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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials \( R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t) \) form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in \( n \) variables over the field \( \mathbb{F} := \mathbb{Q}(q, t) \). They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most \( n \) parts

\[ \mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \} \, . \]

For a partition \( \mu \in \mathcal{P}_n \) we define \( |\mu| = \mu_1 + \cdots + \mu_n \) and write

\[ \overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i} \, . \]
Then \( R_\lambda(x) = R_\lambda(x; q, t) \) is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most \(|\lambda|\) satisfying the vanishing conditions

\[
R_\lambda(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.
\]

The normalization is fixed by requiring that the coefficient of \( x^{\lambda_1} \cdots x^{\lambda_n} \) in the monomial expansion of \( R_\lambda(x) \) is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of \( R_\lambda(x) \) is the Macdonald polynomial \( P_\lambda(x) \) [9] and \( R_\lambda(x) \) satisfies the extra vanishing property \( R_\lambda(\mu) = 0 \) unless \( \lambda \subseteq \mu \) as Ferrer diagrams. Other key properties of \( R_\lambda(x) \), which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of \( R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t) \) in terms of the \( R_\mu(x; q^{-1}, t^{-1}) \)'s over the field \( \mathbb{K} := \mathbb{Q}(q, t, a) \), and the duality or evaluation symmetry, which involves the evaluation points

\[
\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n
\]

and takes the form

\[
\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.
\]

The interpolation polynomials have natural non-symmetric analogs \( G_\alpha(x) = G_\alpha(x; q, t) \), which were also defined in [4, 13]. These are indexed by the set of compositions with at most \( n \) parts, \( \mathcal{C}_n := \mathbb{Z}_{\geq 0}^n \). For a composition \( \beta \in \mathcal{C}_n \) we define

\[
\bar{\beta} := w_\beta(\beta_+),
\]

where \( w_\beta \) is the shortest permutation such that \( \beta_+ = w_\beta^{-1}(\beta) \) is a partition. Then \( G_\alpha(x) \) is, up to normalization, characterized as the unique polynomial of degree at most \(|\alpha| := \alpha_1 + \cdots + \alpha_n\) satisfying the vanishing conditions

\[
G_\alpha(\bar{\beta}) = 0 \text{ for } \beta \in \mathcal{C}_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.
\]

The normalization is fixed by requiring that the coefficient of \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) in the monomial expansion of \( G_\alpha(x) \) is 1.

Many properties of the symmetric interpolation polynomials \( R_\lambda(x) \) admit non-symmetric counterparts for the \( G_\alpha(x) \). For instance, the top homogeneous part of \( G_\alpha(x) \)
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$ 

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \# \{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha|-I(\alpha)}w_0H_{w_0}G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov's duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$ 

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$ 

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

Theorem C. For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n}aw_0x)}{G_\alpha(ax)}.$$ 

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ (1 ≤ $i$ < $n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \to n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n) \in \mathbb{F}^n$ by $\bar{v}_i := q^v t^{-k_i(v)}$ with

$$k_i(v) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$ 

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\bar{v} = (q^{v_1} \tau_1, \ldots, q^{v_n} \tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\bar{v} = w_v(\bar{v}_+)$ with $w_v \in S_n$ the shortest permutation such that $v_+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\bar{v} := -w_0 v$ for $v \in \mathbb{Z}^n$.

Note that $\bar{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for 1 ≤ $i$ < $n$. Consider the $F$-linear operators

$$H_i = ts_i - (1-t)x_i \frac{(1-s_i)}{x_i - x_{i+1}} + t \frac{x_i - tx_{i+1}}{x_i - x_{i+1}} (s_i - 1).$$
on \( \mathbb{F}(x) \) (1 ≤ i < n) called Demazure-Lusztig operators, and the automorphism \( \Delta \) of \( \mathbb{F}(x) \) defined by
\[
\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).
\]

Note that \( H_i \) (1 ≤ i < n) and \( \Delta \) preserve \( \mathbb{F}[x^{\pm 1}] \) and \( \mathbb{F}[x] \). Cherednik [1, 2] showed that the operators \( H_i \) (1 ≤ i < n) and \( \Delta \) satisfy the defining relations of the type A extended affine Hecke algebra,
\[
(H_i - t)(H_i + 1) = 0,
\]
\[
H_iH_j = H_jH_i, \quad |i - j| > 1,
\]
\[
H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1},
\]
\[
\Delta H_{i+1} = H_{i+1}\Delta,
\]
\[
\Delta^2 H_1 = H_{n-1}\Delta^2
\]
for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For \( w \in S_n \) we write \( H_w := H_{i_1}H_{i_2} \cdots H_{i_\ell} \) with \( w = s_{i_1}s_{i_2} \cdots s_{i_\ell} \) a reduced expression for \( w \in S_n \). It is well defined because of the braid relations for the \( H_i \)'s. Write \( \overline{H}_i := H_i + 1 - t = tH_i^{-1} \) and set
\[
\xi_i := t^{1-n}\overline{H}_{i-1} \cdots \overline{H}_1\Delta^{-1}H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \tag{1}
\]
The operators \( \xi_i \)'s are pairwise commuting invertible operators, with inverses
\[
\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1}\Delta H_1 \cdots H_{i-1}.
\]
The \( \xi_i^{-1} \) (1 ≤ i ≤ n) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial \( E_\alpha \in \mathbb{F}[x] \) of degree \( \alpha \in \mathbb{C}n \) is the unique polynomial satisfying
\[
\xi_i^{-1}E_\alpha = \overline{u}_iE_\alpha, \quad i = 1, \ldots, n
\]
and normalized such that the coefficient of \( x^\alpha \) in \( E_\alpha \) is 1.

Let \( \iota \) be the field automorphism of \( \mathbb{K} \) inverting \( q \), \( t \) and \( a \). It restricts to a field automorphism of \( \mathbb{F} \), inverting \( q \) and \( t \). We extend \( \iota \) to a \( \mathbb{Q} \)-algebra automorphism of \( \mathbb{K}[x] \).
and $\mathbb{F}[x]$ by letting $\iota$ act on the coefficients of the polynomial. Write

$$G_\alpha^0 := \iota(G_\alpha), \quad E_\alpha^0 := \iota(E_\alpha)$$

for $\alpha \in C_n$. Note that $\overline{\nu}^{-1} = (\iota(\overline{\nu}_1), \ldots, \iota(\overline{\nu}_n))$.

Put $H_i^0, H_w^0, \overline{H}_i, \Delta^0$ and $\xi_i^0$ for the operators $H_i, H_w, \overline{H}_i, \Delta$ and $\xi_i$ with $q, t$ replaced by their inverses. For instance,

$$H_i^0 = t^{-1} s_i - \frac{(1 - t^{-1}) x_i}{x_i - x_{i+1}} (1 - s_i),$$
$$\Delta^0 \tilde{f}(x_1, \ldots, x_n) = \tilde{f}(q x_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi_i^0 E_\alpha^0 = \overline{\alpha}^i E_\alpha^0$ for $i = 1, \ldots, n$, which characterizes $E_\alpha^0$ up to a scalar factor.

**Theorem 1.** For $\alpha \in C_n$ we have

$$G'_\alpha(x) = t^{(1-n)|\alpha|+1(\alpha)} w_0 H_w^0 G_\alpha^0 (t^{n-1} x)$$

with $1(\alpha) := \# \{ i < j | \alpha_i \geq \alpha_j \}$.

**Remark.** Formally set $t = q^r$, replace $x$ by $1 + (q - 1)x$, divide both sides of (2) by $(q - 1)^{|\alpha|}$ and take the limit $q \to 1$. Then

$$G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n - 1)r; r)$$

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here $\sigma$ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators $H_i$, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}} (1 - s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0) w_0 = w_0 \sigma^0(w_0)$ with $\sigma^0$ the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^0(s_i) = s_i - \frac{r}{x_i - x_{i+1}} (1 - s_i)$$

of $H_i^0$. Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_\alpha$. For the vanishing property, note that
\[ t^{n-1}w_0\tilde{\beta} = \tilde{\beta}^{-1} \] (4)
(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,
\[ (w_0H^\circ_{w_0}G^\circ_{\alpha}(t^{n-1}x))|_{x=\tilde{\beta}} = (H^\circ_{w_0}G^\circ_{\alpha}(x))|_{x=\tilde{\beta}^{-1}}. \]
This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G^\circ_{\alpha}(w_{\beta}^{-1})$ $(w \in S_n)$ by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that
\[ E_\alpha = t^{I(\alpha)}w_0H^\circ_{w_0}E^\circ_{\alpha}. \] (5)

Note that $\Psi := w_0H^\circ_{w_0}$ satisfies the intertwining properties
\[ H_i\Psi = t\Psi H^\circ_i, \]
\[ \Delta\Psi = t^{n-1}\Psi H^\circ_{n-1}\cdots H^\circ_1(\Delta^\circ)^{-1}H^\circ_{n-1}\cdots H^\circ_1 \] (6)
for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi^\circ_i^{-1}\Psi = \Psi\xi^\circ_i$ for $i = 1, \ldots, n$. Therefore,
\[ E_\alpha(x) = c_\alpha\Psi E^\circ_{\alpha}(x) \]
for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-I(\alpha)}$; hence, $c_\alpha = t^{I(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi^\circ_j$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi^\circ_j$ with eigenvalues $\overline{u}_j$ $(1 \leq j \leq n)$, normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that
\[ E_{u+(1^n)} = x_1 \cdots x_nE_u(x). \]
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{I(u)}w_0H_{w_0}\circ E_u$$

with $E_u := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0E_{-w_0u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0u}(x^{-1}) = t^{I(u)}H_{w_0}\circ E_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

### 3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left( t^{1-n-q} t^{1-l'(s)} \right) \prod_{s \in \alpha} (at - q^{a'(s)})$$

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \#\{k > i \mid j \leq \alpha_k \leq \alpha_i\} + \#\{k < i \mid j \leq \alpha_k + 1 \leq \alpha_i\},$$

$$a'(s) := j - 1, \quad l'(s) := \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \geq \alpha_i\}.$$

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|}G_\alpha(a\tau) = \prod_{s \in \alpha} \left( t^{1-n+l'(s)} - q^{a'(s)+1} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.$$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha| + I(\alpha) - \ell(w_0)} G_\alpha^0 (\alpha^{-1}).$$

Proof. Since $t^{n-1} w_0 \tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha| + I(\alpha)} (H_{w_0}^0 G_\alpha^0 (\alpha \overline{\tau}^{-1}))
= t^{(1-n)|\alpha| + I(\alpha) - \ell(w_0)} G_\alpha^0 (\alpha \overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G_\alpha^0(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha| - n(\alpha) q^{n'(\alpha)}} G_\alpha^0 (a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}.$$

(10)

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For \( \alpha \in \mathbb{C}^n \) we have

\[
G'_\alpha(a^{-1}\tau) = \tau a^{-|\alpha|} G_\alpha(a\tau).
\]

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of \( \mathbb{K} \)-valued functions on \( \mathbb{Z}^n \), which is constructed as follows.

For \( v \in \mathbb{Z}^n \) and \( y \in \mathbb{K}^n \) write \( v^\natural := (v_2, \ldots, v_n, v_1 + 1) \) and \( y^\natural := (y_2, \ldots, y_n, qy_1) \).

Denote the inverse of \( ^\natural \) by \( ^\flat \), so \( v^\flat = (v_n - 1, v_1, \ldots, v_n - 1) \) and \( y^\flat = (y_n/q, y_1, \ldots, y_n - 1) \). We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \). Then we have

1. \( s_i(v^\flat) = s_i^\natural v \) if \( v_i \neq v_{i+1} \).
2. \( \overline{v}_i = \overline{t}^\natural_{i+1} \) if \( v_i = v_{i+1} \).
3. \( v^\natural = v^\natural \).

Let \( \mathbb{H} \) be the double affine Hecke algebra over \( \mathbb{K} \). It is isomorphic to the subalgebra of \( \text{End}(\mathbb{K}[x^{\pm 1}]) \) generated by the operators \( H_i (1 \leq i < n) \), \( \Delta^{\pm 1} \), and the multiplication operators \( x_j^{\pm 1} (1 \leq j \leq n) \).

For a unital \( \mathbb{K} \)-algebra \( A \) we write \( \mathcal{F}_A \) for the space of \( A \)-valued functions \( f : \mathbb{Z}^n \rightarrow A \) on \( \mathbb{Z}^n \).

Corollary 6. Let \( A \) be a unital \( \mathbb{K} \)-algebra. Consider the \( A \)-linear operators \( \widehat{H}_i (1 \leq i < n) \), \( \widehat{\Delta} \) and \( \widehat{x}_j (1 \leq j \leq n) \) on \( \mathcal{F}_A \) defined by

\[
(\widehat{H}_i f)(v) := tf(v) + \frac{\overline{v}_i - t^{\natural}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (f(s_i^\natural v) - f(v)),
\]

\[
(\widehat{\Delta} f)(v) := f(v^\flat), \quad (\widehat{\Delta}^{-1} f)(v) := f(v^\natural),
\]

\[
(\widehat{x}_j f)(v) := a\overline{v}_j f(v)
\]

for \( f \in \mathcal{F}_A \) and \( v \in \mathbb{Z}^n \). Then \( H_i \mapsto \widehat{H}_i (1 \leq i < n) \), \( \Delta \mapsto \widehat{\Delta} \) and \( x_j \mapsto \widehat{x}_j (1 \leq j \leq n) \) defines a representation \( \mathbb{H} \rightarrow \text{End}_A(\mathcal{F}_A) \), \( X \mapsto \widehat{X} (X \in \mathbb{H}) \) of the double affine Hecke algebra \( \mathbb{H} \) on \( \mathcal{F}_A \).
Remark 8. Let \( \mathcal{O} \subset \mathbb{K}^n \) be the smallest \( S_n \)-invariant and \( \mathfrak{z} \)-invariant subset that contains \( \{a\mathcal{V} \mid \mathcal{V} \in \mathbb{Z}^n\} \). Note that \( \mathcal{O} \) is contained in \( \{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\} \). The Demazure–Lusztig operators \( H_i (1 \leq i < n) \), \( \Delta^{\pm 1} \) and the coordinate multiplication operators \( x_j (1 \leq j \leq n) \) act \( A \)-linearly on the space \( F_A^O \) of \( A \)-valued functions on \( \mathcal{O} \), and hence turns \( F_A^O \) into an \( \mathbb{H} \)-module. Define the surjective \( A \)-linear map

\[
\text{pr} : F_A^O \to F_A
\]

by \( \text{pr}(g)(\mathcal{V}) := g(a\mathcal{V}) (\mathcal{V} \in \mathbb{Z}^n) \).

We claim that \( \text{Ker}(\text{pr}) \) is an \( \mathbb{H} \)-submodule of \( F_A^O \). Clearly \( \text{Ker}(\text{pr}) \) is \( x_j \)-invariant for \( j = 1, \ldots, n \). Let \( g \in \text{Ker}(\text{pr}) \). Part 3 of Lemma 5 implies that \( \Delta g \in \text{Ker}(\text{pr}) \). To show that \( H_j g \in \text{Ker}(\text{pr}) \) we consider two cases. If \( \mathcal{V}_i \neq \mathcal{V}_{i+1} \) then \( s_i \mathcal{V} = s_i \mathcal{V} \) by part 1 of Lemma 5. Hence,

\[
(H_j g)(a\mathcal{V}) = tg(a\mathcal{V}) + \frac{\mathcal{V}_i - t\mathcal{V}_{i+1}}{\mathcal{V}_i - \mathcal{V}_{i+1}} (g(a\mathcal{V}_i - g(a\mathcal{V})) = 0.
\]

If \( \mathcal{V}_i = \mathcal{V}_{i+1} \) then \( \mathcal{V}_i = t\mathcal{V}_{i+1} \) by part 2 of Lemma 5. Hence,

\[
(H_j g)(\mathcal{V}) = tg(a\mathcal{V}) + \frac{\mathcal{V}_i - t\mathcal{V}_{i+1}}{\mathcal{V}_i - \mathcal{V}_{i+1}} (g(a\mathcal{V}_i - g(a\mathcal{V})) = tg(a\mathcal{V}) = 0.
\]

Hence, \( F_A \) inherits the \( \mathbb{H} \)-module structure of \( F_A^O / \text{Ker}(\text{pr}) \). It is a straightforward computation, using Lemma 5 again, to show that the resulting action of \( H_i (1 \leq i < n) \), \( \Delta \) and \( x_j (1 \leq j \leq n) \) on \( F_A \) is by the operators \( \hat{H}_i (1 \leq i < n) \), \( \hat{\Delta} \) and \( \hat{x}_j (1 \leq j \leq n) \). \( \square \)

Remark 7. With the notations from (the proof of) Corollary 6, let \( \tilde{g} \in F_A^O \) and set \( g := \text{pr}(\tilde{g}) \in F_A \). In other words, \( g(\mathcal{V}) := \tilde{g}(a\mathcal{V}) \) for all \( \mathcal{V} \in \mathbb{Z}^n \). Then

\[
(\hat{X} g)(\mathcal{V}) = (X \tilde{g})(a\mathcal{V}), \quad \mathcal{V} \in \mathbb{Z}^n
\]

for \( X = H_i, \Delta^{\pm 1}, x_j \).

Remark 8. Let \( F_A^+ \) be the space of \( A \)-valued functions on \( \mathcal{O}_n \). We sometimes will consider \( \hat{H}_i (1 \leq i < n) \), \( \hat{\Delta}^{-1} \) and \( \hat{x}_j (1 \leq j \leq n) \), defined by the formulas (11), as linear operators on \( F_A^+ \).

Definition 9. We call

\[
K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a\tau; q, t)} \in \mathbb{K}[x]
\]

the normalized non-symmetric interpolation Macdonald polynomial of degree \( \alpha \).
We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(t) = G_\alpha(t) = 0$ if $\alpha \in \mathbb{C}$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(t)}$$

since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi_1 = (x_n - t^{1-n}) \Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$ 

The operators $H_i, \Xi_j$ and $\Phi$ preserve $K[x]$ (see [4]); hence, they give rise to $K$-linear operators on $F^+_{K[x]}$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in C_n$). Note that the operators $H_i, \Xi_j$ and $\Phi$ on $F^+_{K[x]}$ commute with the hat-operators $\hat{H}_i, \hat{x}_j$ and $\hat{\Delta}^{-1}$ on $F^+_{K[x]}$ (cf. Remark 8). The same remarks hold true for the space $F^+_{K,(x)}$ of $K((x))$-valued functions on $\mathbb{Z}_n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $F^+_{K,(x)}$).

Let $K \in F^+_{K[x]}$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in C_n$).

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $F^+_{K[x]}$:

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a \tilde{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2 \tilde{x}_1^{-1} - 1) \hat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$’s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$’s.

If $\alpha \in C_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t - 1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{s_\alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t - 1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - \overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1} \overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{s_\alpha}(x)$$
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_i G_\alpha(x) = tG_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_i K_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(at) = \frac{\overline{\alpha}_{i+1} - t \overline{\alpha}_i}{\overline{\alpha}_{i+1} - \overline{\alpha}_i} G_{s,0}(at)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_i K = \widehat{H}_i K \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\circ}(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_{\alpha^\circ}(at)}{G_\alpha(at)} = at^{1-n+k_1(a)} - q^\alpha t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n}(a\overline{\alpha}_1^{-1} - 1) K_{\alpha^\circ}(x). \tag*{\blacksquare}
\]

**Remark 11.** Note that

\[
\Phi K_\alpha(x) = (a\overline{\alpha}_n - t^{1-n}) K_{\alpha^\circ}(x)
\]

for \( \alpha \in C_n \) since \( \overline{\alpha}^{-1} = t^{n-1} w_0 \overline{\alpha} \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in \mathcal{F}_{[\overline{\mathbb{K}}(x)]}^+ \) to a map \( K \in \mathcal{F}^+_{[\overline{\mathbb{K}}(x)]} \) such that Lemma 10 remains valid.

**Lemma 12.** For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a\overline{\alpha}_i^{-1}}{1 - q^{n-1} x_i} \right) K_{\alpha+(1^n)}(qx).
\]
Proof. Note that for $f \in \mathbb{K}[x],$

$$\Phi^nf(x) = \left(\prod_{i=1}^{n}(x_i - t^{1-n})\right)f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m}a \vec{v}^{-1}; q)_m}{(qt^{n-1}x_i; q)_m} \quad \forall v \in \mathbb{Z}^n,$$

with $(y; q)_m := \prod_{j=0}^{m-1}(1 - q^jy)$ the $q$-shifted factorial.

**Definition 13.** Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n.$ Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v|-m^2n} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m}t^{1-n}x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v)K_{v+(m^n)}(q^m x),$$

where $m$ is a nonnegative integer such that $v + (m^n) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

**Example 14.** If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0},$

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left(\frac{x}{a}\right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.$$

**Lemma 15.** For all $v \in \mathbb{Z}^n,$

$$K_v(x) = \frac{G_v(x)}{G_v(x)}.$$

**Proof.** Let $v \in \mathbb{Z}^n.$ Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(x) = 1.$ Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n.$ Then

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x) = A_m(x; v) \frac{G_{v+(m^n)}(q^m x)}{G_{v+(m^n)}(x)} = 1,$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $F_{\mathbb{K}(x)}$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^{2\hat{x}_1^{-1}} - 1)\hat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in F_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $F_{\mathbb{K}(x)}$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in F_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $F_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in F_{\mathbb{K}(x)}$,

$$(\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

(15)

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^mx)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = tK_v$ and $H_i K_{v+(m^n)}(q^mx) = tK_{v+(m^n)}(q^mx)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\hat{v}_1^{-1} - 1)K_{v}(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

(16)

where $\Phi(q^m) := (q^mx_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \hat{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = -w_0v$ for $v \in \mathbb{Z}^n$.

**Theorem 17.** (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}).$$

(17)

**Example 18.** If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr}(a^{-1}; q)_{m+r} \frac{(a^{-1}; q)_m}{(a^{-1}; q)_r}$$

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of $m$ and $r$.

**Proof.** We divide the proof of the theorem in several steps.

**Step 1.** If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{s_i u}(a\tilde{v}) = K_v(a\tilde{s}_i\tilde{u})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

$$\frac{(t - 1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{v}) + \frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}} K_u(a\tilde{s}_{n-i}v)$$

$$= \frac{(t - 1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{i+1})} K_u(a\tilde{v}) + \frac{\tilde{u}_i - t\tilde{u}_{i+1}}{\tilde{u}_i - \tilde{u}_{i+1}} K_{s_i u}(a\tilde{v}).$$

(19)

Replacing in (19) the role of $u$ and $v$ and replacing $i$ by $n - i$ we get

$$\frac{(t - 1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a\tilde{u}) + \frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} K_v(a\tilde{s}_i\tilde{u})$$

$$= \frac{(t - 1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a\tilde{u}) + \frac{\tilde{v}_{n-i} - t\tilde{v}_{n+1-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} K_{s_{n-i} v}(a\tilde{u}).$$

(20)

Suppose that $s_{n-i} v = v$. Then $\tilde{v}_{n-i} = t\tilde{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n}w_0\tilde{v}$, that is, $\tilde{v}_i = t^{1-n}\tilde{v}_{n+1-i}$, we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{s_i u}(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_v(a\tilde{s}_i\tilde{u}) = K_v(a\tilde{u})$ if $s_{n-i} v = v$. 
Hence, we obtain
\[ K_{s_i u}(a\tilde{v}) = K_u(a\tilde{v}) = K_{s_i u}(a\tilde{u}) = K_v(a\tilde{s}_i \tilde{u}). \]

If \( s_{n-i} v \neq v \) then (19) and the induction hypothesis can be used to write \( K_{s_i u}(a\tilde{v}) \) as an explicit linear combination of \( K_v(a\tilde{u}) \) and \( K_{s_{n-i} v}(a\tilde{u}) \). Then (20) can be used to rewrite the term involving \( K_{s_{n-i} v}(a\tilde{u}) \) as an explicit linear combination of \( K_v(a\tilde{u}) \) and \( K_v(a\tilde{s}_i \tilde{u}) \). Hence, we obtain an explicit expression of \( K_{s_i u}(a\tilde{v}) \) as linear combination of \( K_v(a\tilde{u}) \) and \( K_v(a\tilde{s}_i \tilde{u}) \), which turns out to reduce to \( K_{s_i u}(a\tilde{v}) = K_v(a\tilde{s}_i \tilde{u}) \) after a direct computation.

\[ \text{Step 2.} \quad K_0(a\tilde{v}) = 1 = K_v(a\tilde{0}) \text{ for all } v \in \mathbb{Z}^n. \]

\[ \text{Proof of Step 2.} \quad \text{Clearly } K_0(x) = 1 \text{ and } K_v(a\tilde{0}) = K_v(\alpha\gamma) = 1 \text{ for } v \in \mathbb{Z}^n \text{ by Lemma 15.} \]

\[ \text{Step 3.} \quad K_u(a\tilde{v}) = K_v(a\tilde{u}) \text{ for } v \in \mathbb{Z}^n \text{ and } \alpha \in C_n. \]

\[ \text{Proof of Step 3.} \quad \text{We prove it by induction. It is true for } \alpha = 0 \text{ by Step 2. Let } m \in \mathbb{Z}_{>0} \text{ and suppose that } K_\gamma(a\tilde{v}) = K_v(a\tilde{y}) \text{ for } v \in \mathbb{Z}^n \text{ and } \gamma \in C_n \text{ with } |\gamma| < m. \text{ Let } \alpha \in C_n \text{ with } |\alpha| = m. \]

\[ \text{We need to show that } K_\alpha(a\tilde{v}) = K_v(a\tilde{u}) \text{ for all } v \in \mathbb{Z}^n. \text{ By Step 1 we may assume without loss of generality that } \alpha_n > 0. \text{ Then } \gamma := a\tilde{x} \in C_n \text{ satisfies } |\gamma| = m - 1, \text{ and } \alpha = \gamma\tilde{z}. \text{ Furthermore, note that we have the formula} \]
\[ (a\tilde{v}_1^{-1} - 1)K_u(a\tilde{v}) = (a\tilde{u}_1^{-1} - 1)K_{u\gamma}(a\tilde{v}) \quad (21) \]

\[ \text{for all } u, v \in \mathbb{Z}^n, \text{ which follows by writing out the formula from part 3 of Lemma 16.} \]

\[ \text{Hence, we obtain} \]
\[ K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{v}) = \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{v}_1^{-1} - 1)} K_\gamma(a\tilde{v}) \]
\[ = \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{v}_1^{-1} - 1)} K_\gamma(a\tilde{v}) = K_v(a\gamma\tilde{z}) = K_v(a\tilde{u}), \]

\[ \text{where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.} \]
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = v - (m^n) \) and \( q^{-m}\tilde{u} = u + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v}) \\
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m}a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n)) = 1,
\]

which follows by a straightforward computation using (4).

\[\blacksquare\]

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(1)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}_{\text{Fix}}^+ \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\overline{K} = \hat{H}_i\overline{K} \).
2. \( \xi_j\overline{K} = \hat{\xi}_j^{-1}\overline{K} \).
3. \( x_n\Delta\overline{K} = t^{1-n}\hat{\Delta}^{-1}\overline{K} \).
Note that

\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),

\[
E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},
\]

\[
\overline{K}_\alpha(x) = q^{\lvert \alpha \rvert} \, t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha_i} x_i)^{-1} \right) \overline{K}_{\alpha+(1^n)}(x). \tag{22}
\]

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula

\[
E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.
\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n\).

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n\). Then \(\overline{K}_\nu(x) := \overline{K}_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) is defined by

\[
\overline{K}_\nu(x) := q^{\lvert \nu \rvert} \, t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-1} \right) \overline{K}_{\nu+(m^n)}(x).
\]

Using

\[
\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2 n} \, t^{(1-n)n} \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-m}
\]

and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that

\[
\lim_{a \to \infty} a^{-\lvert \nu \rvert} G_\nu(ax) = E_\nu(x),
\]

\[
\lim_{a \to \infty} K_\nu(ax) = \overline{K}_\nu(x)
\]

for all \(\nu \in \mathbb{Z}^n\), so in particular

\[
\overline{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.
\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}_{\mathbb{F}[x^{\pm 1}]}$ defined by $\nu \mapsto K_{\nu}$ ($\nu \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\tilde{v}) = K_v(\tilde{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\bar{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\bar{\beta}^{-1}) = K_\alpha(a\bar{\alpha}) = K_\beta(a\bar{\alpha}),$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. ■

### 7.3 Okounkov’s duality

Write $\mathcal{F}[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in \mathcal{F}[x]^{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\alpha^+(x)$$

for $\alpha \in C_n$. Okounkov's [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in P_n$ we have

$$K^+_\lambda(a\bar{\mu}^{-1}) = K^+_\mu(a\bar{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\tilde{C}_+ = \sum_{w \in S_n} \tilde{H}_w$, with $\tilde{H}_w := \tilde{H}_{i_1} \cdots \tilde{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_K$ for the function $f_\mu(u) := K_u(a\tilde{\mu}) (u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\lambda(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\tilde{C}_+ f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{u}) = (Jw_0 K_\mu(t^{1-n}x))|_{x=a^{-1}\pi}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in K(x)$. A direct computation shows that

$$JH_iJ = (H_i^o)^{-1}, \quad w_0H_iw_0 = (H_{n-i}^o)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\tilde{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x=a^{-1}\pi}.\quad (24)$$

By (23) and (4) this simplifies to

$$(\tilde{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\mu(a\tilde{\lambda}).\quad (25)$$

Returning to (24) we conclude that $K^+_\lambda(a\tilde{\mu}) = K^+_\mu(a\tilde{\lambda})$. Since $K^+_\lambda$ is symmetric we obtain from (4) that

$$K^+_\lambda(a\bar{\mu}^{-1}) = K^+_\mu(a\bar{\lambda}^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0}K_u)(a\tilde{v}) = (H_{w_0}K_v)(a\tilde{u}). \quad (27)$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a\tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0}K_u)(a\tilde{v}) = (\hat{H}_{w_0}f_v)(u). \quad (29)$$

Since $f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}\tilde{u})$ by (4), Remark 7 implies that

$$(\hat{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}\tilde{u}). \quad (30)$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$(\hat{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}\tilde{u}) = (H_{w_0}K_v)(a\tilde{u}),$$

which completes the proof. ■

Recall from Theorem 1 that

$$G'_\beta(x) = t^{(1-n)|\beta| + I(\beta)}\Psi \, G^\circ_\beta(t^{n-1}x)$$

with $\Psi := w_0H_{w_0}^\circ$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1}\tau)} = t^{\ell(w_0)}\Psi \, K^\circ_\beta(t^{n-1}x), \quad \beta \in C_n,$$

with $K^\circ_\beta := \iota(K_\beta)$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K'_\nu(x) := t^{\ell(w_0)}\Psi \, K^\circ_\nu(t^{n-1}x). \quad (28)$$

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n)$. Since $H_i \Psi = \Psi H_i^\circ$, part 1 of Proposition 16 gives $H_iK' = \hat{H}_i^\circ K'$. Considering the action of $((x_n - 1)\Delta^\circ)^n$ on $K'_\beta(x)$ we get, using the fact that $((x_n - 1)\Delta^\circ)^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K'_\nu(x) = (\prod_{i=1}^n \frac{(1-a^{-1}\nu_i)}{(1-q^{-1}x_i)})K'_{\nu+(1^n)}(q^{-1}x).$$
in particular
\[
K'_v(x) = \left( \prod_{i=1}^{n} \frac{1}{(q^{-m}x_i; q)_m} \right) K'_{v+(m)}(q^{-m}x).
\]

Example 25. For \( n = 1 \) we have \( K'_v(x) = K'^0_v(x) \) for \( v \in \mathbb{Z} \); hence,
\[
K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m},
\]
\[
K'_m(x) = (ax)^{m} \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}
\]
for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

Proposition 26. For all \( u, v \in \mathbb{Z}^n \) we have
\[
K'_v(a^{-1}u) = K'_u(a^{-1}v).
\]

Proof. Note that
\[
K'_v(a^{-1}u) = t^{\ell(w_0)} \Psi K'^0_v(t^{n-1}x)|_{x = a^{-1}u} = t^{\ell(w_0)} (H_{w_0}^o K_v^o) (a^{-1}\tilde{u}^{-1})
\]
by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in \mathcal{C}_n \) the generalized binomial coefficient by
\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} := \frac{G^o_\beta(\bar{\alpha})}{G^o_\beta(\bar{\beta})}.
\]
(29)

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get
\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} = \frac{G^o_\beta(\bar{\alpha}^{-1})}{G^o_\beta(\bar{\beta}^{-1})},
\]
Theorem 27. For $\alpha, \beta \in \mathbb{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} a|\beta| \left[ \frac{G_\beta'(x)}{G_\beta(ax)} \right]_{q^{-1}, t^{-1}}.$$  \hfill (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} \tau^{-1}_\beta \left[ \frac{\alpha}{\beta} \right]_{q^{-1}, t^{-1}} K_\beta'(x)$$

$$= \sum_{\beta \in \mathbb{C}_n} \frac{K_\beta^\circ(\alpha^{-1})K_\beta'(x)}{\tau^{-1}_\beta K_\beta^\circ(\beta^{-1})} \hfill (31)$$

$$= t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K_\beta^\circ(\alpha^{-1})\Psi K_\beta^0(t^{n-1}x)}{\tau^{-1}_\beta K_\beta^0(\beta^{-1})}$$

with $\Psi = w_0 H_{w_0}$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K_\beta^\circ(x)$ and $K_\beta'(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K_\beta^\circ(\alpha^{-1})K_\beta^0(\beta^{-1})}{\tau^{-1}_\beta K_\beta^0(\beta^{-1})}. \hfill (4)$$

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in \mathbb{C}_n} \frac{K_\beta^\circ(\alpha^{-1})K_\beta^0(\beta^{-1})}{\tau^{-1}_\beta K_\beta^0(\beta^{-1})}. \hfill (5)$$

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G_\alpha'$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

\[
K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).
\] (32)

The starting point of the alternative proof of (32) is the binomial formula in the form

\[
K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_{\beta}^\alpha(\alpha^{-1}) \Psi K^\beta_\alpha(t^{n-1}x)}{\tau_\beta G_{\beta}^\alpha(\beta^{-1})},
\]

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0Hw_0$ on both sides. Since $w_0Hw_0\Psi = \text{Id}$ we obtain

\[
\Psi K^\alpha_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).
\]

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

\[
\Psi K^\alpha_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\alpha)x K_\beta(\bar{\beta}).
\] (33)

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

\[
\sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.
\]

Since \[
\left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0\] unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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