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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials \( R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t) \) form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in \( n \) variables over the field \( \mathbb{F} := \mathbb{Q}(q, t) \). They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most \( n \) parts

\[
\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.
\]

For a partition \( \mu \in \mathcal{P}_n \) we define \(|\mu| = \mu_1 + \cdots + \mu_n\) and write

\[
\overline{\mu} = (q^{\mu_1 \tau_1}, \ldots, q^{\mu_n \tau_n}) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.
\]
Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\mu) = 0 \text{ for } \mu \in P_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$ 

The normalization is fixed by requiring that the coefficient of $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ [9] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n, q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $K := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in P_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.$$ 

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in C_n$ we define

$$\bar{\beta} := w_\beta(\beta_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\bar{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$ 

The normalization is fixed by requiring that the coefficient of $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$  

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)}w_0H_0G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\beta)}{G_\alpha(a\tau)} = \frac{G_\beta(a\bar{\beta})}{G_\beta(a\bar{\tau})}.$$  

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\beta^{-1}) = \frac{G_\beta(a\bar{\beta})}{G_\beta(a\bar{\tau})} \text{ for all } \beta.$$  

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 


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Theorem C. For all compositions \( \alpha \in C_n \) we have

\[
O_\alpha(x) = \frac{G_\alpha(t^{1-n}aw_0x)}{G_\alpha(ax)}.
\]

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials \( G_\alpha'(x) \) in terms of the \( G_\beta(ax) \)’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let \( S_n \) be the symmetric group in \( n \) letters and \( s_i \in S_n \) the permutation that swaps \( i \) and \( i+1 \). The \( s_i \) (\( 1 \leq i < n \)) are Coxeter generators for \( S_n \). Let \( \ell : S_n \to \mathbb{Z}_{\geq 0} \) be the associated length function. Let \( S_n \) act on \( \mathbb{Z}^n \) and \( \mathbb{K}^n \) by \( s_i v := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots) \) for \( v = (v_1, \ldots, v_n) \). Write \( w_0 \in S_n \) for the longest element, given explicitly by \( i \to n+1-i \) for \( i = 1, \ldots, n \).

For \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \) define \( \overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n \) by \( \overline{v}_i := q^{v_i}t^{-k_i(v)} \) with

\[
k_i(v) := \# \{ k < i \mid v_k \geq v_i \} + \# \{ k > i \mid v_k \geq v_i \}.
\]

If \( v \in \mathbb{Z}^n \) has non-increasing entries \( v_1 \geq v_2 \geq \cdots \geq v_n \), then \( \overline{v} = (v_1^{\tau_1}, \ldots, v_n^{\tau_n}) \). For arbitrary \( v \in \mathbb{Z}^n \) we have \( \overline{v} = w_v(\overline{v}_+) \) with \( w_v \in S_n \) the shortest permutation such that \( v_+ := w_v^{-1}(v) \) has non-increasing entries, see [4, Section 2]. We write \( \overline{v} := -w_0 v \) for \( v \in \mathbb{Z}^n \).

Note that \( \overline{\alpha}_n = t^{1-n} \) if \( \alpha \in C_n \) with \( \alpha_n = 0 \).

For a field \( F \) we write \( F[x] := F[x_1, \ldots, x_n] \) and \( F[x^{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) and \( F(x) \) for the quotient field of \( F[x] \). The symmetric group acts by algebra automorphisms on \( \mathbb{F}[x] \) and \( \mathbb{F}(x) \), with the action of \( s_i \) by interchanging \( x_i \) and \( x_{i+1} \) for \( 1 \leq i < n \). Consider the \( \mathbb{F} \)-linear operators

\[
H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1 - s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1).
\]
on $\mathbb{F}(x)$ ($1 \leq i < n$) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ ($1 \leq i < n$) and $\Delta$ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ ($1 \leq i < n$) and $\Delta$ satisfy the defining relations of the type $A$ extended affine Hecke algebra,

$$(H_i - t)(H_i + 1) = 0,$$

$$H_i H_j = H_j H_i, \quad |i - j| > 1,$$

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

$$\Delta H_{i+1} = H_i \Delta,$$

$$\Delta^2 H_i = H_{n-1} \Delta^2$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = t H_i^{-1}$ and set

$$\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.$$

The $\xi_i^{-1}$ ($1 \leq i \leq n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$\xi_i^{-1} E_\alpha = \overline{a}_i E_\alpha, \quad i = 1, \ldots, n$$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$.
and \( F[x] \) by letting \( \iota \) act on the coefficients of the polynomial. Write

\[
G_{\alpha}^{\circ} := \iota(G_{\alpha}), \quad E_{\alpha}^{\circ} := \iota(E_{\alpha})
\]

for \( \alpha \in C_n \). Note that \( \overline{v}^{-1} = (\iota(\overline{v}_1), \ldots, \iota(\overline{v}_n)) \).

Put \( H_i^{\circ}, H_w^{\circ}, \overline{H}_i, \Delta^{\circ} \) and \( \xi_i^{\circ} \) for the operators \( H_i, H_w, \overline{H}_i, \Delta \) and \( \xi_i \) with \( q, t \) replaced by their inverses. For instance,

\[
H_i^{\circ} = t^{-1}s_i - \frac{(1 - t^{-1})x_i}{x_i - x_{i+1}}(1 - s_i),
\]

\[
\Delta^{\circ}f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).
\]

We then have \( \xi_i^{\circ}E_{\alpha}^{\circ} = \overline{\alpha}_iE_{\alpha}^{\circ} \) for \( i = 1, \ldots, n \), which characterizes \( E_{\alpha}^{\circ} \) up to a scalar factor.

**Theorem 1.** For \( \alpha \in C_n \) we have

\[
G_{\alpha}'(x) = t^{(1-n)|\alpha|+I(\alpha)}w_0H_w^{\circ}G_{\alpha}(t^{n-1}x)
\]

with \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \).

**Remark.** Formally set \( t = q^r \), replace \( x \) by \( 1 + (q - 1)x \), divide both sides of (2) by \( (q - 1)^{|\alpha|} \) and take the limit \( q \to 1 \). Then

\[
G_{\alpha}'(x; r) = (-1)^{\sigma\omega}(w_0)w_0G_{\alpha}(-x - (n - 1)r; r)
\]

for the non-symmetric interpolation Jack polynomial \( G_{\alpha}(\cdot; r) \) and its primed version (see [14]). Here \( \sigma \) denotes the action of the symmetric group with \( \sigma(s_i) \) the rational degeneration of the Demazure-Lusztig operators \( H_i \), given explicitly by

\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),
\]

see [14, Section 1]. To establish the formal limit (3) one uses that \( \sigma(w_0)w_0 = w_0\sigma^\circ(w_0) \) with \( \sigma^\circ \) the action of the symmetric group defined in terms of the rational degeneration

\[
\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)
\]

of \( H_i^{\circ} \). Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_{\alpha}$.

For the vanishing property, note that

$$t^{n-1} w_0 \tilde{\beta} = \tilde{\beta}^{-1}$$

(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,

$$\left(w_0 H_{w_0}^o G_{\alpha}^o(t^{n-1} x)\right)|_{x=\bar{\beta}} = (H_{w_0}^o G_{\alpha}^o(x))|_{x=\bar{\beta}^{-1}}.$$

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{\alpha}^o(w \beta^{-1}) (w \in S_n)$ by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_{\alpha} = t^{1(\alpha)} w_0 H_{w_0}^o E_{\alpha}^o.$$  \hspace{1cm} (5)

Note that $\Psi := w_0 H_{w_0}^o$ satisfies the intertwining properties

$$H_i \Psi = t \Psi H_i^o,$$

$$\Delta \Psi = t^{n-1} \Psi H_{n-1}^o \cdots H_1^o (\Delta^o)^{-1} H_{n-1}^o \cdots H_1^o$$

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1} \Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$. Therefore,

$$E_{\alpha}(x) = c_{\alpha} \Psi E_{\alpha}^o(x)$$

for some constant $c_{\alpha} \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_{\alpha} = t^{1(\alpha)}$. \hfill \blacksquare

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\bar{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_{\alpha} \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1 \cdots x_n E_u(x).$$
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{l(u)} w_0 H_{w_0}^\circ E_u$$  \hspace{1cm} (7)$$

with $E_u := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{l(u)} H_{w_0}^\circ E_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

### 3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)} t^{1-l'(s)}}{1 - q^{a(s)} t^{l(s)+1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})$$  \hspace{1cm} (8)$$

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i,j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \},$$

$$a'(s) := j - 1, \quad l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}. $$

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - q^{a'(s)+1} t^{l'(s)+1}}{1 - q^{a(s)+1} t^{l(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}. $$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G^{\circ}_\alpha(a \tau^{-1}).$$

Proof. Since $t^{n-1}w_0 \tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)} (H^\circ_{w_0} G^\circ_\alpha)(a \overline{\tau}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G^\circ_\alpha(a \overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality.

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^{\circ}_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^{n} \left(\begin{array}{c} \alpha \end{array}\right)_{i}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha|-n(\alpha)} q^{n'(\alpha)} G^{\circ}_\alpha(a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 

Corollary 4. For $\alpha \in C_n$ we have
\[ G'_\alpha(a^{-1} \tau) = \tau^{-1} a^{-|\alpha|} G_\alpha(a \tau). \]

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $K$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in K^n$ write $v^\sharp := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\sharp := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $\sharp$ by $\check{\cdot}$, so $v^\check{\sharp} = (v_n - 1, v_1, \ldots, v_1)$ and $y^\check{\sharp} = (y_n/q, y_1, \ldots, y_{n-1})$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have
1. $s_i(v) = s_i^\check{\cdot} v$ if $v_i \neq v_{i+1}$.
2. $v_i = t v_{i+1}$ if $v_i = v_{i+1}$.
3. $v^\check{\sharp} = v^\sharp$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $K$. It is isomorphic to the subalgebra of $\text{End}(K[x^\pm 1])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^\pm 1$, and the multiplication operators $x_j^\pm 1$ ($1 \leq j \leq n$).

For a unital $K$-algebra $A$ we write $F_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $K$-algebra. Consider the $A$-linear operators $\hat{H}_i$ ($1 \leq i < n$), $\hat{\Delta}$ and $\hat{x}_j$ ($1 \leq j \leq n$) on $F_A$ defined by
\[
(\hat{H}_if)(v) := tf(v) + \frac{v_i - t v_{i+1}}{v_i - v_{i+1}} (f(s_i v) - f(v)),
\]
\[
(\hat{\Delta}f)(v) := f(v^\check{\sharp}), \quad (\hat{\Delta}^{-1}f)(v) := f(v^\sharp),
\]
\[
(\hat{x}_jf)(v) := a v_j f(v)
\]
for $f \in F_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \hat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \hat{\Delta}$ and $x_j \mapsto \hat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(F_A), X \mapsto \hat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $F_A$. 
Proof. Let $O \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\mathfrak{z}$-invariant subset that contains \{a\overline{v} \mid v \in \mathbb{Z}^n\}. Note that $O$ is contained in \{y \in \mathbb{K}^n \mid y_i \neq y_j$ if $i \neq j\}. The Demazure–Lusztig operators $H_i$ $(1 \leq i < n)$, $\Delta^{\pm 1}$ and the coordinate multiplication operators $x_j$ $(1 \leq j \leq n)$ act $A$-linearly on the space $F_A^O$ of $A$-valued functions on $O$, and hence turns $F_A^O$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map
\[ \text{pr}: F_A^O \to F_A \]
by $\text{pr}(g)(v) := g(a\overline{v})$ ($v \in \mathbb{Z}^n$).

We claim that $\text{Ker}(\text{pr})$ is an $\mathbb{H}$-submodule of $F_A^O$. Clearly $\text{Ker}(\text{pr})$ is $x_j$-invariant for $j = 1, \ldots, n$. Let $g \in \text{Ker}(\text{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(\text{pr}).$ To show that $H_j g \in \text{Ker}(\text{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \overline{v} = s_i \overline{v}$ by part 1 of Lemma 5. Hence,
\[ (H_i g)(a\overline{v}) = tg(a\overline{v}) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (g(as_i \overline{v}) - g(a\overline{v})) = 0. \]
If $v_i = v_{i+1}$ then $t \overline{v} = t \overline{v}$ by part 2 of Lemma 5. Hence,
\[ (H_i g)(\overline{v}) = tg(a\overline{v}) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (g(as_i \overline{v}) - g(a\overline{v})) = tg(a\overline{v}) = 0. \]

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F_A^O/\text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i$ $(1 \leq i < n)$, $\Delta$ and $x_j$ $(1 \leq j \leq n)$ on $F_A$ is by the operators $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$.

Remark 7. With the notations from (the proof of) Corollary 6, let $\overline{g} \in F_A^O$ and set $g := \text{pr}(\overline{g}) \in F_A$. In other words, $g(v) := \overline{g}(a\overline{v})$ for all $v \in \mathbb{Z}^n$. Then
\[ (\widehat{X} g)(v) = (X \overline{g})(a\overline{v}), \quad v \in \mathbb{Z}^n \]
for $X = H_i$, $\Delta^{\pm 1}$, $x_j$.

Remark 8. Let $F_A^+$ be the space of $A$-valued functions on $\mathcal{C}_n$. We sometimes will consider $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$, defined by the formulas (11), as linear operators on $F_A^+$.

Definition 9. We call
\[ K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a\alpha; q, t)} \in \mathbb{K}[x] \]  \hspace{1cm} (12)
the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 

We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(0) = 0$ if $\alpha \in \mathbb{C}_n$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}$$

since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $/Phi_1 = (x_n - t^{1-n}) \Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$/Xi_j = x_j H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$  

The operators $H_i$, $/Xi_j$ and $\Phi$ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $\mathcal{F}_{\mathbb{K}[x]}^+$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in C_n$). Note that the operators $H_i$, $/Xi_j$ and $\Phi$ on $\mathcal{F}_{\mathbb{K}[x]}^+$ commute with the hat-operators $\hat{H}_i$, $\hat{x}_j$ and $\hat{\Delta}^{-1}$ on $\mathcal{F}_{\mathbb{K}[x]}^+$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_{\mathbb{K}(x)}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $\mathcal{F}_{\mathbb{K}(x)}$).

Let $K \in \mathcal{F}_{\mathbb{K}[x]}^+$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in C_n$).

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $\mathcal{F}_{\mathbb{K}[x]}^+$:

1. $H_i K = \hat{H}_i K$.
2. $/Xi_j K = a \hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2 \hat{x}_i^{-1} - 1) \hat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$'s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$'s.

If $\alpha \in C_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t-1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t-1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - \overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1} \overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{s_i \alpha}(x)$$
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_i G_\alpha(x) = tG_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_i K_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(x) = G_s(\alpha x)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_i K = \hat{H}_i K \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-\alpha_1} G_\alpha^\cdot(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_\alpha^\cdot(x)}{G_\alpha(x)} = a t^{1-n+k_1(a)} - q^{\alpha_1} t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n}(a\bar{\alpha}_1^{-1} - 1)K_\alpha^\cdot(x).
\]

Remark 11. Note that

\[
\Phi K_\alpha(x) = (a\bar{\alpha}_n - t^{1-n})K_\alpha^\cdot(x)
\]

for \( \alpha \in C_n \) since \( \bar{\alpha}^{-1} = t^{n-1}w_0\bar{\alpha} \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in \mathcal{F}_n^+ \) to a map \( K \in \mathcal{F}_n^\cdot \) such that Lemma 10 remains valid.

Lemma 12. For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^n \frac{(1-a\bar{\alpha}_i^{-1})}{(1-q t^{1-n} x_i)} \right) K_{\alpha+(1^n)}(qx).
\]
Proof. Note that for \( f \in \mathbb{K}[x] \),
\[
\Phi^n f(x) = \left( \prod_{i=1}^n (x_i - t^{1-n}) \right) f(q^{-1} x).
\]
The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For \( m \in \mathbb{Z}_{\geq 0} \) we define \( A_m(x; \nu) \in \mathbb{K}(x) \) by
\[
A_m(x; \nu) := \prod_{i=1}^n \left( q^{1-ma} t^{-1} q^{-i}; q \right)_m \prod_{i=1}^n x_i^{m} (q^{-m} t^{1-n} x_i^{-1}; q)_m \quad \forall \nu \in \mathbb{Z}^n,
\]
with \( (y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y) \) the \( q \)-shifted factorial.

Definition 13. Let \( \nu \in \mathbb{Z}^n \) and write \(|\nu| := \nu_1 + \cdots + \nu_n\). Define \( G_\nu(x) = G_\nu(x; q, t) \in \mathbb{F}(x) \) and \( K_\nu(x) = K_\nu(x; q, t; a) \in \mathbb{K}(x) \) by
\[
G_\nu(x) := q^{-m|\nu| - m^2} \frac{G_{\nu+(m^n)}(q^m x)}{\prod_{i=1}^n x_i^{m} (q^{-m} t^{1-n} x_i^{-1}; q)_m},
K_\nu(x) := A_m(x; \nu) K_{\nu+(m^n)}(q^m x),
\]
where \( m \) is a nonnegative integer such that \( \nu + (m^n) \in C_n \) (note that \( G_\nu \) and \( K_\nu \) are well defined by Lemma 12).

Example 14. If \( n = 1 \) then for \( m \in \mathbb{Z}_{\geq 0} \),
\[
K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \frac{x^{-m} (x^{-1}; q)_m}{(a^{-1}; q)_m}.
\]

Lemma 15. For all \( \nu \in \mathbb{Z}^n \),
\[
K_\nu(x) = \frac{G_\nu(x)}{G_\nu(a^\tau)}.
\]
Proof. Let \( \nu \in \mathbb{Z}^n \). Clearly \( G_\nu(x) \) and \( K_\nu(x) \) only differ by a multiplicative constant, so it suffices to show that \( K_\nu(a^\tau) = 1 \). Fix \( m \in \mathbb{Z}_{\geq 0} \) such that \( \nu + (m^n) \in C_n \). Then
\[
K_\nu(a^\tau) = A_m(a^\tau; \nu) K_{\nu+(m^n)}(q^m a^\tau) = A_m(a^\tau; \nu) \frac{G_{\nu+(m^n)}(q^m a^\tau)}{G_{\nu+(m^n)}(a^\tau)} = 1,
\]
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_K(x)$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\hat{x}_j K$.
3. $\Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_K(x)$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_K(x)$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_K(x)$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_K(x)$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_K(x)$,

$$(\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = tK_v$ and $H_i K_{v+(m^n)}(q^m x) = tK_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\bar{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

where $\Phi(q^m) := (q^m x_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \bar{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{v} = -w_0 v \) for \( v \in \mathbb{Z}^n \).

Theorem 17. (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a\tilde{v}) = K_v(a\tilde{u}).
\] (17)

Example 18. If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r}
\] (18)

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

Proof. We divide the proof of the theorem in several steps. ■

Step 1. If \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{s_i}u(a\tilde{v}) = K_v(a\tilde{s_i}u) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

Proof of Step 1. Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{v}) + \left( \frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}} \right) K_u(\tilde{s}_{n-i}v) = \frac{(t-1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{i+1})} K_u(a\tilde{v}) + \left( \frac{\tilde{u}_i - t\tilde{u}_{i+1}}{\tilde{u}_i - \tilde{u}_{i+1}} \right) K_{s_i}u(a\tilde{v}).
\] (19)

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t-1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a\tilde{u}) + \left( \frac{\tilde{v}_{n-i} - t\tilde{v}_{n+1-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} \right) K_v(\tilde{s_i}u) = \frac{(t-1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a\tilde{u}) + \left( \frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} \right) K_{s_{n-i}}v(a\tilde{u}).
\] (20)

Suppose that \( s_{n-i}v = v \). Then \( \tilde{v}_{n-i} = t\tilde{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n}w_0\tilde{v}^{-1} \), that is, \( \tilde{v}_i = t^{1-n}\tilde{v}_{n+1-i} \) we then also have \( \tilde{v}_i = t\tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{s_i}u(a\tilde{v}) = K_u(a\tilde{v}) \) and (20) to \( K_v(\tilde{s_i}u) = K_v(a\tilde{u}) \) if \( s_{n-i}v = v \).
We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v$. We have to show that $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s}_i\tilde{u})$ for all $v$. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that $v$ satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(a\tilde{s}_i\tilde{u}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i}v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s}_i\tilde{u})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s}_i\tilde{u})$, which turns out to reduce to $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s}_i\tilde{u})$ after a direct computation.

**Step 2.** $K_0(a\tilde{v}) = 1 = K_v(a\tilde{0})$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(a\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for $v \in \mathbb{Z}^n$ and $\alpha \in C_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_v(a\tilde{v}) = K_v(a\tilde{y})$ for $v \in \mathbb{Z}^n$ and $\gamma \in C_n$ with $|\gamma| < m$. Let $\alpha \in C_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{u})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\sharp \in C_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\flat$. Furthermore, note that we have the formula

$$(a\tilde{v}^{-1}_1 - 1)K_\alpha(a\tilde{v}) = (a\tilde{u}^{-1}_1 - 1)K_\gamma(a\tilde{v})$$

(21)

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{v}) = \frac{(a\tilde{v}^{-1}_1 - 1)}{(a\tilde{y}^{-1}_1 - 1)}K_\gamma(a\tilde{v})$$

$$= \frac{(a\tilde{v}^{-1}_1 - 1)}{(a\tilde{y}^{-1}_1 - 1)}K_\gamma(a\tilde{v}) = K_v(a\tilde{v}) = K_v(a\tilde{u}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

**Proof of Step 4.** Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = \tilde{v} - (m^n) \) and \( q^{-m}\tilde{u} = \tilde{u} + (m^n) \).

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v}) \\
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(\tilde{v} - (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(\tilde{u} + (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; \tilde{v} - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; \tilde{v} - (m^n)) = 1,
\]

which follows by a straightforward computation using (4).

\[ \blacksquare \]

7 **Some Applications of Duality**

7.1 **Non-symmetric Macdonald polynomials**

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-\|\alpha\|} G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\bar{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \bar{K} \in \mathcal{F}^{+}_{[\text{Fix}]^\ast} \) for the resulting map \( \alpha \mapsto \bar{K}_\alpha \). Taking limits in Lemma 10 we get the following.

**Lemma 19.** We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\bar{K} = \hat{H}_i\bar{K} \).
2. \( \hat{\xi}_j\bar{K} = \hat{\xi}_j^{-1}\bar{K} \).
3. \( x_n\Delta\bar{K} = t^{1-n}\hat{\Delta}^{-1}\hat{\Delta}^{-1}\bar{K} \).
Note that

\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),

\[
E^\alpha(x) = \frac{E_{\alpha + (1^n)}(x)}{x_1 \cdots x_n},
\]

\[
\overline{K}_\alpha(x) = q^{\lvert \alpha \rvert} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha}_i x_i)^{-1} \right) K_{\alpha + (1^n)}(x).
\]

(22)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_v(x) := E_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) for arbitrary \(v \in \mathbb{Z}^n\) to those labeled by compositions through the formula

\[
E_v(x) = \frac{E_{v + (m^n)}(x)}{(x_1 \cdots x_n)^m}.
\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(v \in \mathbb{Z}^n\).

**Definition 20.** Let \(v \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(v + (m^n) \in C_n\). Then \(K_v(x) := \overline{K}_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) is defined by

\[
\overline{K}_v(x) := q^{\lvert v \rvert} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha}_i x_i)^{-m} \right) \overline{K}_{v + (m^n)}(x).
\]

Using

\[
\lim_{a \to \infty} A_m(ax; v) = q^{-m^2n} t^{(1-n)n} \prod_{i=1}^{n} (\overline{\alpha}_i x_i)^{-m}
\]

and the definitions of \(G_v(x)\) and \(K_v(x)\) it follows that

\[
\lim_{a \to \infty} a^{-\lvert v \rvert} G_v(ax) = E_v(x),
\]

\[
\lim_{a \to \infty} K_v(ax) = \overline{K}_v(x)
\]

for all \(v \in \mathbb{Z}^n\), so in particular

\[
\overline{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \quad \forall \ v \in \mathbb{Z}^n.
\]
Lemma 19 holds true for the extension of \( K \) to the map \( K \in \mathcal{F}[x^{\pm 1}] \) defined by \( \nu \mapsto K_\nu \) \((\nu \in \mathbb{Z}^n)\). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all \( u, v \in \mathbb{Z}^n \),

\[
\overline{K}_u(\tilde{v}) = \overline{K}_v(\tilde{u}).
\]

### 7.2 O-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the \( O_\alpha \) (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for \( O_\alpha \) in terms of the non-symmetric interpolation Macdonald polynomial \( K_\alpha \).

**Proposition 22.** For all \( \alpha \in \mathcal{C}_n \) we have

\[
O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).
\]

**Proof.** The polynomial \( \tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x) \) is of degree at most \(|\alpha|\) and

\[
\tilde{O}_\alpha(\tilde{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\tilde{\beta}^{-1}) = K_\alpha(a\tilde{\beta}) = K_\beta(a\tilde{\alpha})
\]

for all \( \beta \in \mathcal{C}_n \) by (4) and Theorem 17. Hence, \( \tilde{O}_\alpha = O_\alpha \).

### 7.3 Okounkov’s duality

Write \( F[x]^{S_n} \) for the symmetric polynomials in \( x_1, \ldots, x_n \) with coefficients in a field \( F \). Write \( C_+ := \sum_{w \in S_n} H_w \). The symmetric interpolation Macdonald polynomial \( R_\lambda(x) \in F[x]^{S_n} \) is the multiple of \( C_+ G_\lambda \) such that the coefficient of \( x^\lambda \) is one (see, e.g., [13]). We write

\[
K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(\alpha \tau)} \in \mathbb{K}[x]^{S_n}
\]

for the normalized symmetric interpolation Macdonald polynomial. Then

\[
C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\alpha^+(x)
\]

for \( \alpha \in \mathcal{C}_n \). Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\widehat{C}_+ = \sum_{w \in S_n} \widehat{H}_w$, with $\widehat{H}_w := \widehat{H}_{i_1} \cdots \widehat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_\lambda$ for the function $f_\mu(u) := K_\mu(a\tilde{\mu})$ ($u \in \mathbb{Z}^n$). Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\widehat{C}_+ f_\mu)(\lambda)$$

(24)

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{\mu}) = (Jw_0 K_\mu(t^{1-n}x))|_{x = a^{-1} \bar{u}}$$

(25)

with $(Jf)(x) := f(x_{\mu,1}^{-1}, \ldots, x_{\mu,n}^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_i J = (H_i^\circ)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^\circ)^{-1}$$

(26)

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x = a^{-1} \bar{\lambda}}.$$  

By (23) and (4) this simplifies to

$$(\widehat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\tilde{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\tilde{\mu}) = K_\mu^+(a\tilde{\lambda})$. Since $K^+_\lambda$ is symmetric we obtain from (4) that

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0}K_u)(a\bar{v}) = (H_{w_0}K_v)(a\bar{u}).$$  

(27)

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a\bar{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0}K_u)(a\bar{v}) = (\tilde{H}_{w_0}f_v)(u).$$

Since $f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}u)$ by (4), Remark 7 implies that

$$(\tilde{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}u).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$(\tilde{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}u) = (H_{w_0}K_v)(a\bar{u}),$$

which completes the proof. □

Recall from Theorem 1 that

$$G_\beta'(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G_\beta^0(t^{n-1}x)$$

with $\Psi := w_0H_{w_0}^0$. We define normalized versions by

$$K_\beta'(x) := \frac{G_\beta'(x)}{G_\beta'(a^{-1}\tau)} = t^{\ell(w_0)}\Psi K_\beta^0(t^{n-1}x), \quad \beta \in C_n,$$

with $K_\beta^0 := \iota(K_\beta)$ for $\nu \in \mathbb{Z}$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K_\nu'(x) := t^{\ell(w_0)}\Psi K_\nu^0(t^{n-1}x).$$  

(28)

We write $K': \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K_\nu'(\nu \in \mathbb{Z}^n)$. Since $H_i\Psi = \Psi H_i^0$, part 1 of Proposition 16 gives $H_iK' = \tilde{H}_i^0K'$. Considering the action of $((x_n - 1)\Delta^\circ)^n$ on $K_\beta'(x)$ we get, using the fact that $((x_n - 1)\Delta^\circ)^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K_\nu'(x) = \left(\prod_{i=1}^n \frac{(1-a^{-1}\bar{\nu}_i)}{(1-q^{-1}x_i)}\right)K_{\nu+(1^n)}(q^{-1}x),$$
in particular

\[ K'_v(x) = \left( \prod_{i=1}^{\mu} \frac{(a^{-1}v_i; q)_{m_i}}{(q^{-m_i}x_i; q)_{m_i}} \right) K'_{v+}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K_v^0(x) \) for \( v \in \mathbb{Z} \); hence,

\[
K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_{m}}{(q^{-1}x; q^{-1})_{m}} = (ax)^{-m} \frac{(qa; q)_{m}}{(qx^{-1}; q)_{m}},
\]

\[
K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_{m}}{(a; q^{-1})_{m}} = \frac{(x; q)_{m}}{(a^{-1}; q)_{m}}
\]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_v(a^{-1}u) = K'_u(a^{-1}v). \]

**Proof.** Note that

\[
K'_v(a^{-1}u) = t^\ell(w_0) \Psi_{\mu} K_v^0(t^{a^{-1}x})|_{x=a^{-1}u} = t^\ell(w_0) (H_{w_0}^\circ K_v)(a^{-1}u^{-1})
\]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \). \( \blacksquare \)

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q, t} := \frac{G_\beta(\alpha)}{G_\beta(\beta)}. \tag{29}
\]

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} = \frac{G_\beta(\alpha^{-1})}{G_\beta(\beta^{-1})}.
\]
Theorem 27. For \( \alpha, \beta \in \mathbb{C}_n \) we have the binomial formula

\[
K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} a^{\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1},t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}.
\]

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless \( \beta \subseteq \alpha \), with \( \beta \subseteq \alpha \) meaning \( \beta_i \leq \alpha_i \) for \( i = 1, \ldots, n \).

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

\[
K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} \tau_{\beta}^{-1} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1},t^{-1}} K'_\beta(x)
= \sum_{\beta \in \mathbb{C}_n} \frac{K_{\beta}(\bar{\alpha}^{-1})K'_{\beta}(x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}
= t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K_{\beta}(\bar{\alpha}^{-1})\Psi K_{\beta}(t^{n-1}x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}
\]

with \( \Psi = w_0 H_{w_0} \) (note that the dependence on \( a \) in the right-hand side of (31) is through the normalization factors of the interpolation polynomials \( K_{\beta}(x) \) and \( K'_{\beta}(x) \)).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of \( K_\alpha \) as follows. By the identity \( H_{w_0} \psi = w_0 \) the binomial formula (31) implies the finite expansion

\[
(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K_{\beta}(\bar{\alpha}^{-1})K_{\beta}(t^{n-1}w_0x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}.
\]

Substituting \( x = \tilde{\gamma} \) and using (4) we obtain

\[
(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in \mathbb{C}_n} \frac{K_{\beta}(\bar{\alpha}^{-1})K_{\beta}(\bar{\gamma}^{-1})}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}.
\]

The right-hand side is manifestly invariant under interchanging \( \alpha \) and \( \gamma \), which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating \( G'_\alpha \) and \( G_\alpha \) is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K_\alpha'(x) = \sum_{\beta \in C_n} \tau_{\beta}^\alpha \binom{\alpha}{\beta}_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta^\alpha(\alpha^{-1})\Psi K_\beta^\alpha(t^{n-1}x)}{\tau_{\beta} G_\beta(\beta^{-1})},$$

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0Hw_0$ on both sides. Since $w_0Hw_0\Psi = \text{Id}$ we obtain

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_{\beta} \binom{\alpha}{\beta}_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \frac{\tau_{\beta} K_\beta(\alpha) K_\beta(ax)}{K_\beta(\beta)}. \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \frac{\tau_{\beta}}{\tau_{\alpha}} \binom{\alpha}{\beta}_{q,t} \binom{\beta}{\gamma}_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}. $$

Since $\binom{\delta}{\epsilon}_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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References


