \mathcal{O}(x_3) \rangle = -\frac{i\Delta}{6\pi} \frac{(x^-_{1,2})^{-n}}{x^-_{12}x^+_{23}(x^-_{1,2} - x^-_{3,2})^3x^-_{13}} \left[ (n+1)(n+2)(x^-_{1,2})^2 - 2(n+2)(n-2)x^-_{1,2}x^-_{3,2} + (n-1)(n-2)(x^-_{3,2})^2 \right].
\] (3.64)
Finally, we have the global operators $L_n$ with $|n| \leq 2$. These operators do not introduce either a pole at zero or a pole at infinity for these three-point functions, in which case we can safely close the contour in either direction. Indeed, one can explicitly check that (3.63) and (3.64) agree for $L_n$ with

$$n = \{ -2, -1, 0, 1, 2 \}. \quad (3.65)$$

For reference, let us explicitly write out the resulting expressions for these five special operators in scalar three-point functions:

$$\langle O(x_1)L_{-2}(x_2)O(x_3) \rangle = -\frac{2i\Delta}{\pi} \frac{1}{x_{12}^+x_{23}^+(x_{1,2}^- - x_{3,2}^-)^3x_{13}^{2\Delta - 2}},$$

$$\langle O(x_1)L_{-1}(x_2)O(x_3) \rangle = -\frac{i\Delta}{\pi} \frac{x_{1,2}^- + x_{3,2}^-}{x_{12}^+x_{23}^+(x_{1,2}^- - x_{3,2}^-)^3x_{13}^{2\Delta - 2}},$$

$$\langle O(x_1)L_0(x_2)O(x_3) \rangle = -\frac{i\Delta}{3\pi} \frac{(x_{1,2}^-)^2 + 4x_{1,2}^-x_{3,2}^- + (x_{3,2}^-)^2}{x_{12}^+x_{23}^+(x_{1,2}^- - x_{3,2}^-)^3x_{13}^{2\Delta - 2}},$$

$$\langle O(x_1)L_1(x_2)O(x_3) \rangle = -\frac{i\Delta}{\pi} \frac{x_{1,2}^-x_{3,2}^-(x_{1,2}^- + x_{3,2}^-)}{x_{12}^+x_{23}^+(x_{1,2}^- - x_{3,2}^-)^3x_{13}^{2\Delta - 2}},$$

$$\langle O(x_1)L_2(x_2)O(x_3) \rangle = -\frac{2i\Delta}{\pi} \frac{(x_{1,2}^-)^2}{x_{12}^+x_{23}^+(x_{1,2}^- - x_{3,2}^-)^3x_{13}^{2\Delta - 2}}. \quad (3.66)$$

Because these operators form a finite-dimensional representation of the collinear subgroup, their correlation functions are related by the action of the generators $J$, in (3.35).

For the rest of this chapter, we will leave the operators with $|n| > 2$ aside and only focus on the five global operators. From now on, when we write a function $f$ we will therefore implicitly mean that the function is $f(x^-) = (x^-)^{n+2}$ for $-2 \leq n \leq 2$.

### 3.5 Four-point functions in free field theory

So far, we have only considered properties of light-ray operators which are fixed by conformal symmetry. In the coming sections, we will now turn to explicit computations of four-point functions involving two generalized ANEC operators. Since such correlators are highly theory-dependent, we must specify the CFT in which we want to compute them. We will start in free field theory, before moving to holographic CFTs in section 3.7.

In $d = 4$, a free field $\phi$ has dimension $\Delta = 1$, whose two-point function we will normalize as in (3.58). The associated Wightman functions can all be constructed via Wick

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6. While this normalization is natural from a CFT perspective, it differs from the usual convention for a free scalar by a factor of $\frac{1}{4\pi^2}$. 

3.5. Four-point functions in free field theory

contraction, and involve derivatives of two-point functions of the field \( \phi \). It will therefore be useful to consider the building block two-point function

\[
\langle \partial^n \phi(x) \partial^m \phi(y) \rangle = \frac{(-1)^n (m + n)! (x^+ - y^+)^{m+n}}{(x - y)^{2(m+n+1)}}. \tag{3.67}
\]

In free-field theory, the stress-energy tensor is given by

\[
T_{\mu\nu}(x) = \frac{1}{6\pi^2} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{24\pi^2} \eta_{\mu\nu} \partial_\sigma \phi(x) \partial^\sigma \phi(x) - \frac{1}{12\pi^2} \phi(x) \partial_\mu \partial_\nu \phi(x). \tag{3.68}
\]

We will mainly be interested in the component \( T_{--} \),

\[
T_{--}(x) = \frac{1}{6\pi^2} \left[ (\partial^2 \phi(x))^2 - \frac{1}{2} \phi(x) \partial^2 \phi(x) \right], \tag{3.69}
\]

which has been normalized according to the three-point function (3.57). We are interested in computing the following correlation functions

\[
\langle \phi(x_1) E_f(x_2) E_g(x_3) \phi(x_4) \rangle = \int dx_{\bar{2}} dx_{\bar{3}} f(x_{\bar{2}}) g(x_{\bar{3}}) \langle \phi(x_1) T_{--}(x_2) T_{--}(x_3) \phi(x_4) \rangle, \tag{3.70}
\]

for

\[
f(x^-), g(x^-) \in \{(x^-)^0, (x^-)^1, (x^-)^2, (x^-)^3, (x^-)^4\}, \tag{3.71}
\]

corresponding to the five global operators \( \{L_2, L_{--}, L_0, L_1, L_2\} \). Note that for any correlation function in this class, the integrand never has poles at zero or infinity. We can therefore safely close each of the integration contours in either the lower or upper half-plane, without worrying about the regularization procedure (3.18). Moreover, the order in which the integrals are performed also does not affect the resulting expressions.

To compute (3.70), we first need the correlator \( \langle \phi T_{--} T_{--} \phi \rangle \). Using (3.69), it is a straightforward Wick contraction exercise to compute the full correlator, and the full expression is shown in appendix B.2. However, it turns out that not all Wick contractions are relevant for computing the correlation functions of light-ray operators, since several of them will vanish once the integrals are evaluated. The simplest way to see this is to close both contours outwards, which we can represent as

\[
\langle \phi(x_1) E_f(x_2) E_g(x_3) \phi(x_4) \rangle = \int dx_{\bar{2}} dx_{\bar{3}} f(x_{\bar{2}}) g(x_{\bar{3}}) \langle \phi(x_1) T_{--}(x_2) T_{--}(x_3) \phi(x_4) \rangle,
\]

where this notation means that we integrate \( x_{\bar{2}} \) by picking up the singularity when \( x_2 \to x_1 \), and integrate \( x_{\bar{3}} \) by picking up the singularity when \( x_3 \to x_4 \). Because of this, it is clear that the only terms that will contribute to the final result are terms in \( \langle \phi T_{--} T_{--} \phi \rangle \) that have a denominator of the form \( x_{12}^a x_{34}^b \) for positive \( a \) and \( b \). All other terms will vanish once we integrate. The different contractions of this four-point function lead to three possible topologies, shown schematically in figure 3.2. Based on the preceding argument, it is clear that only the first topology contributes upon integration.
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\[ \langle \phi(x_1) T^{--}(x_2) T^{--}(x_3) \phi(x_4) \rangle \]

\[ \neq 1 \]

\[ 1 \]

\[ 2 \]

\[ 3 \]


\[ \phi(x_1) \]
\[ \phi(x_4) \]
\[ \phi(x_1) \]
\[ \phi(x_4) \]

Figure 3.2 – The three different topologies of Wick contractions that appear in \( \langle \phi(x_1) T^{--}(x_2) T^{--}(x_3) \phi(x_4) \rangle \), where we wrote \( T(x_2) \equiv T^{--}(x_2) \). The lines indicates how the fields are contracted since \( T^{--}(x) \) is bilinear in \( \phi(x) \). Only the first topology contributes to the integrated correlators for the contours at hand.

To compute a four-point function with light-ray operators, we can therefore concentrate solely on the subset of the full correlator given by the Wick contractions appearing in the first topology. This is explicitly given by

\[
\langle \phi(x_1) T^{--}(x_2) T^{--}(x_3) \phi(x_4) \rangle
\]

\[
\supset 1 \]

\[
1 \]

\[
2 \]

\[
3 \]

\[
\frac{1}{36\pi^4} (x_{23}^+)^4 + \frac{1}{9\pi^4} x_{23}^4 (x_{23}^+)^2 \left( \frac{x_{12}^+}{x_{12}^+} + \frac{x_{23}^+}{x_{23}^+} \right) \left( \frac{x_{23}^+}{x_{23}^+} + \frac{x_{34}^+}{x_{34}^+} \right)
\]

\[
+ \frac{1}{36\pi^4} x_{23}^8 \left( \frac{x_{12}^+}{x_{12}^+} + \frac{x_{23}^+}{x_{23}^+} \right)^2 \left( \frac{x_{23}^+}{x_{23}^+} + \frac{x_{34}^+}{x_{34}^+} \right)^2.
\]

(3.72)

We are now ready to perform the integrals and compute the four-point function \( \langle \phi E_f E_g \phi \rangle \).

3.5.1 The four-point function \( \langle \phi E_f E_g \phi \rangle \) and the commutator

In this section, we compute both the four-point correlator with two distinct light-ray operators \( E_f \) and \( E_g \), as well as the resulting commutator. The relevant part of the local four-point function is given by (3.72) and we now need to integrate it, picking up poles as \( x_2 \to x_1 \) and \( x_3 \to x_4 \).

The very first term of (3.72) has a single pole in both \( x_2^- \) and \( x_3^- \). The next term has both a single pole and a double pole for both coordinates, while the last term has poles up to third order. These integrals are displayed in full glory in appendix B.2 (cf eqs. (B.9) to (B.11)). Combining the results of all the integrals together, we then obtain the full
3.5. Four-point functions in free field theory

light-ray correlator,

\[
\langle \phi(x_1) \mathcal{E}_f(x_2) \mathcal{E}_g(x_3) \phi(x_4) \rangle = (2\pi)^2 \left( \frac{6}{\pi^4} \frac{f(x_{1,2}^-)g(x_{4,3}^-)(x_{23}^+ - ie)^4}{(x_{1,2}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^5} \right.
\]

\[
+ \frac{3}{2\pi^4} \left( \frac{f'(x_{1,2}^-)g(x_{4,3}^-) - f(x_{1,2}^-)g'(x_{4,3}^-)}{(x_{12}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^4} \right)
\]

\[
- \frac{1}{2\pi^4} \left( \frac{f'(x_{1,2}^-)g'(x_{4,3}^-)}{(x_{12}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^3} \right)
\]

\[
+ \frac{1}{12\pi^4} \left( \frac{f''(x_{1,2}^-)g(x_{4,3}^-) + f(x_{1,2}^-)g''(x_{4,3}^-)}{(x_{12}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^3} \right)
\]

\[
- \frac{1}{24\pi^4} \left( \frac{f''(x_{1,2}^-)g'(x_{4,3}^-) - f'(x_{1,2}^-)g''(x_{4,3}^-)}{(x_{12}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^2} \right)
\]

\[
+ \frac{1}{144\pi^4} \left( \frac{f''(x_{1,2}^-)g''(x_{4,3}^-)}{(x_{12}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]} \right). \tag{3.73}
\]

In expression (3.73) we have anticipated which factors of \(ie\) will be necessary to compute the commutator and have suppressed all factors of \(ie\) that are irrelevant. To compute the correlator with the opposite ordering for the light-ray operators, we simply take (3.73) and replace \(f \leftrightarrow g\) and \(x_2 \leftrightarrow x_3\). We can then take the difference of these two expressions to obtain the commutator.

Because we are interested in the case where both light-ray operators are on the same null-plane, we need to be particularly careful with the \(ie\) prescription between \(x_2\) and \(x_3\) when we subtract correlators of the form (3.73). To simplify the analysis, we will study the various terms independently, starting with the term that has no derivatives acting on the functions \(f(x^-)\) and \(g(x^-)\). We denote this specific term as

\[
\langle \phi[\mathcal{E}_f(x_2), \mathcal{E}_g(x_3)]\phi \rangle_{fg} \sim \frac{f(x_{1,2}^-)g(x_{4,3}^-)(x_{23}^+ - ie)^4}{(x_{1,2}^+(x_{34}^+)[-(x_{23}^+ - ie)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^5}
\]

\[
- \frac{f(x_{4,2}^-)g(x_{1,3}^-)(x_{23}^+ + ie)^4}{(x_{1,3}^+(x_{34}^+)[(x_{23}^+ + ie)(x_{1,3}^- - x_{4,2}^-) + |\vec{x}_{23}^+|^2]^5}. \tag{3.74}
\]

If we did not include the correct \(ie\) prescription, then the numerator would naively vanish for \(x_{23}^+ = 0\) and one would conclude that this contribution is zero. Instead, with the

---

7. To reinstate all the \(ie\) where needed, one just needs to perform the following replacements in (3.73): \(x_{1,2}^- \rightarrow x_{1,2}^- - ie\), \(x_{4,3}^- \rightarrow x_{4,3}^- + ie\), \(x_{12}^+ \rightarrow x_{12}^+ - ie\), and \(x_{34}^- \rightarrow x_{34}^- - ie\).
correct \(i\epsilon\) prescription, we obtain the following expression in the limit where \(x_2^+ = x_3^+\),

\[
\langle \phi[\mathcal{E}_f(x_2), \mathcal{E}_g(x_3)]\phi \rangle_{fg} \to \frac{(i\epsilon)^4}{x_{12}^+ x_{24}^+} \left( \frac{f(x_{1,2})g(x_{4,3})}{[|x_{23}^+|^2 + i\epsilon(x_{1,2} - x_{4,3})]^5} - \frac{f(x_{4,2})g(x_{1,2})}{[|x_{23}^+|^2 + i\epsilon(x_{1,3} - x_{4,2})]^5} \right).
\]

(3.75)

Looking at this expression, we see that it vanishes in the limit \(\epsilon \to 0\) provided \(|\vec{x}_{23}^+|^2 \neq 0\). However, if \(|\vec{x}_{23}^+|^2 = 0\), then this expression actually diverges as \(\epsilon \to 0\). It is hence clear that this expression should be thought of as a distribution proportional to a delta function in the transverse separation between the two light-ray operators: \(\delta^{(2)}(\vec{x}_{23}^+)\). To extract this delta function contribution, we need the following relation

\[
\lim_{\epsilon \to 0} \frac{(i\epsilon)^{a-1}}{|[\vec{x}_{23}^+]^2 + i\epsilon y|^a} = \frac{\pi}{(a-1)y^{a-1}} \delta^{(2)}(\vec{x}_{23}^+),
\]

(3.76)

which is valid provided \(a \geq 2\). The derivation of this result is presented in appendix B.3. Using (3.76), the term (3.78) hence becomes

\[
\langle \phi[\mathcal{E}_f(x_2), \mathcal{E}_g(x_3)]\phi \rangle_{fg} \sim \frac{\pi \delta^{(2)}(\vec{x}_{23}^+)}{4x_{12}^+ x_{24}^+ (x_{1,2} - x_{4,2})^4} \left( f(x_{1,2})g(x_{4,2}) - f(x_{4,2})g(x_{1,2}) \right).
\]

(3.77)

Based on this analysis, we can now compute the general commutator contribution, which is valid for \(j \leq 3\) and \(0 \leq k \leq j\).

\[
\frac{(x_{23}^+)^{4-j} f(k)(x_{1,2}^-)g(j-k)(x_{4,3}^-)}{x_{12}^+ x_{34}^+ [-x_{23}^+(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^+|^2]^{5-j}} - \frac{(-x_{23}^+)^{4-j} f(j-k)(x_{4,2}^-)g(k)(x_{1,3}^-)}{x_{13}^+ x_{24}^+ [x_{23}^+(x_{1,3}^- - x_{4,2}^-) + |\vec{x}_{23}^+|^2]^{5-j}}
\]

\[
= \frac{(-1)^j \pi \delta^{(2)}(\vec{x}_{23}^+)}{(4 - j)x_{12}^+ x_{24}^+ (x_{1,2}^- - x_{4,2}^-)^4} \left( f(k)(x_{1,2}^-)g(j-k)(x_{4,2}^-) - f(j-k)(x_{4,2}^-)g(k)(x_{1,2}^-) \right).
\]

(3.78)

Note that the commutator only has support when \(|\vec{x}_{23}^+|^2 = 0\), except for when \(j = 4\), which has a finite contribution when \(|\vec{x}_{23}^+|^2 \neq 0\), and is much more subtle, as we will discuss below.

8. See appendix B.4.4 for a different way of extracting the delta function, by integrating over the transverse coordinates.
We can now present the full commutator, which is given by

\[
\langle \phi(x_1)|\mathcal{E}_f(x_2),\mathcal{E}_g(x_3)|\phi(x_4)\rangle = \frac{1}{\pi x_{12}^+ x_{24}^+} \delta^{(2)}(\vec{x}_{23}^\perp) \left[ -6 \frac{f_{1,2} g_{4,2} - f_{4,2} g_{1,2}}{(x_{1,2}^-, x_{4,2}^-)^4} + 2 \frac{(f_{1,2} g_{4,2} - f_{4,2} g_{1,2}) - (f_{1,2} g_{4,2} - f_{4,2} g_{1,2})}{(x_{1,2}^-, x_{4,2}^-)^3} - \frac{1}{6} \frac{(f_{1,2} g_{4,2} - f_{4,2} g_{1,2}) - (f_{1,2} g_{4,2} - f_{4,2} g_{1,2})}{x_{1,2}^- - x_{4,2}^-} + \frac{1}{36\pi^2 x_{12}^+ x_{24}^+} \left[ -\frac{f_{1,2} g_{4,3}}{(x_{23}^+ - \epsilon)(x_{1,2}^- - x_{4,3}^-) + |\vec{x}_{23}^\perp|^2} - \frac{f_{4,2} g_{1,3}}{((x_{23}^+ + \epsilon)(x_{1,3}^- - x_{4,2}^-) + |\vec{x}_{23}^\perp|^2} \right] \right],
\]

where we have defined the shorthand notation,

\[ f_{i,j} \equiv f(x_{i,j}^-), \quad g_{i,j} \equiv g(x_{i,j}^-). \]

In (3.79), the two terms displayed on the last line which are proportional to \( f'' g'' \), do not admit a representation as a distribution proportional to a delta function. They correspond to (3.76) with \( a = 1 \), whose numerator is finite. In appendix B.3, we demonstrate that this contribution is not integrable and explain why we cannot write a meaningful transverse function \( \delta^{(2)}(\vec{x}_{23}^\perp) \). Nevertheless, these two terms admit a well-defined limit when \( \epsilon \to 0 \), provided \( |\vec{x}_{23}^\perp|^2 \neq 0 \).

Before analyzing this finite-separation contribution, let us first focus our attention on computations where the \( f'' g'' \) term does not contribute. This is the case for all computations where \( \mathcal{E}_f \) or \( \mathcal{E}_g \) are one of the three operators \( \{L_{-2}, L_{-1}, L_0\} \). This subset of operators is special in the sense that they are the local versions of some of the charge operators of the conformal algebra, and are thus constrained by the associated Ward identities [106].

### 3.5.2 The algebra of light-ray operators

We have now obtained the general commutator (3.79) valid for the operators:

\[
\mathcal{E}_f, \mathcal{E}_g \in \{L_{-2}, L_{-1}, L_0, L_1, L_2\}.
\]

In addition, we have seen that there is a subtlety involving a finite piece at spacelike separation when \( f'' g'' \) does not vanish, namely for certain commutators involving \( L_1 \) and \( L_2 \). Let us start by discussing the algebra in the absence of this finite piece.
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Algebra in the absence of the finite separation term

We can use (3.79) to directly evaluate commutators. The simplest scenario is the commutator of two ANEC operators

$$\langle \phi(x_1)[L_2(x_2), L_2(x_3)] \phi(x_4) \rangle = 0,$$

(3.82)

which in fact vanishes not only in free field theory but in arbitrary CFTs [108] leading to CFT sum rules with important consequences for large N theories [1, 108].

Next, consider the commutator $[L_2, L_1]$. Using (3.79), we find

$$\langle \phi(x_1)[L_1(x_2), L_2(x_3)] \phi(x_4) \rangle = -\frac{2}{\pi} \frac{1}{x_{1,2}^+ x_{2,3}^+(x_{1,2}^+ - x_{4,2}^-)^3} \delta^{(2)}(\vec{x}_{23}^+) ,$$

(3.83)

which has a nonzero contact term. Note that here we specifically assumed that $x_2^+ = x_3^+$, but kept the transverse components $\vec{x}_{1,2}^+$, $\vec{x}_{2,3}^+$, and $\vec{x}_{4,2}^+$ arbitrary. Up to a prefactor, the function multiplying the delta function in $\vec{x}_{23}^+$ turns out to be the one-point function on the ANEC operator, namely

$$\langle \phi(x_1)[L_1(x_2), L_2(x_3)] \phi(x_4) \rangle = -i \delta^{(2)}(\vec{x}_{23}^+) \langle \phi(x_1) L_2(x_2) \phi(x_4) \rangle .$$

(3.84)

This easily generalizes to many other commutators as we summarize below (see appendix B.4 for details):

$$[L_2(x), L_2(0)] = 0 \quad [L_1(x), L_1(0)] = 0$$

$$[L_1(x), L_2(0)] = -i \delta^{(2)}(\vec{x}_{1}^+) L_2(0) \quad [L_0(x), L_1(0)] = -i \delta^{(2)}(\vec{x}_{1}^+) L_0(0)$$

$$[L_0(x), L_2(0)] = -2i \delta^{(2)}(\vec{x}_{1}^+) L_1(0) \quad [L_1(x), L_1(0)] = -2i \delta^{(2)}(\vec{x}_{1}^+) L_1(0)$$

$$[L_2(x), L_2(0)] = -3i \delta^{(2)}(\vec{x}_{1}^+) L_0(0) \quad [L_2(x), L_1(0)] = -3i \delta^{(2)}(\vec{x}_{1}^+) L_2(0)$$

$$[L_2(x), L_2(0)] = -4i \delta^{(2)}(\vec{x}_{1}^+) L_1(0) \quad [L_0(x), L_0(0)] = 0 .$$

(3.85)

These relations are not an accident, we are reproducing part of the Witt algebra mentioned in the introduction, namely

$$[L_m(\vec{x}_{1}^+), L_n(\vec{y}_{1}^+)] = -i \delta^{(2)}(\vec{x}_{1}^+ - \vec{y}_{1}^+)/(m - n) L_{m+n+1}(\vec{y}_{1}^+) ,$$

(3.86)

This algebra was advocated in [107], and some commutators were checked in [106] for free field theories for correlators evaluated on the same null-plane (i.e. $x^+ = y^+$). We have now reproduced those results, and generated many more terms in the algebra.\footnote{Note that in (3.86), it naively seems that the quantum numbers don’t match on both sides of the equation, but in fact they do because the transverse delta function $\delta^{(2)}(\vec{x}_{1}^+ - \vec{y}_{1}^+)$ carries the appropriate compensating dimension.}

So what becomes of the commutators not included in (3.85)? These are precisely the ones where the finite piece coming from $f'' g''$ does not vanish. As we will now see, they seem to present an obstruction for the algebra.
3.5. Four-point functions in free field theory

Commutators involving the finite separation term

Let’s finally discuss the commutators that involve a contribution at $|\vec{x}_{23}| \neq 0$. At finite separation we can use the general result derived in (3.79),

$$\langle \phi(x_1)[L_1(x_2), L_0(x_3)]\phi(x_4) \rangle = -\frac{x_{1,2}^- x_{4,2}^-}{3\pi^2 x_{12}^+ x_{24}^+ |\vec{x}_{23}|^2}. \quad (3.87)$$

This expression produces an ambiguity at $\vec{x}_{23} = 0$. The problem is that the finite separation commutator diverges in that limit as $|x|^2$, and this singularity cannot be integrated against arbitrary test-functions. It is therefore not possible to extract a delta function term at coincident points like for previous examples. This spells doom for the proposed operator algebra (3.86). We expand on this point in appendix B.3.

The breakdown of the algebra was already hinted at in [108], where it was argued that Wightman functions which do not converge uniformly under the lightlike integrals may possibly be regularized, but at the cost of potentially destroying commutation at spacelike separation. It is interesting that we find this obstruction at spacelike separations for an integral that does converge uniformly. We will return to this in the discussion section.

For completeness, we also give the other commutators. Restricting to $|\vec{x}_{23}| \neq 0$, where the finite separation contribution is well-behaved, we have

$$\langle \phi(x_1)[L_2(x_2), L_0(x_3)]\phi(x_4) \rangle = -\frac{2}{3\pi^2} \frac{(x_{1,2}^- x_{4,2}^-)(x_{1,2}^- + x_{4,2}^-)}{x_{12}^+ x_{24}^+ |\vec{x}_{23}|^2}, \quad (3.88)$$

$$\langle \phi(x_1)[L_2(x_2), L_1(x_3)]\phi(x_4) \rangle = \frac{2}{\pi^2} \frac{(x_{1,3}^- (x_{4,2}^-)^2 - (x_{1,2}^-)^2 x_{4,3}^-)}{\pi x_{12}^+ x_{24}^+ |\vec{x}_{23}|^2}. \quad (3.89)$$

Once again, we obtain non-integrable finite separation contributions.

The final two commutators are the diagonal ones, for which the delta-function contribution vanishes. The finite-separation piece is thus the only non-zero contribution,

$$\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4) \rangle = -\frac{1}{\pi^2} \frac{x_{1,2}^- x_{4,3}^- - x_{1,2}^- x_{1,3}^-}{x_{12}^+ x_{24}^+ |\vec{x}_{23}|^2}, \quad (3.90)$$

$$\langle \phi(x_1)[L_2(x_2), L_2(x_3)]\phi(x_4) \rangle = -\frac{4}{\pi^2} \frac{(x_{1,2}^-)^2 (x_{4,3}^-)^2 - (x_{1,2}^-)^2 (x_{1,3}^-)^2}{x_{12}^+ x_{24}^+ |\vec{x}_{23}|^2}. \quad (3.91)$$

The divergences are integrable in this case and there is no $|\vec{x}_{23}| = 0$ contribution.

The three off-diagonal commutators pose a serious challenge to any possible algebra of light-ray operators. Moreover, we would like to emphasize that these results did not depend on the choice of the external state: all commutators discussed in this section have finite contributions in other scalar states. For example, we can easily see that

$$\langle \phi^n(x_1)[L_1(x_2), L_1(x_3)]\phi^n(x_4) \rangle = n^2 \langle \phi^{n-1}(x_1)\phi^{n-1}(x_4) \rangle \langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4) \rangle,$$

which obviously also contains a finite separation term. In the next section, we demonstrate how to reproduce this contribution using the OPE in free field theory.
3. On the Stress Tensor Light-ray Operator Algebra

\( T_{-\cdot} \times \phi \) OPE in free field theory

The choice of contours from the previous section is such that the relevant OPE channel to study our correlators is \( T_{-\cdot}(x) \times \phi(y) \). In this section, we explore what becomes of this OPE under the null integral over \( x^{-} \) in free field theory.

Using the definition of \( T_{-\cdot}(x) \) in free field theory (3.69), we can look at the OPE of \( T_{-\cdot}(x) \) with \( \phi(y) \), focusing only on the terms that are proportional to the operator \( \phi \) itself. We find

\[
T_{-\cdot}(x)\phi(y) \sim \left[ \frac{1}{12\pi^2} (\partial_-^2 \phi)(x) - \frac{1}{12\pi^2} \phi \partial_-^2 \phi(x) \right] \phi(y) = \frac{1}{12\pi^2} \left( \frac{-2(x^+ - y^+)^2}{(x-y)^6} + \frac{4(x^+ - y^+)}{(x-y)^4} \partial_- - \frac{1}{(x-y)^2} \partial_-^2 \right) \phi(x). \tag{3.92}
\]

Note that this OPE is not written in the canonical way, because the resulting operator on the right-hand side is evaluated at position \( x \) instead of position \( y \). If we would like an OPE evaluated at \( \phi(y) \), then we still need to expand \( \phi(x) \) around the position \( y \). This implies that \( \phi(x) \) in (3.92) is already the resummed version of that Taylor expansion. Because of this, one should keep in mind that we are really keeping infinitely many descendants.

We can use the OPE (3.92) twice in \( \langle \phi(x_1)T_{-\cdot}(x_2)T_{-\cdot}(x_3)\phi(x_4) \rangle \), as \( x_2 \rightarrow x_1 \) and as \( x_3 \rightarrow x_4 \) to obtain a differential operator acting on the two-point function \( \langle \phi(x_2)\phi(x_3) \rangle \). This implies

\[
\langle \phi(x_1)T_{-\cdot}(x_2)T_{-\cdot}(x_3)\phi(x_4) \rangle = \frac{1}{36\pi^4} \left( \frac{(x_{21}^+)^2(x_{34}^+)^2}{x_{21}^6x_{34}^6} + \ldots \right) \langle \phi(x_2)\phi(x_3) \rangle, \tag{3.93}
\]

where we have written only the most singular term coming from the double OPE, as this is the only term that contributes to the finite separation term. We can then multiply (3.93) by \( (x_{2}^-)^3(x_3^-)^3 \) and perform the two integrals as \( x_2 \rightarrow x_1 \) and \( x_3 \rightarrow x_4 \), we arrive at \( \langle \phi(x_1)L_1(x_2)L_1(x_3)\phi(x_4) \rangle \). Once we send \( \epsilon \rightarrow 0 \), the contribution from (3.93) is

\[
\langle \phi(x_1)L_1(x_2)L_1(x_3)\phi(x_4) \rangle = -\frac{1}{\pi^2} \frac{x_{12}^1x_{43}^-}{x_{12}^+x_{24}^+|x_{23}^1|^2}, \tag{3.94}
\]

which is half the commutator from (3.90). The other half is obtained the same way, but for the other ordering by performing the OPE as \( x_3 \rightarrow x_1 \) and \( x_2 \rightarrow x_4 \).

In addition, we can ask how to reproduce these results in the other OPE channel, namely by fusing \( T_{-\cdot}(x) \) with \( T_{-\cdot}(y) \). This is what we will discuss in the next section.

3.6 The OPE of light-ray operators

In the previous section, we saw that in free field theory commutators involving the light-ray operators \( L_1 \) and \( L_2 \) do not vanish at finite transverse separation. We would now
like to understand whether this nonvanishing commutator can be understood as arising from the integrals of local operators $\mathcal{O}$ in the $T \times T$ OPE,

$$[L_n(x), L_n(y)] = \sum_{\mathcal{O}} \frac{1}{|\vec{x}^\perp - \vec{y}^\perp|^2|\alpha_{\mathcal{O}}|} \int dy^- f_{\mathcal{O}}(y^-) \mathcal{O}(y),$$  \hspace{1cm} (3.95)$$

where, for simplicity, we will focus on the diagonal case where the two generalized ANEC operators are the same.

The OPE of two light-ray operators has been studied several times in the literature [91, 101, 104, 179], but here we will largely not make direct contact with these general results, and instead choose to focus on the particular example of free field theory. It would be very interesting in future work to understand the structure of generalized ANEC commutators more broadly, using the technology developed in those works.

To begin, we need to first rephrase the calculation of the commutator $\langle \phi[L_n, L_n] \phi \rangle$ from the previous section in order to make the connection with the $T \times T$ OPE more manifest. There, we originally computed the commutator by evaluating

$$\langle \phi(x_1)[L_n(x_2), L_n(x_3)]\phi(x_4) \rangle = \int dx^-_2 dx^-_3 (x_2^-)^{n+2}(x_3^-)^{n+2}\langle \phi(x_1)T_--(x_2)T_--(x_3)\phi(x_4) \rangle$$

$$- \int dx^-_2 dx^-_3 (x_2^-)^{n+2}(x_3^-)^{n+2}\langle \phi(x_1)T_--(x_3)T_--(x_2)\phi(x_4) \rangle,$$

where the contraction symbols indicate which poles we considered while performing the integrals over $x^-_2$ and $x^-_3$ (for example, in the first line we integrate $x^-_2$ by picking up the OPE singularity at $x^2_{12} \to 0$). In order to obtain an expression of the form (3.95), we need to evaluate only one of these two integrals (for concreteness, $x^-_2$), such that we can write the commutator in terms of the single remaining light-ray integral (over $x^-_3$).

While the particular contraction shown above was useful in practice, it clearly makes use of the OPE in the $T \times \phi$ channel, obscuring any connection with the $T \times T$ OPE. Instead, we can choose to close the contour for $x^-_2$ in the upper half-plane for both orderings, represented by the contractions

$$\langle \phi(x_1)[L_n(x_2), L_n(x_3)]\phi(x_4) \rangle = \int dx^-_2 dx^-_3 (x_2^-)^{n+2}(x_3^-)^{n+2}\langle \phi(x_1)T_--(x_2)T_--(x_3)\phi(x_4) \rangle$$

$$- \int dx^-_2 dx^-_3 (x_2^-)^{n+2}(x_3^-)^{n+2}\langle \phi(x_1)T_--(x_3)T_--(x_2)\phi(x_4) \rangle,$$

\hspace{1cm} (3.96)

In the first term, we now need to pick up both OPE singularities in $x^2_{23}$ and $x^2_{24}$, while in the second term, we only need to pick up the singularity in $x^2_{24}$ (as before). One can check explicitly that in this correlator the two contributions from the $x^2_{24}$ OPE singularity
cancel when evaluating the commutator and we are left with \(^{10}\)

\[
\langle \phi(x_1)[L_n(x_2), L_n(x_3)]\phi(x_4) \rangle = \int dx_3^- (x_3^-)^{n+2} \\
\int dx_2^- (x_2^-)^{n+2} \langle \phi(x_1)T_{-}(x_2)T_{-}(x_3)\phi(x_4) \rangle.
\]

With this, we can compute the \(x_2^-\) integral, picking up only the singularity at \(x_{23}^2 \to 0\). If we now expand the integrand with the \(T \times T\) OPE, we thus obtain an expression of the form (3.95). We now need to understand this OPE in free field theory, which we do below for the case of \([L_1, L_1]\).

### 3.6.1 Example: \([L_1, L_1]\)

To illustrate how to obtain the finite separation term from the OPE we will focus on the concrete example of \([L_1(x_2), L_1(x_3)]\) (we also explain how to adapt this to other commutators). The goal is to understand the leading divergence in this commutator as \(|x_{23}^1|^2 \to 0\) by identifying which operators in the \(T \times T\) OPE are responsible for it.

One might have hoped that this finite separation contribution arises from the integral of a single local operator. However, we will see that this is not the case and we need to include an infinite number of terms in the sum (3.95). This infinite sum can also be rewritten as the light-ray integral of a single non-local operator, which we construct explicitly in appendix B.5.

The commutator we wish to compute via the OPE is given in (3.90), but we reproduce it here for convenience

\[
\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4) \rangle = -\frac{1}{\pi^2} \left( x_1^- - \frac{|x_{13}^-|^2}{x_{13}^+} \right) \left( x_4^- + \frac{|x_{34}^-|^2}{x_{34}^+} \right) \\
+ \frac{1}{\pi^2} \left( x_4^- + \frac{|x_{34}^-|^2}{x_{34}^+} \right) \left( x_1^- - \frac{|x_{13}^-|^2}{x_{13}^+} \right).
\]

We will not aim to derive the complete contribution from the stress-tensor OPE, but will concentrate solely on the terms that are responsible for the leading singularity in the

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10. As an operator statement, one generally cannot deform the integration contour for \(x_2^-\) to represent the commutator \([O_2, O_3]\) as an integral around \(x_3^-\). Conceptually, this is due to the \(i\epsilon\) prescription for \(x_{23}^+\), where the two orderings on the LHS of (3.96) are functions of \(x_{23}^+ \pm i\epsilon\), while the RHS only depends on \(x_{23}^+ - i\epsilon\). However, for this particular correlation function involving \(\phi\), where we are focused solely on the finite separation piece when both light-ray operators are on the same null plane, one can check explicitly that there is no dependence on this subtlety, and (3.97) holds.
transverse separation $|\vec{x}_{23}^\perp|$,}

$$
\langle \phi(x_1)|L_1(x_2), L_1(x_3)\rangle \phi(x_4) \rangle_{x_2 \to x_3^\perp} \approx \frac{2(x_{23}^\perp)^I}{\pi^2 |x_{23}^\perp|^2 (x_{13}^+:x_{34}^+)^2} \left[-(x_{13}^+)^I x_{34}^- x_{1,3}^- + (x_{34}^+)^I x_{13}^+ x_{1,3}^- \right],}
$$

(3.98)

where $I = 1, 2$ runs over the perpendicular indices. Note that the naive leading singularity $\sim 1/|x_{23}^\perp|^2$ vanishes due to antisymmetry. It is worthwhile to mention that the rest of the expansion truncates, with only one subleading correction which is regular as $|\vec{x}_{23}^\perp| \to 0$.

We now want to understand the leading term (3.98) as arising from the integral of local operators in the $T \times T$ OPE. The only operators which can contribute must satisfy the following set of constraints. First, given the overall factor of $(x_{23}^\perp)^I$, it is clear that the operators must all be vectors in the transverse direction. In addition, the operators that have the required properties and are consistent with scale and boost three-point function would vanish. With this, we can construct the most general set of operators that have the required properties and are consistent with scale and boost invariance, giving us the general sum

$$
[L_1(x_2), L_1(x_3)]_{l.s.} = \frac{(x_{23}^\perp)^I}{|x_{23}^\perp|^2} \sum_{m=0}^{\infty} a_m \int dx^-_{3} (x_{3}^+)^{m+3} \phi \partial_{I} \partial_{m} \phi(x_3),
$$

(3.99)

where the subscript l.s. stands for leading singularity in the $\phi$ correlator. Concretely, the operators written schematically in the RHS correspond to (descendants of) the conserved higher-spin currents,

$$
\phi \leftrightarrow \partial_{I} \partial_{m} \phi(x) \equiv \begin{cases} \partial_{I} J_{-\ldots -}(x) & m \text{ even}, \\ J_{I-\ldots -}(x) & m \text{ odd}, \end{cases}
$$

(3.100)

whose explicit form is given by [192, 193]

$$
J_{I_1 \ldots I_s}(x) = \sum_{k=0}^s (-1)^k \frac{1}{\Gamma^2(k+1)\Gamma^2(s-k+1)} \partial_{i_1} \ldots \partial_{i_k} \phi(x) \partial_{i_{k+1}} \ldots \partial_{i_s} \phi(x) - \text{traces},
$$

(3.101)

where the indices are symmetrized. For example, the term with $m = 0$ corresponds to $\partial_{I} \phi^2$, while the next term with $m = 1$ is the stress tensor component $T_{I-}$.

The overall coefficients $a_m$ can in principle be determined from the OPE coefficients in $T \times T$. For our purposes here, however, we merely wish to understand the structural form of this expansion. To do so, we will consider a simplifying kinematic limit and compute the resulting contributions in the sum. Specifically, we consider the following arrangement:

$$
x_{13}^+ = x_{34}^+ \equiv x^+, \quad \bar{x}_{13}^\perp = \bar{x}_{34}^\perp \equiv \bar{x}^\perp, \quad |\bar{x}^\perp|^2 \gg |\vec{x}_{23}^\perp|^2.
$$

(3.102)

We thus align all four operators on a line in the two transverse directions, with $x_1$ and $x_4$ at large equal distance on either side of the light-ray operators. This simplifying arrangement is displayed in figure 3.3. In this setup, the leading singularity (3.98) takes
the simple form
\[ \langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4) \rangle \bigg|_{x_2^+ \to x_3^+} \approx \frac{-4(x_{23}^+)f(x^+)f|\vec{x}_1^-|^2}{\pi^2(x^+)^4|\vec{x}_{23}^+|^2}. \]  
(3.103)

In appendix B.6, we explicitly compute the terms in the sum (3.99) evaluated inside the correlation function, which we denote as
\[ G^m(x_i) \equiv \left( \frac{x_{23}^+}{x_{23}^+} \right)^f \int dx_3^- (x_3^-)^{m+3} \langle \phi(x_1) \phi(x_3) \rangle \phi(x_4) \phi(x_5). \]  
(3.104)

We then show that in the limit described above these expressions all have the same functional form, up to an overall coefficient,
\[ G^m(x_i) \approx \frac{b_m(x_{23}^+)f(x^+)f|\vec{x}_1^-|^2}{\pi^2(x^+)^4|\vec{x}_{23}^+|^2}, \]  
(3.105)

where \( b_m \) are numerical factors that can be found in the appendix B.6. We can therefore clearly see that all operators in the OPE (3.99) contribute to the leading term, and we are ultimately just resumming the infinite series of coefficients.

We have therefore seen that the nonvanishing commutator \( \langle \phi[L_1, L_1]\phi \rangle \) can be understood directly as an infinite sum of light-ray operators built from the \( T \times T \) OPE. Let us briefly comment on two aspects of this calculation.

First, if we want to understand the finite separation contributions of other commutators, then we just need to modify the sum of operators in (3.99) to account for the dimension and spin of the specific commutator we are considering. For example, in the case of \( [L_2, L_2] \), we just need to replace \( (x_3^-)^3 \to (x_3^-)^5 \) in (3.99) (with new coefficients \( \tilde{a}_m \)).

Second, the integrand of \( \langle \phi[L_1, L_1]\phi \rangle \) does not contain a pole at infinity, as one can explicitly confirm. As we discuss in more detail in appendix B.6, the first three terms of the sum \( m = \{0, 1, 2\} \) individually each contain a pole at infinity in their respective integrands. Nevertheless, in the limit used above, this contribution vanishes and the sum is well-behaved. Because the full expression that this sum is replacing has no pole at infinity, we expect the general infinite sum to also not have any such poles. This concludes our discussion of the generalized ANEC commutators in free field theory.
3.7 Conformal block decomposition at large $N$

So far, we have only studied commutators of light-ray operators in the example of free field theory. In this section, we generalize this analysis by considering the commutator $[L_m, L_n]$ (at finite transverse separation) in a four-point function with a general scalar operator $O$ of dimension $\Delta$,

$$\langle O(x_1)[L_m(x_2), L_n(x_3)]O(x_4) \rangle \quad (x_2^+ \neq x_3^+; |m|, |n| \leq 2).$$  \hfill (3.106)

Specifically, we study the contribution to this commutator from the conformal block associated with $O$ itself,

$$O \times T \rightarrow O \rightarrow T \times O.$$  \hfill (3.107)

This particular contribution is interesting for two reasons. First, it is universal, as the OPE coefficient is fixed by conformal Ward identities. Second, this particular block is the leading contribution to the correlation function of light-ray operators in CFTs with large $N$ and a large gap, as we explain below by slightly generalizing an argument from [101, 108]. We can therefore compare the resulting commutator from $O$ exchange to gravitational computations in AdS, which we do in section 3.8.

In practice, the easiest way to compute the contribution of individual conformal blocks is to map the null plane $x^+ = 0$ to the celestial sphere. After briefly reviewing this map, we demonstrate that $O$ exchange gives rise to a nonzero commutator at finite separation, even for the simplest case of $[L_{-1}, L_{-2}]$, which suggests the existence of a non-trivial sum rule, where subleading conformal blocks must cancel this finite-separation commutator in physical CFTs.

3.7.1 Contributions in holographic CFTs

One of our main motivations in studying the commutators of generalized ANEC operators is to gain insight on possible universal structure in gravitational theories in AdS. We are therefore particularly interested in CFTs with a weakly-coupled bulk dual. Such theories generally must have the structure of generalized free field theory, with corrections suppressed by a large parameter $N$,

$$\langle O(x_1)T(x_2)T(x_3)O(x_4) \rangle = \langle O(x_1)O(x_4) \rangle \langle T(x_2)T(x_3) \rangle + O(1/N^2),$$  \hfill (3.108)

as well as a large gap $\Delta_{\text{gap}}$ in the scaling dimensions of the lowest single-trace operators with spin $j > 2$.

In such theories, the leading correction to the four-point function $\langle OTTO \rangle$ can only come from $O$ itself, as well as the leading $1/N$ corrections to the infinite family of double-trace operators $[OT]_{n,j}$ corresponding to two-particle states in AdS. This is shown schematically in figure 3.4.
Figure 3.4 – Leading contributions to the four-point function $\langle OTTO \rangle$ in CFTs with large $N$ and large gap. When integrating to obtain correlators of the global light-ray operators $L_m$, the contribution from double-trace operators (right) vanishes, leaving only the $O$ exchange.

Note that we are specifically interested in correlation functions of the global light-ray operators $L_m$ with $|m| \leq 2$, which annihilate the vacuum in either direction. In this case, we can rewrite their correlation function as the double-commutator

$$\langle O(x_1) L_m(x_2) L_n(x_3) O(x_4) \rangle = \langle [O(x_1), L_m(x_2)] [L_n(x_3), O(x_4)] \rangle$$

At leading order in $1/N$ and $1/\Delta_{\text{gap}}$, the only contribution to the commutators $[L_m, L_n]$ therefore comes from the exchange of $O$.

3.7.2 Event shapes on the celestial sphere

Up to this point, we have focused on light-ray operators located on a null plane at finite $x^+$. However, we can also consider the limit where these operators are taken to future null infinity (i.e. $x^+ \to \infty$),\footnote{The rescaling by $x^+$ ensures that the resulting correlation functions are finite, while the factor of 2 simply provides a useful normalization for the resulting expressions.}

$$L_m(\infty) \equiv \lim_{x^+ \to \infty} \left( \frac{x^+}{2} \right)^2 L_m(x).$$

More generally, we can use rotations to take light-ray operators to future null infinity in any direction, parametrized by the null vector

$$n^\mu = (1, \hat{n}), \quad n^2 = 0,$$
3.7. Conformal block decomposition at large $N$

![Diagram showing light-ray operators](image)

**Figure 3.5** – Two light-ray operators (indicated by the blue and orange lines) on the null plane $x^+ = 0$ (left) can be mapped via the conformal transformation (3.112) to future null infinity $\mathcal{I}^+$ (right). The transverse separation $\vec{x}_{ij}^\perp$ between the two operators on the plane leads to an angular separation $\theta_{ij}$ on the celestial sphere.

where $\vec{n}$ is a unit-normalized vector indicating a point on the celestial sphere. In the following discussion, we will label these light-ray operators at general positions on the celestial sphere as $L_m(n)$, suppressing the $\infty$ in the argument for notational simplicity.

We can also map the null plane $x^+ = 0$ to future null infinity $\mathcal{I}^+$ via the conformal transformation

$$
\begin{align*}
x^+ &\rightarrow -\frac{1}{x^+}, & x^- &\rightarrow x^- - \frac{|\vec{x}^\perp|^2}{x^+}, & \vec{x}^\perp &\rightarrow \vec{x}^\perp/x^+.
\end{align*}
$$

(3.112)

In this case, light-ray operators at distinct transverse coordinates $\vec{x}_{ij}^\perp$ on the null plane map to light-ray operators inserted in different directions $\vec{n}_i$ on the celestial sphere, as shown in figure 3.5. The relative angle $\vec{n}_i \cdot \vec{n}_j = \cos \theta_{ij}$ between them is given by

$$
\frac{1 - \cos \theta_{ij}}{2} = \frac{|\vec{x}_{ij}^\perp|^2}{(1 + |\vec{x}_i^\perp|^2)(1 + |\vec{x}_j^\perp|^2)}.
$$

(3.113)

We can therefore map correlation functions involving light-ray operators on the same null plane to so-called event shapes \[194–197\] involving light-ray operators at null infinity,

$$
\langle \mathcal{O}|L_{m_1}(x_1) \cdots L_{m_k}(x_k)|\mathcal{O}\rangle \iff \langle \mathcal{O}|L_{m_1}(n_1) \cdots L_{m_k}(n_k)|\mathcal{O}\rangle.
$$

(3.114)

Physically, these event shapes can be thought of as the correlation between detectors located at different points on the celestial sphere.

The important point is that we can study the commutators of light-ray operators at finite separation by computing event shapes for light-ray operators separated by a finite angle. As we will see, the calculation of the resulting commutator will be much simpler for event shapes than on the null plane.
We can already see this simplification in three-point functions involving the insertion of a single light-ray operator. For example, if we consider the three-point function with the ANEC operator $L_{-2}$ from eq. (3.66), at $x^+ \rightarrow \infty$ we obtain

$$\lim_{x_2^+ \rightarrow 0} \left(\frac{x_2^+}{2}\right)^2 \langle O(x_1)L_{-2}(x_2)O(x_3)\rangle = \frac{i\Delta}{2\pi} \frac{1}{(x_{13})^4 x_{13}^{-2}}. \tag{3.115}$$

Generalizing this expression to an arbitrary direction on the celestial sphere, we have

$$\langle O(x_1)L_{-2}(n)O(x_3)\rangle = \frac{i\Delta}{2\pi} \frac{1}{(-n \cdot x_{13})^3 x_{13}^{-2}}, \tag{3.116}$$

which reduces to the above expression for $n^+ = 2$, $n^- = \bar{n}^+ = 0$.

Note that this expression depends only on the relative distance between the two insertions of $O$, which means that if we Fourier transform to momentum space we find that momentum is conserved [198],

$$\langle O(p)L_{-2}(n)O(p')\rangle = \int \! d^4x_1d^4x_3 e^{-i(p \cdot x_1 - p' \cdot x_3)} \langle O(x_1)L_{-2}(n)O(x_3)\rangle = (2\pi)^4 \delta^4(p - p') \frac{\pi^2 (-p^2)^\Delta}{2^{2\Delta-4} (-n \cdot p)^3 \Gamma(\Delta - 1) \Gamma(\Delta)}.$$ \tag{3.117}

As is well-known, if we divide this expression by the norm of the external state,

$$\langle O(p)O(p')\rangle = (2\pi)^4 \delta^4(p - p') \frac{\pi^3 (-p^2)^{\Delta-2}}{2^{2\Delta-6} \Gamma(\Delta - 1) \Gamma(\Delta)}, \tag{3.118}$$

we measure the energy deposited in a given direction on the celestial sphere, which for a spherically-symmetric state with $\vec{p} = 0$ is simply

$$\langle L_{-2}(n) \rangle \equiv \frac{\langle O(p^0)L_{-2}(n)O(p^0)\rangle}{\langle O(p^0)O(p^0)\rangle} = \frac{p^0}{4\pi}. \tag{3.119}$$

We can also consider three-point functions involving the other global light-ray operators at null infinity, such as $L_{-1}$,

$$\langle O(x_1)L_{-1}(n)O(x_3)\rangle = \frac{i\Delta}{4\pi} \frac{(-n \cdot x_1) + (-n \cdot x_3)}{(-n \cdot x_{13})^3 x_{13}^{-2}}. \tag{3.120}$$

As we can see, this expression is almost the same as for $L_{-2}$, except for the positions in the numerator. We can use this fact to simplify the Fourier transform to momentum space,

$$\langle O(p)L_{-1}(n)O(p')\rangle = \frac{i\Delta}{4\pi} \int \! d^4x_1d^4x_3 e^{-i(p \cdot x_1 - p' \cdot x_3)} \frac{(-n \cdot x_1) + (-n \cdot x_3)}{(-n \cdot x_{13})^3 x_{13}^{-2}} \tag{3.121}$$

$$= -\frac{i}{2} \left( n \cdot \partial_{p} - (n \cdot \partial_{p'}) \right) \frac{i\Delta}{2\pi} \int \! d^4x_1d^4x_2 e^{-i(p \cdot x_1 - p' \cdot x_3)} \frac{1}{(-n \cdot x_{13})^3 x_{13}^{-2}}$$

$$= -\frac{i}{2} \left( n \cdot \partial_{p} - (n \cdot \partial_{p'}) \right) \langle O(p)L_{-2}(n)O(p')\rangle.$$
The three-point function with $L_{-1}$ is therefore given by a differential operator acting on the three-point function with $L_{-2}$,

$$
\langle O(p)|L_{-1}(n)|O(p') \rangle = -(2\pi)^4 \delta^4(p - p') \frac{i \pi^2 \Delta (-p^2)^{-1}}{2^{2\Delta - 4}(-n \cdot p)^2 \Gamma(\Delta - 1) \Gamma(\Delta)} - \left( (n \cdot \partial_p)(2\pi)^4 \delta^4(p - p') \right) \frac{i \pi^2 (-p^2)^{\Delta}}{2^{2\Delta - 4}(-n \cdot p)^2 \Gamma(\Delta - 1) \Gamma(\Delta)}.
$$

(3.122)

Notice that the second term gives a singular contribution to the expectation value in a given momentum eigenstate. This is a common feature related to the non-normalizability of definite momentum states [91]. It can easily be solved by spreading the wave function over a small neighbourhood around $p$. By doing this one can integrate the second term by parts. The upshot is that the expectation value becomes independent of the direction $n$, just like that of $L_{-2}$,

$$
\langle L_{-1}(n) \rangle \equiv \frac{\langle O(p^0)|L_{-1}(n)|O(p^0) \rangle}{\langle O(p^0)|O(p^0) \rangle} = \frac{i \Delta}{4\pi}.
$$

(3.123)

A similar calculation can be performed for the remaining global light-ray generators $L_m$ with $m = 0, 1, 2$. We can still connect them to the three point function involving $L_{-2}$ albeit by acting with higher-order differential operators,

$$
\langle O(p)|L_0(n)|O(p') \rangle = -\frac{1}{6} \left( (n \cdot \partial_p)^2 - 4(n \cdot \partial_p)(n \cdot \partial_{p'}) + (n \cdot \partial_{p'})^2 \right) \langle O(p)|L_{-2}(n)|O(p') \rangle,
$$

$$
\langle O(p)|L_1(n)|O(p') \rangle = -\frac{i}{2} (n \cdot \partial_p)(n \cdot \partial_{p'}) \left( (n \cdot \partial_p) - (n \cdot \partial_{p'}) \right) \langle O(p)|L_{-2}(n)|O(p') \rangle,
$$

$$
\langle O(p)|L_2(n)|O(p') \rangle = (n \cdot \partial_p)^2(n \cdot \partial_{p'})^2 \langle O(p)|L_{-2}(n)|O(p') \rangle.
$$

(3.124)

### 3.7.3 Commutators at finite separation

Let’s now consider event shapes involving two light-ray operators. In general, we can compute this correlation function by inserting a complete set of intermediate states, which in momentum space takes the simple form

$$
\langle O(p)|L_{m_1}(n_1)L_{m_2}(n_2)|O(p') \rangle = \sum_{O'} \int \frac{d^4 q}{(2\pi)^4} \frac{\langle O(p)|L_{m_1}(n_1)|O'(q) \rangle \langle O'(q)|L_{m_2}(n_2)|O(p') \rangle}{\langle O'(q)|O'(q) \rangle},
$$

(3.125)

where the sum is over primary operators $O'$ in the $O \times T$ OPE. Here we will specifically focus on the case where the only contribution is from $O$ itself. In this case, we can easily obtain the resulting four-point function from the three-point functions computed in the previous section.

To start, let’s consider the case where both light-ray operators are the ANEC operator $L_{-2}$. In this case, the delta function in eq. (3.117) fixes the intermediate momentum $q$,
leaving us with the straightforward result

\[
\langle \mathcal{O}(p)|L_{-2}(n_1)L_{-2}(n_2)|\mathcal{O}(p') \rangle = (2\pi)^4 \delta^4(p-p') \frac{\pi(-p^2)^{\Delta+2}}{2^{2\Delta-2}(-n_1 \cdot p)^3(-n_2 \cdot p)^3 \Gamma(\Delta-1) \Gamma(\Delta)}. \tag{3.126}
\]

This expression is symmetric under the exchange \( n_1 \leftrightarrow n_2 \), which means that the ANEC operators commute for any relative position on the celestial sphere [91, 108],

\[
\langle \mathcal{O}(p)|[L_{-2}(n_1), L_{-2}(n_2)]|\mathcal{O}(p') \rangle \big|_{\mathcal{O}} = 0, \tag{3.127}
\]

where the subscript indicates that this is specifically the contribution of the \( \mathcal{O} \) conformal block to the commutator.

Next, we can study the commutator between \( L_{-1} \) and \( L_{-2} \). Using the differential operator derived in the previous section, we can write the resulting commutator as

\[
\langle \mathcal{O}(p)|[L_{-1}(n_1), L_{-2}(n_2)]|\mathcal{O}(p') \rangle \\
= \int \frac{d^4q}{(2\pi)^4} \left\{ \left( -\frac{i}{2} (n_1 \cdot \partial_p - n_1 \cdot \partial_q) \langle \mathcal{O}(p)|L_{-2}(n_1)|\mathcal{O}(q) \rangle \right) \langle \mathcal{O}(q)|L_{-2}(n_2)|\mathcal{O}(p') \rangle \\
- \int \frac{d^4q}{(2\pi)^4} \langle \mathcal{O}(p)|L_{-2}(n_2)|\mathcal{O}(q) \rangle \left( -\frac{i}{2} (n_1 \cdot \partial_q - n_1 \cdot \partial_p) \langle \mathcal{O}(q)|L_{-2}(n_1)|\mathcal{O}(p') \rangle \right) \right\}. \tag{3.128}
\]

Using integration by parts, we can massage this expression into the form

\[
\langle \mathcal{O}(p)|[L_{-1}(n_1), L_{-2}(n_2)]|\mathcal{O}(p') \rangle \\
= -\frac{i}{2} \left( (n_1 \cdot \partial_p) + (n_1 \cdot \partial_q) \right) \langle \mathcal{O}(p)|L_{-2}(n_1)L_{-2}(n_2)|\mathcal{O}(p') \rangle \\
- \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \langle \mathcal{O}(p)|L_{-2}(n_1)|\mathcal{O}(q) \rangle \langle \mathcal{O}(q)|L_{-2}(n_2)|\mathcal{O}(p') \rangle \left( (n_1 \cdot \partial_q) \frac{1}{\langle \mathcal{O}(q)|\mathcal{O}(q) \rangle} \right) \\
- \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \langle \mathcal{O}(p)|L_{-2}(n_2)|\mathcal{O}(q) \rangle \langle \mathcal{O}(q)|L_{-2}(n_1)|\mathcal{O}(p') \rangle \langle \mathcal{O}(q)|\mathcal{O}(q) \rangle \\
+ \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \langle \mathcal{O}(p)|L_{-2}(n_2)|\mathcal{O}(q) \rangle \langle \mathcal{O}(q)|L_{-2}(n_1)|\mathcal{O}(p') \rangle \langle \mathcal{O}(q)|\mathcal{O}(q) \rangle. \tag{3.129}
\]

While this expression is admittedly somewhat complicated, it makes a few useful facts manifest. First, all derivatives of the overall momentum-conserving delta function cancel, which is not true in general correlation functions involving \( L_{-1} \). Next, the last two terms cancel if the two null directions are the same, \( n_1^\mu = n_2^\mu \), which means their contribution must depend solely on the angular dependence of the ANEC correlator (3.126) and is proportional to \( n_1 \cdot n_2 \). Finally, any proportionality to the scaling dimension \( \Delta \) of the external operator comes from the first two terms, and this dependence clearly cancels.

Evaluating this expression, we then obtain the resulting commutator

\[
\langle \mathcal{O}(p)|[L_{-1}(n_1), L_{-2}(n_2)]|\mathcal{O}(p') \rangle \tag{3.130}
= (2\pi)^4 \delta^4(p-p') \frac{-i\pi(-p^2)^{\Delta+1}}{2^{2\Delta-4}(-n_1 \cdot p)^2(-n_2 \cdot p)^3 \Gamma(\Delta-1) \Gamma(\Delta)} \left( 1 - \frac{3}{4} \frac{(-n_1 \cdot n_2)(-p^2)}{(-n_1 \cdot p)(-n_2 \cdot p)} \right).
\]
3.7. Conformal block decomposition at large $N$

Figure 3.6 – Schematic representation of the leading Regge trajectory at large $N$ for the example of $N = 4$ super Yang-Mills at zero (blue), strong (red), and infinite (gray) coupling. In free theory, the Regge trajectory corresponds to the tower of higher-spin conserved currents (blue), with intercept $j_0 = 1$. At infinite coupling, the higher-spin currents are lifted to $\Delta_{\text{gap}} \to \infty$, resulting in a flat Regge trajectory (gray) with intercept $j_0 = 2$.

In a spherically symmetric state with $\vec{p} = 0$, this expression reduces to the expectation value

$$
\langle [L_{-1}(n_1), L_{-2}(n_2)] \rangle = -\frac{ip^0}{4\pi^2} \left( 1 - \frac{3}{4}(-n_1 \cdot n_2) \right) = -\frac{ip^0}{16\pi^2} \left( 1 + 3\cos \theta_{12} \right). \quad (3.131)
$$

Using the map from null infinity to the null plane in eq. (3.112), we therefore find

$$
\langle \O(x_1) [L_{-1}(x_2), L_{-2}(x_3)] \O(x_4) \rangle \bigg|_{\O \neq 0} \neq 0 \quad (\vec{x}_{12}^+ \neq \vec{x}_{34}^+). \quad (3.132)
$$

We confirm this directly in position space in appendix B.7 (see (B.95)).

To sum up, $L_{-1}$ and $L_{-2}$ do not commute perturbatively at leading order in $1/N$ and $1/\Delta_{\text{gap}}$. In section 3.8 we reproduce this result, as well as the exact form of the non-vanishing commutator, explicitly from the AdS side.

On the one hand, this result is not surprising. In [108], it was argued that the commutator $[L_{m_1}, L_{m_2}]$ is controlled by the intercept $j_0$ of the leading Regge trajectory in the $T \times T$ OPE, with the expectation

$$
m_1 + m_2 < -(j_0 + 1) \quad \Rightarrow \quad [L_{m_1}(x_1), L_{m_2}(x_2)] = 0 \quad (\vec{x}_{12}^+ \neq \vec{x}_{34}^+). \quad (3.133)
$$

In the limit $\Delta_{\text{gap}} \to \infty$, the Regge intercept $j_0 = 2$, as shown schematically in figure 3.6. The commutator $[L_{-1}, L_{-2}]$ therefore lies outside the bounds of guaranteed commutation.

The authors of [108] do not commit on the fate of such operators since the integral of the Wightman function needs to be regularized, but they mention the possibility that this leads to non-commutativity at spacelike separation, as we have now confirmed.

However, it was also shown in [108] that nonperturbatively any CFT should satisfy $j_0 \leq 1$. This indicates that in a physical theory with finite $N$ and $\Delta_{\text{gap}}$, the nonperturbative con-
tributions from additional intermediate states must perfectly cancel the $O$ contribution, such that the commutator vanishes for any finite separation. This parallels the discussion in section 3.5.2: if we had considered $\phi^2$ as an external state, the contribution from the infinite tower of higher-spin conserved currents in the $\phi^2 \times T$ OPE would have exactly cancelled the contribution from $\phi^2$ itself,

$$\langle \phi^2 | \left[ L_{-1}(x_2), L_{-2}(x_3) \right] | \phi^2 \rangle \bigg| _{\phi^2} + \sum_{j=2}^{\infty} \langle \phi^2 | \left[ L_{-1}(x_2), L_{-2}(x_3) \right] | \phi^2 \rangle \bigg| _{\phi^2} = 0, \quad (\vec{x}_2 \neq \vec{x}_3).$$

(3.134)

It would be interesting to study this sum rule in more detail in future work, especially in the context of holography, as it requires nonperturbative effects in a UV complete theory of gravity to contribute an $O(1)$ correction to an inherently IR observable.

One can now repeat this procedure for other commutators of global operators, such as $[L_{-1}, L_{-1}]$. While the exact form of the final expression is not important, the crucial point is that those commutators are also nonzero at finite separation. The naive expectation from (3.133), along with the bound $j_0 \leq 1$ from [108], is that in general physical CFTs only $[L_{-2}, L_{-2}]$ and $[L_{-1}, L_{-2}]$ vanish nonperturbatively. However, we have seen that in the case of free field theory all global commutators with $m_1 + m_2 \leq 0$ vanish at finite separation. It would be useful to determine whether this is specifically a property of free field theory, or if such correlators are well-behaved in a larger class of CFTs.

### 3.8 Generalized shockwaves in AdS

In this section, we discuss the gravitational counterpart of the generalized ANEC operators in holographic CFTs. It is not surprising that linearized solutions, representing the insertion of generalized ANEC operators in the boundary, can be obtained both by solving the Einstein equations directly or by performing the bulk dual of the conformal transformations discussed for the CFT formulation. Maybe more surprisingly, we present exact solutions in the bulk which contain the information of higher n-point correlators in an analogous fashion to known results for the ANEC operator [91, 92]. This is a consequence of the collinear algebra explored in section 3.3. Even more unexpectedly, there exists a gauge where the linearized solutions can be made exact. We explain why this is the case and why it fails for the shock dual to the $L_0$ operator.

Using these results we discuss correlators and commutators in the gravitational theory. We see that the naive commutation relations found in free theories fail to materialize in this setup. This is expected from the fact that the gravitational theory is only keeping track of a single block running between shockwaves in the scattering process. As shown in section 3.7 this is also the case in the CFT if only one block contributes. Nonperturbative corrections at finite $N$ should presumably fix this glitch. We comment on this in the conclusions.
3.8. Generalized shockwaves in AdS

3.8.1 AdS isometries and collinear transformations

Throughout this section, we will work with AdS in Poincaré coordinates with metric
\[ ds^2 = \bar{g}_{\mu \nu} dx^\mu dx^\nu = \frac{dz^2 - dx^+ dx^- + d\vec{x}^\perp \vec{x}^\perp}{z^2/\ell^2}. \] (3.135)

We will also set \( \ell = 1 \) for convenience. As explained in sec. 3.3, the collinear transformations that map a light-ray onto itself are generated by a scale transformation, translations in the \( x^- \) direction, and a special conformal transformation. While the scale and translational isometries of the metric (3.135) are manifest, the special conformal transformations are less obvious. They can be obtained by the usual inversion/translation/inversion procedure. In the bulk inversion is simply \( z^\mu \rightarrow z^\mu/(z^\nu z_\nu) \) [199]. Therefore, the most general collinear transformation in the bulk is given by
\[ x^- \rightarrow \frac{ax^- + b}{cx^- + d}, \quad x^+ \rightarrow x^+ - c \frac{z^2 + |\vec{x}^\perp|^2}{cx^- + d}, \] (3.136)
\[ z \rightarrow \frac{z}{cx^- + d}, \quad x^i \rightarrow \frac{x^i}{cx^- + d}, \]
with \( ad - bc = 1 \). In what follows, the \( S \) transformation will be particularly relevant so we give it here explicitly
\[ x^- \rightarrow -\frac{1}{x^-}, \quad x^+ \rightarrow x^+ - \frac{z^2 + |\vec{x}^\perp|^2}{x^-}, \] (3.137)
\[ z \rightarrow \frac{z}{x^-}, \quad x^i \rightarrow \frac{x^i}{x^-}. \]
As we will see, the isometries (3.136) can be used to obtain new linearized shockwave solutions corresponding to generalized ANEC operators from known exact solutions (e.g. the usual ANEC shockwave). Moreover, the \( S \) transformation (3.137) can in some cases be further exploited to obtain new exact solutions. We explain this in detail below.

3.8.2 Review of AdS shockwaves

Shockwave geometries [200–202] (see [92] for an AdS perspective) are solutions to the Einstein equations of the form
\[ g_{\mu \nu} = \bar{g}_{\mu \nu} \delta g_{\mu \nu} = \bar{g}_{\mu \nu} + \epsilon h_{\mu \nu}, \] (3.138)
where \( \bar{g}_{\mu \nu} \) is the AdS metric. They have remarkable properties: they are full non-linear (i.e. to all orders in \( \epsilon \)) solutions to Einstein’s equations and remain so even when higher-derivative terms are added to the action. The shockwave solution in AdS takes the form
\[ h_{++} = \mathcal{H}(x^+), \] (3.139)
for any function \( \mathcal{H}(x^+) \). In what follows we will be mostly interested in the case where the shock is localized to a null-plane, namely
\[ \mathcal{H}(x^+) = \delta(x^+). \] (3.140)
From the form of the metric, it is manifest that for any $\vec{x}^\perp \neq 0$, there is no source turned on. At $\vec{x}^\perp = 0$, there is a source for the stress-tensor with delta function support in $x^+$. The CFT operator that couples to the $++$ component of the metric is $T_{--}$ and so this metric describes the insertion of the following operator in the path integral

$$S_{\text{CFT}} \to S_{\text{CFT}} + \epsilon \int d^4x \; T_{--} \delta(x^+) \delta^{(2)}(\vec{x}^\perp). \quad (3.141)$$

In other words, shockwave geometries are the gravitational duals of exponentiated ANEC operators. These shocks can be superposed non-linearly giving access to higher $n$-point functions for the ANEC operators by taking the appropriate $\epsilon$ derivatives. Alternatively we can compute correlators of ANEC operators by propagating wavefunctions for particular states in the bulk of AdS and expanding the result in a power series in $\epsilon$.

One perspective that we will exploit further below to obtain more general solutions in Einstein gravity is to understand the action of the collinear conformal group on this operator. When the source is localized in the $x^+$ direction this can be of great help. Concretely, the operator above is invariant under $x^-$ translations and has collinear twist $\bar{J}_0 = 1$ and collinear weight $J_0 = 2$ as can be seen from the transformations in section 3.3. It is easy to see that the most general ansatz for the metric given these properties is:

$$\delta g = \epsilon \frac{\delta (x^+)}{z^4} f (\zeta) \; dx^+ dx^+, \quad \text{with} \quad \zeta = \frac{|\vec{x}^\perp|^2}{z^2}. \quad (3.142)$$

This scaling ansatz turns Einstein equations into an ordinary differential equation that can be solved to yield (3.139), $f = \frac{1}{(1+\zeta)^2}$. This procedure, for sources localized in $x^+$ can be generalized to obtain exact solutions for other generalized ANEC sources. We explain these techniques in detail below.

### 3.8.3 Generalized shocks

In this section, we write down explicit metrics corresponding to the insertion of generalized ANEC operators $L_{-1}$, $L_0$, $L_1$ and $L_2$. We will see that the story for $L_0$ is more intricate and discuss this fact. We present a linearized solution in this case. The most efficient method to obtain these metrics depends on the particular operator at hand. We consider each one individually.

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12. Note that because the source is localized in the $\vec{x}^\perp$ directions it is contained in the $z^{-4}$ term of the metric rather than the ordinary $z^{-2}$.  

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3.8. Generalized shockwaves in AdS

$L_{-1}$ Shocks

Here, a form of the scaling ansatz above produces an exact solution. We are searching for a metric which corresponds to the insertion of the following operator in the path integral

$$\epsilon \int d^4x^- T_{--} \delta(x^+) \delta^{(2)}(\vec{x}^\perp) .$$

(3.143)

This means we want to find a metric with collinear twist $\tilde{J}_0 = 1$ and collinear weight $J_0 = 1$. The most general ansatz with this scaling that remains regular in $x^-$ is

$$\delta g = \epsilon \left( \frac{x^+}{z^4} f(\zeta) + \frac{\delta(x^+)}{z^2} q(\zeta) + \frac{\delta(x^+)^2}{z^2} s(\zeta) \right) dx^+ dx^- + \epsilon^2 \left( \frac{\delta'(x^+)}{z^4} t(\zeta) + \frac{\delta(x^+)^2}{z^2} r(\zeta) \right) dx^+ dx^- + \epsilon \delta(x^+)^2 \left( \frac{k(\zeta)}{z^4} \vec{x}^\perp \cdot d\vec{x}^\perp + \frac{g(\zeta)}{z^2} z dz dx^+ \right) ,
$$

(3.144)

with

$$\zeta = \frac{|\vec{x}^\perp|^2}{z^2} .$$

(3.145)

Notice that in this case the scaling properties allow an $\epsilon^2$ contribution to the exact shockwave solution. In principle this is the situation. It turns out that the functions $t(\zeta)$ and $s(\zeta)$ can be set to zero as they represent sources for independent ANEC operators. This solution has some gauge freedom. In particular the general form above is not in Fefferman-Graham gauge. We can remedy the situation right away by fixing $g(\zeta) = 0$. The system then becomes a fully determined system of coupled ODEs that can be solved explicitly. We can fix the integration constants by demanding that their only source is given by the boundary operator (3.143). The procedure consists in demanding that away from $\vec{x}^\perp = 0$ the metric contains no components which go as $1/z^2$. To fix the remaining constants we demand that integrating the source over the whole transverse plane yields the uniform shockwave solution (up to an overall normalization):

$$\delta g_{\text{uniform}} \sim \epsilon \frac{x^-}{z^2} \delta(x^+) dx^+ dx^- .$$

(3.146)

The result of this procedure produces the exact solution:

$$f(\zeta) = \frac{1}{(1 + \zeta)^3} ,$$

(3.147)

$$k(\zeta) = -\frac{1}{\zeta(1 + \zeta)^2} ,$$

(3.148)

$$q(\zeta) = -\frac{1}{2(1 + \zeta)^2} \left( -\frac{1}{2(1 + \zeta)} - \frac{1}{2} \log \frac{\zeta}{1 + \zeta} \right) ,$$

(3.149)

$$r(\zeta) = \frac{1}{2(1 + \zeta)^5} + \frac{3}{4(1 + \zeta)^4} + \frac{3}{8(1 + \zeta)^3} \left( 1 + 2 \log \frac{\zeta}{1 + \zeta} \right) .$$

(3.150)


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As advertised, this solution is quadratic in $\epsilon$. This term is proportional to the square of a delta function in $x^+$. This feature might seem unpleasant. Notice, however, that this property is actually gauge dependent. We used our gauge freedom to go to Fefferman-Graham gauge but we could have just as easily used it to go to a gauge where $r(\zeta) = 0$, yielding an exact linear in $\epsilon$ solution.\(^{14}\) We will discuss this momentarily. In any case, calculations that depend only on the linear properties in $\epsilon$ will not be sensitive to the $\delta(x^+)^2$ term.

More worrisome seems to be the case that this solution has a coordinate singularity at $\vec{x}^\perp = 0$ visible in $k(\zeta)$. This singularity is solely due to the choice of coordinates, which can be verified by computing the Kretschmann scalar $R_{abcd}R^{abcd} = 40$, just like empty AdS. In any case this coordinate singularity is mild and disappears if one integrates the source over a small area. Carrying out this procedure is useful if one wants to explore the properties of this metric near $\vec{x}^\perp = 0$. As expected, if we do so in a rotationally invariant manner, we obtain that the smeared value of the $dx^i dx^+\cdot$ component of the metric is indeed zero at $\vec{x}^\perp = 0$. The upshot is that this singularity can also be gauged away by a change of coordinates as we now show.

Lastly, one might be interested in going to coordinates where the term proportional to $\delta'(x^+)$ disappears. This is particularly useful when computing scattering past this shock. If we are only interested in the insertion of the dual operator in a correlation function we only care about terms that are linear in $\epsilon$. Looking at the metric (3.144), this is the only offensive term that might complicate the calculation, see section 3.8.4.

We can deal with all these issues simultaneously by considering the following change of coordinates which is consistent with our scaling Ansatz:

$$x^- \rightarrow x^- - \epsilon \delta(x^+) a(\zeta), \quad \text{with} \quad \zeta = \frac{||\vec{x}^\perp||^2}{z^2}. \quad (3.151)$$

Under this change of coordinates:

$$f(\zeta) \rightarrow f(\zeta), \quad (3.152)$$
$$k(\zeta) \rightarrow k(\zeta) + 2a'(\zeta), \quad (3.153)$$
$$g(\zeta) \rightarrow g(\zeta) - 2\zeta a'(\zeta), \quad (3.154)$$
$$q(\zeta) \rightarrow q(\zeta) + a(\zeta), \quad (3.155)$$
$$r(\zeta) \rightarrow r(\zeta) - f(\zeta) a(\zeta). \quad (3.156)$$

Having figured out the sources in the more physical Fefferman-Graham gauge, the transformations above allow us to go other useful gauges. As promised above, it is trivial to see that the choice

$$a = \frac{1}{2(1 + \zeta)^2} + \frac{3}{4(1 + \zeta)} + \frac{3}{8} \left(1 + 2\log \frac{\zeta}{1 + \zeta}\right), \quad (3.157)$$

\(^{14}\) Actually, it is the inverse metric that determines the propagation of waves in this AdS geometry and the actual source for the energy-momentum tensor, non-linearly. One could try instead to go to coordinates where the $\delta(x^+)^2$ term disappears from $g^{-1}$.
3.8. Generalized shockwaves in AdS

takes us to an exact linear in $\epsilon$ solution. As we will see momentarily this procedure extends to the $L_1$ and $L_2$ shockwaves while it fails for $L_0$.

More useful for our purposes will be the choice

$$a = \frac{1}{2(1 + \zeta)^2} + \frac{1}{2(1 + \zeta)} + \frac{1}{2} \log \frac{\zeta}{1 + \zeta}.$$ \hspace{1cm} (3.158)

In these coordinates we simultaneously remove the $\delta'(x^+)\text{ term in the metric and the artificial singularity at } \vec{x}^+ = 0$. The price to pay was to depart our beloved Fefferman-Graham gauge. As we will use this metric in our scattering experiments we quote the result below for the new metric components.

$$f(\zeta) = \frac{1}{(1 + \zeta)^3},$$ \hspace{1cm} (3.159)

$$k(\zeta) = -\frac{2}{(1 + \zeta)^3},$$ \hspace{1cm} (3.160)

$$g(\zeta) = \frac{\zeta - 1}{(1 + \zeta)^3},$$ \hspace{1cm} (3.161)

$$q(\zeta) = 0,$$ \hspace{1cm} (3.162)

$$r(\zeta) = \frac{1}{4(1 + \zeta)^4} + \frac{3}{8(1 + \zeta)^3} \left(1 + \frac{2}{3} \log \frac{\zeta}{1 + \zeta}\right).$$ \hspace{1cm} (3.163)

$L_0$ Shocks

The $L_0$ shocks turn out to be the most complicated. We have not found an explicit solution in this case. The reason is that if one imposes $J_0 = 0$ and $\bar{J}_0 = 1$ scaling, the resulting dimensions for $\epsilon$ allow the appearance of factors of the form $\epsilon^n \delta^{(n)}(x^+)$ or $\epsilon^n (\delta(x^+))^n$ for any $n > 0$. Our scaling ansatz is therefore not guaranteed to produce a solution at a finite order in $\epsilon$.

While the most general solution will certainly not be linear in $\epsilon$ one can hope that there is a gauge where that is possible, as was the case for $L_{-1}$. This can be checked explicitly. We have done so and found only complex solutions to this order. One can hope that introducing terms quadratic in $\epsilon$ the equation can be solved for a real metric.

There is also a good argument to explain why we would not expect exact solutions that are linear in $\epsilon$ for $L_0$. Remember that $L_{\{-2,-1,0,1,2\}}$ form a multiplet under the action of the $SL(2)$ generated by $J_{\{-1,0,1\}}$. The light-ray operators transform in a five-dimensional representation of the collinear group. One can compute the invariant $SL(2)$ norm for this (non-unitary) representation. In our conventions the transformation properties of the $L$’s under the action of finite $SL(2)$ group elements is given by (3.34). The invariant norm of a generic vector

$$v = \sum_{i=-2}^{2} a_i L_i$$ \hspace{1cm} (3.164)

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is given by

\[ |v|^2 = \left( a_0^2 - \frac{4}{3} a_1 a_{-1} + \frac{1}{3} a_2 a_{-2} \right). \quad (3.165) \]

Now, the fact that an exact solution can truncate to linear order is directly related to the associated vector in the algebra squaring to zero, suppressing higher order corrections. We see in the above expression that while \( L_{-2}, L_{-1}, L_1 \) and \( L_2 \) are indeed lightlike, \( L_0 \) has a non-zero norm. This explains the lack of exact linear solutions in this case, but gives us hope that this will be possible for \( L_1 \) and \( L_2 \). We will confirm this expectation shortly.

An interesting comment relates to CFTs in odd dimensions. Here we expect the multiplet of \( L \)'s to lie in an even-dimensional representation of \( SL(2) \). In this case, we expect there always exists a complete basis of null-operators. Therefore, we expect, for example, that all associated shocks in AdS\(_4\) can be made linear in \( \epsilon \).

While we leave for future work the task of finding an exact solution we now present a linearized (i.e. non-exact) solution representing the \( L_0 \) shock. While this could be obtained by brute force, we present here a method based on the \( SL(2) \) algebra that will be crucial to find exact solutions for \( L_1 \) and \( L_2 \).

The \( SL(2) \) algebra acts on a vector space and its action is therefore linear. This is very clear in the CFT as can be seen in Fig. 3.1. In the gravitational setting this translates to the fact that the algebra cannot act directly on the space of bulk solutions which can be non-linear and, hence, do not manifestly exhibit the properties of a vector space. Of course, this would be the case for linearized solutions but we will learn something by thinking about the action of the group on exact solutions.

Given an exact solution, we can generate a new one by the action of a finite symmetry transformation parameterized by an \( SL(2) \) group element. Notice that this technique only allows us to access solutions within the same conjugacy class. As we can see by the form of the norm (3.165), solutions sourced by the \( L_0 \) operator have to necessarily be in a different conjugacy class than those sourced by \( L_{-2} \) and \( L_{-1} \). This problem does not affect \( L_1 \) and \( L_2 \). In the next subsection we will use this method and the S transformation (3.137) to obtain those solutions. Here we will have to content ourselves with a linear solution.

We act with a one-parameter family of \( SL(2) \) transformations connected to the identity on an exact solution that is linear in \( \epsilon \), for example the one sourced by \( L_{-2} \). Notice that this produces a family of exact solutions, still linear in \( \epsilon \). We can expand this solution in powers of the parameter. If the transformation is generated by, say \( e^{\lambda J_1} \), then we will have a solution of the form

\[ g = \bar{g} + \epsilon h + \epsilon \lambda h_1 + \epsilon \lambda^2 h_2 + \cdots \quad (3.166) \]

This must be an exact solution, so it must also be a linear solution. Furthermore, it must be a solution for all \( \lambda \) which means that each \( h_j \) for \( j = 1, 2, \cdots \) must be a linear
perturbation that solves Einstein equations at this order. Matching to the expansion of \( e^{\lambda_1 J_1} \) we learn that:

\[
h_j = (J_1)^j (h) .
\]

One can check explicitly that each one of these solutions has the following source at the boundary (if \( h \) corresponds to the \( L_{-2} \) source (3.139)):

\[
\text{source}(h_j) = [J_1, \text{source}(h_{j-1})] .
\]

Because the \( L_n \)'s fall in a five-dimensional representation, after acting with \( J_1 \) five times we obtain solutions that have no sources at the boundary and are pure gauge. This gives us an efficient method to obtain linearized solutions for all \( L_n \)'s. Here we quote the linearized solution for an \( L_0 \) source:

\[
\begin{align*}
h_{zz} &= \frac{4 \delta(x^+)}{z^2(1 + \zeta)^3} , \\
h_{z+} &= -\frac{6x^- \delta(x^+)}{z^3(1 + \zeta)^3} + \frac{2\delta'(x^+)}{z(1 + \zeta)^2} , \\
h_{zi} &= \frac{4x^i \delta(x^+)}{z^3(1 + \zeta)^3} , \\
h_{++} &= \frac{6}{z^4(1 + \zeta)^3} \frac{(x^-)^2 \delta(x^+)}{2(1 + \zeta)} - \frac{3x^- \delta'(x^+)}{z^3(1 + \zeta)^2} + \frac{\delta''(x^+)}{2(1 + \zeta)} , \\
h_{+-} &= \frac{\delta(x^+)}{z^3(1 + \zeta)^2} , \\
h_{+i} &= -\frac{6x^i x^- \delta(x^+)}{z^4(1 + \zeta)^3} + \frac{2x^i \delta'(x^+)}{z^2(1 + \zeta)^2} , \\
h_{ij} &= \frac{4x^i x^j \delta(x^+)}{z^4(1 + \zeta)^3} .
\end{align*}
\]

In the following section we will obtain exact solutions by acting with the \( S \) transformation (3.137) on the \( L_{-2} \) and \( L_{-1} \) shocks. Notice that the exact solution for \( L_0 \) must be self-dual under \( S \). As expected, we have failed in finding an exact solution linear in \( \epsilon \) with this property but hopefully this fact can be used to find a solution containing higher orders in \( \epsilon \). We leave this for future work.

**\( L_1 \) and \( L_2 \) Shocks**

Having learned our lessons in the previous cases we are now ready to obtain exact linear in \( \epsilon \) solutions for \( L_1 \) and \( L_2 \) sources. As for \( L_0 \), the scaling Ansatz fails to produce a finite order in \( \epsilon \) guess\(^\text{15}\) for the solution. We will instead use the transformation properties of the generalized ANEC operators under the collinear transformations (3.34) as was done in the previous case. Here, however, we can use the finite group element corresponding to

\(^{15}\) Although it gives us a systematic way to organize the solution order by order in \( \epsilon \).
the $S$ transformation (3.137) to obtain exact solutions. This is possible as $L_1$ and $L_2$ are, correspondingly, in the same orbit as $L_{-1}$ and $L_{-2}$. The procedure is straightforward and metrics for both $L_1$ and $L_2$ share similar properties. Here we quote the $L_2$ metric only for succinctness, starting from the usual $L_{-2}$ shock (3.139)

$$h_{zz} = 4\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{(x^-)^2}{z^2(1 + \zeta)^3},$$

$$h_{z+} = -2\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{(x^-)^3}{z^3(1 + \zeta)^3},$$

$$h_{z-} = -2\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{x^-}{z(1 + \zeta)^2},$$

$$h_{z+i} = 4\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{x^i(x^-)^2}{z^3(1 + \zeta)^3},$$

$$h_{++} = \delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{(x^-)^4}{z^4(1 + \zeta)^3},$$

$$h_{+-} = \delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{(x^-)^2}{z^2(1 + \zeta)^2},$$

$$h_{-+} = -2\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{x^i(x^-)^3}{z^3(1 + \zeta)^3},$$

$$h_{-i} = -2\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{x^i x^-}{z^2(1 + \zeta)^2},$$

$$h_{ij} = 4\delta \left( x^+ - (1 + \zeta) \frac{z^2}{x^-} \right) \frac{x^i x^j}{z^4(1 + \zeta)^3}.$$

The first surprise is perhaps that the shockwave is no longer localized directly on a null-plane everywhere in the bulk. This is a natural consequence of the way that we obtained the metric through a change of coordinates. Note however that at the boundary, the source is still localized on the light-ray $x^+ = 0$, $\vec{x}^\perp = 0$. The functional form of the metric restricts support to $\vec{x}^\perp = 0$. In the bulk, the situation is more complicated. Notice that if we expanded the delta function in its derivatives we would obtain all powers of $\left( \frac{z^2}{x^-} \right)^n \delta^{(n)}(x^+)$ which is dimensionless by our scalings. In our previous scaling ansatz this was forbidden by demanding only positive powers of $x^-$ appear in the solution. Once a term like this shows up, the expansion cannot truncate to preserve regularity. There might be other gauges where only regular terms appear and truncate to finite order in $x^-$. The price to pay, however, will probably be the inclusion of arbitrarily high powers of $\epsilon$. It would be nice to see if a simpler solution exists.

To see the way the shockwave propagates in the bulk, it is slightly more convenient to
3.8. Generalized shockwaves in AdS

Figure 3.7 – The support of the shock inside the bulk of AdS, plotted at $\vec{x}^\perp = 0$. The magnitude of the delta function is $x^-$ dependent such that it vanishes (this is displayed in blue) on the lightlike directions $x^+$ at the boundary.

We can rewrite

$$ \delta \left( x^+ - \frac{z^2 + |\vec{x}^\perp|^2}{x^-} \right) = |x^-| \delta \left( -x^+ x^- + z^2 + |\vec{x}^\perp|^2 \right), \quad (3.177) $$

and we see that the shock lies on two hyperbolae, i.e. at all points null-separated from the origin, but that the delta function has a magnitude that is $x^-$ dependent. This is represented in Fig. 3.7.

Note that the fact that the magnitude of the delta function is $x^-$ dependent is what allows the limit towards the boundary to give the correct result yielding support on a single light-ray of the light-cone at the boundary.

### 3.8.4 Superposing shocks

Having obtained shockwave solutions, we are now ready to discuss the superposition of them and the propagation of particles in these backgrounds. This procedure will allow us to compute correlators in the holographic setup. In some cases (i.e. when the support of the shockwaves does not overlap in the bulk), it is trivial to obtain exact solutions by linear superposition of the shockwaves discussed in the previous section. For example two usual $L_{-2}$ shockwaves can be trivially superposed by placing them at different $x^+$ and $\vec{x}^\perp$ positions as:

$$ \delta g = \epsilon_1 \frac{\delta(x^+)}{z^4} f \left( \frac{|\vec{x}^\perp|^2}{z^2} \right) dx^+ dx^+ + \epsilon_2 \frac{\delta(x^+ - y^+)}{z^4} f \left( \frac{|\vec{x}^\perp - \vec{y}^\perp|^2}{z^2} \right) dx^+ dx^+. \quad (3.178) $$
Furthermore, since the solution above is completely smooth one can easily take the limit \( y^+ \to 0 \) to obtain a shockwave localized on a single null plane. The limit is clearly independent of the sign of \( y^+ \), which determines the time ordering of these perturbations for an incoming particle. This might make us think that the commutator of the sources immediately vanishes as a consequence. While this is true for the case above, we will see explicitly that this reasoning is incorrect for more general sources. In particular we will use the results above to compute the commutator \([L_{-2}, L_{-1}]\).\(^{16}\)

Let us now go over these computations for three different examples of shockwave superpositions: \( L_{-2} \oplus L_{-2} \), \( L_{-2} \oplus L_{-1} \) and \( L_{2} \oplus L_{2} \).

\( L_{-2} \oplus L_{-2} \) superposition

It is known that ANEC operators commute \([108]\). This was already observed in the gravitational setting in \([91]\). This fact has important consequences for the space of allowed gravitational theories \([1]\). Let us review this computation following mostly \([91]\).

We consider scalar perturbations in a shockwave background created by \( L_{-2} \) insertions. All we need is the form of the Laplace operator in this curved space \(^{17}\). Consider first the metric (3.178) with \( \epsilon_2 \) turned off

\[
4\partial_-\partial_+ \phi + \frac{3}{z} \partial_z \phi - \partial_z \phi - \nabla^2 \phi + 4\epsilon_1 \frac{\delta (x^+)}{z^2} f \left( \frac{|\vec{x}^\perp|^2}{z^2} \right) \partial_\phi \phi = 0,
\]

where \( \nabla^2 \) is the Laplace operator in flat transverse space \( \vec{x}^\perp \). This equation can be solved exactly. Away from the shock, the equation is trivially solved by AdS evolution. At the shock we just need to integrate across the delta function. There, the only coordinate that varies rapidly is \( x^+ \), so we can disregard regular terms that do no involve \( x^+ \) derivatives. Integrating the resulting equation we obtain:

\[
\partial_- \phi(x^+ = 0^+) = e^{-\frac{3}{2z^2} f \left( \frac{|\vec{x}^\perp|^2}{z^2} \right)} \partial_- \phi(x^+ = 0^-).
\]

It turns out this is all we need to compute correlators of \( L_{-2} \) insertions in scalar states. Assume we know the wave function corresponding to scalar states on the null surface \( x^+ = 0 \). Then the expectation value of the exponentiated ANEC operator is computed as \([91]\):

\[
\langle \phi_{\text{out}} | e^{\epsilon L_{-2}} | \phi_{\text{in}} \rangle \sim \int dx^- dz dx^1 dx^2 \frac{dz}{z^3} i \phi_{\text{out}}^* e^{\epsilon L_{-2}} \partial_- \phi_{\text{in}} + \text{c.c.}.
\]

Notice that this expression amounts to the integral over the light-ray parametrized by \( x^- \) and the three dimensional hyperboloid given by \( (z, x^1, x^2) \). We write the symbol \(~\)
as we are disregarding overall normalizations that can be obtained easily by knowing
the charges of the states involved. We will be mostly interested in the transverse space
dependence of the observables above.

If one is interested in the expectation value of $L^{-2}$ all one needs to do is to expand the
expression above and keep only the linear term in $\epsilon$. Concretely:

$$\langle \phi | L^{-2}(\vec{y}^\perp) | \phi \rangle \sim \int dx^- dz dx^1 dx^2 i\phi^* \frac{z}{(z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^3} \partial^2 \phi + c.c. ,$$

(3.182)

where above we have inserted the operator $L^{-2}$ at an arbitrary position $\vec{y}^\perp$ in transverse
space. If we are interested in computing this for conformal collider experiments where
we imagine the shockwave is sourced at the conformal boundary of Minkowski space, this
calculation amounts to the computation of the energy flux at infinity. The wave functions
for scalar states with definite timelike momentum $q^0$ are delta-function localized in the
hyperboloid at $z = 1$ and $\vec{x}^\perp = 0$ and are plane wave-like in $x^-$ going as $e^{iq^0 x^-}$, see [91].
In this case we obtain:

$$\langle \phi | L^{-2}(\vec{y}^\perp) | \phi \rangle \sim \frac{1}{(1 + |\vec{y}^\perp|^2)^3} .$$

(3.183)

where we have stripped above overall coefficients not depending on $\vec{y}^\perp$. The map between
the transverse coordinates and the $S^2$ at infinity in the collider experiment is:

$$y^1 = \frac{n^1}{1 + n^3}, \quad y^2 = \frac{n^2}{1 + n^3}, \quad \text{with} \quad (n^1)^2 + (n^2)^2 + (n^3)^2 = 1 ,$$

(3.184)

and the surface elements are related by:

$$d^2 y^\perp = \frac{d^2 \Omega}{(1 + n^3)^2} .$$

(3.185)

This implies that the operators on the plane and the sphere are related as[91]:

$$L^{-2}(\vec{y}^\perp) = (1 + n^3)^3 \mathcal{E}(n^i) .$$

(3.186)

The power of 3 above can be understood as coming from the fact that the ANEC operator
has collinear twist 1 adding to the two powers coming from the transformation of the
measure.

Plugging these results in (3.183) above we find

$$\langle \phi | \mathcal{E}(n^i) | \phi \rangle \sim 1 .$$

(3.187)

The result is independent of the angle in the celestial sphere, as it should be for a scalar
operator evaluated on a scalar state. The actual normalization is $\frac{q^0}{4\pi}$ to reproduce the
total energy of the state upon integration.

We can now tackle the insertion of two shocks as in (3.178). We will be interested in the
computation of the commutator $[L^{-2}(\vec{y}^\perp), L^{-2}(0)]$ so we will be considering shockwaves
inserted at an infinitesimal distance from each other in light-cone time $y^+$. Notice that
while the metric is smooth under \( y^+ \to 0 \), the solution for the propagation of perturba-
tions on top of it depends generically on the ordering of the shocks. This is because the
formal solution to the Laplace equation across the shock is

\[
\partial_\phi(x^+ = 0^+) = e^{-\frac{z^2}{2}} f\left(\frac{|x^+ - y^+|^2}{z^2}\right) \partial_- e^{-\frac{z^2}{2}} f\left(\frac{|x^+|^2}{z^2}\right) \partial_- \phi(x^+ = 0^-). \tag{3.188}
\]

This is completely analogous to solutions in gauge theory given by path ordered exponen-
tials. In this simple case, however, we see right away that the action of both exponential
operators commute and the ordering is not important. Concretely,

\[
\langle \phi | [L_{-2}(\vec{y}^\perp), L_{-2}(0)] | \phi \rangle \sim \int dx^- dz d^2 x^+ i \phi^* \left[ \frac{z^4}{(z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^3} \partial_-, \frac{z^4}{(z^2 + |\vec{x}^\perp|^2)^3} \partial_- \right] \partial_- \phi + \text{c.c.} \tag{3.189}
\]

as expected.

**\( L_{-2} \oplus L_{-1} \) superposition**

Let us now perform the equivalent computation for this more interesting case. Here we
consider the shockwave metric:

\[
\delta g = \epsilon_1 \delta(x^+ - (z^2 + |\vec{x}^\perp|^2)^3) \partial_- x^+ - \epsilon_1 \delta(x^+ - \epsilon_1 \delta(x^+ - y^+)^2) \partial_- \partial_+ dx^+ + \epsilon_1 \delta(x^+ - y^+) \partial_+ dx^+ \tag{3.190}
\]

We have chosen to represent the \( L_{-1} \) shock in the coordinates where the \( \delta'(x^+) \) term is
absent. Furthermore we have only kept track of terms linear in \( \epsilon_1 \) as they will be the
only ones of importance for the calculation at hand.

For now, let us set \( \epsilon_2 \to 0 \) and consider the resulting Laplace equation:

\[
4 \partial_- \partial_+ \phi + \frac{3}{z} \partial_+ \phi - \partial_+ \partial_+ \phi - \nabla^2 \phi \tag{3.191}
\]

Going through the same steps as before (and moving the source to an arbitrary point \( \vec{y}^\perp \)) we can calculate:

\[
\langle \phi | L_{-1}(\vec{y}^\perp) | \phi \rangle \sim \int dx^- dz d^2 x^+ i \phi^* \left[ 1 + (x^+ - y^+)^2 \partial_+ - \frac{|\vec{x}^\perp - \vec{y}^\perp|^2 - z^2}{2 \epsilon_1} z \partial_\phi \right] \partial_- \phi + \text{c.c.} \tag{3.192}
\]
Once again, considering a localized wave function on the hyperboloid and integrating by parts we get:

$$\langle \phi | L_{-1} (\vec{y}^\perp) | \phi \rangle \sim \left( \# - \partial_i \frac{(x^i - y^i)}{2} + \partial_z \frac{\sqrt{\vec{x}^\perp - \vec{y}^\perp}^2 - z^2}{4z} \right) \frac{z}{(z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^3} \bigg|_{z=1, \vec{x}^\perp = 0} \sim \frac{1}{(1 + |\vec{y}^\perp|^2)^3} .$$

(3.193)

In the above computation, the delta-function localized momentum states need to be regularized to compute the action of $x^- \partial_-$ on the wave function. The physics is completely equivalent to that of the term including a derivative of the momentum delta function in (3.122). For our purposes, it suffices to say that this term produces a constant, independent of $\vec{y}^\perp$. Fixing the normalization would amount to demanding that, upon integration over transverse space, the dilatation charge is reproduced. As in (3.123) the correct factor is $\frac{i\Delta}{4\pi}$.

This result, once again, corresponds to a uniform flux in the celestial sphere. In this case, the associated charge is the Lorentzian boost symmetry in the plane $x^- \rightarrow \lambda^{-1} x^-$, $x^+ \rightarrow \lambda x^+$. Under a conformal transformation this symmetry maps to the dilatation symmetry in the conformal collider picture [91]. This is also a scalar operator, so we do not expect any angular dependence.

It is now straightforward to compute the commutator of $L_{-1}$ and $L_{-2}$ shocks.

$$\langle \phi | [L_{-2} (\vec{y}^\perp), L_{-1} (0)] | \phi \rangle \sim \int dx^- d^2 x^\perp dz \frac{z^4}{(z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^3} \left( 1 + x^i \partial_i + x^- \partial_- - \frac{|\vec{x}^\perp - z^2 - z\partial z|^2}{2z^2} \right) \partial^- \phi + c.c. \sim \int dx^- d^2 x^\perp dz \frac{z^6}{(z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^4} \left( 2 |\vec{x}^\perp|^2 |\vec{z}^\perp - \vec{y}^\perp|^2 + (2 |\vec{x}^\perp - \vec{y}^\perp|^2 - 2 |\vec{y}^\perp|^2) z^2 + 2z^4 \right) \partial^2 \phi + c.c \sim \frac{2 - |\vec{y}^\perp|^2}{(1 + |\vec{y}^\perp|^2)^4} .$$

(3.194)

This commutator is not zero in contrast to free field theory computations in previous sections of this work. It matches, however, the computation in conformal field theory when only one block propagates between the shockwaves in section 3.7. A clear way to state this result is in conformal collider variables. Defining

$$L_{-1} (\vec{y}^\perp) = (1 + n^3)^3 D(n^i) ,$$

(3.195)

18. While we have disregarded the mass of the bulk scalar in this section, as it does not affect the scattering of the shockwave, it does control the scaling of the wave function with the energy $q^0$. This is where the $\Delta$ originates in the normalization at hand.

19. All operators in the global five dimensional multiplet of generalized ANEC operators have the same conformal twist, so they pick up the same factor $(1 + n^3)^3$ when mapping to collider variables.
we obtain:

\[ \langle \phi | [ \mathcal{E}(m'), \mathcal{D}(n')] | \phi \rangle \sim (1 + 3 \vec{m} \cdot \vec{n}) . \]  

(3.196)

Once again, the normalization can be easily obtained by integrating over the \( S^2 \). The angular dependence however is striking and matches the CFT result (3.131).

Notice that by looking at the result (3.194) we see that, as far as the scalar field is concerned, one can consider the evolution across the commutator as provided by an effective metric. It is given by:

\[
\delta g = \epsilon \delta(x^+) \frac{z^4 \left( 2|\vec{x}^\perp|^2 |\vec{y}^\perp|^2 - 2|\vec{y}^\perp|^2 \right) + \left( |2\vec{x}^\perp - \vec{y}^\perp|^2 - 2|\vec{y}^\perp|^2 \right) z^2 + 2z^4}{(z^2 + |\vec{x}^\perp|^2)^3 (z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2)^4} \, dx^+ dx^+ . \]  

(3.197)

This metric does not satisfy the Einstein equations. This implies that this commutator cannot be expressed in terms of sources for the boundary energy momentum components alone. An interesting direction here would be to compute the bulk energy momentum tensor that could support this solution. This way, one could understand if composite operators related to \( \phi \) could account for this commutator by back-reacting on the metric. We will not pursue this here.

\( L_2 \oplus L_2 \) superposition

Here, we include a short discussion on the superposition properties of \( L_2 \) shocks. A similar discussion would apply to \( L_1 \) shocks as well. The novelty in this case is that two
Figure 3.9 – The blue and orange rays represent two $L_{-2}$ operators located on the same null-plane. If we act with the $S$ transformation on this setup, the blue ray maps to itself, while the orange ray maps to the green one. We can see that it has left the null-plane and now intersects the blue ray at an angle. Since the original blue and orange $L_{-2}$ operators commute, the configuration after the $S$ transformation must also commute.

$L_2$ sources located at different $\vec{x}^\perp$ and $x^+$ do have intersecting support in the bulk as displayed in figure 3.8.

Therefore these solutions cannot be superposed. One could of course solve for the linear propagation of one shock on top of the other. This would be enough to repeat the type of calculations from the previous section. Let us focused instead in a particular type of configuration of these shocks that can be superposed. Let us start with the superposition of two usual $L_{-2}$ shockwaves

$$\delta g = \delta(x^+) \left( \epsilon_1 \left( \frac{z^2}{z^2 + |\vec{x}^\perp|^2} \right)^3 + \epsilon_2 \left( \frac{z^2}{z^2 + |\vec{x}^\perp - \vec{y}^\perp|^2} \right)^3 \right),$$

(3.198)

which is an exact solution of Einstein’s equation. Now, we can apply the diffeomorphism (3.137). The transformation of the shockwave inserted at $\vec{x}^\perp = 0$ is the solution given in (3.176). The part proportional to $\epsilon_2$ ends up being more complicated. Looking at the sources proportional to $\epsilon_2$ as $z \to 0$, one can easily check that the new operator is located at

$$x^i = -y^i x^-,$$

$$x^+ = |\vec{y}^\perp|^2 x^-.$$

(3.199)

The operator therefore intersects the previous light-ray at $x^- = 0$ with an angle dependent on $\vec{y}^\perp$. This is represented in Fig. 3.9. Because the original two ANEC shocks commute, the resulting two shocks must also commute, even though they intersect.

The $(x^-)^4$ dependence of the source here seems to make the amount of energy near $x^- = 0$ soft enough that it allows for the crossing with another operator. It would
be interesting to understand better why such operators can cross, and whether it is interesting from a more phenomenological point of view. One could imagine applications to quark-gluon plasma physics where these operators correspond to dragging of nucleons in the boundary gauge theory, see [203] for example.

3.9 Conclusion and future directions

The problem of bootstrapping non-trivial $d > 2$ CFTs remains one of the most interesting open problems in high-energy physics. While the solution of this problem for generic theories might very well be out of reach, one could hope that the addition of extra simplifying assumptions, like supersymmetry, large $N$ and/or large gap might provide a lamppost where this program can be carried out to completion. In recent years, important (non-trivial) constraints coming from unitarity of UV complete QFTs have proven very helpful in reducing the landscape of allowed consistent theories. These come in the form of positivity bounds or, more generally, sum rules that all consistent QFTs must satisfy. When applied to holographic (i.e. large $N$, large gap) CFTs these tools become quite powerful.

A crucial role in this program has been played by light-ray operators. They appear behind constraints in central charges [1, 91, 93], computations of entanglement entropy [89, 107], unitarity constraints in QFT [90, 94, 95] and recent sum rules [108]. While computing all correlation functions of a CFT might not be possible even at large $N$, one could ask if the subsector spanned by light-ray operators can be solved in some form. Some hope that this might be possible was presented in [107] and [1]. In a similar manner that all $n$-point functions of the energy momentum are fixed for $d = 2$ CFTs, how much does the algebra of operators and unitarity constrain the correlation functions of light-ray operators?

We have explored this problem in this present work. In particular, we have studied the algebra of $L_n = \int dx^{-} (x^{-})^{n+2} T_{-\cdot}(x)$ operators both in free field theories and holographic CFTs both in a QFT setup and from the point of view of AdS bulk gravity. We list our findings and comment on them, including a discussion on some future directions.

We have proposed a formalism to compute correlation functions of light-ray operators in states created by some CFT operator. Throughout this work we have focused on states created by scalar operators. The technique amounts to computing light-ray integrals by complex contour techniques taking into consideration the time ordering of operators through $i\epsilon$ prescriptions. While this technique is just an efficient calculational method when the real integrals involved are convergent, it amounts to a regularization prescription when they are not. This is an important point of contact to keep in mind when comparing the results presented here to those of [104, 108]. A physical interpretation is to consider matrix elements of these operators on states localized enough in the $x^{-}$ direction. From the conformal collider experiment perspective this relates to the assumption
that most of the radiation will be captured at the calorimeters after a finite time.

We then considered the action of the collinear conformal group that leaves the light-ray invariant on generalized ANEC operators. We found that there exists a five-dimensional subalgebra spanned by $L_n$ with $n \in \{-2, -1, 0, 1, 2\}$ which is closed under the action of this group. They annihilate the conformal vacuum and therefore have vanishing 2-point functions.

Operators outside this finite set can have non-vanishing two-point correlators in the vacuum giving rise to a central term for the infinite dimensional algebra. We find that this central term is infinite at vanishing $x^+$ separation, in agreement with suggestions in [107]. It is important to remark that this term is not the naive central term expected by the form of the Virasoro algebra suggested in [107] (see however [181] for a previous appearance of this term). Our finite complex contour integrals cannot produce that term as it is forbidden by the collinear group. Said differently, in our computations there is no IR divergence. The lack of this extra scale severely constrains the form of a central term to the one presented in (3.56).

One might attempt to change the normalization of the generalized ANEC operators to absorb this divergence as $L_n \to \epsilon^{n+1} L_n$. This regularizes the central term while preserving the form of (3.3). This amounts to the insertion of an explicit UV cutoff scale $\epsilon^{-1}$. Furthermore, the expectation value of the $L_n$‘s themselves might become trivial or divergent under this prescription. It would be interesting to pursue this in future work.

We further computed correlators involving one and two insertions of operators in the five-dimensional global subalgebra in scalar states. We first considered this in free field theory. We found that commutators involving $L_1$ and $L_2$ failed to commute at finite spacelike separations. This non-commutativity behaved as $|\vec{x}^\perp|^2$ at short distances for non identical operators and is therefore non-integrable. This implies that it is not possible to have a well defined algebra of light-ray operators for free field theories. In comparing with [104, 108] we should point out that for these particular states under consideration, the integrals involved are manifestly convergent.

One future direction to consider is the inclusion of other components of the energy-momentum tensor in the definition of the light-ray algebra. Notice, from (3.35), that the action of the collinear algebra, away from $\vec{x}^\perp = 0$, mixes the other components of $T_{\mu\nu}$ with $T_{--}$. If one included those terms in the definition of the new light-ray operators one might be able to soften (or even cancel) the finite separation contribution to commutators. From the point of view of conformal colliders this amounts to considering the flux related to other charges beside the ones associated to translations and dilatations. This is worth exploring further.

It would be interesting to understand how our results change by including interactions. The obvious arena to push this agenda is to consider generalized ANEC operators in weakly-coupled $\mathcal{N} = 4$ Super Yang-Mills theory, building on [204–207].

For holographic CFTs, there is only one conformal block that can propagate between
the insertions of generalized ANEC operators. This has the effect of further enhancing non-commutativity all the way down to $[L_{-2}, L_{-1}]$ which we compute in (3.131) and is in conflict with expectations for any finite $N$ CFT [106]. This is explained in terms of a discontinuity in the infinite $N$ limit of Regge trajectories. In this case the non-commutativity is integrable. This is the familiar behavior at strong coupling, where the short distance singularities get softened as a consequence of operators acquiring large anomalous dimensions [91]. It would be interesting to see if this feature persists generically and allows the construction of a light-ray algebra in this case. We will return to this briefly when we discuss our holographic results.

Of course, the commutativity of $[L_{-2}, L_{-1}]$ should be restored by non-perturbative effects. This is quite interesting as we see that an IR sensitive observable is affected at order 1 in the large $N$ limit. It seems irresistible to suggest an analogy with the black hole information paradox. In that case it is the fact that we care about late time observables that complicates the situation. Is the present discussion a bootstrap version of this type of phenomenon? Sum rules of the form (3.134) and the general discussion of [108] can provide a hint on how to control this problem explicitly. This is yet another interesting direction to pursue in the future.

We now turn to our computations in AdS gravity.

We have found new exact shockwave solutions that are dual to the insertions of exponentiated generalized global ANEC operators. The operator $L_0$ has resisted producing an exact dual shockwave. A potential way forward would be to consider an $S$ self-dual scaling ansatz at finite order in $\epsilon$. This problem seems tractable and we leave it for future work.

Using these shockwaves we computed the propagation of perturbations in their background. We have used this to compute the holographic commutators of generalized ANEC operators obtaining full agreement with the results in section 3.7.

We also computed the effective metric created by the commutator of shockwaves and stated that it does not satisfy the Einstein equations. It would be nice to understand what type of (multi-trace) operators are responsible for the bulk energy-momentum tensor producing these solutions. Further understanding here could shed light on the operators involved in the holographic light-ray algebra and could help making progress in understanding the sum-rules that give the non-perturbative completion to these calculations.

We left for the future the computation of commutators involving $L_1$ and $L_2$ in the gravitational setup. The fact that these shocks intersect in the bulk changes qualitatively the nature of this experiment. Finally, it would be nice to explore phenomenological applications of the shockwaves presented here. For example, in the understanding of nucleon scattering in Quark-Gluon plasma physics [203]

In conclusion, the study of light-ray operators has already provided important results in constraining the space of consistent UV complete QFTs. While their properties are
3.9. Conclusion and future directions

strikingly simpler than those of local operators, and particularly so for holographic CFTs, the understanding of their algebra and bootstrapping of their correlation functions remain yet out of reach. Still, the simple geometric action of the conformal group on them and their inherent Lorentzian nature make these objects the ideal avenue to further the understanding of the landscape of allowed theories. We hope to see important progress in this area in the coming years.
In this chapter, which is based on [3], we recast superfluid hydrodynamics as the hydrodynamic theory of a system with an emergent anomalous higher-form symmetry. The higher-form charge counts the winding planes of the superfluid – its constitutive relation replaces the Josephson relation of conventional superfluid hydrodynamics. This formulation puts all hydrodynamic equations on equal footing. The anomalous Ward identity can be used as an alternative starting point to prove the existence of a Goldstone boson, without reference to spontaneous symmetry breaking. This provides an alternative characterization of Landau phase transitions in terms of higher-form symmetries and their anomalies instead of how the symmetries are realized. This treatment is more general and, in particular, includes the case of BKT transitions. As an application of this formalism we construct the hydrodynamic theories of conventional (0–form) and 1–form superfluids.

4.1 Preliminaries and framework

Consider a superfluid, i.e. a system that spontaneously breaks a $U(1)$ symmetry. If it is Lorentz invariant, it can be described by the low energy effective field theory (EFT) [208, 209]

$$S = \int d^d x P \left( \sqrt{-D_\mu \phi D^\mu \phi} \right) + \cdots ,$$

(4.1)

with $D_\mu \phi = \delta_\mu \phi - q A_\mu$. Here, $P(\cdot)$ is a smooth function away from zero and the ellipses denote higher derivative terms. From the high energy perspective, $\phi$ represents the phase of the charged operator that condenses, $q$ is its charge and $A_\mu$ is a background $U(1)$ gauge field $^1$. The $U(1)$ current is given by

$$J_\mu = \frac{\delta S}{\delta(D_\mu \phi)} = P' \frac{D_\mu \phi}{\sqrt{-D_\nu \phi D^\nu \phi}} + \cdots .$$

(4.2)

$^1$ The ultraviolet reader uncomfortable with this EFT will find a discussion in section 4.1.2 on how this theory emerges as the low-energy description of certain microscopic models.
Now, following the nomenclature from [112], notice that the EFT also enjoys a \((d - 2)\)-form symmetry \(U(1)^{(d-2)}\) carried by the current \(K_{\mu_1\mu_2...\mu_{d-1}}\) given, compactly, by

\[
(\star K)_\mu = D_\mu \phi. \tag{4.3}
\]

where \(\star\) is the Hodge dual operator. Charged objects under this symmetry are winding planes of the superfluid phase \(\phi\). This higher form symmetry is explicitly broken by the proliferation of vortices as one returns to the normal phase, and typically will not be a symmetry of the microscopic theory: it is an emergent symmetry of the superfluid phase. In the presence of non-trivial background gauge fields \(F = dA \neq 0\), the conservation of the higher form current is also broken at low energies by an anomaly as

\[
d \star K = -a F. \tag{4.4}
\]

where \(a\) is the anomaly coefficient. It can be connected to UV data by \(a = q\). Notice that even without invoking UV arguments relating to charge quantization, it is easy to see that flux quantization implies \(a \in \mathbb{Z}\) for a compact \(U(1)\) symmetry,

Unlike axial-type anomalies, this mixed anomaly between \(U(1)\) and \(U(1)^{(d-2)}\) symmetries can occur in any dimension. It is similar to the axial anomaly in \(d = 2\) (e.g. in the Schwinger model) and generalizes it to higher dimensions. This anomaly has a simple physical interpretation: without background fields, the number of winding planes (or the supercurrent in a superconductor) is conserved. In an external electric field the number of winding planes (or the supercurrent) will increase linearly in time. Winding planes in any direction can be added or removed by turning on an appropriate electric field.

Spontaneous symmetry breaking (SSB) leads, therefore, to an emergent \((d - 2)\)-form symmetry with anomaly (4.4). In section 4.1.1 below, we show that there exists an (almost) converse statement, namely a system with \(U(1) \times U(1)^{(d-2)}\) symmetry with anomaly (4.4) contains a massless Goldstone boson transforming non-linearly in its spectrum. As a consequence, SSB phases of systems enjoying abelian symmetries can equivalently be formulated in terms of mixed anomalies (4.4). It is tempting to adjust Landau’s paradigm for classifying phases by only specifying which generalized symmetries each phase has, along with their anomalies, disregarding how they are realized – linearly or non-linearly (i.e. whether the symmetries are spontaneously broken or not). For example, the BKT transition in 2 + 1 dimensional superfluids is sometimes said to be non-Landau because there is only quasi-long range order at low but finite temperatures. Generalized symmetries however distinguish both phases. Conservation of the emergent higher form symmetry in the superfluid phase has tangible consequences: it leads, in particular, to

2. The current associated with the higher-form symmetry is \((\star K)_\mu = \partial_\mu \phi\). Nevertheless, this current is not invariant under \(U(1)\) gauge transformations, and the gauge invariant combination is given in (4.3), which is not conserved. The non-conservation of (4.3) leads to the anomalous conservation equation (4.4).

3. Strictly speaking, this is weaker than SSB as there does not need to be a charged operator that acquires an expectation value. Therefore, the symmetry structure presented here, including the anomaly, is a weaker assumption than SSB, making it more general.
an infinite dc conductivity \(\sigma(\omega) \sim i/\omega\), observed experimentally even in 2 + 1 dimensions e.g. in thin superconducting films or superfluids when \(T < T_{\text{BKT}}\) [210, 211].

The philosophy of insisting on symmetries alone rather than how they are realized on microscopic fields plays a central role in hydrodynamics. In this chapter, we propose to recast superfluid hydrodynamics as a the hydrodynamical theory of a system with \(U(1) \times U(1)^{(d-2)}\) symmetry with a mixed anomaly (4.4). This formulation puts all hydrodynamic equations on equal footing – as conservation laws and constitutive relations for the various currents. The ‘Josephson relation’ in the standard treatment of superfluid hydrodynamics [212] is replaced by the constitutive relation for the higher form current (see Ref. [118] for the analogous statement in the context of spontaneous breaking of translation symmetry). This streamlines the hydrodynamics algorithm along the lines of what was done recently with magnetohydrodynamics [117]. How anomalies enter hydrodynamics has been understood since the seminal work of Son and Surowka [213] where it was shown that the chiral anomaly fixes terms in the constitutive relations at first order in derivatives. Furthermore, the understanding of the interplay between hydrodynamics and anomalies has led to many important results in the field, e.g. see [214–221]. The case at hand is in some sense the simplest anomaly in hydrodynamics, since it enters at zeroth order in derivatives.

Mixed anomalies of higher form symmetries have also been discussed recently in the context of 2-groups [222, 223] and for discrete symmetries in the context of topological phases [224].

### 4.1.1 An alternative to Goldstone’s theorem

The standard input for the Nambu-Goldstone theorem is that a symmetry breaking order parameter acquires a vacuum expectation value. Here, we obtain the equivalent result for relativistic QFTs with a different starting point; namely that the theory has a global symmetry \(U(1) \times U(1)^{(d-2)}\) with mixed anomaly

\[
\delta_{\mu} \langle J^\mu \rangle = 0, \quad \delta_{\mu} \langle (\star K)_\nu \rangle = -a F_{\mu \nu}. \tag{4.5}
\]

The Fourier transform of the mixed correlator is constrained by Lorentz invariance to take the form

\[
\Pi_{\mu \nu} p^\nu = 0 \quad \Rightarrow \quad f(p^2) + g(p^2) = 0. \tag{4.7}
\]

The anomalous Ward identity for the \((d-2)\)-form current (4.5) reads

\[
p_{[\alpha} \Pi_{\mu] \nu} = -a p_{[\alpha g_{\mu] \nu} \quad \Rightarrow \quad g(p^2) = -\frac{a}{p^2}. \tag{4.8}
\]
4. Superfluids as Higher-form Anomalies

The mixed correlator is therefore completely fixed by the anomaly

$$\Pi_{\mu\nu}(p) = a \frac{p_\mu p_\nu - p^2 g_{\mu\nu}}{p^2}. \quad (4.9)$$

An important remark here is that the $U(1)^{(d-2)}$ symmetry is emergent, and is broken by vortices. This will lead to corrections to (4.9) which are non-singular as $p^2 \to 0$ and vanish when $p_\mu \to 0$, since the vortices are gapped. The $p^2 = 0$ pole in (4.9) is therefore a robust consequence of the anomaly (4.5) of the emergent symmetry, and is all that is needed for this proof.

We now proceed along lines similar to current-algebra proofs of Goldstone’s theorem [208]. The Källén-Lehmann representation of the time-ordered correlator (4.9) is

$$\Pi_{\mu\nu}(p) = \int_0^\infty d\mu^2 \rho_{KJ}(\mu^2) \frac{p_\mu p_\nu - p^2 g_{\mu\nu}}{p^2 - \mu^2 + i\epsilon}, \quad (4.10)$$

where the spectral density $\rho_{KJ}(\mu^2)$ (non-sign definite since it involves two different operators) is defined as

$$\sum_n (2\pi)^d \delta^d(p - p_n)\langle 0 | (\star K)_{\mu}(0)|n\rangle \langle 0 | J_{\nu}(0)|n\rangle^* \equiv \rho_{KJ}(p^2)p_\mu p_\nu. \quad (4.11)$$

Comparing with (4.9) one immediately concludes that there exists a massless state $p_n^2 = 0$ that is created by both currents, i.e.

$$\rho_{KJ}(\mu^2) = a \delta(\mu^2) + \cdots, \quad (4.12)$$

where $\cdots$ are contributions that are finite as $\mu \to 0$.

Notice that, up to contact terms (see footnote 4) and identifying $(\star K_{\mu}) = \partial_\mu \phi$, we can interpret (4.9) as coming from the momentum space correlation function

$$\langle \phi J_{\nu} \rangle = a \frac{p_\nu}{p^2}. \quad (4.13)$$

This precisely satisfies the Ward identity for a field $\phi$ transforming non-linearly under the $U(1)$ induced by $J_\mu$, indicating that this symmetry is spontaneously broken in $d > 2$. While there is no spontaneous symmetry breaking in $d \leq 2$, our arguments do go through even in that case showing that our setup is more general than the usual classification of phases by the realization of symmetries and includes more exotic cases, such as BKT transitions.

A short but important conclusion from this analysis is that it is really anomalies that are responsible for the existence of massless modes, as a more general statement than symmetry breaking. This discussion connects with the study of topological phases [226, 227], where topological insulators accommodate massless modes at their boundaries.

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4. A non-covariant contact term has to be added to make the time-ordered correlator of spin-1 operators covariant. This can be done while preserving Ward identities, see [225].
4.1. Preliminaries and framework

stabilized by anomaly inflow. The present anomaly (4.4) can be canceled by inflow from a bulk with the term $S_{\text{bulk}} = a \int B \wedge F$, where $B$ is the $(d-1)$-form source for the current $K$.

A mixed anomaly can similarly be seen to protect the masslessness of the photon or higher-form gauge fields – the hydrodynamics of such a system is discussed in Sec. 4.4. The proof above can easily be generalized to the case where an anomalous $U(1)^{(p)} \times U(1)^{(d-p-2)}$ is present, leading to the presence of p-form massless gauge fields in the spectrum for $0 \leq p \leq d-2$. When $d = 2p + 2$, we expect the emergence of a conformal phase at low energies, as the gauge coupling constant is dimensionless. In this very special case the converse of this statement was proven in [116]: a conformal theory enjoying a $U(1)^{(p)}$ symmetry in $d = 2p + 2$ dimensions must also have an anomalous $U(1)^{(d-p-2)}$.

4.1.2 Scales and defects in the superfluid EFT

Before studying the hydrodynamics we pause to make a few comments on the effective field theory (4.1). A paradigmatic microscopic model that leads to it is the Landau-Ginzburg model for a complex scalar

$$\mathcal{L} = -\frac{1}{2} |D_\mu \Phi|^2 - V(|\Phi|^2), \quad (4.14)$$

where $D_\mu \Phi \equiv \delta_\mu \Phi + iA_\mu \Phi$ and with potential

$$V(\rho^2) = -\frac{1}{2} m^2 \rho^2 + g \frac{4}{3} \rho^4 = \frac{g}{4}(\rho^2 - v^2)^2 + \text{const}, \quad (4.15)$$

where $v = m/\sqrt{g}$. Expanding around the saddle $\Phi = (v + r)e^{i\phi}$ and integrating out the radial mode at tree level this leads to (for energies $E \ll m$)

$$S_{\text{eff}} = -v^2 \int d^d x \frac{1}{2} (D_\mu \phi)^2 + \frac{a_4}{m^2} (D_\mu \phi)^4 + \frac{a_6}{m^4} (D_\mu \phi)^6 + \cdots, \quad (4.16)$$

where the $a_n$ are combinatorial factors. This is clearly a special case of (4.1), where we can understand $P(\mu)$ as an analytic series expansion in $\mu^2$. The strong coupling scale in this model is $\Lambda_{\text{sc}} \sim \left( \frac{m^4}{g} \right)^{\frac{1}{2}}$, which, as usual, is parametrically larger than the scale of new physics $m$ if the UV is weakly coupled. In this cases we can imagine resumming the series to obtain a function $P(\mu)$ and still retain perturbative control. The leads to the treatment discussed around (4.1).

From the point of view of the higher form current (4.3), the UV scale $\Lambda_{\text{sc}}$ has clear physical meaning. The theory (4.14) admits vortex solutions which can be constructed (up to logarithmically IR divergent terms) as soon as we hit the symmetry restoration scale given by $\Lambda_{\text{sc}}$. When this happens, the winding planes charged under $U(1)^{(d-p-2)}$ can end on the vortices and the symmetry becomes explicitly broken.

5. This notation is natural if one notices that for constant field configurations, in a background time-like field, the argument of $P(\mu)$ is indeed the chemical potential.
4. Superfluids as Higher-form Anomalies

Of course this weakly coupled description does not have to be valid. While the superfluid system described above exists even with zero chemical potential, so that one can consistently take $\mu \ll \Lambda_{\text{sc}}$, certain superfluid phases only occur at finite chemical potential (such as in QCD). In this case the strong coupling scale is typically of order the chemical potential. A simple example of such a situation is a conformally invariant superfluid, where $P(\mu) = \alpha \mu^d$ by scale invariance so $\Lambda_{\text{sc}} \sim \mu$ (in this case $P(\mu)$ might not be analytic in $\mu^2$). More generally the equation of state $P(\mu)$ entirely fixes the EFT at leading order in gradients – the equation may however be complicated away from the conformal situation. See [228] for an extended discussion.

4.1.3 Decoupling of currents in the superfluid EFT

The reader may wonder to what extent the two currents (4.2) and (4.3) should be treated as independent vectors in the hydrodynamic description. Although the operators $J_\mu$ and $(\star K)_\mu$ are different (for example in (4.16) they differ by terms suppressed in $m^2$), they remain collinear when evaluated in any given field configuration. This is no longer the case in a thermal ensemble. $T \neq 0$ introduces a preferred vector ($u^\mu = \delta^\mu_0$ in the rest frame of the fluid) which together with the superfluid winding distinguishes both currents. Since $u_\mu$ is even under charge conjugation\(^6\), the decoupling of currents can only happen when also at finite chemical potential $\mu \neq 0$. The difference between the currents corresponds to the ‘normal density’ in the two-fluid picture of finite temperature superfluids, coming from thermally populated superfluid phonons. In this section we show how a thermal 1-loop computation in the effective theory (4.1) distinguishes the two currents at finite (but small) temperature $T$. Although this calculation has been done in certain microscopic models that exhibit superfluidity (see e.g. [229] for a field theory calculation in the Landau-Ginzburg model), we are not aware of a calculation in the universal EFT. This decoupling of the currents at finite temperature justifies why they should be treated as independent vectors in the hydrodynamic setup of section 4.3.

Taking $A_\mu = \mu \delta^0_\mu$ and expanding the action in fields around a solution with finite superfluid winding\(^7\) $\delta_\mu \phi \to \delta^I_\mu \tilde{\rho}_I + \delta_\mu \phi$ leads to

$$S = \int d^d x P(\sqrt{-D_\mu \phi D^\mu \phi}) + \cdots$$

$$= \int d^d x P - \frac{P'}{2X_0}(\delta_\mu \phi)^2 + \frac{1}{2} \left( \frac{P''}{X_0^3} - \frac{P'}{X_0^2} \right) (\mu \phi + \tilde{\rho} \cdot \nabla \phi)^2 + O(\delta \phi)^3 + \cdots,$$

where the functions without argument ($P$, $P'$, etc.) are evaluated at $X_0 \equiv \sqrt{\mu^2 - \tilde{\rho}^2}$. We can identify these with the pressure $P$, charge density $\rho = P'$ and susceptibility $\chi = P''$ at zero temperature. The speed of superfluid sound is clearly anisotropic, see [229] for an extended discussion.

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\(^6\) Charge conjugation acts in the EFT (4.1) as $\phi \to -\phi$, $\mu \to -\mu$.

\(^7\) The index $I = 1, 2, \ldots, d - 1$ runs over the spatial dimensions.
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The currents in this theory are given by (4.2) and (4.3); expanding again in fields gives

\[ (\ast K)_\mu = \delta_\mu \phi + \delta_\mu I - \mu \delta_\mu I \quad \text{(4.18)} \]

\[ J_\mu = \frac{(\ast K)_\mu}{X_0} \left[ P' \left( \frac{P''}{X_0} - \frac{P'}{X_0^2} \right) (\mu \dot{\phi} + \tilde{\rho} \cdot \nabla \phi) + O((\delta \phi)^2 + \cdots) \right]. \quad \text{(4.19)} \]

In the ground state the normal ordered operators above have no expectation value and we have the densities

\[ \langle J_0 \rangle = P' = \rho \quad \text{and} \quad \langle (\ast K)_I \rangle = \tilde{\rho} I \quad \text{at} \quad T = 0 \quad \text{as expected.} \]

For any single field configuration, the two currents are manifestly parallel. However they are distinguished in the finite temperature ensemble, where \( \dot{\phi}^2 \) and \( \nabla \phi^2 \) acquire thermal expectation values at 1-loop. Here we will work at small temperature for simplicity so that the equation of state can be expanded around \( T = 0 \). The expectation value of the dual current is simply

\[ \langle (\ast K)_\mu \rangle_\beta = \delta_\mu I - \mu \delta_\mu I. \quad \text{(4.20)} \]

The finite temperature correction in the direction of the regular current is

\[ \langle J_\mu \rangle_\beta - \langle J_\mu \rangle = - \left( \frac{P''}{X_0^3} - \frac{P'}{X_0^2} \right) \langle \delta_\mu \phi (\mu \dot{\phi} + \tilde{\rho} \cdot \nabla \phi) \rangle_\beta. \quad \text{(4.21)} \]

Here we neglected two contributions to \( \langle J_\mu \rangle_\beta \), coming from the temperature dependence of \( P' \) and the thermal expectation value of the \( O((\delta \phi)^2) \) term in (4.19). Both of these will give corrections to the magnitude of \( \langle J_\mu \rangle_\beta \), but not to its direction which we are interested in. In the traditional language, they are finite temperature corrections to the superfluid density, instead of contributions to the normal density.

In order to prove that the two currents are independent, one must compute the thermal expectation value (4.21) in the linearized theory (4.17) and show that it is not parallel to (4.20). It turns out to be sufficient to do so at leading order in \( \tilde{\rho} \), which simplifies the calculation because the linearized action (4.17) can be taken to be isotropic. One then has

\[ \langle \delta_\mu \phi (\mu \dot{\phi} + \tilde{\rho} \cdot \nabla \phi) \rangle_\beta = \left( \mu \delta_\mu \langle \dot{\phi}^2 \rangle_\beta + \frac{1}{d-1} \delta_\mu I \langle \nabla \phi^2 \rangle_\beta \right) (1 + O(\tilde{\rho})). \quad \text{(4.22)} \]

A 1-loop calculation with appropriate UV regulator gives

\[ \langle \dot{\phi}^2 \rangle_\beta = c_s^2 \langle \nabla \phi^2 \rangle_\beta = \frac{2(d-1)f_d}{\beta d c_s^2 - 1 P'}, \quad \text{with} \quad f_d = \frac{\Gamma\left(\frac{d}{2}\right)\zeta(d)}{\pi^{d/2}}, \quad \text{(4.23)} \]

where the isotropic speed of sound is given by \( c_s^2 = \frac{P'}{\mu P'} \). We therefore find that the contribution (4.21) to the current is

\[ \langle J_\mu \rangle_\beta - \langle J_\mu \rangle = \frac{1 - c_s^2}{\mu c_s^2 (d-1)f_d} \left[ \delta_\mu \frac{1}{d-1} \left( \frac{\tilde{\rho} I \delta_\mu}{(d-1)c_s^2} \right) \right], \quad \text{(4.24)} \]

which is never parallel to (4.20). At low temperatures, we see that as long as \( c_s^2 < 1 \) the deviation between the two currents (and therefore the ‘normal density’) is \( \sim T^d \), in agreement with standard results (see e.g. [230]).
4. Superfluids as Higher-form Anomalies

4.2 An incoherent superfluid appetizer

The existence of the anomaly (4.4) applies to any local system with a spontaneously broken $U(1)$ symmetry and does not rely on translational or boost invariance. This leads us to consider a system where conservation of energy and momentum can be ignored\(^8\) and focus on the hydrodynamics of the conserved currents (4.2) and (4.3) alone. This constitutes the simplest instance of a hydrodynamic system with the (anomalous) symmetry structure discussed in the introduction. The XY model on a lattice is a simple microscopic realization of such a system.

The full hydrodynamics, including energy-momentum, is treated in the next section in a systematic manner (including a careful study of the role of anomalies); we take advantage of the simpler setting in this section to make the conceptual issues more clear. We therefore consider the hydrodynamics of a system satisfying the conservation laws

\begin{align}
    d \star J &= 0, \quad (4.25) \\
    d \star K &= -aF, \quad (4.26)
\end{align}

with $a \in \mathbb{Z}$ where $J$ and $(\star K)$ are one forms and $F = dA$ is a background two-form for the $U(1)$ gauge field that couples to $J$. At finite temperature, there exists a preferred rest frame for this system given by a (non-dynamical) velocity field $u^\mu$. This allows us to discuss the physics in manifest $SO(d - 1)$ non-relativistic notation in what follows. We will consider the case where there are no background sources, $A = 0$.

The hydrodynamic variables are the charge densities $J^0 = \rho$ and $(\star K)^I = \tilde{\rho}^I$. We will denote their conjugate dynamical potentials by $\mu$ and $\tilde{\mu}^I$ and the corresponding susceptibilities\(^9\) $\chi$ and $\tilde{\chi}$. As a further simplification in this section, we will assume the background dual potential vanishes $\tilde{\mu}^I = 0$ (corresponding to the absence of a background winding of the superfluid). This assumption is lifted in the general treatment of Sec. 4.3.

The most general constitutive relations up to first order in derivatives are

\begin{align}
    J^I &= a\tilde{\mu}^I - \sigma \delta^I \mu + \cdots, \quad (4.27) \\
    (\star K)^0 &= a\mu - \tilde{\sigma} \nabla \cdot \tilde{\mu} + \cdots. \quad (4.28)
\end{align}

Onsager relations require both terms that are zeroth order in derivatives to have the same coefficient, which is fixed to be the anomaly coefficient $a$ by Luttinger’s argument\(^{10}\). There are only two transport parameters $\sigma, \tilde{\sigma} \geq 0$, which are positive by the second law

---

\(^8\) Strictly speaking, one would also have to consider the conservation of energy. For simplicity we disregard this contribution which would just lead to an additional diffusive mode.

\(^9\) The dual susceptibility is related to the superfluid stiffness $f_s$ as $\tilde{\chi} = 1/f_s$.

\(^{10}\) The argument \cite{231} is as follows: in equilibrium the charge densities respond to background fields $\delta J_0 = \chi A_0$, and the currents vanish. Their constitutive relations should therefore be functions of $\frac{1}{\chi} \delta J_0 - A_0 = \delta \mu - A_0$. 

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of thermodynamics. Identifying temporarily \( * K \) with the gradient of the superfluid phase reproduces the equations of conventional superfluid hydrodynamics, in particular (4.28) gives rise to the Josephson relation (see e.g. Eqs. (11) and (13) in Ref. [232]).

The conservation equations read

\[
0 = \chi \partial_t \mu + a \partial_I \tilde{\mu}^I - \sigma \nabla^2 \mu + \ldots,
\]

\[
0 = \tilde{\chi} \partial_t \tilde{\mu}^I + a \partial_I \mu - \tilde{\sigma} \nabla^2 \tilde{\mu}^I + \ldots,
\]

\[
0 = \partial_I \tilde{\mu}_J - \partial_J \tilde{\mu}_I .
\]

These equations represent two physical (first order) modes, as the third equation above is a constraint. They combine into a single (second order) damped sound mode

\[
\omega = \pm \frac{a}{\sqrt{\chi \tilde{\chi}}} |k| - \frac{i}{2} \left( \frac{\sigma}{\chi} + \frac{\tilde{\sigma}}{\tilde{\chi}} \right) k^2 + O(k^3).
\]

Introducing a background dual potential \( \tilde{\mu}^I \neq 0 \) would lead to anisotropies in the speed of sound (as was shown in the non-dissipative treatment of section 4.1.3) and in the sound attenuation rate.

4.2.1 Phase relaxation and vortices

In this language, phase relaxation due to proliferating vortices is naturally captured as explicit breaking of the higher form symmetry. If the explicit breaking is weak (i.e. if the relaxation rate is small in units of the hydrodynamics cutoff), it can be incorporated in the hydrodynamics by replacing the higher form conservation equation with

\[
\delta \mu K^\mu = \Gamma u^\mu K^\mu + \ldots,
\]

to leading order in derivatives. Here we specialized to \( 2 + 1 \) dimensions so that vortices do not break isotropy. As usual with weak explicit breaking of symmetries, the relaxation rate \( \Gamma \) can be related to microscopic relaxation mechanisms via a Kubo formula [233]

\[
\Gamma \delta_{IJ} = \lim_{\omega \to 0} \frac{1}{\omega} \Re G^R_{K_{0I} K_{0J}}(\omega),
\]

See Refs. [232, 234] for applications of this Kubo formula to thin film incoherent superconductors. Generalized symmetries therefore allow to recast weak phase relaxation as weak breaking of higher form symmetries.

4.3 Relativistic superfluid hydrodynamics

In this section we will systematically construct the complete hydrodynamics of a system enjoying an (anomalous) \( U(1) \times U(1)^{(d-2)} \) symmetry, including its energy-momentum
sector, to first non-trivial order in derivatives. We will show that the result agrees precisely with previous results in the literature \([129, 235, 236]\), without invoking any extra assumptions except for the symmetries but with no reference to the character of their realization.

The way to construct a hydrodynamic theory is to write down the most general constitutive relations for all conserved quantities in the system in terms of equilibrium thermodynamic functions and the tensor structure that represents the (explicit) breaking of space-time symmetries.

### 4.3.1 Zeroth-order hydrodynamics

We want to build the hydrodynamical theory for \(d\)-dimensional relativistic superfluids. Therefore we must have a conserved energy-momentum tensor \(T^\mu_\nu\), a conserved current \(J^\mu\) and a second, anomalous, conserved current \(K^{\mu_1\cdots\mu_{d-1}}\) that is associated to the dual symmetry. We expect that the system can be completely described in terms of three scalars and two vectors that we take to be the temperature \(T\), two chemical potentials \(\mu\) and \(\tilde{\mu}\), a velocity vector \(u^\mu\) that specifies the rest frame and a vector \(h^\mu\) that specifies the orientation of the co-dimension 1 charged objects (i.e. planes) under \(K\). Since it is the codimension of these objects that is fixed, it is more convenient to write the hydrodynamics in terms of the Hodge dual \(\star K\) instead of \(K\). Furthermore we have the freedom to consider orthonormalized vectors as

\[
\begin{align*}
    u^\mu u_\mu &= -1, & h^\mu h_\mu &= 1, & u^\mu h_\mu &= 0. \\
\end{align*}
\]

In addition, we can define a projector onto the plane orthogonal to both \(u^\mu\) and \(h^\mu\) as

\[
\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u_\nu - h^\mu h_\nu,
\]

whose trace is \(\Delta^{\mu\mu} = d - 2\).

The most general expressions for the conserved tensors in terms of these quantities at zeroth order in derivatives are

\[
\begin{align*}
    T^{\mu\nu} &= (\epsilon + p - \tau)u^\mu u_\nu + (p - \tau)\eta^{\mu\nu} + \tau h^\mu h_\nu + \gamma u^{(\mu} h^{\nu)}, \\
    J^\mu &= \rho u^\mu + \sigma h^\mu, \\
    (\star K)^\mu &= \tilde{\sigma} u^\mu + \tilde{\rho} h^\mu,
\end{align*}
\]

where all scalar functions are understood to depend on \(T\), \(\mu\) and \(\tilde{\mu}\). We define symmetrization and antisymmetrization without the conventional factor of two, i.e \(u^{(\mu} h^{\nu)} = u^\mu h_\nu + u^\nu h_\mu\).

In the presence of a background field \(A_\mu\) for the current \(J^\mu\), the conservation equations
4.3. Relativistic superfluid hydrodynamics

read

\[ \partial_\mu T^{\mu \nu} = F^{\mu \rho} J_\rho, \] (4.40)
\[ \partial_\mu J^\mu = 0, \] (4.41)
\[ \partial_\mu (\ast K)_\nu = -a F_{\mu \nu}, \] (4.42)

where

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \] (4.43)

and we have allowed for an anomaly coefficient \(a\).

As a sanity check we see that we have 2\(d\) dynamical equations\(^{11}\) for 2\(d\) degrees of freedom contained in \(T, \mu, \tilde{\mu}, u^\mu, h^\mu\). While we expect one equation of state to fix the scalar functions \(p, \epsilon, \rho, \tilde{\rho}\) in (4.40-4.42) in terms of \(T, \mu, \tilde{\mu}\), the remaining four scalars must be fixed by other means.

First, the function \(\gamma\) can be fixed to any desired value by boosting the system in the \((u, h)\) plane. This preserves the norms, so it is an ambiguity of the parametrization. Normally we would like to pick \(\gamma = 0\), but we will keep it arbitrary for now as it simplifies the discussion of the anomaly. We will fix it by demanding that the entropy current is at rest in the frame given by \(u^\mu\) later on.

Second, \(\tau\) corresponds to the tension of the charged planes in the fluid. It can be uniquely fixed using a thermodynamic argument equivalent to the one displayed in [117] in the case of magnetohydrodynamics. It amounts to showing that this tension has to be a particular fixed function of \(\tilde{\mu}\) and \(\tilde{\rho}\) in order for our system to show the thermodynamic volume scaling characteristic of local theories. We reproduce this argument in appendix C.1, but we will shortly show that this is not necessary as this coefficient is fixed uniquely, as well, by entropy conservation.

Lastly, \(\sigma\) and \(\tilde{\sigma}\) will be fixed by the anomaly and its effect in the conservation of the entropy current. If there were no anomaly, we would just set \(\sigma = \tilde{\sigma} = 0\) as we would consider a frame where the charges are at rest simultaneously with the entropy. Once the anomaly is included we will see this is no longer possible.

This discussion makes the system of equations closed. The thermodynamics is hence completely fixed by a single function that is the pressure \(p(T, \mu, \tilde{\mu})\), and the relevant relations\(^{12}\)

\[ \epsilon + p = sT + \rho \mu + \tilde{\rho} \tilde{\mu}, \] (4.44)
\[ d\epsilon = Tds + \mu d\rho + \tilde{\mu} d\tilde{\rho}. \] (4.45)

\(^{11}\) Only \(d - 1\) dynamical equations can be derived from (4.42). The rest are constraints on the initial conditions.

\(^{12}\) Note that our definition of the pressure differs from the one in [236] and [237] because we want it to be symmetric in terms of tilde and non-tilde quantities. This also explains the difference in (4.37).
4. Superfluids as Higher-form Anomalies

Notice that here we are discussing co-dimension 1 charged planes as opposed to [117] where strings were present. This explains why the tension appears differently in (4.37) compared to [117].

Entropy current conservation and anomaly

We now want to show that the entropy current is conserved at this order in the hydrodynamic expansion. In the process, we will obtain the values of the yet undefined scalar functions. We consider the following combination of the equations of motion:

\[ \Omega = u_\nu \partial_\mu T^{\mu \nu} + \tilde{\mu} \partial_\mu J^\mu + \tilde{\mu} u^\nu h^\nu \left( \partial_\mu (\ast K)_\mu - \partial_\nu (\ast K)_\mu \right). \] (4.46)

This quantity can be computed using the constitutive relations (4.37-4.39) as well as the thermodynamic relations (4.44) and (4.45) but leaving \( \tau, \gamma, \sigma \) and \( \tilde{\sigma} \) arbitrary. We obtain

\[ \Omega = -T \partial_\mu (su^\mu) + (\tau - \tilde{\mu} \tilde{\rho}) \Delta^{\mu \nu} \partial_\mu u_\nu - (\gamma - \mu \sigma) \partial_\mu h^\mu + (\gamma - \tilde{\mu} \tilde{\sigma}) u^\mu u^\nu \partial_\mu h_\nu \]

\[ -h^\mu \partial_\mu \gamma + \mu h^\mu \partial_\sigma + \tilde{\mu} h^\mu \partial_\mu \tilde{\sigma}. \] (4.47)

On the other hand, we can also compute \( \Omega \) using the conservation equations (4.40), (4.41) and (4.42). In this case, we obtain

\[ \Omega = u^\mu h^\nu F_{\mu \nu} (\sigma - a \tilde{\mu}). \] (4.48)

If there were no anomaly \( (a = 0) \), it is trivial to see that \( \tau - \tilde{\mu} \tilde{\rho} = \gamma = \sigma = \tilde{\sigma} = 0 \) is the only possibility that yields a conserved entropy current which is at rest in the frame defined by \( u^\mu \). This is the statement that charges must be at rest in the same frame as the energy/entropy. With the anomaly, the equation (4.48) fixes \( \sigma = a \tilde{\mu} \) since it must be zero for arbitrary \( F_{\mu \nu} \). For the entropy to be conserved in arbitrary flows and backgrounds, we have to impose

\[ \tau = \tilde{\mu} \tilde{\rho}, \quad \gamma = a \mu \tilde{\mu}, \quad \sigma = a \tilde{\mu}, \quad \tilde{\sigma} = a \mu. \] (4.49)

Allowing for the identifications described in appendix C.2, this agrees exactly with the results from [236] and [235].

Notice that this manifestation of the interplay between anomalies and entropy current conservation is, in some way, a simpler version of the first example discussed in [213]. There the effect appeared at first order in the derivative expansion, while here it is already present at zeroth order.

4.3.2 First order hydrodynamics

Hydrodynamics is organised as a derivative expansion. The constitutive equations (4.37-4.39) are only the zeroth-order term in this expansion. Here, we construct the next order
4.3. Relativistic superfluid hydrodynamics

\[ T_{\mu\nu} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + \ldots , \]  
\[ J^\mu = J^\mu_{(0)} + J^\mu_{(1)} + \ldots , \]  
\[ (\ast K)^\mu = (\ast K)^\mu_{(0)} + (\ast K)^\mu_{(1)} + \ldots , \]

The first order corrections are parametrized in terms of scalar quantities called transport coefficients and dissipation appears at this order. The requirement that the entropy has to increase over time strongly constrains these corrections.

In constructing first order corrections, discrete symmetries such as charge conjugation \((C)\) and parity \((P)\) play an important role. Notice that, because of the anomaly, there is only one notion of charge conjugation that changes the signs of \(J\) and \(K\) simultaneously.

In this chapter, we assume the charge assignments displayed in Table 4.1.

<table>
<thead>
<tr>
<th>(T_{\mu\nu})</th>
<th>(J^\mu)</th>
<th>((\ast K)^\mu)</th>
<th>(u^\mu)</th>
<th>(h^\mu)</th>
<th>(\epsilon, \rho, \tau, \gamma)</th>
<th>(\rho, \mu, \sigma)</th>
<th>(\tilde{\rho}, \tilde{\mu}, \tilde{\sigma})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(C)</td>
<td>+</td>
<td>–</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 4.1 – Charges under discrete symmetries for 0-form symmetry.

The most general corrections that we can write for the first-order terms are

\[ T_{\mu\nu}^{(1)} = \delta \epsilon u^\mu u^\nu + \delta f \Delta^{\mu\nu} + \delta \tau h^\mu h^\nu + \ell^{(\mu} h^{\nu)} + m^{(\mu} u^{\nu)} + t^{\mu\nu} , \]  
\[ J^\mu_{(1)} = \delta \rho u^\mu + \delta \sigma h^\mu + j^\mu , \]  
\[ (\ast K)^\mu_{(1)} = \delta \tilde{\sigma} u^\mu + \delta \tilde{\rho} h^\mu + k^\mu . \]

In this decomposition, \(l^\mu, m^\mu, j^\mu\) and \(k^\mu\) are transverse vectors to both \(u^\mu\) and \(h^\mu\), and \(t^{\mu\nu}\) is a symmetric traceless tensor. Note that in (4.53), we have not added a term \(\delta \gamma u^{(\mu} h^{\nu)}\). This is because as explained previously, we can always boost our system in the \((u, h)\) plane to modify the value of \(\gamma\). Our frame is fixed once and for all at zeroth order by choosing the entropy current to remain at rest.

In hydrodynamics, we have the freedom to change the hydrodynamical frame. This is because the fluid variables \(\{u^\mu, h^\mu, \mu, \tilde{\mu}, T\}\) have no intrinsic microscopic definition out of equilibrium. The currents and the stress-energy tensor must be invariant under such redefinition. We use the scalar redefinitions of \(\mu, \tilde{\mu},\) and \(T\) to set \(\delta \rho = \delta \tilde{\rho} = \delta \epsilon = 0\) and the two vector redefinitions of \(u^\mu\) and \(h^\mu\) to set \(l^\mu = m^\mu = 0\). We end up with the simpler first order expansion:

\[ T_{\mu\nu}^{(1)} = \delta f \Delta^{\mu\nu} + \delta \tau h^\mu h^\nu + t^{\mu\nu} , \]  
\[ J^\mu_{(1)} = \delta \sigma h^\mu + j^\mu , \]  
\[ (\ast K)^\mu_{(1)} = \delta \tilde{\sigma} u^\mu + k^\mu . \]
To proceed, we need to determine the most general form of the first order corrections \( \{ \delta f, \delta \tau, \delta \sigma, \delta \tilde{\sigma}, j^\mu, k^\mu, t^\mu_{\nu} \} \) in terms of derivatives of fluid variables. This is done in appendix C.3. In any case, most possible structures do not appear as a consequence of the second law of thermodynamics to which we turn now.

The entropy current needs to be modified to first order in derivatives as

\[
S^\mu = su^\mu - \frac{1}{T} T^\mu_{\nu} u_\nu - \frac{\mu}{T} J^\mu_{(1)} - \frac{\tilde{\mu}}{T} (\ast K)_{(1)\nu} h^\nu u^\mu .
\]  

(4.59)

One can easily check that this combination is invariant under frame redefinitions as required [122]. Note that, in (4.59), we could have expected corrections coming from the anomaly as in [213]. However, this is not the case as the anomaly was already included at zeroth order.

We can now compute the divergence of this quantity. After some algebra, we obtain

\[
\partial_\mu S^\mu = -T^\mu_{\nu} \partial_\mu \left( \frac{u_\nu}{T} \right) - J^\mu_{(1)} \left( \partial_\mu \left( \frac{u_\nu}{T} \right) - \frac{u^\nu F_{\nu\mu}}{T} \right) - (\ast K)_{(1)\nu} \partial_\mu \left( \frac{\tilde{\mu}}{T} h^\nu u^\mu \right) .
\]  

(4.60)

The second law of thermodynamics implies that the right hand side of (4.60) must always be positive. Because the contributions to the divergence of the entropy current decompose is scalar, vector and tensor channels, we can impose positivity on each sector separately. This fixes completely the form the first order correction to the constitutive equations up to a number of transport coefficients. Concretely, in the tensor sector,

\[
t^\mu_{\nu} = -\eta (\Delta^\mu_\alpha \Delta^\nu_\beta - \frac{1}{d - 2} \Delta^\mu_{\nu} \Delta^\alpha_\beta) \partial_\alpha u_\beta ,
\]  

(4.61)

where \( \eta \) is the shear viscosity and must be positive. The vector sector yields,

\[
\begin{pmatrix} j^\mu \\ k^\mu \end{pmatrix} = -\Delta^\mu_\rho \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \partial_\rho \left( \frac{u_\nu}{T} \right) - \frac{u^\nu F_{\nu\rho}}{T} \\ \partial_\sigma \left( \frac{\tilde{\mu}}{T} h^\rho u^\sigma \right) \end{pmatrix} .
\]  

(4.62)

The matrix \( \Sigma \) of conductivities must be positive semi-definite implying \( \Sigma_{11} \geq 0 \) and \( \Sigma_{11} \Sigma_{22} \geq \Sigma_{12}^2 \). Onsager relations enforce that this matrix must be symmetric [122]. A small detail is that the vector structures used above contain terms that include time derivatives. This term can easily be removed by the considerations of appendix C.3 and written in terms of other structures if one wanted to preserve the nature of the initial value problem.

In the scalar sector,

\[
\begin{pmatrix} \delta f \\ \delta \tau \\ \delta \sigma \\ \delta \tilde{\sigma} \end{pmatrix} = - \begin{pmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} \\ \zeta_{12} & \zeta_{22} & \zeta_{23} & \zeta_{24} \\ \zeta_{13} & \zeta_{23} & \zeta_{33} & \zeta_{34} \\ \zeta_{14} & \zeta_{24} & \zeta_{34} & \zeta_{44} \end{pmatrix} \begin{pmatrix} \Delta^\mu_\nu \partial_\mu \left( \frac{u_\nu}{T} \right) \\ h^\mu h^\nu \partial_\mu \left( \frac{u_\nu}{T} \right) - \frac{u^\nu F_{\nu\mu}}{T} \\ h^\mu \partial_\mu \left( \frac{\tilde{\mu}}{T} \right) - \frac{u^\nu F_{\nu\mu}}{T} \\ h^\mu \partial_\mu \left( \frac{\tilde{\mu}}{T} \right) + \left( \frac{\tilde{\mu}}{T} \right) \Delta^\mu_\nu \partial_\mu h^\nu \end{pmatrix} .
\]  

(4.63)

Once again, the matrix of transport coefficients in equation (4.63) has to be symmetric due to Onsager relations on mixed correlation functions [122] as well as positive definite. This matrix contains terms such as bulk viscosities and components of the conductivity.
4.4 Generalization to higher-form superfluids

All in all, we have fourteen transport coefficients that are split as $1 + 3 + 10 = 14$ in the tensor, vector and scalar sectors respectively. This completely agrees with [236].

4.4 Generalization to higher-form superfluids

Using the technology of the previous section one can easily build the equivalent hydrodynamic theories for systems enjoying a $p$-form abelian symmetry $U(1)^{(p)}$ that is spontaneously broken. All the physics is, in this case, contained in an anomalous emergent $U(1)^{(d-p-2)}$ symmetry. We sketch an example of this construction for $p=1$ and $d=4$; generalizations to other cases are straightforward. This particular system describes the hydrodynamic behavior of Quantum Electrodynamics (QED) at energy scales below the electron mass. The Goldstone mode is none other than the (partially screened) photon.

The results obtained here match the construction in [238] in terms of an effective action.

4.4.1 1-form superfluid hydrodynamics

Consider a system with a $U(1)^{(1)} \times U(1)^{(1)}$ symmetry in $d=4$ in the presence of a background two-form gauge field $B$ that couples to one of the $U(1)$ currents (which we call magnetic, keeping the QED example in mind). The conservation equations for this system read

\[ \partial_\mu T^{\mu\nu} = H^{\nu\alpha\beta} J_{\alpha\beta}, \]
\[ \partial_\mu J^{\mu\nu} = 0, \]  
\[ \partial_\mu K^{\mu\nu} = -\frac{a}{3} \epsilon^{\nu\alpha\beta\gamma} H_{\alpha\beta\gamma}, \]

where

\[ H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} - \partial_\beta B_{\alpha\gamma} + \partial_\gamma B_{\alpha\beta}, \]

and $B_{\mu\nu}$ is a two-form gauge potential. Notice that this system, in a non-trivial state, possesses no continuous space-time symmetries, as the magnetic and electric field can point in arbitrary directions. In these conditions a new situation arises: the charges, even in equilibrium, need not be collinear with the chemical potentials. As all space-time symmetries are broken we can pick a basis of orthonormal vectors:

\[ u_\mu u^\mu = -1, \quad h_\mu h^\mu = e_\mu e^\mu = 1, \quad u^\mu h_\mu = u^\mu e_\mu = h^\mu e_\mu = 0, \]

and write

\[ \mu^\mu = \mu h^\mu, \quad \tilde{\mu}^\mu = \tilde{\mu}_\parallel h^\mu + \tilde{\mu}_\perp e^\mu. \]

Here, $u^\mu$ is the fluid velocity as in conventional hydrodynamics, $h^\mu$ indicates the direction of the magnetic chemical potential related to $J$ while the electrical analog quantity related
to $K$ is contained in the $(h, e)$ plane. The most general constitutive relations for the currents are

$$J^{\mu\nu} = \rho u^{[\mu} e^{\nu]} + \sigma || e^{\mu\rho\sigma} u_{\rho} h_{\sigma} + \sigma \perp e^{\mu\rho\sigma} u_{\rho} e_{\sigma},$$

(4.70)

$$K^{\mu\nu} = \tilde{\rho} || u^{[\mu} h^{\nu]} + \tilde{\rho} \perp u^{[\mu} e^{\nu]} + \tilde{\sigma} e^{\mu\rho\sigma} u_{\rho} h_{\sigma} + \tilde{\sigma} \times e^{\mu\rho\sigma} u_{\rho} e_{\sigma}.$$  

(4.71)

Because charges and chemical potentials don’t need to be aligned, we cannot remove any of the structures above. Notice that equations (4.70-4.71) allow the inclusion of a parity odd structure. This follows from the existence of a parity odd scalar in this system. In QED this is the familiar scalar product between the electric and magnetic field. In Table 4.2 we display our conventions for charge conjugation ($C$), which as in the previous section reverses the sign of both $J$ and $K$, and parity ($P$), appropriate for QED.

$$\begin{array}{cccccccccc}
T^{\mu\nu} & J^{\mu\nu} & K^{\mu\nu} & u^{\mu} & h^{\mu} & e^{\mu} & \epsilon, p, \tau, \tilde{\tau}, \gamma & \varphi \\
P & + & - & + & + & - & + & - \\
C & + & - & - & + & - & + & + \\
\end{array}$$

Table 4.2 — Charges under discrete symmetries for 1-form symmetry.

With these charges under discrete symmetries, we can write the constitutive equation for the stress-energy tensor

$$T^{\mu\nu} = (\epsilon + p)u^{\mu} u^{\nu} + p \eta^{\mu\nu} - \tau h^{\mu} h^{\nu} - \tilde{\tau} e^{\mu} e^{\nu} - \varphi h^{(\mu} e^{\nu)} - \gamma \epsilon^{\mu\sigma\rho\gamma} u_{\rho} h_{\sigma} e_{\gamma} u^{\nu)},$$

(4.72)

where $\epsilon$ is the energy density, $p$ is the pressure and $\tau, \tilde{\tau}, \varphi$ parameterize the stress tensor of magnetic and electric strings\textsuperscript{13}. The $\gamma$ term contains the effect of the anomaly. While the symmetries allow another term quadratic in $\epsilon^{\mu\nu\rho\sigma}$, this term in not linearly independent from the $\eta^{\mu\nu}$ term.

The thermodynamics is completely specified by an equation of state and the relevant thermodynamic relations are

$$\epsilon + p = s T + \rho_{\mu} u^{\mu} + \tilde{\rho}_{\mu} \tilde{u}^{\mu},$$

(4.73)

$$d\epsilon = T d s + \mu_{\mu} d \rho^{\mu} + \tilde{\mu}_{\mu} d \tilde{\rho}^{\mu}.$$  

(4.74)

\textsuperscript{13} In principle, techniques similar to those displayed in appendix C.1 can be used to find the values of $\tau, \tilde{\tau}, \varphi$. In this case one needs to consider more general volume preserving deformations of the fluid element, not present for the 0-form case. These are important in the theory of elasticity and have been considered in a modern setup recently in [119].

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where
\[ \rho^\mu = \rho h^\mu + \rho_\times e^\mu, \quad \tilde{\rho}^\mu = \tilde{\rho}_|| h^\mu + \tilde{\rho}_\perp e^\mu. \] (4.75)

In a less covariant, but more transparent notation these equation can be rewritten as:
\[ \epsilon + p = sT + \rho \mu + \tilde{\rho}_|| \tilde{\mu}_|| + \tilde{\rho}_\perp \tilde{\mu}_\perp, \] (4.76)
\[ \vartheta = sT d\sigma + \mu d\rho + \mu \rho_\times h \mu d\mu + \tilde{\mu}_|| d\tilde{\rho}_|| + \tilde{\mu}_\perp d\tilde{\rho}_\perp + \tilde{\mu}_\perp \tilde{\rho}_|| e_\mu dh^\mu. \] (4.77)

Provided we can use the conservation of the entropy current and the anomaly argument from the previous section to fix uniquely \( \sigma_||, \sigma_\perp, \tilde{\sigma}, \tilde{\sigma}_\times, \tilde{\tau}, \varphi, \gamma, \rho_\times, \) the above is a closed system of equations. In total, there are ten hydrodynamic variables in \( \mu, \tilde{\mu}_||, \tilde{\mu}_\perp, T, u^\mu, h^\mu, e^\mu \) and twelve equations of motion of which two are constraints, making the system closed.

Notice that one of the densities, which we choose to be \( \rho_\times \) needs to be fixed. This is consistent with the fact that there are (due to rotational symmetry) only 3 chemical potentials \( (\mu, \tilde{\mu}_||, \tilde{\mu}_\perp) \) as, covariantly, the pressure \( p \) can only be a function of \( (\mu \cdot \mu, \tilde{\mu}_|| \cdot \tilde{\mu}_||, \mu \cdot \tilde{\mu}_\perp) \). As a consequence, only 3 densities can be independent. This is equivalent to the final expressions
\[ \epsilon + p = sT + \rho \mu + \tilde{\rho}_|| \tilde{\mu}_|| + \tilde{\rho}_\perp \tilde{\mu}_\perp, \] (4.78)
\[ d\epsilon = T ds + \mu d\rho + \tilde{\mu}_|| d\tilde{\rho}_|| + \tilde{\mu}_\perp d\tilde{\rho}_\perp, \] (4.79)
\[ dp = sdT + \rho d\mu + \tilde{\mu}_|| d\tilde{\rho}_|| + \tilde{\mu}_\perp d\tilde{\rho}_\perp, \] (4.80)
\[ \rho_\times = \frac{\tilde{\mu}_\perp \tilde{\rho}_|| - \tilde{\mu}_|| \tilde{\rho}_\perp}{\mu}. \] (4.81)

We now proceed as with the 0-form case and demand the entropy current to be conserved. Consider
\[ \Omega = u_\nu \partial_\mu T^{\mu\nu} + \mu h_\nu \partial_\mu J^{\mu\nu} + \tilde{\mu}_|| h_\nu \partial_\mu K^{\mu\nu} + \tilde{\mu}_\perp e_\nu \partial_\mu K^{\mu\nu}. \] (4.82)

This quantity can be computed from the conservation equations (4.64-4.66) to give
\[ \Omega = H^{\nu\alpha\beta} \left[ u_\nu \epsilon_{\alpha\beta\rho\sigma} u^\rho \left( \sigma_|| h^\sigma + \sigma_\perp e^\sigma \right) - \frac{a}{3} \epsilon_{\lambda\nu\alpha\beta} \left( \tilde{\mu}_|| h^\lambda + \tilde{\mu}_\perp e^\lambda \right) \right]. \] (4.83)

This must vanish for arbitrary \( H^{\nu\alpha\beta} \) in order for the entropy to be conserved. On the other hand, using constitutive relations (4.70), (4.71) and (4.72) we must obtain \( \Omega = -T \partial_\mu (su^\mu) \). Both these conditions can only be satisfied provided
\[ \tau = \mu \rho + \tilde{\mu}_|| \tilde{\rho}_||, \] (4.84)
\[ \varphi = \tilde{\mu}_\perp \tilde{\rho}_||, \] (4.85)
\[ \rho_\times = \frac{\tilde{\mu}_\perp \tilde{\rho}_|| - \tilde{\mu}_|| \tilde{\rho}_\perp}{\mu}, \] (4.86)
\[ \sigma_|| = a \tilde{\mu}_||, \] (4.87)
\[ \tilde{\sigma}_\times = 0, \] (4.88)

14. This result also follows direct from entropy conservation as explained below.
This results in a hydrodynamic system equivalent to the one presented in [238], if one considers the set of identifications presented in appendix C.2.

With this information, it is straightforward to follow the standard procedure outlined in the previous section and construct higher order corrections to the constitutive relations in the derivative expansion. We will not do this in this present chapter, but refer the reader instead to [238] for the general structure of these corrections, albeit in a different formalism.
4.5 Outlook

In this chapter, we have shown that the masslessness of bosons coming from spontaneous breaking of abelian symmetries (superfluids, photons, etc.) can be interpreted as being protected by an anomaly, analogously to the masslessness of fermions. This observation was upgraded into a fundamental principle by reversing the logic and classifying certain phases of matter by their (higher-form) symmetries and their anomalies without reference to how the symmetries are realized. As an example of this program we have presented constructions of 0-form and 1-form superfluids in a systematic fashion in a formalism that puts the Josephson relation on equal footing with the other conservation equations.

It is tempting to explore the consequences of this new paradigm. For example, are all gapless phases protected by anomalies? Goldstones for non-abelian symmetries are parametrized by an element of a coset \( g \in G/H \). The natural generalization of the current \( \star K = d\phi \) to this situation is the Maurer-Cartan form \( \star K = g^{-1}dg \). This current fails to be conserved but instead satisfies the Cartan structure equation

\[
\frac{d\star K}{\star K} = -(\star K) \wedge (\star K). \tag{4.89}
\]

Since sigma models are IR-free in \( d \geq 3 \), the theory abelianizes at low energies where one can neglect the non-linear term in (4.89). A mixed abelian anomaly of the form (4.4) can therefore also be said to protect the massless modes of this non-abelian theory.

There are other generalizations to non-abelian groups that are possible in certain circumstances. Although higher-form symmetries are always abelian (because the objects counting higher-form charges have enough codimensions to be swapped nonviolently), the 0-form symmetry can be non-abelian. The anomaly (4.4) is canceled by inflow from a bulk term \( \int B \wedge F \). One natural generalization is for the bulk term to be replaced with \( \int B \wedge \text{Tr} F^n \). For \( n \) even, a theory with an anomaly of this form is given by the \( G_L \times G_R/G_{\text{diag}} \) sigma-model. Focusing on \( n = 2 \) for concreteness, this theory can have a closed Wess-Zumino (WZ) 3-form \( \omega^{(3)} \) (which can be used to add a WZ term to the theory in 2 dimensions), whose closedness is spoiled if a certain subgroup \( F \subset G_L \times G_R \) is gauged [239] – a simple example is \( F = G_L \). The best improvement of \( \omega^{(3)} \) that one can construct then satisfies

\[
d\omega^{(3)} \propto \text{Tr} \left( F_L^2 - F_R^2 \right). \tag{4.90}
\]

The theory therefore contains an anomalous \( U(1)^{(d-4)} \) symmetry carried by the \( (d-3) \)-from current \( \star K \equiv \omega^{(3)} \) – this is the skyrmion number current [240] \(^{15}\). See Ref. [241] for further obstructions to gauging WZ terms. Interesting generalizations that can accommodate a non-abelian structure in a more fundamental fashion, such as connections to 2-groups [222], should be pursued. We leave this for future work.

\(^{15}\) Note that in the context of chiral symmetry breaking, this symmetry is not emergent but carries the baryon number \( U(1) \) present in the UV, as required by anomaly matching.
The ideas discussed here are in line with recent progress in incorporating higher form symmetries in hydrodynamics [117–119, 238, 242–246] and we expect to see further developments in this area.

The identification of higher form symmetries in this chapter also revealed the fact that the BKT transition is a regular Landau transition between two phases with different symmetries (and anomalies). One can then ask: which phases are truly non-Landau, after generalized symmetries (continuous and discrete) and their anomalies are taken into account? Certain fractional quantum hall phases\(^\text{16}\) can also be distinguished by the symmetries of their effective Chern-Simons descriptions. A discrete higher-form symmetry similarly distinguishes both phases of the Ising model, obviating the need to specify whether the symmetries are spontaneously broken or not. It is important to understand if and under which conditions the Landau paradigm effectively fails.

It would also be of interest to use this new point of view to shed light on the traditional treatment of superfluids within the gauge/gravity duality [237, 247, 248]. The proper treatment of higher form symmetries within holography requires the inclusion of bulk Chern-Simons terms and a careful consideration of the boundary conditions in some cases [120]. A second look at this system might provide a holographic version of the anomaly inflow mechanism described in section 4.1.1, giving a clearer connection to the study of topological phases.

\(^{16}\) Interestingly, the simpler case of integer quantum hall states is more resistant to the Landau paradigm treatment. D.H. thanks Anton Kapustin and David Tong for discussions on this issue.
In this final chapter, we want to summarize the main findings that we presented in this thesis, and put them into perspective. We will finally provide an outlook, where we mention several interesting directions for future work.

5.1 Summary

5.1.1 Light-ray operators

In chapters 2 and 3 of this thesis, we investigated different open problems in high-energy physics using non-local operators in CFT, which are commonly known as light-ray operators. In particular, we focused on the ANEC operator as well as generalizations thereof to be able to decipher higher-dimensional conformal field theories. The goal was twofold: First, we wanted to understand what are the necessary and sufficient conditions that a four-dimensional CFT has to satisfy to admit a weakly-coupled Einstein gravity dual in the bulk. This was the motivation for chapter 2. Second, we investigated a possible algebra amongst light-ray operators that was conjectured to hold in previous literature. This proposed algebra is some sort of higher-dimensional analog of the Virasoro algebra. This is the content of chapter 3. Such an algebra would be extremely powerful because it could provide a new set of constraints that need to be obeyed in consistent higher-dimensional CFTs.

Einstein gravity from the CFT

The positive-definiteness of the ANEC operator in quantum field theory can be used to bound OPE coefficients in higher-dimensional CFTs, as we reviewed in section 1.4.2 [91]. In particular, no further bounds can be derived from the three-point function of the ANEC operator but the conformal collider bounds [91]. To bring this program further, and derive new bounds, it is necessary to be able to evaluate higher-point correlation functions of the ANEC operator.
In chapter 2, we studied correlation functions of the ANEC operator with external states created by local operators in a CFT with large $N$ and large $\Delta_{\text{gap}}$. We developed a concise way to evaluate such correlators with multiple ANEC insertions. The main idea is to recast the OPE between the ANEC operator and a local operator as a differential operator acting on the same local operator, which holds within correlation functions.

The starting point is the local three-point function of the stress-energy tensor with external local operators, which is fixed by conformal symmetry. We performed the light-ray integral to obtain an ANEC operator from the local stress-energy tensor. We then expanded this correlator and mapped it order by order to the integrated version of the OPE between the stress-energy tensor and a given local operator. This allowed us to derive differential operators that replace the action of the ANEC operator within correlation functions, and depend on the external states that are considered (or the states that are considered in the OPE). These differential operators were obtained as infinite series expansions, but we were able to resum them. Once the ANEC operator is sent far away, it can be interpreted as a calorimeter measuring the energy on the celestial sphere at infinity. In this limit, these differential operators admit a particularly simple expression that can be used to compute correlation functions recursively.

In the case of three-point functions with a single ANEC insertion, the differential operator is exact because the remaining two-point functions are orthogonal. Nevertheless, in a large $N$, large gap CFT, the contributions that originate from double-trace operators completely dropped out from our computations at leading order in a large $N$ expansion. Moreover, the large gap assumption ensures that only the operator $\mathcal{O}$ contributes in the $T_{\mu\nu}\mathcal{O}$ OPE. This implies that the differential operators that we derived using three-point functions can be used to access higher-point functions of multiple ANEC insertions by acting recursively with a sequence of differential operators on the two-point functions of local operators. This is the main benefit of this differential operator representation of the ANEC operator. In addition, this is similar to what happens in two-dimensional CFTs, where higher-point functions of local stress-energy tensors can be accessed using the conformal Ward identities, as we reviewed in section 1.3.3.

With this technology, we computed four-point functions involving two ANEC operators in states created with local stress-energy tensors. By subtracting two such appropriately ordered four-point functions, we obtained the ANEC commutator in external stress-energy tensor states. This commutator must vanish. Requiring this implies strong constraints on the OPE coefficients of the theory (or equivalently the anomaly coefficients $a$ and $c$) such that we obtained

$$a = c.$$  \hfill (5.1)

This is the CFT version of the following statement: any large $N$, large gap theory must have a holographic dual with Einstein gravity minimally coupled to matter, and the conformal collider bounds, which are a consequence of the ANEC, must get strengthened. We have thus shown that in four dimensions, large $N$ and large gap are sufficient to obtain a holographic CFT.
A possible algebra amongst light-ray operators

Studying the space of CFTs and trying to systematically solve and classify them is one of the most important and interesting open problems in high-energy physics, as we have already explained in the introduction. While this program is way more advanced in two dimensions, the higher-dimensional case is still more mysterious. At this stage, it is not even clear whether this task can be carried to completion in full generality. One might hope that some simplifying assumptions could help us decipher some of these theories. In particular, some possible assumptions could be large $N$, large gap, the conjunction of both, or even supersymmetry. In these cases, the odds of being able to bootstrap these theories are higher.

In recent years, this question has triggered a lot of attention, and powerful constraints that need to be true in any consistent QFT have appeared in the literature. They usually follow from unitarity of UV complete QFTs and materialize as sum rules (or positivity bounds similar to the conformal collider bounds reviewed in section 1.4.2).

In two dimensions, Virasoro symmetry fixes all higher-point functions of the stress tensor. In particular, they can be computed using a differential operator and acting on lower-point functions (you can see section 1.3.3). We have seen in chapter 2 that the same was true for the ANEC operator in large $N$ and large $\Delta_{\text{gap}}$ CFTs once the action of the ANEC is recast as a differential operator. This hints towards the fact that it might be possible to compute the higher-point functions of light-ray operators in higher-dimensional CFTs, in a way similar to the two-dimensional case. It might thus be possible to solve the sector of higher-dimensional CFTs that is spanned by light-ray operators. This would be a major step forward in the program of bootstrapping higher-dimensional CFTs, and light-ray operators play a crucial role in this program.

In chapter 3, we investigated a possible algebra amongst light-ray operators built as $L_n = \int dx^- (x^-)^{n+2} T_{--} (x^-)$ that we presented in section 3.3. These operators are the natural higher-dimensional generalizations of the usual two-dimensional Virasoro generators. For concreteness, we started our computations in free field theories. In this case, computing the correlation functions necessary for our analysis did not require knowing the conformal blocks explicitly, as the correlators are just given by Wick contractions. We continued with holographic CFTs at large $N$ and large gap. For the case of holographic CFTs, we first performed the computations on the QFT side before reproducing our results in the AdS bulk.

To compute the commutators of interest in CFT, we evaluated the four-point function of two stress-energy tensors in states created by local scalar states. The strategy was to compute the local four-point functions and then to perform the light-ray integrals (introducing the appropriate weighting factor) to obtain light-ray operators from local stress-energy tensors. The operators within correlation functions needed to be time ordered, and this was achieved using the usual $i\epsilon$ prescription in Lorentzian time. This technique is just an efficient calculational method when the integrals that we needed to evaluate are convergent and boils down to a regularization scheme when they are not.
Physically, this amounts to considering matrix elements for these operators on states localized enough in the lightlike direction \( x^- \).

We uncovered the collinear conformal group, which is a subgroup of the conformal group. It contains the subset of conformal transformations that leave the light-ray invariant. There is a five-dimensional subalgebra, spanned by the operators \( L_n \) with \( n \in \{ -2, -1, 0, 1, 2 \} \) which closes under the action of this collinear conformal group. These five operators, which we called global operators, annihilate the conformally-invariant vacuum, as their two-dimensional analogs, and thus have vanishing two-point functions. For operators outside this set, the two-point function is non-vanishing. We then computed a possible central term for the infinite-dimensional algebra. This central term is nevertheless not the naive one that we would have expected, as this one is forbidden by the collinear subgroup.

The upshot of the analysis of chapter 3 is that even in free field theories, we stumbled upon discrepancies with the proposed algebra. For example, \( L_1 \) and \( L_2 \), despite being members of the global subalgebra, failed to commute at finite spacelike separation. This non-commutativity is non-integrable, which already spells doom of the proposed algebra in free field theory.

For holographic CFTs, we might have expected better behaving commutators. We performed a conformal block expansion in momentum space and concluded that even \([L_{-2}, L_{-1}]\) is not commuting at finite spacelike separation. In this case, the finite separation contribution was integrable. This non-commutativity is a consequence of the fact that only one conformal block propagates between the insertions of generalized ANEC operators in the channel that we considered. This conflicts with the expectation that this commutator needs to vanish at finite spacelike separation for any finite \( N \) CFT \([106]\). This discrepancy can be explained by a discontinuity in the infinite \( N \) limit of Regge trajectories. The commutativity of \([L_{-2}, L_{-1}]\) should ultimately be restored by non-perturbative effects, and we will comment on this further in the next section.

We also compared our holographic CFT results with the appropriate bulk computations. In the bulk, we first needed to derive exact shockwave solutions that are dual to the insertions of exponentiated generalized global ANEC operators in the CFT. These previously unknown geometries are solutions to the Einstein equation. With these solutions at hand, we computed the propagation of perturbations in their background, which allowed us to compute the holographic commutators of generalized ANEC operators in the bulk. We found perfect agreement between these results and the holographic CFT computations.

Finally, we see that studying light-ray operators can provide important results that constrain the space of consistent UV complete QFTs. The main advantage of light-ray operators compared to their local counterparts is that the properties of integrated operators are dramatically simpler than the one of local operators, as we have seen in chapter 2 and 3. This is because the light-ray integral projects out a lot of information, and radically simplifies the correlation functions. For higher-dimensional CFTs, bootstrapping all the correlation functions in full generality remains out of reach. Nevertheless,
5.1. Summary

Light-ray operators are the ideal candidate to decipher the landscape of allowed consistent higher-dimensional CFTs, and we expect that they will provide important results in the coming years.

5.1.2 Higher-form symmetries and hydrodynamics

In chapter 4, we investigated the interplay between higher-form symmetries and the mechanism of spontaneous symmetry breaking. In particular, when a system experiences the spontaneous breaking of an Abelian symmetry (in our case superfluids) there are bosons in the spectrum that are massless. This is just the statement of Goldstone’s theorem. In chapter 4, we showed that we can understand the masslessness of these bosons using an anomaly that connects a regular and a higher-form symmetry. This anomaly protects the massless phase of the theory.

Spontaneous symmetry breaking and the appearance of a second emergent conserved current are related, as we already explained in section 1.5. In our case, the spontaneous breaking of a regular $U(1)$ symmetry led to an emergent $(d-2)$-form $U(1)^{(d-2)}$ symmetry. These two symmetries are connected through a mixed anomaly given by the curvature of the regular $U(1)$. This anomaly can occur in any dimension $d$. This is the starting point for our alternative proof of Goldstone’s theorem. In particular, we reversed the logic and proved that a system with a regular $U(1)$ and a higher form $U(1)^{(d-2)}$ symmetries that are connected through a mixed anomaly of the type we just described contains a massless Goldstone boson that transforms non-linearly in its spectrum. The starting point for such a proof of the Goldstone theorem is the anomalous Ward identity, and we never needed to make explicit reference to spontaneous symmetry breaking, nor have an operator acquiring an expectation value. This implies that the symmetry structure that we considered is a weaker assumption than spontaneous symmetry breaking and that it is thus more general. As a consequence, we can reformulate phases of systems where there is an Abelian symmetry that gets spontaneously broken in terms of mixed anomalies.

With this point of view in mind, it is then tempting to adjust Landau’s paradigm, which is the usual way we understand continuous phase transitions. The idea of Landau’s theory is to see the free energy as an analytic function and to perform a Taylor expansion of the free energy in terms of the order parameter $o(T)$ in the vicinity of the critical temperature $T_c$ where the phase transition happens. Because the free energy must be bounded from below, the highest term in the expansion should be positive. We can then minimize the free energy with respect to the order parameter and we have two solutions. For $T < T_c$, the minimum is at $o(T) \neq 0$, and we have a stable phase. For $T > T_c$, the solution is at $o(T) = 0$. Hence, a phase transition can be characterized by the temperature where the order parameter deviates from zero.

With our point of view, we aim at classifying phases by identifying which generalized symmetries are present in each phase, along with their anomalies, but disregarding how they...
are realized. This means that we do not need to know whether the symmetries are linearly realized (no spontaneous symmetry breaking) or non-linearly realized (spontaneous symmetry breaking). We have illustrated this in the introduction, where we considered a toy example of a two-dimensional effective field theory where we could distinguish both phases by their symmetry structure only. This was done in section 1.5.3.

In the second part of chapter 4, we reconsidered the superfluid hydrodynamics. As we reviewed in section 1.6, in hydrodynamics, we insisted on the global symmetries present in our problem, and not on how they are realized. We recast superfluid hydrodynamics as the hydrodynamics for a theory with $U(1) \times U(1)^{(d-2)}$ symmetries that are connected through a mixed anomaly. In this formulation, there are three conserved currents, the stress-energy tensor, the $U(1)$ current, and the $U(1)^{(d-2)}$ current for which we wrote conservation laws and constitutive equations. This puts all hydrodynamics equations on equal footing. In particular, the Josephson condition which needs to be added by hand in the usual treatment of the subject is now contained in the conservation equation of the higher-form symmetry current. This provided the simplest example of how anomalies enter the hydrodynamics expansion. In this case, the anomaly fixed terms in the expansion already at ideal order. We also concluded by discussing the case of 1–form superfluid hydrodynamics.

5.2 Outlook

In this final section, we would like to mention some future directions in which the results presented in this thesis can be either improved or used to derive new constraints on quantum gravity and strongly-coupled QFTs. These have already been discussed to some extent in the conclusion of each chapter of this thesis, but we would like to give more details here.

5.2.1 Light-ray operators

Einstein gravity from the CFT

The coefficients of the three-point function of stress-energy tensors in CFT (or equivalently the coefficients of the higher-derivative terms in the bulk effective action) have to be suppressed in a large gap theory. This follows from a careful study of causality of the graviton three-point vertex. The precise statement is the one of equation (1.33), which reads

$$c_n \lesssim \Delta^{-n}_{gap}. \quad (5.2)$$

In four dimensions, this translates to the following parametric bound in terms of anomaly coefficients $a$ and $c$

$$\frac{|a - c|}{c} \lesssim \Delta^{-2}_{gap}. \quad (5.3)$$
This bound, despite being true, is somewhat unsatisfactory. In particular, it lacks a quantitative understanding of the relation between anomaly coefficients. Finding a more quantitative bound that relates the anomaly coefficients in four dimensions would be a particularly interesting direction to pursue. The goal is to find a relation of the form (5.3) that would be a strict inequality. Ultimately, we would like to obtain a bound of the form

$$\frac{|a - c|}{c} \leq \# \Delta_{\text{gap}}^{-2},$$

for some number (or possibly function) #. Getting a quantitative understanding of this kind of bound is a natural future direction to consider. At the time of writing this thesis, a new paper made great progress in this direction [249].

In addition, a related important direction to pursue is the relaxing of the infinite gap to higher spin operators assumption. In chapter 2, we have considered an effectively infinite $\Delta_{\text{gap}}$ theory, which implies that $a = c$. One could keep a large but finite value of $\Delta_{\text{gap}}$ and study the subsequent corrections to $a = c$. This would require an expansion in $\Delta_{\text{gap}}^{-1}$, and would provide precise expressions of the form (5.4). In particular, this would require a precise understanding of which operators can now contribute in the OPE between an ANEC and a local operator in terms of $\Delta_{\text{gap}}$. It still seems unlikely that a finite number of operators could modify the rigid constraints that follow from requiring the commutator of two ANEC operators to vanish. If we perform this computation in the channel that we used throughout chapter 2, we would just conclude that non-minimal couplings to these new heavy operators must also vanish. Hence, to be able to derive useful quantitative results, we need to add an infinite number of heavy operators, whose contribution to $a = c$ must also vanish. Hence, to be able to derive useful quantitative results, we need to add an infinite number of heavy operators, whose contribution to the OPE needs to be determined. This is a technically really difficult task, but this is equivalent to the inclusion of a finite number of heavy operators in the cross-channel. This is where this analysis should be performed. This would require solving the crossing equations (1.48) at each order in the expansion.

Moreover, we could also try to get a better quantitative understanding of the bound (5.3) in terms of large $N$. We would also consider a large but finite $N$ theory and study corrections to these bounds in terms of $N^{-1}$. This problem seems more tractable, but we would need to understand how to account for double trace operators propagating in the four-point function, which are effectively suppressed at infinite $N$. A possible way to do this would be to use the construction of [250], where they considered a slight breaking of higher spin symmetry and computed subleading corrections in $1/N$. A related question is the sign of these corrections. Do the corrections to the ratio $a/c = 1$ have a universal sign? If so, this would be useful to characterize couplings at loop level and stringy corrections in the bulk.

A final direction that we would like to mention is the one of six-dimensional CFTs. In this case, there are four anomaly coefficients which are $a$, $c_1$, $c_2$ and $c_3$. For non-supersymmetric theories, they are all independent but supersymmetry relates the $c_i$ between each other. For example, there is effectively only one $c$ in six-dimensional theories with $(2, 0)$ supersymmetry. These central charges are notoriously hard to obtain as some of them are hidden in the three-point functions but appear explicitly in the four-point...
functions. Using our technology with differential operators, it might be possible to access these anomaly coefficients in six-dimensional correlation functions. The reason why we think it is possible is that light-ray operators tend to project out a lot of information, and this is why they are inherently simpler to handle than local operators. Using this technology to study six-dimensional CFTs could then yield important results on the central charges (or anomaly coefficients). We can also hope to be able to derive bounds relating them, similarly to the conformal collider bounds in four dimensions.

A possible algebra amongst light-ray operators

A first natural direction in which to extend the work of chapter 3 is centered on the existence and well-definiteness of light-ray operators of the form considered in this chapter. Evaluating the correlation functions involving operators that are not global (i.e. are outside of the five-dimensional subalgebra that was mainly considered in chapter 3) is more involved. In particular, operators \( L_n \) with \( n \leq -3 \) have an extra pole at zero, which is introduced by the weighting factor of the local operators that we integrate. We need to decide on a prescription to evaluate the correlation functions involving these operators. In chapter 3, we picked a prescription that amounts to a specific choice of \( i\epsilon \) such that they mimic their two-dimensional analogs. For operators \( L_n \) with \( n \geq 3 \), this is more involved. The reason is that these operators have a pole at infinity that is introduced by the factor \((x^-)^{n+2}\) that needs to be integrated. For global operators, the integrals are convergent and hence this subtlety is not important. In other words, for these integrals, we can close the contour in any possible way and obtain the correct answer. For non-global operators, this is not the case, and we need to be careful in evaluating these integrals. At first sight, it is not even clear that the operators that are outside of the global family are well-defined in CFT and that their correlation functions are not infinite in most external states. Hence, investigating this question more thoroughly is a natural way to expand on the work presented in chapter 3. Note that another set of possible four-dimensional light-ray operators with better behaving commutators was presented in [4]. A more careful study of these operators could also yield a well-defined algebra in higher-dimensional CFTs.

A related question concerns the central charge. In chapter 3, we determined a possible central charge for the operators that lie outside the subalgebra spanned by \( L_n \) with \( n \in \{-2, -1, 0, 1, 2\} \). We realized that this central term is divergent when the operators lie on the same null-plane at a given \( x^+ \) in the limit \( \epsilon \to 0 \). To regularize this divergence, we could modify the definition of our operators. A natural guess is to modify the definition of the operators we considered as \( L_n \to \epsilon^{n+1}L_n \). While this regularizes the central term that we found and makes it finite, this also preserves the form of the proposed algebra. Nevertheless, this also trivializes most of the correlators involving \( L_n \)'s which are now zero. Some of them even become divergent. This redefinition of our operators amounts to inserting an explicit UV cut-off scale in the definition of the operators. Investigating whether this renders the algebra better behaved, or whether another regularization that does not spoil most of the correlator exists is an interesting direction in which to expand
5.2. Outlook

on the work presented in chapter 3.

Moreover, we have shown in chapter 3 that the proposed algebra does not hold in free field theory nor holographic CFTs. We have discovered non-zero commutators at finite spacelike separations. It may nevertheless be possible to find a well-defined algebra by modifying the definition of the operators \( L_n \). Away from \( \vec{x}^\perp = 0 \), the action of the collinear algebra mixes different components of the stress-energy tensor with \( T_{--} \) (this is contained in equation (3.34)). It is possible that using a clever redefinition of the operators \( L_n \), where we would add other components of the stress-energy tensor, we could be able to cancel (or just soften) the finite separation contributions that we derived in chapter 3. In this way, we might be able to have a well-defined algebra with properly defined commutators. Provided we could find an algebra of this sort, this would have all the implications we already discussed. Another indication that we could find a well-defined algebra by redefining the operators we are considering comes from the bulk computation, and we will return to it shortly.

For holographic CFTs, we witnessed that the finite separation contributions are present in even simpler commutators. The simplest example is the commutator between \( L_{-2} \) and \( L_{-1} \). These two operators fail to commute at spacelike separation. This is a statement at leading order in \( 1/N \) and \( 1/\Delta_{\text{gap}} \). We have explained in chapter 3 why this result is not so surprising given the analysis of [108]. The commutator \([L_{-1}, L_{-2}]\) lies right outside the bounds of guaranteed commutation that they derived. In the same reference, it was nevertheless also shown that the commutators of interest should not have any finite spacelike separation contribution in a physical theory at finite \( N \) and finite \( \Delta_{\text{gap}} \). In a holographic CFT, only \( O \) (which is the external state in which we evaluated commutators) propagates in this channel at infinite \( N \) and \( \Delta_{\text{gap}} \) and the contribution from all other operators is suppressed. This implies that nonperturbative contributions from all other possible intermediate states must perfectly cancel the \( O \) contribution, such that the complete commutator vanishes. This provides a sum rule, that admits the following schematic form:

\[
\langle O|[L_{-1}(x_2), L_{-2}(x_3)]|O\rangle|_O + \sum_j \langle O|[L_{-1}(x_2), L_{-2}(x_3)]|O\rangle|_{O_j} = 0, \tag{5.5}
\]

at finite spacelike separation \( \vec{x}_2^\perp \neq \vec{x}_3^\perp \). In (5.5), \( O_j \) is any operator that is not \( O \) and that is contained in the \( T \times O \) OPE. This includes spinning operators. This sum rule would be worth studying further because it requires nonperturbative effects in a UV complete theory of gravity to contribute to correct an inherently IR observable.

For physical CFTs, we should expect only \([L_{-2}, L_{-2}]\) and \([L_{-2}, L_{-1}]\) to vanish nonperturbatively at finite spacelike separation [108]. For free field theory, we nevertheless reached the conclusion that all commutators \([L_{m_1}, L_{m_2}]\) involving global operators (members of the five-dimensional subalgebra) vanish at finite spacelike separation provided \( m_1 + m_2 \leq 0 \). It would be interesting to understand why this is the case, and whether it is a special property of free field theory that is responsible for this "enhanced" commutativity. This resonates with the question about the definition of the operators \( L_n \).
themselves. It can be that these operators have better properties in free field theories, but maybe this extends to a large class of CFTs.

As we described in chapter 3 and the conclusion, we have also computed the commutators of interest in the bulk of AdS. To be able to do this, we first derived the necessary geometries that are dual to the insertion of (exponentiated) generalized ANEC operators in the CFT. The operator \( L_0 \) resisted the Ansatz method that we used to find these geometries. While this is natural in the way we performed the computation, it is still reasonable to believe that finding the general geometry dual to the insertion of \( L_0 \) could be done with the appropriate Ansatz, and this is an interesting problem to solve, as this would provide access to more commutators in the bulk. In addition, we did not compute commutators involving \( L_1 \) and \( L_2 \) in the bulk, despite having the necessary geometries. These commutators behave differently from the one we computed because these shocks intersect in the bulk. Nevertheless, computing these commutators in the bulk would also be interesting, as we would be able to compare them with the holographic CFT computations in this more involved setup. We should find agreement with the CFT, where this complication is not manifest. Understanding the precise mechanism that reproduces the CFT result from the bulk would thus be interesting.

In addition, we can also think about the well-definiteness of the non-global operators in the AdS bulk. In particular, we have seen that the geometries that are dual to the global operators are shockwave geometries. It would be interesting to see whether we would be able to obtain the geometries dual to the insertion of the operators \( L_n \) with \( n \leq 3 \) or \( n \geq 3 \). It is not even clear that these would be shockwave geometries, and they could well be more exotic solutions, which do not have the very nice properties of shockwave solutions. This would be worth studying further.

Finally, we explained how we think that it could be possible to find a well-defined algebra by changing the definition of the generalized light-ray operators. In practice, we would like to add other components of the stress-energy tensor in the integrand. This resonates with the observation of section 3.8.4. There, we computed the superposition of two \( L_2 \) shocks. We realized that we can obtain this by starting with the commutator \([L_{-2}, L_{-2}] = 0\), which is just the commutator of two ANEC operators in the bulk. We can then act with an appropriate conformal transformation on this commutator to reach \([L_2, L_2] \). While we see in figure 3.9 that these two shocks cross in the bulk, they also need to commute as they are just a conformal transformation of the ANEC commutator. This implies that it could be possible to build a well-defined algebra with more generic operators, that maybe even cross in the bulk. To do this we would need to add other components of the stress-energy tensor. This is clear from figure 3.9, where we see that the green line (which is obtained by a conformal transformation of the ANEC operator \( L_{-2} \) away from \( \vec{x}^+ = 0 \)) is not living on the same null-plane as the starting \( L_{-2} \) operator. This is an interesting direction to study further.
5.2. Outlook

5.2.2 Higher-form symmetries and hydrodynamics

Regarding the work presented in chapter 4, there are a few generalizations that are worth discussing here. First, we can wonder whether all gapless phases are protected by anomalies, as we described in 1.5.3 and chapter 4. In other words, can we prove a version of the Goldstone theorem that holds in all the cases where there is a massless phase? We already saw that this was more general than the usual spontaneous symmetry breaking mechanism. For example, this can be applied in two dimensions while the Mermin-Wagner-Coleman theorem forbids spontaneous symmetry breaking in this case. A natural starting point to investigate this is to understand how the superfluid story of chapter 4 can be generalized in the case of non-Abelian symmetries. We already mentioned that higher-form symmetries can only be Abelian, and as such, we will not be able to have non-Abelian higher-form symmetries that would get spontaneously broken. Nevertheless, we could break a regular non-Abelian symmetry and investigate the emergent Abelian higher-form symmetry that would arise in the symmetry broken phase. We expect that these two symmetries would be connected by a mixed anomaly, similar to what happens in the superfluid case.

In the case where the symmetry that gets spontaneously broken is non-Abelian, the Goldstone theorem states that for a symmetry breaking pattern where $G$ gets broken down to $H$ (this implies that the ground state is invariant under a subgroup $H$ of the full group $G$), then the spectrum contains $\dim(G) - \dim(H)$ massless (Goldstone) bosons. As they are in one-to-one correspondence with the generators of the coset space $G/H$, the usual way to parametrize them is with an element of a coset space $g \in G/H$. The natural generalization of the higher-form current (which is $\star K = d\phi$ in the $U(1)$ superfluid case) is the Maurer-Cartan form $\star K \equiv g^{-1}dg$. This current is not conserved, but its non-conservation is given by the usual Cartan structure equation. It would be interesting to study this case further, and prove that this symmetry structure together with its anomaly is sufficient to prove the Goldstone theorem. In particular, we should be able to account for the appropriate number of Goldstone modes for an arbitrary symmetry breaking pattern.

A related question concerns the extension of the Landau paradigm. In chapter 4, we have explained how certain phase transitions that are believed to fit outside of the usual Landau classification of phase transitions can be incorporated in this framework provided we take into account higher-form symmetries and their anomalies. An example is the BKT phase transition, which is now a usual Landau transition between two phases that have different global symmetry structures and where both symmetries are connected by a mixed anomaly in the phase where they are both present. This is very similar to the toy example given in section 1.5.3. A natural direction in which to extend the work presented in chapter 4 is then to try to understand which phase transitions are genuinely non-Landau once we take into account generalized symmetries, both continuous and discrete, as well as their anomalies. An example where this program can be applied is the fractional quantum hall phases. In this case, certain phases can be distinguished by the generalized symmetries of their effective Chern-Simons descriptions. We can also
apply this reasoning to the Ising model, where both phases can be distinguished by their discrete higher-form symmetry structure.

Moreover, we have described in section 1.5.2 how the masslessness of the photon can be explained by realizing that it is really the Goldstone mode associated with a 1-form symmetry that gets spontaneously broken in free electromagnetism. This shows how thinking about higher-form symmetries in well-understood physical systems can help decipher well-known phenomena. Another natural question where we could try to mimic what happens in electromagnetism is the case of gravity. In this case, the gapless mode in the spectrum is the graviton, and we would like to interpret the graviton as the Goldstone mode associated with the breaking of a higher-form symmetry. To be able to do this, we should repeat the analysis, understanding the correct symmetry structure in this case and how different symmetries are connected through anomalies. Then, we should try to mimic the proof of Goldstone’s theorem that we did in the case of superfluids in chapter 4.

Even though the ultimate goal would be to answer this question in the full non-linear general relativity theory, the natural starting point is to consider linearized gravity. In this case, the theory is way easier to deal with compared to the full non-linear theory. In linearized gravity, we want to consider, as the current for our regular symmetry, the linearized Riemann tensor. This tensor can be written, schematically, as $R_{\mu\nu\rho\sigma} \sim (\partial^2 R)_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}$, where $h_{\alpha\beta}$ is the metric perturbation and $(\partial^2 R)$ is a differential operator that is quadratic in derivatives. This quadratic operator takes $h_{\alpha\beta}$ and produces the linearized Riemann tensor, which is the simplest gauge invariant operator in the theory.

It is also known that there is a unique action for a symmetric two-tensor $h_{\mu\nu}$ that is invariant under the diffeomorphisms of general relativity, which are the transformations $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ for an arbitrary vector $\xi^\mu$. This is the Fierz-Pauli action. Varying the Fierz-Pauli action with respect to the metric gives the linearized Einstein equation, as we would expect. Nevertheless, this action can also be rewritten, using integration by parts, as

$$S_{FP} \sim \int d^4 x \ h_{\mu\nu} G^{\mu\nu} = \int d^4 x \ h^{\mu\nu} (\partial_G)_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}, \quad (5.6)$$

where in the second equality, we defined the differential operator $(\partial_G)_{\mu\nu}^{\alpha\beta}$ as an operator quadratic in derivatives that produces the linearized Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R$ when acting on $h_{\alpha\beta}$. We see that this way, we have obtained an action that is quadratic in $h^{\mu\nu}$ with a differential operator that is quadratic in derivatives instead of being linear in derivative. This is the theory for which we need to introduce a background gauge field and understand the symmetry structure.

We introduce a background gauge field for the linearized Riemann tensor and write a gauge-invariant action with this background gauge field turned on. As usual, we expect that the equations of motion that are derived from this action do not get modified compared to their non-gauged counterparts while the Bianchi identities become anomalous. This is the standard way in which introducing a background gauge field modifies the equations of our system (cf section 1.5.2). We thus expect that by computing the
two-point function between the linearized Riemann tensor and the emergent current for the other higher-form symmetry, we can mimic our proof of the Goldstone theorem and explain the masslessness of the graviton. Once we have understood this for the linearized gravity (where the current (the linearized Riemann tensor) and the emergent current are the same at leading order), we hope to be able to generalize this for the full non-linear theory.

Finally, let me mention a more open-ended direction, which seems more speculative. It is not completely unreasonable to believe that the two main objects that were considered in this thesis, namely generalized symmetries and light-ray operators can be connected one way or another. In particular, generalized symmetries help to classify extended objects, and this is precisely what light-ray operators are. In a four-dimensional CFT with a 1–form symmetry, it is possible to construct infinitely many conserved charges, which form an algebra that can be thought of as a higher-form generalization of the Abelian Kac-Moody algebra in two-dimensional CFTs [116]. These charges are related to the charges of the BMS algebra, which also appear as the conserved charges of a particular algebra of light-ray operators (which is the one we investigated in chapter 3 [106]. This might suggest that generalized symmetries are the natural way to understand the essential nature of light-ray operators, and this is yet an extra direction that is worth pursuing.
A Details on the derivation of differential operators

A.1 Notation and Conventions

We start by setting up some notation. We will work in \( d = 4 \), and work mostly in lightcone coordinates

\[
x^\pm = t \pm z
\]

with metric

\[
ds^2 = -dx^+ dx^- + dx^2 + dy^2.
\]

This also fixes the specification of the vectors \( \xi^\mu_\pm \) from section 2.3, and the vector \( n^i \) would point in the \( z \) direction, namely on the north pole of the celestial sphere. We will also use the more compact notation

\[
\vec{x}_\pm = (x, y)
\]

\[
x^2_\pm = x^2 + y^2
\]

A.2 \( T_{\mu\nu}O \) OPE

In this appendix, we want to compute the operator product expansion of the stress-tensor \( T \) when fusing with a scalar field \( O(x) \) of conformal weight \( \Delta \). We obtain it by expanding the exact result (2.49) when the distance between two points is getting small. We follow [40] and give more terms in the expansion. Also, we use the standard metric on \( \mathbb{R}^4 \), i.e \( g_{\mu\nu} = \delta_{\mu\nu} \). First, let us define

\[
s^\mu = x^\mu - y^\mu, \quad X^\mu = \frac{s^\mu}{s^2} - \frac{x^\mu - z^\mu}{(x - z)^2}.
\]

From (2.49) and (A.5), the short distance limit of the three-point function in 4 dimensions is given by

\[
\langle T_{\mu\nu}(x)O(y)O(z) \rangle = \frac{1}{s^4(x - z)^2(y - z)^{2\Delta - 4}}t_{\mu\nu}(X),
\]

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with $s$ and $X$ as in (A.5) and with $t_{\mu\nu}(X)$ given as in (2.51). The two-point function of two scalar operators is given by

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \frac{1}{(x - y)^{2\Delta}}. \quad (A.7)$$

As $x \to y$, the three-point function can be expressed, using the OPE, as

$$\langle T_{\mu\nu}(x)\mathcal{O}(y)\mathcal{O}(z) \rangle \sim A_{\mu\nu}(s) \frac{1}{(y - z)^{2\Delta}} + B_{\mu\nu\alpha}(s) \frac{\partial}{\partial y^\alpha} \frac{1}{(x - y)^{2\Delta}} + C_{\mu\nu\alpha\beta}(s) \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} \frac{1}{(y - z)^{2\Delta}} + \ldots. \quad (A.8)$$

Expanding (A.6) and matching order by order, one is able to determine the first coefficients $A_{\mu\nu}, B_{\mu\nu\alpha}, \ldots$. They can be built only out of two building blocks, namely the metric $\delta_{\mu\nu}$ and the vector $s_\mu$. The zeroth order term in $s$ is given by

$$A_{\mu\nu}(s) = \frac{a}{s^4} \left( \frac{s_\mu s_\nu}{s^2} - \frac{1}{4} \delta_{\mu\nu} \right) = \frac{a}{s^4} t_{\mu\nu}(s). \quad (A.9)$$

The first order term in $s$ is given by

$$B_{\mu\nu\alpha}(s) = \frac{a}{2\Delta s^4} \left( s_\mu \delta_{\alpha\nu} + s_\nu \delta_{\alpha\mu} - s_\alpha \delta_{\mu\nu} + 4 \frac{s_\mu s_\nu s_\alpha}{s^2} \right). \quad (A.10)$$

The second order term in $s$ is given by

$$C_{\mu\nu\alpha\beta}(s) = C_1 s^2 \delta_{\mu\alpha} \delta_{\nu\beta} + C_2 s^2 \left( \delta_{\beta\nu} \delta_{\mu\alpha} + \delta_{\beta\mu} \delta_{\nu\alpha} \right) + C_3 (s_\beta s_\alpha \delta_{\mu\nu} + s_\mu s_\nu \delta_{\beta\alpha}) + C_4 (s_\nu s_\alpha \delta_{\beta\mu} + s_\mu s_\alpha \delta_{\beta\nu} + s_\beta s_\nu \delta_{\mu\alpha} + s_\beta s_\mu \delta_{\nu\alpha}) + C_5 \frac{s_\beta s_\mu s_\nu s_\alpha}{s^2} \delta_{\mu\nu}, \quad (A.11)$$

with

$$C_1 = \frac{a}{8(\Delta^2 + \Delta) s^4}, \quad (A.12)$$
$$C_2 = \frac{a}{8(\Delta^2 + \Delta) s^4}, \quad (A.13)$$
$$C_3 = -\frac{3a}{4(\Delta^2 + \Delta) s^4}, \quad (A.14)$$
$$C_4 = \frac{a}{2(\Delta^2 + \Delta) s^4}, \quad (A.15)$$
$$C_5 = \frac{a}{(\Delta^2 + \Delta) s^4}. \quad (A.16)$$

The third order term in $s$ is given by

$$D_{\mu\nu\alpha\beta \chi}(s) = D_1 s^2 \left( s_\nu \delta_{\alpha\chi} \delta_{\beta\mu} + s_\mu \delta_{\alpha\chi} \delta_{\beta\nu} + s_\nu \delta_{\alpha\mu} \delta_{\beta\chi} + s_\mu \delta_{\alpha\nu} \delta_{\beta\chi} + s_\alpha \delta_{\mu\beta} \delta_{\nu\chi} + s_\mu \delta_{\alpha\beta} \delta_{\nu\chi} + s_\beta \delta_{\mu\alpha} \delta_{\nu\chi} + s_\nu \delta_{\mu\beta} \delta_{\alpha\chi} \right) + D_2 s^2 \left( \delta_{\mu\beta} \delta_{\nu\alpha} + s_\beta \delta_{\mu\alpha} \delta_{\nu\chi} + s_\nu \delta_{\mu\beta} \delta_{\alpha\chi} \right) + D_3 s^2 \left( s_\beta \delta_{\mu\alpha} \delta_{\nu\chi} + s_\alpha \delta_{\mu\beta} \delta_{\nu\chi} + s_\beta \delta_{\mu\alpha} \delta_{\nu\chi} + s_\nu \delta_{\mu\beta} \delta_{\alpha\chi} \right) + D_4 \left( s_\mu s_\nu s_\chi \delta_{\alpha\beta} + s_\alpha s_\beta s_\nu \delta_{\mu\chi} + s_\beta s_\mu s_\nu \delta_{\alpha\chi} \right) + D_5 \left( s_\beta s_\mu s_\chi \delta_{\alpha\nu} + s_\alpha s_\beta s_\nu \delta_{\mu\chi} + s_\beta s_\mu s_\nu \delta_{\alpha\chi} \right) + D_6 \left( s_\alpha s_\beta s_\nu \delta_{\mu\chi} + s_\alpha s_\beta s_\mu \delta_{\nu\chi} \right) + D_7 \frac{s_\alpha s_\beta s_\mu s_\nu s_\chi}{s^2}, \quad (A.17)$$

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where

\[
D_1 = - \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_2 = \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_3 = \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_4 = - \frac{a}{2\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_5 = \frac{a}{2\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_6 = - \frac{a}{\Delta(\Delta + 1)(\Delta + 2)s^4}, \\
D_7 = \frac{a}{\Delta(\Delta + 1)(\Delta + 2)s^4},
\]

(A.18)  
(A.19)  
(A.20)  
(A.21)  
(A.22)  
(A.23)  
(A.24)

The fourth order term is the last one we will write explicitly, and it is given by

\[
E_{\mu\nu\alpha\beta\chi\delta} = E_{1}s^4 (\delta_{\beta\delta}\delta_{\alpha\chi}\delta_{\mu\nu} + \delta_{\alpha\delta}\delta_{\beta\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu}) \\
+ E_{2}s^2 (s\delta\alpha\gamma\delta_{\mu\nu} + \delta_{\alpha}\delta\delta_{\gamma\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu}) \\
+ E_{3}s^4 (\delta_{\mu\alpha}\delta_{\chi}\delta_{\beta\nu} + \delta_{\alpha}\delta_{\nu}\delta_{\chi}\delta_{\beta\mu} + \delta_{\mu}\delta_{\nu}\delta_{\chi}\delta_{\beta\mu} + \delta_{\alpha}\delta_{\beta}\delta_{\nu}\delta_{\chi\mu} + \delta_{\mu}\delta_{\beta}\delta_{\nu}\delta_{\chi\mu}) \\
+ E_{4}s^2 (s\delta\alpha\gamma\delta_{\mu\nu} + \delta_{\alpha}\delta\delta_{\gamma\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\mu}\delta_{\beta}\delta_{\nu}\delta_{\chi\mu}) \\
+ E_{5}(s\delta\alpha\gamma\delta_{\mu\nu} + \delta_{\alpha}\delta\delta_{\gamma\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu}) \\
+ E_{6}s^2 (\delta_{\alpha\beta}\delta_{\chi\gamma}\delta_{\mu\nu} + \delta_{\alpha\beta}\delta_{\chi\mu\nu} + \delta_{\alpha\beta}\delta_{\chi\nu\mu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\nu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\mu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\nu}) \\
+ E_{7}(s\delta\alpha\gamma\delta_{\mu\nu} + \delta_{\alpha}\delta\delta_{\gamma\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu}) \\
+ E_{8}s^2 (\delta_{\alpha\beta}\delta_{\chi\gamma}\delta_{\mu\nu} + \delta_{\alpha\beta}\delta_{\chi\mu\nu} + \delta_{\alpha\beta}\delta_{\chi\nu\mu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\nu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\mu} + \delta_{\alpha\beta}\delta_{\mu\nu}\delta_{\chi\nu}) \\
+ E_{9}(s\delta\alpha\gamma\delta_{\mu\nu} + \delta_{\alpha}\delta\delta_{\gamma\chi}\delta_{\mu\nu} + \delta_{\delta\chi}\delta_{\alpha\beta}\delta_{\mu\nu}) \\
+ E_{10}\frac{s\delta\alpha\gamma\delta_{\mu\nu}}{s^2},
\]

(A.25)
A. Details on the derivation of differential operators

with

\begin{align}
E_1 &= -\frac{a}{64\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_2 &= \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_3 &= -\frac{a}{64\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_4 &= \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_5 &= \frac{a}{2\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_6 &= -\frac{3a}{32\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_7 &= -\frac{3a}{8\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_8 &= \frac{a}{8\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_9 &= -\frac{5a}{4\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}, \\
E_{10} &= \frac{a}{\Delta(\Delta + 1)(\Delta + 2)(\Delta + 3)s^4}.
\end{align}

To compute the OPE with the ANEC operator, we can integrate these terms order by order in the OPE expansion. The first two vanish upon integration and \(C_{\mu\nu\rho\sigma}\) is the first order that contributes. Performing the integrals and setting \(s^\perp = 0\) gives the differential operator (2.64).

A.3 Differential operator acting on \(T\) and \(J\)

A.3.1 \(U(1)\) currents

In this section, we explain the structure of the OPE between the ANEC operator and conserved currents. Following the rules established in section 2.3.1, the most general
differential operator that one can write down is of the following form

\[ \mathcal{E} \epsilon^\mu J_\mu = \left( \sum_{q_1,r_1,s_1,t_1} c_{q_1,r_1,s_1,t_1}^1 (x_12 \cdot \xi_-)^{q_1} (\bar{\epsilon} \cdot \xi_+)^{r_1} (\bar{\epsilon} \cdot \xi_-)^{s_1} (\bar{\epsilon} \cdot \partial)^{t_1} \right) \epsilon \cdot J \] (A.36)

\[ + \left( \sum_{q_2,r_2,s_2,t_2} c_{q_2,r_2,s_2,t_2}^2 (x_12 \cdot \xi_-)^{q_2} (\bar{\epsilon} \cdot \xi_+)^{r_2} (\bar{\epsilon} \cdot \xi_-)^{s_2} (\bar{\epsilon} \cdot \partial)^{t_2} \right) (\epsilon \cdot \xi_+) (\xi_+ \cdot J) \]

\[ + \left( \sum_{q_3,r_3,s_3,t_3} c_{q_3,r_3,s_3,t_3}^3 (x_12 \cdot \xi_-)^{q_3} (\bar{\epsilon} \cdot \xi_+)^{r_3} (\bar{\epsilon} \cdot \xi_-)^{s_3} (\bar{\epsilon} \cdot \partial)^{t_3} \right) (\epsilon \cdot \xi_+) (\xi_- \cdot J) \]

\[ + \left( \sum_{q_4,r_4,s_4,t_4} c_{q_4,r_4,s_4,t_4}^4 (x_12 \cdot \xi_-)^{q_4} (\bar{\epsilon} \cdot \xi_+)^{r_4} (\bar{\epsilon} \cdot \xi_-)^{s_4} (\bar{\epsilon} \cdot \partial)^{t_4} \right) (\epsilon \cdot \xi_-) (\xi_+ \cdot J) \]

\[ + \left( \sum_{q_5,r_5,s_5,t_5} c_{q_5,r_5,s_5,t_5}^5 (x_12 \cdot \xi_-)^{q_5} (\bar{\epsilon} \cdot \xi_+)^{r_5} (\bar{\epsilon} \cdot \xi_-)^{s_5} (\bar{\epsilon} \cdot \partial)^{t_5} \right) (\epsilon \cdot \partial) (\xi_+ \cdot J) \]

\[ + \left( \sum_{q_6,r_6,s_6,t_6} c_{q_6,r_6,s_6,t_6}^6 (x_12 \cdot \xi_-)^{q_6} (\bar{\epsilon} \cdot \xi_+)^{r_6} (\bar{\epsilon} \cdot \xi_-)^{s_6} (\bar{\epsilon} \cdot \partial)^{t_6} \right) (\epsilon \cdot \partial) (\xi_- \cdot J) \]

\[ + \left( \sum_{q_7,r_7,s_7,t_7} c_{q_7,r_7,s_7,t_7}^7 (x_12 \cdot \xi_-)^{q_7} (\bar{\epsilon} \cdot \xi_+)^{r_7} (\bar{\epsilon} \cdot \xi_-)^{s_7} (\bar{\epsilon} \cdot \partial)^{t_7} \right) (\epsilon \cdot \partial) (\xi_- \cdot J) \],

with the conditions

\[-q_1 + r_1 + s_1 + 2t_1 = 3, \quad q_1 - r_1 + s_1 = 1 \]
\[-q_2 + r_2 + s_2 + 2t_2 = 3, \quad q_2 - r_2 + s_2 = 3 \]
\[-q_3 + r_3 + s_3 + 2t_3 = 3, \quad q_3 - r_3 + s_3 = 1 \]
\[-q_4 + r_4 + s_4 + 2t_4 = 3, \quad q_4 - r_4 + s_4 = 1 \]
\[-q_5 + r_5 + s_5 + 2t_5 = 3, \quad q_5 - r_5 + s_5 = -1 \]
\[-q_6 + r_6 + s_6 + 2t_6 = 2, \quad q_6 - r_6 + s_6 = 2 \]
\[-q_7 + r_7 + s_7 + 2t_7 = 2, \quad q_7 - r_7 + s_7 = 0 \] (A.37)

Note that due to conservation of the current, we have not allowed contraction between \( \partial^\mu \) and \( J^\mu \), since it vanishes. We can compute the values of the coefficients \( c^i \) at any given order by expanding the integrated three-point function, and find a similar expression to (2.64). It is then easy to resum the operator and look at the large distance limit. When acting on zero-momentum eigenstates, the operator is

\[ E \epsilon^\mu J_\mu = \frac{-\pi q^0}{4} \left( \frac{18\bar{\epsilon}}{c_v} \epsilon_- J_+ + 3 \frac{\bar{\epsilon} - 2\bar{\epsilon}}{c_v} \epsilon_+ J_- + 3 \frac{\bar{\epsilon} - 8\bar{\epsilon}}{c_v} \epsilon_+ J_+ - \frac{3(\bar{\epsilon} - 2\bar{\epsilon})}{2c_v} \epsilon_+ J_i \right) \]

\[ = \frac{\pi q^0}{4} \left( 3(\bar{\epsilon} - 2\bar{\epsilon}) \frac{\epsilon \cdot J - 3 \frac{\bar{\epsilon} - 8\epsilon}{c_v} (\xi_+ \cdot J) (\xi_+ - \xi_-) \cdot \epsilon} {2c_v} \right) \] (A.38)

\[ c_v \] is the coefficient appearing in the two-point function and reads

\[ c_v = \pi^2 (\bar{\epsilon} + \bar{\epsilon}) \] (A.39)
A.3.2 The stress-tensor

The most general operator that one can write down for the stress-tensor is

$$
\mathcal{E}_\mu^\nu T_{\mu\nu}
= \left( \sum_{q_1, r_1, s_1, t_1} c_{q_1, r_1, s_1, t_1} \right)
+ \left( \sum_{q_2, r_2, s_2, t_2} c_{q_2, r_2, s_2, t_2} \right)
+ \left( \sum_{q_3, r_3, s_3, t_3} c_{q_3, r_3, s_3, t_3} \right)
+ \left( \sum_{q_4, r_4, s_4, t_4} c_{q_4, r_4, s_4, t_4} \right)
+ \left( \sum_{q_5, r_5, s_5, t_5} c_{q_5, r_5, s_5, t_5} \right)
+ \left( \sum_{q_6, r_6, s_6, t_6} c_{q_6, r_6, s_6, t_6} \right)
+ \left( \sum_{q_7, r_7, s_7, t_7} c_{q_7, r_7, s_7, t_7} \right)
+ \left( \sum_{q_8, r_8, s_8, t_8} c_{q_8, r_8, s_8, t_8} \right)
+ \left( \sum_{q_9, r_9, s_9, t_9} c_{q_9, r_9, s_9, t_9} \right)
+ \left( \sum_{q_{10}, r_{10}, s_{10}, t_{10}} c_{q_{10}, r_{10}, s_{10}, t_{10}} \right)
+ \left( \sum_{q_{11}, r_{11}, s_{11}, t_{11}} c_{q_{11}, r_{11}, s_{11}, t_{11}} \right)
+ \left( \sum_{q_{12}, r_{12}, s_{12}, t_{12}} c_{q_{12}, r_{12}, s_{12}, t_{12}} \right)
+ \left( \sum_{q_{13}, r_{13}, s_{13}, t_{13}} c_{q_{13}, r_{13}, s_{13}, t_{13}} \right)
$$

(A.40)
A.3. Differential operator acting on $T$ and $J$

\[
+ \sum_{q_{14}, r_{14}, s_{14}, t_{14}} c_{q_{14}, r_{14}, s_{14}, t_{14}}^{14} (x_{12} \cdot \xi_{-})^{q_{14}} (\partial \cdot \xi_{+})^{r_{14}} (\partial \cdot \xi_{-})^{s_{14}} (\partial \cdot \partial_{-})^{t_{14}} (\epsilon_{\mu\nu} \xi_{-}^{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{15}, r_{15}, s_{15}, t_{15}} c_{q_{15}, r_{15}, s_{15}, t_{15}}^{15} (x_{12} \cdot \xi_{-})^{q_{15}} (\partial \cdot \xi_{+})^{r_{15}} (\partial \cdot \xi_{-})^{s_{15}} (\partial \cdot \partial_{-})^{t_{15}} (\epsilon_{\mu\nu} \xi_{-}^{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{16}, r_{16}, s_{16}, t_{16}} c_{q_{16}, r_{16}, s_{16}, t_{16}}^{16} (x_{12} \cdot \xi_{-})^{q_{16}} (\partial \cdot \xi_{+})^{r_{16}} (\partial \cdot \xi_{-})^{s_{16}} (\partial \cdot \partial_{-})^{t_{16}} (\epsilon_{\mu\nu} \xi_{-}^{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{17}, r_{17}, s_{17}, t_{17}} c_{q_{17}, r_{17}, s_{17}, t_{17}}^{17} (x_{12} \cdot \xi_{-})^{q_{17}} (\partial \cdot \xi_{+})^{r_{17}} (\partial \cdot \xi_{-})^{s_{17}} (\partial \cdot \partial_{-})^{t_{17}} (\epsilon_{\mu\nu} \partial_{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{18}, r_{18}, s_{18}, t_{18}} c_{q_{18}, r_{18}, s_{18}, t_{18}}^{18} (x_{12} \cdot \xi_{-})^{q_{18}} (\partial \cdot \xi_{+})^{r_{18}} (\partial \cdot \xi_{-})^{s_{18}} (\partial \cdot \partial_{-})^{t_{18}} (\epsilon_{\mu\nu} \partial_{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{19}, r_{19}, s_{19}, t_{19}} c_{q_{19}, r_{19}, s_{19}, t_{19}}^{19} (x_{12} \cdot \xi_{-})^{q_{19}} (\partial \cdot \xi_{+})^{r_{19}} (\partial \cdot \xi_{-})^{s_{19}} (\partial \cdot \partial_{-})^{t_{19}} (\epsilon_{\mu\nu} \partial_{\mu} \partial_{\nu}) (\xi_{+}^{\beta} T_{\alpha \beta}) \\
+ \sum_{q_{20}, r_{20}, s_{20}, t_{20}} c_{q_{20}, r_{20}, s_{20}, t_{20}}^{20} (x_{12} \cdot \xi_{-})^{q_{20}} (\partial \cdot \xi_{+})^{r_{20}} (\partial \cdot \xi_{-})^{s_{20}} (\partial \cdot \partial_{-})^{t_{20}} (\xi_{+} \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}) \\
+ \sum_{q_{21}, r_{21}, s_{21}, t_{21}} c_{q_{21}, r_{21}, s_{21}, t_{21}}^{21} (x_{12} \cdot \xi_{-})^{q_{21}} (\partial \cdot \xi_{+})^{r_{21}} (\partial \cdot \xi_{-})^{s_{21}} (\partial \cdot \partial_{-})^{t_{21}} (\xi_{+} \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}) \\
+ \sum_{q_{22}, r_{22}, s_{22}, t_{22}} c_{q_{22}, r_{22}, s_{22}, t_{22}}^{22} (x_{12} \cdot \xi_{-})^{q_{22}} (\partial \cdot \xi_{+})^{r_{22}} (\partial \cdot \xi_{-})^{s_{22}} (\partial \cdot \partial_{-})^{t_{22}} (\xi_{-} \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}) \\
+ \sum_{q_{23}, r_{23}, s_{23}, t_{23}} c_{q_{23}, r_{23}, s_{23}, t_{23}}^{23} (x_{12} \cdot \xi_{-})^{q_{23}} (\partial \cdot \xi_{+})^{r_{23}} (\partial \cdot \xi_{-})^{s_{23}} (\partial \cdot \partial_{-})^{t_{23}} (\xi_{-} \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}) \\
+ \sum_{q_{24}, r_{24}, s_{24}, t_{24}} c_{q_{24}, r_{24}, s_{24}, t_{24}}^{24} (x_{12} \cdot \xi_{-})^{q_{24}} (\partial \cdot \xi_{+})^{r_{24}} (\partial \cdot \xi_{-})^{s_{24}} (\partial \cdot \partial_{-})^{t_{24}} (\partial \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}) \\
+ \sum_{q_{25}, r_{25}, s_{25}, t_{25}} c_{q_{25}, r_{25}, s_{25}, t_{25}}^{25} (x_{12} \cdot \xi_{-})^{q_{25}} (\partial \cdot \xi_{+})^{r_{25}} (\partial \cdot \xi_{-})^{s_{25}} (\partial \cdot \partial_{-})^{t_{25}} (\partial \epsilon_{\mu\gamma} T_{\gamma\nu} \xi_{+}^{\nu}),
\]
A. Details on the derivation of differential operators

with

\[-q_1 + r_1 + s_1 + 2t_1 = 3, \quad q_1 - r_1 + s_1 = 1,\]
\[-q_2 + r_2 + s_2 + 2t_2 = 3, \quad q_2 - r_2 + s_2 = 5,\]
\[-q_3 + r_3 + s_3 + 2t_3 = 3, \quad q_3 - r_3 + s_3 = 3,\]
\[-q_4 + r_4 + s_4 + 2t_4 = 3, \quad q_4 - r_4 + s_4 = 1,\]
\[-q_5 + r_5 + s_5 + 2t_5 = 3, \quad q_5 - r_5 + s_5 = 3,\]
\[-q_6 + r_6 + s_6 + 2t_6 = 3, \quad q_6 - r_6 + s_6 = 1,\]
\[-q_7 + r_7 + s_7 + 2t_7 = 3, \quad q_7 - r_7 + s_7 = -1,\]
\[-q_8 + r_8 + s_8 + 2t_8 = 3, \quad q_8 - r_8 + s_8 = 1,\]
\[-q_9 + r_9 + s_9 + 2t_9 = 3, \quad q_9 - r_9 + s_9 = -1,\]
\[-q_{10} + r_{10} + s_{10} + 2t_{10} = 2, \quad q_{10} - r_{10} + s_{10} = -3,\]
\[-q_{11} + r_{11} + s_{11} + 2t_{11} = 2, \quad q_{11} - r_{11} + s_{11} = 4,\]
\[-q_{12} + r_{12} + s_{12} + 2t_{12} = 2, \quad q_{12} - r_{12} + s_{12} = 2,\]
\[-q_{13} + r_{13} + s_{13} + 2t_{13} = 2, \quad q_{13} - r_{13} + s_{13} = 0,\]
\[-q_{14} + r_{14} + s_{14} + 2t_{14} = 2, \quad q_{14} - r_{14} + s_{14} = 2,\]
\[-q_{15} + r_{15} + s_{15} + 2t_{15} = 2, \quad q_{15} - r_{15} + s_{15} = 0,\]
\[-q_{16} + r_{16} + s_{16} + 2t_{16} = 2, \quad q_{16} - r_{16} + s_{16} = -2,\]
\[-q_{17} + r_{17} + s_{17} + 2t_{17} = 1, \quad q_{17} - r_{17} + s_{17} = 3,\]
\[-q_{18} + r_{18} + s_{18} + 2t_{18} = 1, \quad q_{18} - r_{18} + s_{18} = 1,\]
\[-q_{19} + r_{19} + s_{19} + 2t_{19} = 3, \quad q_{19} - r_{19} + s_{19} = -1,\]
\[-q_{20} + r_{20} + s_{20} + 2t_{20} = 3, \quad q_{20} - r_{20} + s_{20} = 3,\]
\[-q_{21} + r_{21} + s_{21} + 2t_{21} = 3, \quad q_{21} - r_{21} + s_{21} = 1,\]
\[-q_{22} + r_{22} + s_{22} + 2t_{22} = 3, \quad q_{22} - r_{22} + s_{22} = 1,\]
\[-q_{23} + r_{23} + s_{23} + 2t_{23} = 3, \quad q_{23} - r_{23} + s_{23} = -1,\]
\[-q_{24} + r_{24} + s_{24} + 2t_{24} = 2, \quad q_{24} - r_{24} + s_{24} = 0,\]
\[-q_{25} + r_{25} + s_{25} + 2t_{25} = 2, \quad q_{25} - r_{25} + s_{25} = 2.\]

Note that due to the stress-tensor conservation equation, we have not allowed contraction between $\partial^\mu$ and $T^{\mu\nu}$ since it vanishes. We can once again extract the exact coefficient by a comparison with the three-point function. At large distance and when acting on zero-momentum eigenstates, we find

\[
E\epsilon^{\mu\nu}T_{\mu\nu} = \frac{\pi q^0}{4} \left( \frac{5}{3} \frac{7\dot{a} + 2\dot{b}}{c_T} \xi^{\mu} T_{\mu\nu} + 10 \frac{13\dot{a} + 4\dot{b}}{c_T} \xi^\mu T_{\mu\nu} \epsilon^{\nu\rho}(\xi^-_\rho - \xi^+_\rho) 
- \frac{15}{6} \frac{81\dot{a} + 32\dot{b}}{c_T} \xi^\mu T_{\mu\nu} \epsilon^{\rho\sigma}(\xi^-_\rho - \xi^+_\rho)(\xi^-_\sigma - \xi^+_\sigma) \right), \tag{A.41}
\]

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A.4. Two dimensions: Two scalars and two stress tensors

with

\[ c_T = \pi^2 \frac{14\hat{a} - 2\hat{b} - 5\hat{c}}{3}. \]  

(A.42)

This reproduces the expectation of the transfer matrix (2.45), with the coefficients in agreement with [91].

A.4 Two dimensions: Two scalars and two stress tensors

The \( n \)-point functions of stress-tensors with themselves or with scalar fields can be computed exactly using Ward identities [33] that are recalled here for convenience

\[
\langle T(\xi) \ldots T(x_M) \phi_1(z_1) \ldots \phi_N(z_N) \rangle = \sum_{i=1}^{N} \left[ \frac{\Delta_i}{(\xi - z_i)^2} + \frac{1}{\xi - z_i} \frac{\partial}{\partial z_i} \right] \langle T(x_1) \ldots T(x_M) \phi_1(z_1) \ldots \phi_N(z_N) \rangle \\
+ \sum_{j=1}^{M} \left[ \frac{2}{(\xi - x_j)^2} + \frac{1}{\xi - x_j} \frac{\partial}{\partial x_j} \right] \langle T(x_1) \ldots T(x_M) \phi_1(z_1) \ldots \phi_N(z_N) \rangle \\
+ \sum_{j=1}^{M} \left[ \frac{c/2}{(\xi - x_j)^2} \right] \langle T(x_1) \ldots T(x_{j-1}) T(x_{j+1}) \ldots T(x_M) \phi_1(z_1) \ldots \phi_N(z_N) \rangle.
\]

(A.43)

We can use the two-point function of two scalars of conformal weight \( h \)

\[
\langle \phi(z_1) \phi(z_2) \rangle = \frac{1}{(z_1 - z_2)^{2h}},
\]

(A.44)
as well as the Ward identity (??? to compute \( \langle T(z_1) \phi(z_2) T(z_3) \phi(z_4) \rangle \), which is given as

\[
\langle T(z_1) \phi(z_2) T(z_3) \phi(z_4) \rangle = \frac{1}{2} \frac{c}{z_{24}^{2h+4} z_{13}^{4h+4}} + \frac{h(h z_{13}^2 z_{24}^2 - 2 z_{12} z_{23} z_{14} z_{24})}{z_{24} z_{12} z_{13} z_{23} z_{34} z_{24} z_{13} z_{23} z_{34}}.
\]

(A.45)

It is then straightforward to integrate twice to obtain

\[
\langle E_1 \phi(z_2) E_3 \phi(z_4) \rangle = \int \langle T(z_1) \phi(z_2) T(z_3) \phi(z_4) \rangle dz_1 dz_3 = - (4\pi^2) \frac{2h(2h + 1)}{(z_2 - z_4)^{2h+2h}}.
\]

(A.46)

A.4.1 Conformal block expansion

The result (A.45) can be recast as

\[
\langle T(z_1) \phi(z_2) T(z_3) \phi(z_4) \rangle = z_{12}^{-2-h} \left( \frac{z_{24}}{z_{13}} \right)^{2-h} z_{34}^{-2-h} F_{\phi_1 \phi_2 \phi_3 \phi_4}(\eta),
\]

(A.47)

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with
\[ F_{T\phi T\phi}(\eta) = \frac{1}{2}c\eta^{h+2} + \frac{\eta^h}{(1-\eta)^2}(h^2 - 2h\eta(1-\eta)) \]  
(A.48)
\[ = \sum_{n=0}^{\infty} C_n \eta^{h+n} F(2h+n-2,n+2;2h+2n;\eta) . \]  
(A.49)

Here, (A.48) is the result one get by direct computation using the Ward identity (?), as we did in (A.45) while (A.49) is the conformal block expansion. The coefficients \( C_n \) can be found in [168]. When expanding \( F_{T\phi T\phi} \) for small \( \eta \), we get contributions of the form
\[ z_{12}^{-2-h} \left( \frac{z_{24}}{z_{13}} \right)^{2-h} z_{34}^{-2-h} \eta^m = \frac{1}{z_{12}^{2+h-m} z_{34}^{2+h-m} z_{24}^{2-h-m} z_{13}^{2+h-m}} . \]  
(A.50)

When extracting the residues as \( z_1 \to z_2 \) and \( z_3 \to z_4 \), only the terms with \( m = h \) and \( m = h+1 \) will be non-vanishing while all contributions with \( m \geq h+2 \) vanish.

Expanding the exact Ward identity result (A.48) for small \( \eta \) yields
\[ F_{T\phi T\phi}(\eta) = \eta^h \left( h^2 + (2h^2 - 2h) \eta + \left( 3h^2 - 2h + \frac{c}{2} \right) \eta^2 + (4h^2 - 2h) \eta^3 + O(\eta^4) \right) , \]  
(A.51)
which once integrated gives
\[ \langle E_1 \phi(z_2) E_3 \phi(z_4) \rangle = \int \langle T(z_1)\phi(z_2)T(z_3)\phi(z_4) \rangle dz_1 dz_3 = - \left( 4\pi^2 \right)^2 \frac{2h(2h+1)}{(z_2 - z_4)^{2h+2}} . \]  
(A.52)

This result naturally matches the one obtained by directly integrating the exact result (A.48).

### A.4.2 OPE computation

We can also reproduce this result using the OPE of the stress-tensor with a scalar field \( \phi \), which is
\[ T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{(z-w)} + \ldots \]  
(A.53)
The OPE between \( \epsilon(z) \) and \( \phi(w) \) will simply project onto \( \partial \phi(w) \). We then obtain
\[ \langle \phi(z_2) E_1 E_3 \phi(z_4) \rangle = 4\pi^2 \partial_{z_2} \partial_{z_4} \langle \phi(z_2) \phi(z_4) \rangle = - \left( 4\pi^2 \right)^2 \frac{2h(2h+1)}{(z_2 - z_4)^{2h+2}} . \]  
(A.54)

This result is identical to (A.52).
B Details on the Stress Tensor Light-ray Operator Algebra

B.1 Useful contour integrals for three-point functions

When we compute three-point correlators involving light-ray operators in section 3.4.3, we encounter various integrals over $x^-$, which can be evaluated by closing the contour in the upper or lower half-plane. In general, the pole structure of the correlation function implies that one direction is easier to evaluate than the other. For global operators, the way we close the contour does not matter and every possible contour yields the correct answer.

**Lower Half-Plane (Operator to Left)**

First, we can close the contour in the lower half-plane, in which case we pick up the OPE singularity with the operator to the left in the correlator (i.e. the singularity where $x_2$ hits $x_1$):

$$x^-_2 = x^-_{1,2} - i\epsilon.$$ (B.1)

For correlators involving the stress tensor, the pole will be at most third order, which means we only need to evaluate three integrals:

$$\int_{-\infty}^{\infty} dx^-_2 \frac{1}{x^{12}_3 x^{23}_2} f(x^-_2) = \frac{2\pi i(-1)^{b+1}}{x^{12}_2(x^+_{23})^b} \left[ \frac{f(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})} \right],$$ (B.2)

$$\int_{-\infty}^{\infty} dx^-_2 \frac{1}{x^{34}_1 x^{23}_2} f(x^-_2) = \frac{2\pi i(-1)^{b+1}}{(x^{+1}_2)^2(x^+_{23})^b} \left[ \frac{f'(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})} - b \frac{f(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})^{b+1}} \right],$$

$$\int_{-\infty}^{\infty} dx^-_2 \frac{1}{x^{67}_1 x^{23}_2} f(x^-_2) = \frac{2\pi i(-1)^{b+1}}{(x^{+1}_2)^3(x^+_{23})^b} \left[ \frac{1}{2} \frac{f''(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})^b} - \frac{bf'(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})^{b+1}} \right] + \frac{1}{2} \frac{b(b+1)f(x^+_{1,2})}{(x^+_{1,2} - x^-_{3,2})^{b+2}}.$$
Upper Half-Plane (Operator to Right)

If we instead close the contour in the upper half-plane, we pick up the OPE singularity
with the operator to the right in the correlator (where $x_2$ hits $x_3$):

$$x_2^{-} = x_{3,2}^{-} + i\epsilon.$$  \hspace{1cm} (B.3)

We then need to evaluate three integrals that are similar to the previous case:

$$\int_{-\infty}^{\infty} dx_2^{-} f(x_2^{-}) \frac{1}{x_{12}^2 x_{23}^2} = \frac{2\pi i (-1)^{a+1}}{(x_{12})^{a}(x_{23})^a} \left[ \frac{f(x_{3,2}^{-})}{(x_{1,2}^{-} - x_{3,2}^{-})^a} \right],$$  \hspace{1cm} (B.4)

$$\int_{-\infty}^{\infty} dx_2^{-} f(x_2^{-}) \frac{1}{x_{12}^2 x_{23}^2} = \frac{2\pi i (-1)^{a}}{(x_{12})^{a}(x_{23})^a} \left[ \frac{f'(x_{3,2}^{-})}{(x_{1,2}^{-} - x_{3,2}^{-})^a} + \frac{a f(x_{3,2}^{-})}{(x_{1,2}^{-} - x_{3,2}^{-})^{a+1}} \right],$$

$$\int_{-\infty}^{\infty} dx_2^{-} f(x_2^{-}) \frac{1}{x_{12}^2 x_{23}^2} = \frac{2\pi i (-1)^{a+1}}{(x_{12})^{a}(x_{23})^a} \left[ \frac{f''(x_{3,2}^{-})}{2 (x_{1,2}^{-} - x_{3,2}^{-})^a} + \frac{a f'(x_{3,2}^{-})}{2 (x_{1,2}^{-} - x_{3,2}^{-})^{a+1}} + \frac{1}{2} a(a+1) f(x_{3,2}^{-}) \right].$$  \hspace{1cm} (B.5)

**B.2 Useful contour integrals for four-point functions in free field theory**

To compute the four-point functions involving two light-ray operators, the starting point is the four-point function of two local stress-energy tensors $\langle \phi(x_1) T_{--} (x_2) T_{--} (x_3) \phi(x_4) \rangle$.

In free field theory, this is just a Wick contraction exercise, starting from

$$T_{--}(x) = \frac{1}{6\pi^2} \left[ (\partial_{-} \phi(x))^2 - \frac{1}{2} \phi(x) \partial^2 \phi(x) \right],$$  \hspace{1cm} (B.6)

that yields

$$\langle \phi(x_1) T_{--} (x_2) T_{--} (x_3) \phi(x_4) \rangle$$

$$= \frac{1}{3(2\pi^2)^2} \left[ \frac{(x_{23}^+)^4}{x_{14}^2 x_{23}^{12}} + \frac{1}{(2\pi^2)^2} \left[ \left( \frac{\partial}{\partial x_2^{-}} \frac{1}{x_{12}^2} \right) \left( \frac{\partial^2}{\partial x_2^{-} \partial x_3^{-}} \frac{1}{x_{23}^2} \right) \left( \frac{\partial}{\partial x_3^{-}} \frac{1}{x_{34}^2} \right) + (2 \leftrightarrow 3) \right] \right]$$

$$+ \frac{1}{(12\pi^2)^2} \left[ \left( \frac{\partial^2}{\partial (x_2^{-})^2 \partial (x_3^{-})^2} \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2} \right) + (2 \leftrightarrow 3) \right]$$

$$- \frac{1}{6(2\pi^2)^2} \left[ \left( \frac{\partial^2}{\partial (x_2^{-})^2} \left( \frac{1}{x_{12}^2} \left( \frac{\partial}{\partial x_2^{-}} \frac{1}{x_{23}^2} \right) \left( \frac{\partial}{\partial x_3^{-}} \frac{1}{x_{34}^2} \right) \right) + (2 \leftrightarrow 3) \right] + (1 \leftrightarrow 4) \right].$$

This result is identical, up to an overall normalization, to the one in [106].

Because we are interested in the four-point functions that involve global light-ray operators, we already explained how the final answer is insensitive to the way we close the contours to evaluate the integrals. We thus evaluate these integrals by closing both contours outwards, meaning we integrate $x_2^{-}$ by picking up the singularity when $x_2 \rightarrow x_1$, $x_3 \rightarrow x_2$, and $x_4 \rightarrow x_3$. 

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and $x_3^-$ by picking up the singularity when $x_3 \to x_4$. It implies that only the terms with a denominator of the form $x_{12}^a x_{34}^b$ for arbitrary $a$ and $b$ will survive integration. These are the terms that have the first topology of figure 3.2. This implies that to perform the integrals, we can concentrate solely on the following terms in the four-point function

\begin{equation}
\langle \phi(x_1) T_- (x_2) T_- (x_3) \phi(x_4) \rangle \\
\supset \frac{1}{x_{12} x_{23} x_{34}} \left( \frac{1}{36\pi^4} (x_{23}^+)^4 + \frac{1}{9\pi^4} x_{23}^+ (x_{23}^+)^2 \left( \frac{x_{12}^+}{x_{12}^2} + \frac{x_{23}^+}{x_{23}^2} \right) \left( \frac{x_{23}^+}{x_{23}^2} + \frac{x_{34}^+}{x_{34}^2} \right) \right) + \frac{1}{36\pi^4} x_{23}^8 \left( \frac{x_{12}^+}{x_{12}^2} + \frac{x_{23}^+}{x_{23}^2} \right)^2 \left( \frac{x_{23}^+}{x_{23}^2} + \frac{x_{34}^+}{x_{34}^2} \right)^2 \right). \tag{B.7}
\end{equation}

To compute the four-point function involving two global light-ray operators

\begin{equation}
\langle \phi(x_1) E_f(x_2) E_g(x_3) \phi(x_4) \rangle = \int_{-\infty}^{\infty} dx_2^{-} f(x_2^-) \int_{-\infty}^{\infty} dx_3^{-} g(x_3^-) \langle \phi(x_1) T_- (x_2) T_- (x_3) \phi(x_4) \rangle, \tag{B.8}
\end{equation}

we just need to integrate (B.7). Let us step through the various terms in eq. (B.7) separately, using the general expressions in (B.2) and (B.4). The different terms have poles up to third order in both $x_2^-$ and $x_3^-$. The integrals are

\begin{equation}
\int dx_2^- f(x_2^-) \int dx_3^- g(x_3^-) \left( \frac{(x_{23}^+)^4}{x_{12} x_{23} x_{34}} \right) = (2\pi i)^2 \frac{(x_{23}^+)^4 f(x_{1,2}) g(x_{4,3})}{x_{12} x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]}. \tag{B.9}
\end{equation}

and

\begin{equation}
\begin{align*}
\int dx_2^- f(x_2^-) \int dx_3^- g(x_3^-) \left( \frac{(x_{23}^+)^2}{x_{12} x_{23} x_{34}} \right) &= (2\pi i)^2 \left( 21 f(x_{1,2}) g(x_{4,3}) \frac{(x_{23}^+)^4}{x_{12} x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]5} \\
+ 4 \left( f'(x_{1,2}) g(x_{4,3}) - f(x_{1,2}) g'(x_{4,3}) \right) \frac{(x_{23}^+)^3}{x_{12} x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]4} \\
- f'(x_{1,2}) g'(x_{4,3}) \frac{(x_{23}^+)^2}{x_{12} x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]3} \right). \tag{B.10}
\end{align*}
\end{equation}

Finally, we have the third term, which has poles up to third order in both $x_2^-$ and $x_3^-$.\[189]
leading to the resulting expression

\[
\int dx_{\perp} f(x_{\perp}) \int dx_{\perp} g(x_{\perp}) \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2} \left( \frac{x_{12}^+}{x_{12}^2} + \frac{x_{23}^+}{x_{23}^2} \right)^2 \left( \frac{x_{23}^+}{x_{23}^2} + \frac{x_{34}^+}{x_{34}^2} \right)^2
\]

\[
= (2\pi)^2 \left( 131 f(x_{1,2}) g(x_{4,3}) \frac{(x_{23}^+)^4}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]^5} + 38 \left( f'(x_{1,2}) g(x_{4,3}) - f(x_{1,2}) g'(x_{4,3}) \right) \frac{(x_{23}^+)^3}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]^4}
- 14 f'(x_{1,2}) g'(x_{4,3}) \frac{(x_{23}^+)^2}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]^3} + 3 \left( f''(x_{1,2}) g(x_{4,3}) + f(x_{1,2}) g''(x_{4,3}) \right) \frac{(x_{23}^+)^2}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]^3}
- \frac{3}{2} \left( f''(x_{1,2}) g'(x_{4,3}) - f'(x_{1,2}) g''(x_{4,3}) \right) \frac{x_{23}^+}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]^2} + \frac{1}{4} f'''(x_{1,2}) g''(x_{4,3}) \frac{1}{x_{12}^+ x_{34}^+ [-x_{23}^+ (x_{1,2} - x_{4,3}) + |x_{23}^+|^2]}. \right)
\]

Combining these results yields the four-point function of light-ray operators.

### B.3 Aside on delta functions

In this section, we want to review how to extract a perpendicular delta function \( \delta^{(2)}(\vec{x}_{23}) \) from the expression with \( i\epsilon \) that we encounter as commutators. We need to evaluate the \( \epsilon \to 0 \) limit of expressions of the schematic form:

\[
\frac{(i\epsilon)^{a-1}}{(|x|^2 + i\epsilon q)^a}.
\]

Let us now try to systematically evaluate such expressions, in order to extract the delta function contribution.

To start, let us quickly review how to evaluate the familiar expression

\[
y(x) = \frac{1}{x + i\epsilon}. \quad (B.12)
\]

We can think of \( y(x) \) as a distribution satisfying the relation

\[
x \cdot y(x) = 1. \quad (B.13)
\]

The general solution to this constraint is clearly

\[
y(x) = \mathcal{P} \frac{1}{x} + c_0 \delta(x), \quad (B.14)
\]

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where $\mathcal{P}$ indicates the principal value. We can then fix the coefficient $c_0$ by integrating over the region $-b \leq x \leq b$, take the limit $\epsilon \to 0$, then take $b \to 0$ so that we only pick up the delta function. Evaluating this integral and taking $\epsilon \to 0$, we find

$$
\int_{-b}^{b} dx \frac{1}{x + i \epsilon} = \log(b + i \epsilon) - \log(-b + i \epsilon) = -i \pi.
$$

We therefore find $c_0 = -i \pi$, giving us the familiar identity

$$
\frac{1}{x + i \epsilon} = \mathcal{P} \frac{1}{x} - i \pi \delta(x).
$$

Integrating this expression against a test function shows that this relation is correct in the distribution sense.

Let us now try to generalize this analysis to evaluate the expression

$$
\lim_{\epsilon \to 0} \frac{(i \epsilon)^{a-1}}{(|\vec{x}|^2 + i \epsilon q)^a},
$$

where $\vec{x} = (r \cos \theta, r \sin \theta)$ is a vector in $\mathbb{R}^2$. We expect that this expression contains a delta function $\delta^{(2)}(\vec{x})$. To determine whether this is the case, and to determine the coefficient, we can follow the procedure above, though now we’ll integrate over a disk of radius $b$ around the origin,

$$
\int d^2 x \frac{(i \epsilon)^{a-1}}{(|\vec{x}|^2 + i \epsilon q)^a} = \int_0^b dr \int_0^{2\pi} d\theta \frac{r(i \epsilon)^{a-1}}{(r^2 + i \epsilon q)^a} = \frac{\pi}{(n-1)q^{a-1}}.
$$

We thus obtain the identity

$$
\lim_{\epsilon \to 0} \frac{(i \epsilon)^{a-1}}{(|\vec{x}|^2 + i \epsilon q)^a} = \frac{\pi}{(a-1)q^{a-1}} \delta^{(2)}(\vec{x}) \quad (a > 1).
$$

Integrating (B.18) against a test function proves that this is the correct relation and that it stands as a distribution.

What happens when $a = 1$? In this case, we want to understand the following quantity

$$
\lim_{\epsilon \to 0} \frac{1}{|\vec{x}|^2 + i \epsilon} = \lim_{\epsilon \to 0} \left( \frac{|\vec{x}|^2}{|\vec{x}|^2 + \epsilon^2} - i \frac{\epsilon}{|\vec{x}|^2 + \epsilon^2} \right).
$$

If we consider the real part of (B.19) and integrate it on a disk of radius $b$, we get

$$
\int_{-\infty}^{\infty} d^2 x \frac{|\vec{x}|^2}{|\vec{x}|^2 + \epsilon^2} = 2\pi \int_0^b dr \frac{r^3}{r^4 + \epsilon^2} = \pi \log \left( 1 + \frac{b^4}{\epsilon^2} \right).
$$

In the limit $\epsilon \to 0$, it diverges and we conclude that the real part is not integrable. For the imaginary part, we get

$$
\int_{-\infty}^{\infty} d^2 x \frac{\epsilon}{|\vec{x}|^2 + \epsilon^2} = \int_0^b dr \int_0^{2\pi} d\theta \frac{r \epsilon}{r^4 + \epsilon^2} = \pi \arctan \left( \frac{b^2}{\epsilon} \right).
$$

and the limit $\epsilon \to 0$ gives $\pi^2/2$. We thus see that the imaginary part is integrable and has a well-defined limit when $\epsilon \to 0$. This is the part responsible for our finite separation contribution. It is thus impossible to extract a meaningful transverse delta function in this case for $a = 1$. 

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B. Details on the Stress Tensor Light-ray Operator Algebra

B.4 Commutators of global light-ray operators

In this appendix, we want to list the results of the commutator of two global light-ray operators for the cases where there is no finite transverse separation contribution and where \(x^+_3 = x^+_4\). We will present two examples in some details and list the results of the other computations.

B.4.1 The commutator \([L_{-1}, L_{-2}]\)

In terms of transverse delta functions, the two orderings are

\[
\langle \phi(x_1) L_{-1}(x_2) L_{-2}(x_3) \phi(x_4) \rangle = \frac{\delta^{(2)}(\vec{x}_{23}^4)}{\pi x_{12}^+ x_{24}^+} \left[ -\frac{6x_{1,2}^-}{(x_{1,2}^- - x_{4,2}^-)^4} + \frac{2}{(x_{1,2}^- - x_{4,2}^-)^3} \right], \quad (B.22)
\]

\[
\langle \phi(x_1) L_{-2}(x_3) L_{-1}(x_2) \phi(x_4) \rangle = \frac{\delta^{(2)}(\vec{x}_{23}^4)}{\pi x_{12}^+ x_{24}^+} \left[ -\frac{6x_{4,2}^-}{(x_{1,2}^- - x_{4,2}^-)^4} - \frac{2}{(x_{1,2}^- - x_{4,2}^-)^3} \right]. \quad (B.23)
\]

Combining them results in the commutator

\[
\langle \phi(x_1)[L_{-1}(x_2), L_{-2}(x_3)] \phi(x_4) \rangle = -\frac{2}{\pi x_{12}^+ x_{24}^+ (x_{1,2}^- - x_{4,2}^-)^3} \delta^{(2)}(\vec{x}_{23}^4), \quad (B.24)
\]

which is also what (3.79) yields if you use \(f(x^-) = x^-\) and \(g(x^-) = 1\). This exactly matches the 3-pt function involving \(L_{-2}\) that we computed in (3.66) (multiplied by \(-i\delta^{(2)}(\vec{x}_{23}^4)):

\[
\langle \phi(x_1)[L_{-1}(x_2), L_{-2}(x_3)] \phi(x_4) \rangle = -i\delta^{(2)}(\vec{x}_{23}^4) \langle \phi(x_1) L_{-2}(x_2) \phi(x_4) \rangle. \quad (B.25)
\]

B.4.2 The commutator \([L_0, L_{-2}]\)

In terms of transverse delta functions, the two orderings are

\[
\langle \phi(x_1) L_0(x_2) L_{-2}(x_3) \phi(x_4) \rangle = \frac{\delta^{(2)}(\vec{x}_{23}^4)}{\pi x_{12}^+ x_{24}^+} \left[ -\frac{6(x_{1,2}^-)^2}{(x_{1,2}^- - x_{4,2}^-)^4} + \frac{4x_{1,2}^-}{(x_{1,2}^- - x_{4,2}^-)^3} - \frac{1}{3(x_{1,2}^- - x_{4,2}^-)^2} \right], \quad (B.26)
\]

\[
\langle \phi(x_1) L_{-2}(x_3) L_0(x_2) \phi(x_4) \rangle = \frac{\delta^{(2)}(\vec{x}_{23}^4)}{\pi x_{12}^+ x_{24}^+} \left[ -\frac{6(x_{4,2}^-)^2}{(x_{1,2}^- - x_{4,2}^-)^4} - \frac{4x_{4,2}^-}{(x_{1,2}^- - x_{4,2}^-)^3} - \frac{1}{3(x_{1,2}^- - x_{4,2}^-)^2} \right]. \quad (B.27)
\]

Combining them results in the commutator

\[
\langle \phi(x_1)[L_0(x_2), L_{-2}(x_3)] \phi(x_4) \rangle = -\frac{2}{\pi x_{12}^+ x_{24}^+ (x_{1,2}^- + x_{4,2}^-)^3} \delta^{(2)}(\vec{x}_{23}^4), \quad (B.27)
\]

which matches the three-point function with \(L_{-1}\):

\[
\langle \phi(x_1)[L_0(x_2), L_{-2}(x_3)] \phi(x_4) \rangle = -2i\delta^{(2)}(\vec{x}_{23}^4) \langle \phi(x_1) L_{-1}(x_2) \phi(x_4) \rangle. \quad (B.28)
\]
B.4. Commutators of global light-ray operators

B.4.3 Remaining commutators

The different cases are given by

\[
\langle \phi(x_1)[L_1(x_2), L_{-2}(x_3)]\phi(x_4)\rangle = -\frac{1}{\pi} \left( \frac{x_{1,2}^-}{x_{12}^+} \right)^2 + 4\frac{x_{1,2}^- x_{4,2}^- + (x_{4,2}^-)^2}{x_{12}^+ x_{42}^+(x_{1,2}^- - x_{4,2}^-)^3} \delta^{(2)}(\vec{x}_{23}^+) , \tag{B.28}
\]

and

\[
\langle \phi(x_1)[L_2(x_2), L_{-2}(x_3)]\phi(x_4)\rangle = -4i\delta^{(2)}(\vec{x}_{23}^+) \langle \phi(x_1)L_0(x_2)\phi(x_4)\rangle . \tag{B.29}
\]

and

\[
\langle \phi(x_1)[L_0(x_2), L_{-1}(x_3)]\phi(x_4)\rangle = -i\delta^{(2)}(\vec{x}_{23}^+) \langle \phi(x_1)L_0(x_2)\phi(x_4)\rangle . \tag{B.30}
\]

and

\[
\langle \phi(x_1)[L_1(x_2), L_{-1}(x_3)]\phi(x_4)\rangle = -\frac{2}{\pi} \frac{x_{1,2}^- x_{4,2}^- + (x_{4,2}^-)^2}{x_{12}^+ x_{42}^+(x_{1,2}^- - x_{4,2}^-)^3} \delta^{(2)}(\vec{x}_{23}^+) , \tag{B.31}
\]

and

\[
\langle \phi(x_1)[L_2(x_2), L_{-1}(x_3)]\phi(x_4)\rangle = -2i\delta^{(2)}(\vec{x}_{23}^+) \langle \phi(x_1)L_0(x_2)\phi(x_4)\rangle . \tag{B.32}
\]

and finally

\[
\langle \phi(x_1)[L_2(x_2), L_{-1}(x_3)]\phi(x_4)\rangle = -6i\delta^{(2)}(\vec{x}_{23}^+) \langle \phi(x_1)L_0(x_2)\phi(x_4)\rangle . \tag{B.33}
\]

This concludes the discussion of all the cases where the algebra (3.86) is satisfied in free field theory, and where the \( f''(x^-)g''(x^-) \) contribution to the commutator is vanishing.

B.4.4 Integrating commutators in the perpendicular direction

Instead of extracting a transverse delta function for our commutators, we can also evaluate integrals in the perpendicular direction instead. Evaluating the \( \bar{x}_3^\perp \) integral allows us to read of the coefficient multiplying \( \delta^{(2)}(\bar{x}_{23}^\perp) \), thus providing a useful consistency check. We will do this in some details for the case \([L_{-2}, L_{-2}]\). We need to compute both
orderings, obtaining
\[
\langle \phi(x_1)L_{-2}(x_2)L_{-2}(x_3)\phi(x_4) \rangle = -\frac{24}{\pi^2} \frac{(x_{23}^+ - i\epsilon)^4}{(x_{12}^- + i\epsilon)(x_{34}^- + i\epsilon)} \times \frac{1}{[-(x_{23}^+ - i\epsilon)(x_{1,2}^- - x_{4,3}^- - i\epsilon) + |x_{23}^+|^2]^5},
\]
for \(B.38\)
\[
\langle \phi(x_1)L_{-2}(x_3)L_{-2}(x_2)\phi(x_4) \rangle = -\frac{24}{\pi^2} \frac{(x_{23}^+ + i\epsilon)^4}{(x_{13}^- - i\epsilon)(x_{24}^- - i\epsilon)} \times \frac{1}{[(x_{23}^+ + i\epsilon)(x_{1,3}^- - x_{4,2}^- - i\epsilon) + |x_{23}^+|^2]^5}.
\]
for \(B.39\).

Prior to evaluating the commutator, we will first integrate over \(\vec{x}_3^\perp\). To make the resulting integral simpler, we’ll set the transverse components of the remaining three operators to zero:
\[
\vec{x}_1^\perp = \vec{x}_2^\perp = \vec{x}_4^\perp = 0.
\]
for \(B.40\).

We can then write \(\vec{x}_3^\perp = (r \cos \theta, r \sin \theta)\), and the coordinates \(x_{i,j}^\perp\) become
\[
x_{1,2}^- = x_1^-,
\quad
x_{4,3}^- = -\frac{r^2}{x_{34}^-},
\quad
x_{1,3}^- = x_1^- - \frac{r^2}{x_{13}^-},
\quad
x_{4,2}^- = x_4^-.
\]
for \(B.41\).

When inserting \(B.41\) into both orderings, we get two expressions that have no dependence on \(\theta\), and very simple dependence on \(r\), so we only need to evaluate the general integral
\[
\int d^2 x_\perp \frac{1}{(A + B|\vec{x}_3^\perp|^2)^n} = \int_0^\infty dr^2 \frac{\pi}{(A + B r^2)^n} = \frac{\pi}{(n - 1) A^{n-1} B}.
\]
for \(B.42\).

Using this general integral, the two orderings after integrating over \(\vec{x}_3^\perp\) are given by
\[
\int d^2 x_3^\perp \langle \phi(x_1)L_{-2}(x_2)L_{-2}(x_3)\phi(x_4) \rangle = -\frac{6}{\pi} \frac{1}{(x_{12}^+ - i\epsilon)(x_{24}^+ - i\epsilon)(x_{14}^+ - i\epsilon)^4},
\]
\[
\int d^2 x_3^\perp \langle \phi(x_1)L_{-2}(x_3)L_{-2}(x_2)\phi(x_4) \rangle = -\frac{6}{\pi} \frac{1}{(x_{12}^+ - i\epsilon)(x_{24}^+ - i\epsilon)(x_{14}^+ - i\epsilon)^4}.
\]

As we can see, these two orderings give the exact same expression, such that the commutator vanishes
\[
\int d^2 x_3^\perp \langle \phi(x_1)[L_{-2}(x_2), L_{-2}(x_3)]\phi(x_4) \rangle = 0.
\]

Note that this integrated commutator vanishes even when \(x_{23}^+ \neq 0\) (i.e. when the light-ray operators are not on the same null surface). It is clear that the same procedure can be repeated for any of the correlators we already presented. It gives the correct answer for
all these cases, and we present

\[ \int d^2 x_3^+ \langle \phi(x_1)[L_{-1}(x_2), L_{-2}(x_3)]\phi(x_4)\rangle = \frac{2}{\pi} \frac{1}{x_{12}^+ x_{24}^+ (x_{14}^-)^3} = -i \langle \phi(x_1)L_{-2}(x_2)\phi(x_4)\rangle, \]

\[ \int d^2 x_3^+ \langle \phi(x_1)[L_0(x_2), L_{-2}(x_3)]\phi(x_4)\rangle = -\frac{2}{\pi} \frac{x_1^- + x_4^-}{x_{12}^+ x_{24}^+ (x_{14}^-)^3} = -2i \langle \phi(x_1)L_{-1}(x_2)\phi(x_4)\rangle, \]

\[ \int d^2 x_3^+ \langle \phi(x_1)[L_1(x_2), L_{-2}(x_3)]\phi(x_4)\rangle = \frac{1}{\pi} \frac{(x_1^-)^2 + 4x_1^- x_4^- + (x_4^-)^2}{x_{12}^+ x_{24}^+ (x_{14}^-)^3} = -3i \langle \phi(x_1)L_0(x_2)\phi(x_4)\rangle. \]

**B.5 Non-local operator from the \( T \times T \) OPE**

**B.5.1 Derivation of the non-local operator**

In this appendix, we want to explain how we can get the non-local operator that reproduces the commutator when inserted into correlation functions with scalar external states. This allows us to write the leading singularity of the \([L_1(x_2), L_1(x_3)]\) commutator as a non-local operator. This is the resummed version of the infinite sum we presented in section 3.6.1. The computations are done in free field theory, where the stress energy tensor is given as in equation (B.5). The \( TT \ OPE \) follows from considering

\[ T_{-}(x)T_{-}(y) \sim \alpha^2 \left[ (\partial_- \phi(x))^2 - \frac{1}{2} \phi \partial_-^2 \phi(x) \right] \left[ (\partial_- \phi(y))^2 - \frac{1}{2} \phi \partial_-^2 \phi(y) \right], \quad (B.43) \]

where \( \alpha = \frac{1}{6\pi} \). We want to do one Wick contraction in each product of four fields above while leaving two fields uncontracted and normal ordered. Doing this, we obtain

\[ T_{-}(x)T_{-}(y) = 4\alpha^2 \langle \partial_- \phi(x)\partial_- \phi(y) \rangle \partial_- \phi(x)\partial_- \phi(y) : \]

\[ - \alpha^2 \left[ \langle \partial_- \phi(x)\phi(y) \rangle \partial_- \phi(x)\partial_-^2 \phi(y) : + \langle \partial_- \phi(x)\partial_-^2 \phi(y) \rangle \partial_- \phi(x)\phi(y) : \right] \]

\[ - \alpha^2 \left[ \langle \phi(x)\partial_- \phi(y) \rangle \partial_-^2 \phi(x)\partial_- \phi(y) : + \langle \partial_-^2 \phi(x)\partial_- \phi(y) \rangle \phi(x)\partial_- \phi(y) : \right] \]

\[ + \frac{\alpha^2}{4} \left[ \langle \phi(x)\phi(y) \rangle \partial_-^2 \phi(x)\partial_-^2 \phi(y) : + \langle \phi(x)\partial_-^2 \phi(y) \rangle \partial_-^2 \phi(x)\phi(y) : + \langle \partial_-^2 \phi(x)\partial_-^2 \phi(y) \rangle \phi(x)\phi(y) : \right] \]

To derive the commutator (once inserted into three-point functions with appropriate external states), we need to integrate this as \( x^- \) goes to \( y^- \). Using (B.44), and integrating
where all products of fields are normal ordered. Let us explain the notation in the last equation. \( \phi(x_{3,2}) \) is the field \( \phi \) evaluated at the position \( \phi(x_{3,2}) \equiv \phi(x_1^+, x_3^-, \vec{x}_2^+) \). This happens because we evaluated the \( dx^-_3 \) integral at the location of the pole where \( x_{3,2}^2 = 0 \), which is \( x_{3,2}^- \). In addition, the minus derivatives are \( x_3^- \) minus derivatives.

For \([L_1(x_2), L_1(x_3)]\), which is the easiest case with a contribution at finite separation, we get

\[
[L_1(x_2), L_1(x_3)] = - \frac{(2\pi i)}{\pi^4(x_{23}^- + i\epsilon)} \int dx^-_3 (x^-_3)^3 \tag{B.46}
\]

\[
\left\{ \frac{1}{3} \partial_- \phi(x^-_3) + \frac{19}{24} x_{3,2}^2 \partial^2 \phi(x^-_3) + \frac{3}{8} (x^-_{3,2})^2 \partial^3 \phi(x^-_{3,2}) + \frac{1}{24} (x^-_{3,2})^3 \partial^4 \phi(x^-_{3,2}) \right\} \phi(x_3)
\]

\[
- \left[ \frac{1}{6} \phi(x^-_{3,2}) + \frac{7}{6} (x^-_{3,2}) \partial_- \phi(x^-_{3,2}) + (x^-_{3,2})^2 \partial^2 \phi(x^-_{3,2}) + \frac{1}{6} (x^-_{3,2})^3 \partial^3 \phi(x^-_{3,2}) \right] \partial_- \phi(x_3)
\]

\[
+ \left[ \frac{1}{24} (x^-_{3,2}) \phi(x^-_{3,2}) + \frac{1}{8} (x^-_{3,2})^2 \partial_- \phi(x^-_{3,2}) + \frac{1}{24} (x^-_{3,2})^3 \partial^2 \phi(x^-_{3,2}) \right] \partial^2 \phi(x_3) \right\} .
\]

Once inserted into three-point functions with scalar external states, the non-local operator (B.46) reproduces the result (3.90) for \( \langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4) \rangle \). Let us see how this works.

Once we compute the three-point function of (B.46) with \( \phi \) as external states, every term in the three-point function is of the form \( \langle \phi(x_1) : \partial^\alpha \phi(x_{3,2}) \partial^\beta \phi(x_3) \phi(x_4) \rangle \), which has two different Wick contractions

\[
\langle \phi(x_1) \partial^{\alpha_1} \phi(x^-_{3,2}) \partial^{\alpha_2} \phi(x_3) \phi(x_4) \rangle = \langle \phi(x_1) \partial^{\alpha_1} \phi(x^-_{3,2}) \rangle \langle \partial^{\alpha_2} \phi(x_3) \phi(x_4) \rangle \tag{B.47}
\]

\[
+ \langle \phi(x_1) \partial^{\alpha_2} \phi(x_3) \rangle \langle \partial^{\alpha_1} \phi(x^-_{3,2}) \phi(x_4) \rangle .
\]

When performing the \( dx^-_3 \) integral of (B.46), we want to close the contour in the upper-
half plane. It means that the two poles are at positions

$$x_3^- = x_{4,3}^- + i\epsilon = x_4^- + \frac{|x_{4,3}^+|^2}{x_{4,3}^-} + i\epsilon,$$

$$x_3^- = x_{4,2}^- - \frac{|x_{23}^-|^2}{x_{23}^- - i\epsilon} + i\epsilon = x_4^- - \frac{|x_{23}^-|^2}{x_{23}^- - i\epsilon} + \frac{|x_{24}^-|^2}{x_{24}^-} + i\epsilon.$$  \hspace{1cm} (B.48, 49)

If we insert (B.46) in a three-point function and use only the first Wick contraction (B.47), performing the integral by computing the residue at the location of the pole (B.48), we arrive to

$$\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4)\rangle|_1 = -\frac{1}{\pi^2} \frac{x_{1,2}^- x_{4,-3}}{|x_{23}^-|^2|x_{23}^+|^2}. \hspace{1cm} (B.50)$$

On the other hand, if we use the second line of (B.47) and perform the integral picking up the pole as (B.49), we get

$$\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4)\rangle|_2 = \frac{1}{\pi^2} \frac{x_{4,2}^- x_{1,3}^-}{|x_{23}^-|^2|x_{24}^-|^2}. \hspace{1cm} (B.51)$$

Adding both results yields the commutator we already derived using the four-point function $\langle \phi(x_1)T_{-+}(x_2)T_{-+}(x_3)|\phi(x_4)\rangle$ that is (3.90).

If the goal is to reproduce only the leading singularity in an expansion as $x_2^+ \to x_3^+$, which is given by

$$\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4)\rangle|_{x_2^+ \to x_3^+} = \frac{2(x_{23}^+)^4}{\pi^2 |x_{23}^-|^2 (2x_{34}^\perp)^2} \left[ -\frac{1}{x_{34}^+} \phi(x_1) \phi(x_{3,2}^-) \partial^2 \phi(x_3) \phi(x_4) \right] + \ldots,$$

then only the last line of equation (B.46) is needed. This implies that you can get the whole finite separation contribution to the commutator by just considering

$$\langle \phi(x_1)[L_1(x_2), L_1(x_3)]\phi(x_4)\rangle|_{l.s.}, \hspace{1cm} (B.53)$$

$$= -\int dx_3^- (2\pi i) (x_{3,2}^-)^3 \left[ \frac{1}{24} (x_{3,2}^-)^3 \langle \phi(x_1) \phi(x_{3,2}^-) \partial^2 \phi(x_3) \phi(x_4) \rangle \right] + \frac{1}{8} (x_{3,2}^-)^2 \langle \phi(x_1) \partial_- \phi(x_{3,2}^-) \partial^2 \phi(x_3) \phi(x_4) \rangle + \frac{1}{24} (x_{3,2}^-)^3 \langle \phi(x_1) \partial^2 \phi(x_{3,2}^-) \phi(x_3) \phi(x_4) \rangle,$$

where the two central operators of each term are normal ordered.

Finally, we want to comment on the following. We can expand (B.53) as $x_2$ goes to $x_3$. This produces an infinite sum that is similar to the one we proposed in (3.99). This suggests that the non-local operator (B.53) is the resummed version of the infinite sum (3.99). Let us explain this in more details. If we consider (B.53) and expand $\phi(x_{3,2}^-)$ around $x_3$, this will produce a bilocal operator built out of $\phi(x_3)$ and its derivatives. Because throughout this work, we have evaluated correlators at $x_2^+ = x_3^+$, the Taylor expansion of $\phi(x_{3,2}^-)$ cannot have $\partial_+$ derivatives, and is given by

$$\phi(x_2) = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k=0}^{p} \binom{p}{k} (x_{23}^- \partial_-)^k ((x_{23}^+)^l \partial_1)^{p-k} \phi(x_3). \hspace{1cm} (B.54)$$
The leading singularity in the commutator $[L_1(x_2), L_1(x_3)]|_{l.s.}$ is of the form $(x_{23}^\perp)^I/|x_{23}^\perp|^2$, and we thus want to focus on terms in the expansion (B.54) that have exactly one $\partial_I$ derivative. They are given by $k = p - 1$ in (B.54). Moreover, $x_{23}^- = x_{3,2}^-$ such that $x_{23}^- = -i|x_{23}^\perp|^2$, and the expansion becomes

$$\phi(x_{2,3})\phi(x_3) := \sum_{p=0}^\infty \frac{1}{p!} \left(-\frac{|x_{23}^\perp|^2}{i\epsilon}\right)^p (x_{23}^\perp)^I : \partial^p \partial_I \phi(x_3) \phi(x_3) :.$$  

(B.55)

Inserting this expansion in the last line of (B.53), we obtain an infinite sum that resembles (3.99). In principle, one can obtain the coefficients $a_m$ by explicitly comparing these two sums.

Finally, note that it is clear that one can also just resum (B.53) with (B.55). Unsurprisingly, this reproduces the full contribution at finite separation of the $[L_1(x_2), L_1(x_3)]$ commutator.

### B.6 Details of the $[L_1, L_1]$ expansion

In this appendix, we want to give more details on the computation of the $L_1 L_1$ OPE to investigate the leading singularity in the commutator of $[L_1(x_2), L_1(x_3)]$ at finite transverse separation. We are going to compute the leading operators in the infinite sum of equation (3.99), that we remind here for convenience

$$[L_1(x_2), L_1(x_3)]|_{l.s.} = \sum_{m=0}^{\infty} a_m G^m.$$  

(B.56)

with

$$G^m(x_i) = \frac{(x_{23}^\perp)^I}{|x_{23}^\perp|^2} \int dx_3^- (x_3^-)^{m+3} \phi(x_1) \phi(\partial^m \partial_I \phi(x_3) \phi(x_4)).$$  

(B.57)

#### B.6.1 $m = 0$

When $m = 0$ in (B.57), the leading operator is $\partial_I (\phi^2(x_3))$ and the three-point function is

$$G^0(x_i) = \int dx_3^- \frac{4(x_{23}^\perp)^I}{|x_{23}^\perp|^2} (x_3^-)^3 \left[ (x_{13}^\perp)^I \frac{1}{x_{13}^4 x_{34}^4} - (x_{34}^\perp)^I \frac{1}{x_{13}^4 x_{34}^4} \right].$$  

(B.58)

These two terms are never going to mix so we can compute them independently, and we denote the term we are considering by a subscript indicating which perpendicular vector will be summed over with $(x_{23}^\perp)^I$. Let us consider

$$G^0(x_i)|_{13} = \int dx_3^- \frac{4(x_{23}^\perp)^I}{|x_{23}^\perp|^2} \left[ (x_{13}^\perp)^I \frac{(x_{34}^-)^3}{x_{13}^4 x_{34}^4} \right].$$  

(B.59)
This expression has three poles in $x_3^-$. The first one is when $x_{34}^2 = 0$, the second when $x_{34}^2 = 0$ and the last one when $x_3^- \to \infty$. We will come back to the contributions from the pole at infinity shortly. Let us compute all of these and indicate which pole we are considering by a superscript

\[
\begin{align*}
G_0^{0}(x_i)_{13}^{x_3 \to x_1} & = \frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{13}^+) f \left[ \frac{(x_{13}^-)^2(-2x_{13}^- + 3x_{4,3}^-)}{x_{34}^+(x_{13}^-)^2(x_{13}^- - x_{4,3}^-)^2} \right], \\
G_0^{0}(x_i)_{13}^{x_3 \to x_4} & = \frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{13}^+) f \left[ \frac{-(x_{4,3}^-)^3}{x_{34}^+(x_{13}^-)^2(x_{13}^- - x_{4,3}^-)^2} \right], \\
G_0^{0}(x_i)_{13}^{x_3 \to \infty} & = -\frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{13}^+) f \left[ \frac{(2x_{13}^- + x_{4,3}^-)}{x_{34}^+(x_{13}^-)^2} \right].
\end{align*}
\]

Summing the three contributions gives zero as expected, but this indicates that we need to take care of the pole at infinity for this operator. We can do the same computation with the second term of (B.59). It gives

\[
\begin{align*}
G_0^{0}(x_i)_{34}^{x_3 \to x_1} & = \frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{34}^+) f \left[ \frac{(x_{1,3}^-)^3}{x_{13}^+(x_{34}^+)^2(x_{1,3}^- - x_{4,3}^-)^2} \right], \\
G_0^{0}(x_i)_{34}^{x_3 \to x_4} & = \frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{34}^+) f \left[ \frac{(x_{4,3}^-)^3(2x_{4,3}^- - 3x_{1,3}^-)}{x_{13}^+(x_{34}^+)^2(x_{1,3}^- - x_{4,3}^-)^2} \right], \\
G_0^{0}(x_i)_{34}^{x_3 \to \infty} & = -\frac{4(2\pi i)}{|x_{23}|^2} (x_{23}^+) f(x_{34}^+) f \left[ -\frac{(x_{1,3}^- + 2x_{4,3}^-)}{x_{13}^+(x_{34}^+)^2} \right].
\end{align*}
\]

The limit we described in the main text, and that we remind here, has been designed such that the contribution of the pole at infinity vanishes (at least at leading order), such that the subtlety at infinity disappears. The limit is the following:

\[
|x_{13}^-| = |x_{34}^-| = |x|^+ \to \infty, \quad x_{34}^+ = x_{13}^+ = x^+.
\]

Let us add the two terms for the pole at infinity, and take the limit we just described. This yields

\[
G_0^{0}(x_i)_{13+34}^{x_3 \to \infty} = (2\pi i) \frac{(x_{23}^+) f(x_{13}^+) f}{|x_{23}|^2} \left[ -(x_{13}^+) f + (x_{34}^+) f \right] + \ldots
\]

Because $(x_{13}^+) f = (x_{34}^+) f = (x^+) f$, this vanishes. The other poles give

\[
G_0^{0}(x_i)_{13+34}^{x_3 \to x_4} = -G_0^{0}(x_i)_{13+34}^{x_3 \to x_1} = -6(2\pi i) \frac{(x_{23}^+) f(x^+) f}{|x_{23}|^2} \left[ |x|^+ \right] + \ldots
\]

They have exactly the same functional form as the leading singularity of the commutator in this limit $[L_1(x_2), L_1(x_3)]_{\text{l.s.}}$ (cf (3.103)). We will now present the general terms in this sum, splitting $m$ into even and odd numbers and explaining the general features.
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B.6.2 Even $m$

The leading operators that appear in the sum are given by (3.101). We remind them here for convenience

$$J_{i_1...i_s}(x) = \sum_{k=0}^{s} h(k, s) \partial_{i_1} \ldots \partial_{i_k} \phi(x) \partial_{i_{k+1}} \ldots \partial_{i_s} \phi(x) - \text{traces}, \quad (B.61)$$

with

$$h(k, s) = \frac{(-1)^k}{\Gamma(k) \Gamma(s-k)} . \quad (B.62)$$

For even $m = 2p$, we want to compute

$$G^{2p}(x_i) = \int dx_3^- (\frac{x_3^-}{|x_{23}|^2})^f (x_3^-) 3^{+2p} \partial_I \langle \phi(x_1) J_{(2p)}(x_3) \phi(x_4) \rangle . \quad (B.63)$$

The operator $J_{(2p)}$, where the subscript denote the number of $-$ indices, can be constructed as

$$J_{(2p)} = \sum_{\ell=0}^{p-1} 2 h(\ell, 2p) : \partial^{2p+\ell-1} \phi \phi^{2p+\ell} : + \sum_{\ell=0}^{p-1} h(\ell, 2p) : \partial^{\ell} \phi \partial^{2p+\ell} \phi : . \quad (B.64)$$

The location of the poles is the same as for the case $m = 0$, but the pole at infinity is present in the sum only for $p = \{0, 1\}$. We will come back to this shortly. As for the $m = 0$ case, their contribution vanishes in the limit we are considering. For generic $p$ and in the limit we are interested in, the outcome is

$$G^{2p}(x_i) \big|_{x_3^- \rightarrow x_4^-} = (2\pi i) \frac{(-1)^p 2^{1-2p} (1 + 2p) (3 + 2p) (\frac{x_{23}^-}{|x_{23}|^2})^f (\frac{x_i^+}{x_4^+})^f \frac{|x_i^+|^2}{(x^+)^4}}{2p - 1} \Gamma(1 + p)^2 \quad (B.65)$$

We then see that all the even terms contribute to $[L_1(x_2), L_1(x_3)]|_{L_A}$ in the limit we consider.

B.6.3 Odd $m$

We can now consider the case where $m = 2q + 1$ is odd. We want to compute

$$G^{2q+1}(x_i) = \int dx_3^- (\frac{x_3^-}{|x_{23}|^2})^f (x_3^-)^{4+2q} \langle \phi(x_1) J_{(2p+1)I}(x_3) \phi(x_4) \rangle . \quad (B.66)$$

The operator $J_{(2p+1)I}$ can be constructed as

$$J_{(2p+1)I} = \sum_{\ell=0}^{q+1} (2q + 2 - \ell) h(\ell, 2q + 2) : \partial^{2q+1-\ell} \phi \partial^{\ell} \phi : + : \partial^{\ell} \phi \partial^{2q+1-\ell} \phi : + \sum_{\ell=1}^{q} \ell h(\ell, 2q + 2) : \partial^{2q+2-\ell} \phi \partial^{\ell} \phi : + : \partial^{\ell} \phi \partial^{2q+2-\ell} \phi : . \quad (B.67)$$
Reproducing the computation we did for arbitrary \( q \) in the limit we are interested in yields

\[
G^0(x_i)|_{x_3 \to x_4}^{x_3 \to x_4} = (2\pi i) \frac{(-1)^q 2^{-2q}(3 + 2q)}{(1 + 2q)\Gamma(q + 1)\Gamma(q + 2)} \frac{(x_{23}^\perp)^I (x_{23}^\perp)^I |x_{23}^\perp|^2}{|x_{23}^\perp|^2 (x^\perp)^4}. \tag{B.68}
\]

We then see that all the odd terms all contribute at leading order to \([L_1(x_2), L_1(x_3)]|_{l.s.}\) in the limit we consider. In addition, only \( q = 0 \) has a contribution coming from a pole at infinity. We conclude that all terms in the expansion contribute at the same order and thus the infinite sum does not truncate. We are just resumming an infinite number of coefficients.

### B.6.4 Poles at infinity

Let us now explain why only the first three terms with \( m = 0, 1, 2 \) have a contribution coming from evaluating a residue at infinity. For this discussion, we use our intuition from two-dimensional CFT.

In this setup, the relevant scaling dimension is the collinear weight \( h \). When performing a light-ray integral on an arbitrary operator \( O(x) \) with collinear weight \( h \), the resulting light-ray operator has collinear weight \( h - 1 \). In two dimensions, the first operator that acts non-trivially on the vacuum on the right is schematically

\[
\int dx^- \frac{1}{x-} O(x), \tag{B.69}
\]

which has collinear weight \( h \). Using inversion, the first operator that acts non-trivially on the vacuum on the left is then

\[
\int dx^- (x^-)^{2h-1} O(x). \tag{B.70}
\]

Because acting non-trivially on the vacuum is directly related with having a pole at infinity, this implies that for any operator \( O(x) \) dressed with a power of \( (x^-) \) smaller than \( 2h - 1 \), there will be no poles at infinity.

Let us now do some collinear weight counting. The rules are the following: \( \phi(x) \) has \( h = 1/2 \), \( \partial^- \) has \( h = 1 \) and \( \partial_I \) has \( h = 1/2 \). In addition, when considering operators with even \( m \) that are of the form \( \partial_I J_{(2p)} \) we do not count the \( \partial_I \) because it can be stripped off from the integral without changing the behaviour at infinity.

For a given \( m \), the collinear weight of the leading operator is

\[
h(m) = 1 + m + \frac{1}{2} (m \mod 2). \tag{B.71}
\]

We have a pole at infinity if \( 2h - 1 \leq m - 3 \), which is the power of \( (x^-) \) that appears in our infinite sum. This implies

\[
2 + 2m + (m \mod 2) \leq m - 3 \quad \rightarrow \quad m \leq 2 - (m \mod 2). \tag{B.72}
\]
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This inequality is satisfied provided \( m \leq 2 \), which indicates that only the three first terms in our sum have a contribution coming from a pole at infinity. This is exactly the behaviour we witnessed when performing these integrals explicitly.

B.7 Conformal blocks at finite position

In section 3.7, we analyzed the contribution of the \( \mathcal{O} \) conformal block to the commutators of light-ray operators. However, we specifically considered the case where the light-ray operators were all inserted at future null infinity, which simplified the calculation significantly. In this appendix, we consider the case where the two light-ray operators are instead inserted on the same null slice at some finite \( x^+ \), to confirm the non-vanishing commutator at finite transverse separation. Because this setup is more complicated, we will focus on the specific case where \( \mathcal{O} \) has dimension \( \Delta = 2 \), which corresponds to \( \phi^2 \) for the case of free field theory. Our computation of the \( \mathcal{O} \) conformal block will use the Mathematica package CFTs4D presented in [251] and will largely follow the same methodology and notation introduced there.

B.7.1 Lightning review of spinning correlators

A general four-point function of operators in traceless symmetric representations of the Lorentz group (each labeled by their spin \( j_i \)) can always be written in the form

\[
\langle \mathcal{O}_{j_1}(x_1)\mathcal{O}_{j_2}(x_2)\mathcal{O}_{j_3}(x_3)\mathcal{O}_{j_4}(x_4) \rangle = \mathcal{K}_4(x_i) \sum_I g^I(u,v) T^I(x_i), \tag{B.73}
\]

where the RHS is a sum over all possible four-point function tensor structures \( T^I \), which are completely fixed by conformal symmetry, based on the spins \( j_i \). Each tensor structure is multiplied by a corresponding scalar function \( g^I(u,v) \), which is a function of the standard conformally-invariant cross-ratios

\[
u \equiv \frac{x^{12}x^{34}}{x^{13}x^{24}} = z\bar{z}, \quad v \equiv \frac{x^{14}x^{23}}{x^{13}x^{24}} = (1 - z)(1 - \bar{z}). \tag{B.74}
\]

Finally, the overall kinematic factor \( \mathcal{K}_4 \) is fixed by the scaling dimensions and spins of the external operators,

\[
\mathcal{K}_4(x_i) = \left(\frac{x_{24}}{x_{14}}\right)^{\kappa_1-\kappa_2} \left(\frac{x_{14}}{x_{13}}\right)^{\kappa_3-\kappa_4} \frac{1}{x^{\kappa_1+\kappa_2}_{12}x^{\kappa_3+\kappa_4}_{34}}, \tag{B.75}
\]

where \( \kappa_i \equiv \Delta_i + j_i \).

The set of tensor structures \( T^I \) depends on the spins of the four operators, but they can all be constructed from the two building blocks

\[
H_{i,j}^{\mu\nu} = x_{i,j}^{2}T_{\mu\nu} - 2x^{\mu}_{i}x^{\nu}_{j}, \quad V_{k,l}^{\mu} = \frac{x^{2}_{k,l}x^{2}_{k,l}}{x^{2}_{i,j}} \left( \frac{x^{\mu}_{k,l}}{x^{\mu}_{i,j}} - \frac{x^{\mu}_{l,k}}{x^{\mu}_{l,k}} \right). \tag{B.76}
\]
B.7. Conformal blocks at finite position

We can compute the four-point function by inserting a complete set of intermediate states. These can be arranged into irreducible representations of the conformal group, each associated with a primary operator \( \mathcal{O}' \), resulting in the conformal partial wave decomposition

\[
\langle \mathcal{O}_{j_1}(x_1)\mathcal{O}_{j_2}(x_2)\mathcal{O}_{j_3}(x_3)\mathcal{O}_{j_4}(x_4) \rangle = \sum_{\mathcal{O}'} \sum_{a,b} \lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}'\mathcal{O}_4}^{a}\lambda_{\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}^{b} W_{\mathcal{O}'}^{ab}(x_i). \tag{B.77}
\]

The indices \( a, b \) label the set of allowed tensor structures for three-point functions involving \( \mathcal{O}' \) with the external operators, and \( \lambda^a \) are the associated OPE coefficients,

\[
\langle \mathcal{O}_{j_1}(x_1)\mathcal{O}_{j_2}(x_2)\mathcal{O}_{j_3}(x_3) \rangle = K_3(x_i) \sum_{a} \lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{a} T_3^a(x_i), \tag{B.78}
\]

with the familiar three-point function kinematic factor

\[
K_3(x_i) = \frac{1}{x_1^{\kappa_1+\kappa_2-\kappa_3} x_2^{\kappa_2+\kappa_3-\kappa_1} x_3^{\kappa_1+\kappa_3-\kappa_2}}. \tag{B.79}
\]

The functions \( W_{\mathcal{O}'}^{ab} \) in eq. (B.77) are known as \textit{conformal partial waves}, and encode the contribution to a four-point function from a given pair of three-point function tensor structures associated with \( \mathcal{O}' \). These individual conformal partial waves can each be decomposed into four-point function tensor structures,

\[
W_{\mathcal{O}'}^{ab}(x_i) = K_4(x_i) \sum_{I} G_{\mathcal{O}',AB}^{I,ab}(u,v) T_4^I(x_i). \tag{B.80}
\]

The functions \( G_{\mathcal{O}',AB}^{I,ab} \) are referred to as \textit{conformal blocks}, and encode the contribution of a pair of three-point function tensor structures for \( \mathcal{O}' \) to a particular four-point function tensor structure. Their structure is completely fixed by conformal symmetry, with many efficient techniques for computing their exact expressions.

In this work, we are specifically interested in the case where two of the external operators are scalars \( (j_1 = j_4 = 0) \), and the other two are the stress tensor \( (j_2 = j_3 = 2) \).

B.7.2 Computing the conformal partial wave

Concretely, we would like to compute the contribution of \( \mathcal{O} \) to the four-point function

\[
\langle \mathcal{O}(x_1)T_-(x_2)T_-(x_3)\mathcal{O}(x_4) \rangle.
\]

As discussed in the previous section, this contribution can be decomposed into a sum over four-point function tensor structures \( T_4^I \), which can all be built from the five building blocks:

\[
H_{23}^{++}, \ V_{2,13}^{+}, \ V_{2,14}^{+}, \ V_{3,14}^{+}, \ V_{3,24}^{+}. \tag{B.81}
\]

Schematically, we have three types of combinations,

\[
(H_{23}^{++})^2, \ H_{23}^{++}V_{2}^{+}V_{3}^{+}, \ (V_{2}^{+})^2(V_{3}^{+})^2, \tag{B.82}
\]

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with two options each for $V_2$ and $V_3$. There are therefore $1+4+9 = 14$ different four-point tensor structures.

In general, there is a distinct conformal partial wave $W^{ab}_O$ for every incoming and outgoing three-point function tensor structure. Fortunately, because the exchanged operator $O$ is a scalar, for this case there is only one three-point tensor structure, and therefore only a single conformal partial wave.

The typical strategy for constructing conformal partial waves for external states with spin is to act with particular differential operators, known as weight-shifting operators, on the known conformal blocks for scalar external states [67, 252]. For our particular case, where the spinning external operator is the stress tensor, with $\Delta_T = 4$ and $j_T = 2$, the seed conformal block is

$$G_{seed}^{O}(u,v) = \frac{z \bar{z}}{z - \bar{z}} \left( k_{\Delta}^{(\Delta-2,2-\Delta)}(z) k_{\Delta-2}^{(\Delta-2,2-\Delta)}(\bar{z}) - k_{\Delta-2}^{(\Delta-2,2-\Delta)}(z) k_{\Delta}^{(\Delta-2,2-\Delta)}(\bar{z}) \right),$$

(B.83)

where $k_{\Delta}^{(\alpha,\beta)}(x)$ is defined as

$$k_{\Delta}^{(\alpha,\beta)}(x) = x^{\frac{\Delta}{2}} \text{F}_1 \left( \frac{1}{2}(\Delta - \alpha), \frac{1}{2}(\Delta + \beta); \Delta; x \right).$$

(B.84)

Because the exchanged operator is a scalar, we can rewrite this seed block in the more useful form [66]

$$G_{seed}^{O}(u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\Delta - 1)n}{n!} \frac{\Gamma^2(n + m + 1)}{m!(\Delta)_{2n+m}} u^{\frac{\Delta}{2} + n}(1 - v)^m,$$

(B.85)

which can be partially resummed to obtain

$$G_{seed}^{O}(u,v) = \sum_{n=0}^{\infty} \frac{n!(\Delta - 1)n}{(\Delta)_{2n}} u^{\frac{\Delta}{2} + n} \text{F}_1(n + 1, n + 1; \Delta + 2n; 1 - v).$$

(B.86)

We then need to construct the appropriate weight-shifting operator and act on the seed conformal block to obtain the conformal partial wave. The result is remarkably complicated, but fortunately we are only interested in a subset of the full expression. In particular, we are interested in terms which are nonzero when we integrate over both $x_2^-$ and $x_3^-$ to obtain a correlation function involving light-ray operators. This means we only need to focus on terms with poles when $x_2 \to x_1$ and $x_3 \to x_4$. Such poles come from the overall kinematic factor

$$K_4(x_i) = \frac{1}{x_{12}^{\Delta+6} x_{34}^{\Delta+6}} \left( \frac{x_{24}}{x_{14}} \right)^{\Delta-6} \left( \frac{x_{14}}{x_{13}} \right)^{6-\Delta},$$

(B.87)

multiplied by powers of $u^{\frac{\Delta}{2} + n}$ coming from derivatives of the seed block. Because higher-order terms in $u$ are less singular, in practice we therefore only need the first three terms in the block expansion (B.86).

In addition, we are interested only in contributions to the finite separation commutator of operators on the same null slice. We can therefore set $x_{23}^- = 0$, in which case all tensor structures containing $H_{23}$ vanish.

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For general $\mathcal{O}$, the most singular term has a third-order pole in both $x_2^-$ and $x_3^-$, and takes the simple form

$$
\langle \mathcal{O}(x_1)T_{--}(x_2)T_{--}(x_3)\mathcal{O}(x_4) \rangle \supset 4(x_{12}^+)^2(x_{34}^+)^2 x_{14}^{4-2\Delta} \frac{\, _2F_1(1,1;\Delta;1-v)}{x_{12}^+ x_{34}^+ x_{14}^+ x_{24}^+},
$$

(B.88)

which corresponds to the first term in the expansion of the seed block (B.86). The remaining less singular terms have the same basic structure, but include sums of multiple hypergeometric functions with various arguments.

For the rest of this appendix, we will focus on the specific case $\Delta = 2$, which corresponds to the operator $\phi^2$ in free field theory, though these results hold for any scalar operator with the same scaling dimension in any CFT. In this case, we obtain the full set of singular terms:

$$
\langle \phi^2(x_1)T_{--}(x_2)T_{--}(x_3)\phi^2(x_4) \rangle \phi^2
\supset - \frac{4(x_{12}^+)^2(x_{24}^+)^2 \log v}{(1-v)^3 x_{12}^+ x_{13}^+ x_{24}^+ x_{34}^+} - 8(x_{12}^+)(x_{24}^+)(1-v + \log v) \left( \frac{(x_{12}^+)^2}{x_{12}^+ x_{13}^+} + \frac{(x_{24}^+)^2}{x_{24}^+ x_{34}^+} \right) - \frac{2(3 - 4v + v^2 + 2\log v)}{(1-v)^3 x_{12}^+ x_{13}^+ x_{24}^+ x_{34}^+} \left( \frac{(x_{12}^+)^4}{x_{12}^+ x_{13}^+} + \frac{(x_{24}^+)^4}{x_{24}^+ x_{34}^+} \right) - \frac{52(x_{12}^+)^2(x_{24}^+)^2(2(1-v) + (1+v)\log v)}{(1-v)^5 x_{12}^+ x_{13}^+ x_{24}^+ x_{34}^+} - \frac{40(x_{12}^+)(x_{24}^+)((1-v)(5+v) + 2(1+2v)\log v)}{(1-v)^5 x_{12}^+ x_{13}^+ x_{24}^+ x_{34}^+} \left( \frac{(x_{12}^+)^2}{x_{12}^+ x_{13}^+} + \frac{(x_{24}^+)^2}{x_{24}^+ x_{34}^+} \right) - \frac{292(x_{12}^+)^2(x_{24}^+)^2 \left( 3(1-v^2) + (1+4v+v^2)\log v \right)}{(1-v)^7 x_{12}^+ x_{13}^+ x_{24}^+ x_{34}^+},
$$

(B.89)

where we have suppressed any overall OPE coefficient.

### B.7.3 Light-ray operator commutators

Now that we have the singular terms from the $\phi^2$ partial wave, we can integrate to obtain the contribution to correlators of light-ray operators. As a simple example, let’s first consider the case where both operators are the ANEC operator $L_{-2}$. In this case, we simply need to evaluate the integral

$$
\langle \phi^2(x_1)L_{-2}(x_2)L_{-2}(x_3)\phi^2(x_4) \rangle \phi^2 = \int dx_2^- dx_3^- \left( \langle \phi^2(x_1)T_{--}(x_2)T_{--}(x_3)\phi^2(x_4) \rangle \phi^2 \right).
$$

(B.90)

In practice, this integration is rather straightforward, as we simply pick up the poles

$$
x_2^- = x_{1,2}^-, \quad x_3^- = x_{4,3}^-.
$$

(B.91)

As we can see from eq. (B.90), the singular terms are largely functions of $v$, so the resulting expression is mostly dependent on $v$ evaluated at the singular points, which we
B. Details on the Stress Tensor Light-ray Operator Algebra

Indicate by

\[ \tilde{v} \equiv v \bigg|_{x_2^+ = x_{1,2}^- \atop x_3^+ = x_{4,3}^-} = \frac{x_{14}^2 |x_{23}^+|^2}{x_{12}^- x_{24}^- (x_{1,2}^- - x_{4,2}^-)(x_{1,3}^- - x_{4,3}^-)}. \]  

(B.92)

Note that \( \tilde{v} \) is simplified somewhat by the fact that \( x_{23}^+ = 0 \).

Evaluating this integral, we then obtain the resulting partial wave contribution to a light-ray operator correlator (up to an overall numerical coefficient),

\[ \langle \phi^2(x_1) L_{-2}(x_2) L_{-2}(x_3) \phi^2(x_4) \rangle \bigg|_{\phi^2} = -\frac{864 \left( 3(1 - \tilde{v}^2) + (1 + 4\tilde{v} + \tilde{v}^2) \log \tilde{v} \right)}{(x_{12}^+)^2 (x_{24}^+)^2 (x_{1,2}^- - x_{4,2}^-)^3 (x_{1,3}^- - x_{4,3}^-)^3 (1 - \tilde{v})^5}. \]  

(B.93)

One important feature of this expression, apart from its notable simplicity relative to the full partial wave, is that it is clearly symmetric under the exchange \( x_2 \leftrightarrow x_3 \), due largely to the symmetric nature of \( \tilde{v} \). Because of this symmetry, the resulting commutator is clearly zero, as expected,

\[ \langle \phi^2(x_1) [L_{-2}(x_2), L_{-2}(x_3)] \phi^2(x_4) \rangle \bigg|_{\phi^2} = 0, \]  

(B.94)

In fact, the integral of each individual singular term in eq. (B.89) has this same structure, such that no cancellation between distinct terms is needed to ensure that ANEC operators commute for \( \phi^2 \) exchange.

Finally, let’s repeat this procedure for \([L_{-1}, L_{-2}]\). Using (B.89), we can compute the two orderings, then take the difference to obtain:

\[ \langle \phi^2(x_1) [L_{-1}(x_2), L_{-2}(x_3)] \phi^2(x_4) \rangle \bigg|_{\phi^2} = -\frac{216 \left( (1 - \tilde{v})(7 + 16\tilde{v} + \tilde{v}^2) + 2(1 + 7\tilde{v} + 4\tilde{v}^2) \log \tilde{v} \right)}{(x_{12}^+)^2 (x_{24}^+)^2 (x_{1,2}^- - x_{4,2}^-)^2 (x_{1,3}^- - x_{4,3}^-)^3 (1 - \tilde{v})^5}, \]  

(B.95)

again, up to an overall coefficient. We therefore find that \( \phi^2 \) exchange leads to a \textit{nonzero} commutator at finite transverse separation, as seen in section 3.7 from correlators on the celestial sphere. In free field theory, this nonzero contribution must therefore cancel with the infinite tower of two-particle operators in the \( \phi^2 \times T \) OPE to ensure that the commutator vanishes in the full correlator, as we would have seen in section 3.5.2 had we considered the state \( \phi^2 \).
C

Details on Superfluids as Higher-form

C.1 Thermodynamic argument to fix $\tau$

Consider a system that contains surfaces of area $A$ running perpendicular to lines of length $L$, with an associated tension $\tau$ and a conserved charge $\tilde{Q}$ given by the number of planes through the line. The variation of the internal energy for this system is

$$dU = TdS - pdV + \tau LdA + \bar{\mu}Ad\tilde{Q}. \quad (C.1)$$

Now, since $\tilde{Q}$ is defined by a line integral, it is given by $\tilde{Q} = \rho L$. Consider a Legendre transform to the Landau grand potential:

$$\Phi = U - TS - \bar{\mu}A\tilde{Q}, \quad (C.2)$$

$$d\Phi = -sVdT - pdV - \bar{\rho}Vd\bar{\mu} + (\tau - \bar{\rho}\bar{\mu})LdA, \quad (C.3)$$

where $s$ is the entropy density. This quantity is naturally calculated by the on-shell action; we thus expect this to scale with the volume. This scaling is spoiled unless $\tau = \bar{\rho}\bar{\mu}$.

C.2 Conversion between conventions

We provide here the map between our results and the well established zeroth order superfluid hydrodynamics results from [236]. They are given, in the form (theirs = ours) by

$$\sqrt{\xi^2 + a^2\mu^2} = \tilde{\rho}, \quad f_s = \tilde{\rho}/\bar{\mu}, \quad \xi^\mu = a\mu u^\mu + \tilde{\rho}h^\mu, \quad \xi^\mu = a\mu u^\mu + \tilde{\rho}h^\mu, \quad (C.4)$$

$$n + f_s a^2 \mu = \rho, \quad \epsilon + f_s a^2 \mu^2 = \epsilon, \quad P = P - \bar{\mu}\bar{\rho}. \quad (C.5)$$

Below we list the map between the conventions in [238] and our results from section 4. As before, they are given in the form (theirs = ours)
C. Details on Superfluids as Higher-form

\[ \tilde{\zeta}^\mu = -\tilde{\rho}_\parallel h^\mu - \tilde{\rho}_\perp e^\mu, \quad \zeta^\mu = -\mu h^\mu, \quad (C.6) \]

\[ \tilde{q} = -\frac{\tilde{\mu}_\perp}{\tilde{\rho}_\perp}, \quad q_x = -\frac{\tilde{\mu}_\parallel}{\mu} + \frac{\tilde{\mu}_\parallel \tilde{\rho}_\parallel}{\tilde{\rho}_\perp \mu}, \quad q = \frac{\rho}{\mu} + \frac{\mu \tilde{\rho}_\parallel}{\mu^2} - \frac{\tilde{\mu}_\parallel \tilde{\rho}_\parallel^2}{\tilde{\rho}_\perp \mu^2}, \quad (C.7) \]

\[ \epsilon_{\text{them}} = \epsilon_{us}, \quad p_{\text{them}} = p_{us}, \quad -1 = a_{us}. \quad (C.8) \]

C.3 First order tensor structures in 0-form superfluids

Scalars

We are looking for all the scalars that we can construct out of \{T, \mu, \tilde{\mu}, u^\mu, h^\mu\} with exactly one derivative. Let us start by listing all linearly independent scalars:

\[ u^\lambda \partial_\lambda T, \quad h^\lambda \partial_\lambda T, \quad (C.9) \]

\[ u^\lambda \partial_\lambda \frac{\mu}{T}, \quad h^\lambda \partial_\lambda \frac{\mu}{T}, \quad (C.10) \]

\[ u^\lambda \partial_\lambda \tilde{\mu}, \quad h^\lambda \partial_\lambda \tilde{\mu}, \quad (C.11) \]

\[ \Delta^{\mu \nu} \partial_\mu u_\nu, \quad \Delta^{\mu \nu} \partial_\mu h_\nu, \quad (C.12) \]

\[ h^\mu h^\nu \partial_\mu u_\nu, \quad u^\mu h^\nu \partial_\mu u_\nu, \quad (C.13) \]

Now, we can also use the conservation equations

\[ \partial_\mu J^\mu = 0, \quad \partial_\mu (\ast K)_\nu = 0, \quad \partial_\mu T^{\mu \nu} = 0, \quad (C.14) \]

with which we can build four scalar equations, namely

\[ \partial_\mu J^\mu = 0, \quad (C.15) \]

\[ u^{[\mu} h^{\nu]} \partial_\mu (\ast K)_{\nu]} = 0, \quad (C.16) \]

\[ u_\nu \partial_\mu T^{\mu \nu} = 0 \]

\[ h_\nu \partial_\mu T^{\mu \nu} = 0. \quad (C.17, C.18) \]

We use these equations to zeroth order to further remove terms containing time derivatives. This preserves the nature of the initial value problem. This way we remove \( u^\lambda \partial_\lambda T, u^\mu h^\nu \partial_\mu u_\nu, u^\lambda \partial_\lambda \mu \) and \( u^\lambda \partial_\lambda \tilde{\mu} \). Finally, we take our set of independent scalars to be

\[ h^\lambda \partial_\lambda \frac{\mu}{T}, \quad (C.19) \]

\[ h^\lambda \partial_\lambda \tilde{\mu}, \quad (C.20) \]

\[ h^\mu h^\nu \partial_\mu u_\nu, \quad (C.21) \]

1. In this section we turn off the background gauge field A.
where the first line is charge odd and the two remaining lines are charge even. Of these structures, only four will be allowed by the second law of thermodynamics (4.63).

Vectors

The transverse vector conservation equations are
\[
\Delta_{\rho\alpha} \partial_{\mu} T^{\mu\alpha} = 0, 
\]
\[
\Delta^{\mu\alpha} u_\alpha \partial_{[\mu} (\star K)_{\nu]} = 0, 
\]
\[
\Delta^{\mu\alpha} h_\alpha \partial_{[\mu} (\star K)_{\nu]} = 0.
\]

All the transverse vector quantities that we can consider are
\[
\Delta_{\mu\nu} \partial_{\nu} T, \quad \Delta_{\mu\nu} \partial_{\nu} \frac{\mu}{T}, 
\]
\[
\Delta_{\mu\nu} \partial_{\nu} \tilde{\mu}, \quad \Delta^{\mu\nu} h^\lambda \partial_{\nu} u_\lambda, 
\]
\[
\Delta^{\mu\nu} u^\lambda \partial_{\lambda} u_\nu, \quad \Delta^{\mu\nu} h^\lambda \partial_{\lambda} u_\nu, 
\]
\[
\Delta^{\mu\nu} u^\lambda \partial_{\lambda} h_\nu, \quad \Delta^{\mu\nu} h^\lambda \partial_{\lambda} h_\nu,
\]
where we have used (C.22-C.24) to get rid of the three last components. Only two structures are allowed by the second law of thermodynamics (4.62).

Tensors

In this case there are no transverse symmetric tensor equations. The transverse traceless symmetric tensors are given by
\[
\sigma^{\mu\nu} = (\Delta^{\mu\alpha} \Delta_{\nu\beta} - \frac{1}{d-2} \Delta^{\mu\nu} \Delta^{\alpha\beta}) \partial_{(\alpha} u_{\beta)}, 
\]
\[
\zeta^{\mu\nu} = (\Delta^{\mu\alpha} \Delta_{\nu\beta} - \frac{1}{d-2} \Delta^{\mu\nu} \Delta^{\alpha\beta}) \partial_{(\alpha} h_{\beta)}.
\]
Only one structure is allowed by the second law of thermodynamics (4.61).
C. Details on Superfluids as Higher-form
Quantum field theory (QFT) is the language of theoretical physics. Vastly different phenomena, all the way from the scattering of elementary particles to the dynamics of boiling water permit a field-theoretic description. Despite these successes of QFT, there remain important systems where our standard tools fail. For example, quarks and gluons interact strongly at low energies, and the theoretical tools necessary to understand their dynamics are still in development. In this thesis, we studied such strongly-coupled theories by combining three tools: holography, the conformal bootstrap, and generalized symmetries.

Holography relates a non-gravitational QFT to a higher-dimensional gravitational theory. It is a particularly powerful tool to gain intuition for how strongly-coupled QFTs behave because it allows us to relate a strongly-coupled QFT to a gravitational theory that is weakly interacting, and hence tractable using perturbation theory. We can thus use weakly-coupled gravity as a guidepost to probe the space of strongly-coupled QFTs and discuss universal features. The long-distance dynamics of QFTs is captured by Conformal Field Theories (CFTs), which are QFTs with an additional symmetry related to scale invariance. Hence, all QFTs can be thought of as deformed CFTs, which therefore provide a natural starting point to probe the space of strongly-coupled QFTs. Moreover, the holographic duality implies that understanding the space of CFTs will also shed light on quantum gravity, which is one of the most challenging problems in modern physics.

The most promising approach to map the space of CFTs is the conformal bootstrap. This program uses the full power of conformal symmetry, as well as the existence of an associative operator algebra, to derive powerful constraints on the observables of the theory. In two dimensions, these constraints are powerful enough to fully solve minimal models and have led to the complete classification of phase transitions. Their application to higher-dimensional CFTs is an ongoing program, and impressive progress has been made in understanding the space of three-dimensional CFTs. Much of the work done on the conformal bootstrap has focused on constraints that arise from purely local physics. In general, however, non-local operators may probe physics that is invisible to these local probes. Much of this thesis focused on a special class of such operators, called light-ray operators. An example is the Averaged Null Energy Condition (ANEC) operator, a non-local energy operator with a remarkable property: its expectation value is positive.
in any state of the theory. Imposing this positivity constrains allowed parameters in CFTs. In this thesis, we used light-ray operators combined with the conformal bootstrap to carve out the space of holographic four-dimensional CFTs. Recent research provided strong evidence that light-ray operators satisfy an infinite-dimensional algebra in four dimensions. Uncovering such an algebra in higher-dimensional CFTs would have far-reaching consequences, as an infinite number of consistency conditions (similar to the ones that are very powerful in two dimensions) would follow.

Another important tool that we employed in this thesis is generalized global symmetries. Symmetries are one of the most powerful guiding principles to understand a physical system, and they tightly constrain most of the world we experience. The recently formalized so-called generalized symmetries are a generalization of regular symmetries. They act on extended objects and are ubiquitous in QFT. Most of the known results for regular symmetries have analogs for generalized symmetries. They have also proven to be extremely powerful to probe strongly-coupled QFTs, where they provide a new organization principle suited for extended objects. The second focus of this thesis was to use them to understand phase transitions in systems with higher-form symmetries.

**Light-ray operators**

In chapters 2 and 3 of this thesis, we investigated different open problems in high-energy physics using non-local operators in CFT, which are commonly known as light-ray operators. In particular, we used the ANEC operator as well as generalizations thereof to be able to decipher some open questions in CFT. The goal was twofold: First, we wanted to understand what are the necessary and sufficient conditions that a CFT has to satisfy to admit a weakly-coupled Einstein gravity dual in the bulk. This was the motivation for chapter 2. Second, we investigated a possible algebra amongst light-ray operators that was conjectured to hold in the previous literature. This algebra is some sort of higher-dimensional analog of the Virasoro algebra. This is the content of chapter 3.

**Einstein gravity from the CFT**

In chapter 2, we studied correlators of ANEC operators in states that are created by local operators. Since the seminal work [91], whose main idea we reviewed in section 1.4.2, it is understood how the positive-definiteness of the ANEC operator in quantum field theory can be used to bound OPE coefficients in higher-dimensional CFTs. To bring this program further, we developed a concise way to evaluate such correlators with multiple ANEC insertions. The main idea was to recast the OPE between the ANEC operator and a local operator as a differential operator acting on the same local operator in a holographic CFT with large $N$ and large $\Delta_{gap}$. This allowed us to compute higher-point functions by acting with a sequence of differential operators on lower-point functions. When the ANEC is sent far away, it can be interpreted as a calorimeter that measures the energy on the celestial sphere at infinity. In this case, the differential operator admits
a particularly simple expression.

Using this technology, we computed four-point functions involving two ANEC operators in states created with local stress-energy tensors. In particular, we subtracted two such correlators in the appropriate ordering to obtain the ANEC commutator, which must vanish. Requiring this put strong constraints on the OPE coefficients on the theory (or equivalently the anomaly coefficients $a$ and $c$) such that we obtained

$$a = c.$$  \hspace{1cm} (5.31)

This is the CFT version of the following statement: any large $N$, large gap theory must have a holographic dual with Einstein gravity minimally coupled to matter, and the conformal collider bounds, which are a consequence of the ANEC, must get strengthened.

**A possible algebra amongst light-ray operators**

In chapter 3, we investigated a possible algebra amongst a specific set of light-ray operators. These operators are the natural higher-dimensional generalizations of the usual two-dimensional Virasoro generators.

We uncovered a subgroup of the conformal group, that is the collinear conformal group. This is the set of transformations that leaves the light-ray invariant. There is a five-dimensional subalgebra that closes under the action of this collinear conformal group. These five operators, which we called global operators, annihilate the conformally-invariant vacuum, as their two-dimensional analogs.

To investigate the algebra in CFT, we computed the four-point function of two stress-energy tensors in states created by a local operator. We considered external scalar states for concreteness. The four-point functions of light-ray operators can then be obtained by performing the light-ray integrals of the local correlators.

Already in free field theories, we stumbled upon discrepancies with the proposed algebra, and even members of the global subalgebra failed to commute at finite spacelike separation. This spelled doom of the proposed algebra in this case. For holographic CFTs, where we might have expected better behaving commutators, we found that the problem even worsens.

In the bulk, we first derived exact shockwave solutions that are dual to the insertions of exponentiated generalized global ANEC operators in the CFT. These geometries are new, previously unknown, solutions to the Einstein equation. With these solutions at hand, we computed the propagation of perturbations in their background, which allowed us to compute the holographic commutators of generalized ANEC operators in the bulk. We found perfect agreement between these results and the holographic CFT computations.

We have thus shown that the proposed algebra did not naively hold, and that extra investigations are necessary.
Higher-form symmetries and hydrodynamics

In chapter 4, we investigated the interplay between higher-form symmetries and the mechanism of spontaneous symmetry breaking. In particular, we know that when a system experiences the spontaneous breaking of an Abelian symmetry there are bosons in the spectrum that are massless. This is just the statement of Goldstone’s theorem. In chapter 4, we showed that the masslessness of these bosons can be interpreted as being protected by an anomaly between a regular and a higher-form symmetry.

The spontaneous breaking of a regular $U(1)$ symmetry leads to an emergent $(d-2)$-form $U(1)^{(d-2)}$ symmetry with an anomaly given by the curvature of the regular $U(1)$. This anomaly can occur in any dimension $d$. This is the starting point for our alternative proof of Goldstone’s theorem. In particular, we reversed the logic and proved that a system with a regular $U(1)$ and a higher form $U(1)^{(d-2)}$ symmetries that are connected through a mixed anomaly of the type we just described contains a massless Goldstone boson that transforms non-linearly in its spectrum.

We can adjust the usual Landau paradigm to classify phase transitions. In our case, we would like to classify phases by looking at which generalized symmetries are present in each phase, along with their anomalies, but disregarding how they are realized. This means that we do not need to know whether the symmetries are spontaneously broken or not.

In the second part of chapter 4, we reconsidered the superfluid hydrodynamics. We recast superfluid hydrodynamics as the hydrodynamics for a theory with $U(1) \times U(1)^{(d-2)}$ symmetries that are connected through a mixed anomaly. In this formulation, we thus have three conserved currents for which we wrote conservation laws and constitutive equations. This puts all hydrodynamics equations on equal footing, and the Josephson condition is now contained in the conservation equation of the higher-form symmetry current. It provided the simplest example of how anomalies enter the hydrodynamics expansion. We also concluded by discussing the case of 1-form superfluid hydrodynamics.
Quantumveldentheorie (QFT, naar het Engelse quantum field theory) is de taal van de theoretische natuurkunde. Allerlei verschillende processen, van botsingen tussen elementaire deeltjes tot het koken van water, kunnen beschreven worden in termen van velden. Naast deze succesverhalen van QFT zijn er echter nog belangrijke systemen waar ons standaardgereedschap onvoldoende is. Een voorbeeld hiervan is de sterke interactie tussen quarks en gluonen bij lage energie: de theoretische gereedschappen die we nodig hebben om hun wisselwerking te begrijpen zijn nog steeds in ontwikkeling. Dit proefschrift richt zich op zulke sterk gekoppelde theorieën en combineert daarbij drie gereedschappen: holografie, de conforme bootstrap, en gegeneraliseerde symmetrie.

Holografie relateert een QFT zonder zwaartekracht aan een hogerdimensionele theorie met zwaartekracht. Dit is vooral waardevol als we willen begrijpen hoe sterk gekoppelde QFT’s zich gedragen, omdat deze ‘duaal’ zijn aan een gravitationele theorie met zwakke interacties. Die laatste kunnen we goed beschrijven met storingsrekening (perturbatietheorie). Op die manier kunnen we zwak gekoppelde zwaartekracht gebruiken als gids in onze zoektocht naar een beschrijving van sterk gekoppelde QFT’s, en universele eigenschappen daarvan. De langeafstandsdynamica van QFT’s wordt beschreven met zogenaamde conforme veldentheorieën (CFT’s): QFT’s met een extra symmetrie gerelateerd aan schaalinvariantie. Dit betekent dat we alle QFT’s kunnen zien als gedeformeerd CFT’s; die laatste zijn daarmee een natuurlijk vertrekpunt voor onze ontdekkingstocht in de verzameling van sterk gekoppelde QFT’s. Holografie impliceert daarnaast dat het begrijpen van de verzameling van CFT’s ons ook meer zal leren over quantumzwaartekracht: één van de grootste uitstaande problemen in de moderne natuurkunde.

De meest veelbelovende aanpak om de verzameling van CFT’s in kaart te brengen is de conforme bootstrap. Dit onderzoeksveld maakt gebruik van de volledige kracht van de conforme symmetrie, in combinatie met een associatieve algebra van operatoren, om restricties op te leggen op de observabelen van de theorie in kwestie. In twee dimensies zijn deze restricties krachtig genoeg om minimale modellen volledig op te lossen, en ze hebben geleid tot een complete classificatie van faseovergangen. De toepassing op hogerdimensionele CFT’s is een levendig onderzoeksveld, en er is indrukwekkende voortgang geboekt in het begrijpen van de verzameling van driedimensionele CFT’s. Veel van het onderzoek in de conforme bootstrap is gericht op de restricties die worden opgelegd door
lokale fysica. Niet-lokale operatoren zijn echter gevoelig voor processen die lokale probes niet kunnen bereiken, en daarmee een relevante aanvulling. Dit proefschrift focust op een specifieke klasse van zulke operatoren: lichtstraaloperatoren. Een voorbeeld van zo’n lichtstraaloperator is de Averaged Null Energy Condition (ANEC)-operator, een niet-lokale energie-operator met een bijzondere eigenschap: de verwachtingswaarde is positief in elke willekeurige toestand van de theorie. Deze positiviteit begrenst de toegestane parameters van de CFT. In dit proefschrift gebruiken we lichtstraaloperatoren in combinatie met de conforme bootstrap om de ruimte van holografische vierdimensionele CFT’s in kaart te brengen. Recent onderzoek duidt erop dat lichtstraaloperatoren voldoen aan een oneindigdimensionele algebra in vier dimensies. Het ontkennen van zo’n algebra in hogerdimensionele CFT’s zou verstrekende gevolgen hebben: dit zou leiden tot een oneindig aantal consistentievoorwaarden, vergelijkbaar met die in twee dimensies. In dit proefschrift maken we gebruik van nog een belangrijk instrument: gegeneraliseerde globale symmetrie. Symmetriëën zijn één van de meest krachtige grondbeginselen die we kunnen gebruiken om een natuurkundig systeem te beschrijven, en ze leggen sterke restricties op aan de wereld die we waarnemen. De recent geformuleerde gegeneraliseerde symmetriëën zijn een veralgemenisering van gewone symmetriëën. Ze werken op uitgebreide objecten en zijn alomtegenwoordig in QFT’s. De meeste eigenschappen van gewone symmetriëën hebben een analogon voor gegeneraliseerde symmetriëën. Ze komen goed van pas bij het bestuderen van sterk gekoppelde QFT’s: ze vormen een nieuw rangschikkingsevenement geschikt voor uitgebreide objecten. De tweede focus van dit proefschrift is het gebruik van gegeneraliseerde symmetriëën om faseovergangen te begrijpen in systemen met zogeheten hogere-vorm symmetriëën.

Lichtstraaloperatoren

In hoofdstuk 2 en 3 van dit proefschrift hebben we verschillende onopgeloste problemen in hoge-energiefysica onderzocht met behulp van niet-lokale operatoren in CFT, bekend als lichtstraaloperatoren. Specifiek gebruikten we de ANEC-operator en generalisaties daarvan om enkele CFT-gerelateerde openstaande problemen te ontcijferen. Het doel hiervan was tweeledig. Ten eerste wilden we begrijpen wat de noodzakelijke en voldoende voorwaarden zijn waar een CFT aan moet voldoen om in de bulk duaal te zijn aan zwak gekoppelde Einstein zwaartekracht; dit was de motivatie voor hoofdstuk 2. Ten tweede onderzochten we een mogelijke algebra van lichtstraaloperatoren die in eerdere literatuur gepostuleerd was. Deze algebra is een soort hogerdimensionele versie van de Virasoro-algebra. Dit vormt de inhoud van hoofdstuk 3.

Van CFT naar Einsteinzwaartekracht

In hoofdstuk 2 bestudeerden we correlatiefuncties van ANEC-operatoren in toestanden gecreëerd door lokale operatoren. Sinds het baanbrekende werk [83], waarvan we de kerngedachte hebben beschreven in sectie 1.4.2, weten we hoe het feit dat de
ANEC-operator in QFT positief-definiet is kunnen gebruiken om operatorproductexpansiecoëfficiënten (ofwel OPE-coëfficiënten) te begrenzen in hogedimensionele CFT’s. Hierop voortbordurend hebben we een manier ontwikkeld om correlatiefuncties met meerdere ANEC-invloedingen uit te rekenen. Het idee is om de OPE van een ANEC-operator en een lokale operator uit te drukken als een differentiaaloperator die werkt op dezelfde lokale operator in een holografische CFT met $N$ groot en een grote $\Delta_{\text{gap}}$. Dit maakt het mogelijk om meerpuntsinteracties uit te rekenen door met meerdere differentiaaloperatoren op correlatiefuncties tussen minder operatoren te werken. Deze differentiaaloperator neemt een simpele vorm aan in de limiet waarin de ANEC-operator op grote afstand werkt: die krijgt daardoor de interpretatie van een calorimeter die de energie meet op een hemelsfeer op oneindig.

Met deze technologie is het mogelijk om vierpuntsinteracties uit te rekenen die twee ANEC-operatoren bevatten in toestanden gecreëerd door lokale stress-energieoperatoren. Door twee van zulke correlatiefuncties in de juiste volgorde van elkaar af te trekken krijgt men de ANEC-commutator, die gelijk aan nul moet zijn. Dit plaatst sterke beperkingen op de OPE-coëfficiënten van de theorie (of op de anomaliecoëfficiënten $a$ en $c$), zodat we vinden

$$a = c.$$  

Dit is de CFT-versie van de volgende stelling: elke willekeurige theorie op grote $N$ en met een grote gap is holografisch duaal aan Einsteinzwartekracht minimaal gekoppeld aan materie, en de conformal collider bounds, die een direct gevolg zijn van de ANEC, moeten gelden.

**Een mogelijke algebra van lichtstraaloperatoren**

In hoofdstuk 3 onderzochten we een mogelijke algebra van een specifieke verzameling van lichtstraaloperatoren. Deze operatoren zijn de natuurlijke hogedimensionele generalisaties van de gebruikelijke tweedimensionele Virasoro-generatoren.

We vonden een deelgroep van de conforme groep, de collinaire conforme groep, die bestaat uit de verzameling transformaties die de lichtstraal invariant laten. Er is een vijfdimensionele deelalgebra die sluit onder de werking van deze collinaire conforme groep. Deze vijf operatoren, die we globale operatoren noemen, annihileren het conform-invariante vacuüm, net als hun tweedimensionele broertjes.

Om deze algebra te onderzoeken hebben we de vierpuntsfunctie van twee stress-energietensoren uitgerekend, in een toestand die is gecreëerd door een lokale operator. Om het concreet te maken bekeken we het geval van externe scalaire toestanden. De vierpuntsfunctie van lichtstraaloperatoren kan dan verkregen worden door lichtstraalintegralen van lokale correlatiefuncties.

Zelfs in vrije veldentheorieën vinden we al discrepenties met de voorgestelde algebra; ook
blijkt dat sommige elementen van de globale deelalgebra niet commuteren op eindige ruimteachtige afstand. In dit geval lijkt de voorgestelde algebra dus niet te kloppen. Voor holografische CFT’s, waar we naïef gezien juist beter gedrag van de commutatoren zouden verwachten, vinden we dat het probleem erger wordt.

In de bulk hebben we eerst exacte schokgolfoplossingen afgeleid die duaal zijn aan de invoegingen van exponenten van gegeneraliseerde globale ANEC-operatoren in de CFT. Deze geometriën zijn nieuwe, voorheen onbekende oplossingen van de Einsteinvergelijkingen. Met deze oplossingen hebben we de voortplanting van perturbaties op deze achtergrond uitgerekt; hiermee konden we de holografische commutatoren van de gegeneraliseerde ANEC-operatoren in de bulk uittrekken. Deze resultaten kwamen perfect overeen met die van de holografische CFT-berekening.

Hogere-vorm symmetriën en hydrodynamica

In hoofdstuk vier onderzochten we de wisselwerking tussen hogere-vorm symmetriën en het mechanische van spontane symmetriebreking. In het bijzonder weten we dat de spontane breking van een Abelse symmetrie leidt tot massaloze bosonen in het spectrum; dit is Goldstone’s theorema. In hoofdstuk 4 hebben we laten zien dat de massaloosheid van deze bosonen geïnterpreteerd kan worden als een eigenschap die wordt beschermd door een anomalie tussen een reguliere en een hogere-vorm symmetrie.

De spontane breking van een reguliere $U(1)$-symmetrie leidt tot een emergente $(d-2)$-vorm $U(1)^{(d-2)}$-symmetrie met een anomalie die wordt gegeven door de kromming van de reguliere $U(1)$. Deze anomalie kan in elke willekeurige dimensie optreden. Dit is het vertrouk punkt voor ons alternatieve bewijs van Goldstone’s theorema. In het bijzonder draaien we de logica om, en bewijzen we dat een systeem met een reguliere $U(1)$- en een hogere-vorm $U(1)^{(d-2)}$-symmetrie die zijn verbonden door een dergelijke gemengde anomalie een massaloos Goldstoneboson bevat die niet-lineair transformeert in het spectrum.

We kunnen het gebruikelijke Landauparadigma aanpassen om faseovergangen te classificeren. In ons geval willen we fases classificeren aan de hand van de aanwezige gegeneraliseerde symmetriën in elke fase, samen met hun anomalieën – maar onafhankelijk van hoe ze gerealiseerd worden. Dit betekent dat het niet nodig is om te weten welke symmetriën spontaan gebroken worden of niet.

In het tweede deel van hoofdstuk 4 vatten we superfluide hydrodynamica samen, en formuleerden die als de hydrodynamica voor een theorie met $U(1) \times U(1)^{(d-2)}$-symmetriën die verbonden zijn door een gemengde anomalie. In deze formulering hebben we drie behouden stromen, waar we behoudswetten en toestandsvergelijkingen voor opstellen. Dit zet alle hydrodynamica-vergelijkingen op hetzelfde voetstuk, en de Josephsonvoorwaarde is nu ingebouwd in de behoudswet voor de stroom die hoort bij de hogere-vorm symmetrie. Dit geeft het simpelste voorbeeld van hoe anomalieën opduiken in de hydrodynamica. Tot slot bespraken we het geval van een 1-vorm superfluide hydrodynamica.
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