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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$ 

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$
Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$ 

The normalization is fixed by requiring that the coefficient of $x^{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\lambda)}{R_\mu(a\tau)}.$$ 

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $\mathcal{C}_n := \mathbb{Z}_{\geq 0}^n$. For a composition $\beta \in \mathcal{C}_n$ we define

$$\bar{\beta} := w_\beta(\beta_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\bar{\beta}) = 0 \text{ for } \beta \in \mathcal{C}_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$ 

The normalization is fixed by requiring that the coefficient of $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0 \beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$ 

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \# \{i < j | \alpha_i \geq \alpha_j \}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - 1(\alpha)}w_0H_wG_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}$$

for all $\beta$.

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

Theorem C. For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n}aw_0x)}{G_\alpha(at)}.$$  

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ (1 ≤ $i$ < $n$) are Coxeter generators for $S_n$. Let $\ell : S_n \rightarrow \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (\ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \rightarrow n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i}t^{-k_i(v)}$ with

$$k_i(v) := \# \{ k < i \mid v_k \geq v_i \} + \# \{ k > i \mid v_k > v_i \}.$$

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{v} = (q^{v_1}\tau_1, \ldots, q^{v_n}\tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_v(\overline{v})$ with $w_v \in S_n$ the shortest permutation such that $v_+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\overline{v} := -w_0v$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^\pm 1] := F[x_1^\pm, \ldots, x_n^\pm]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for 1 ≤ $i$ < $n$. Consider the $F$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i-x_{i+1}}(1-s_i) = t + \frac{x_i-tx_{i+1}}{x_i-x_{i+1}}(s_i - 1)$$
on $\mathbb{F}(x)$ ($1 \leq i < n$) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ ($1 \leq i < n$) and $\Delta$ preserve $\mathbb{F}[x^\pm 1]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ ($1 \leq i < n$) and $\Delta$ satisfy the defining relations of the type A extended affine Hecke algebra,

$$(H_i - t)(H_i + 1) = 0,$$

$$H_i H_j = H_j H_i, \quad |i - j| > 1,$$

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

$$\Delta H_{i+1} = H_i \Delta,$$

$$\Delta^2 H_1 = H_{n-1} \Delta^2$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = tH_i^{-1}$ and set

$$\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.$$

The $\xi_i^{-1}$ ($1 \leq i \leq n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$\xi_i^{-1} E_\alpha = \overline{\alpha}_i E_\alpha, \quad i = 1, \ldots, n$$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$
and $\mathbb{F}[x]$ by letting $\iota$ act on the coefficients of the polynomial. Write

$$G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)$$

for $\alpha \in \mathbb{C}_n$. Note that $\overline{\nu}^{-1} = (\iota(\overline{\nu}_1), \ldots, \iota(\overline{\nu}_n))$.

Put $H_i^\circ, H_w^\circ, \overline{H}_i, \Delta^\circ$ and $\xi_i^\circ$ for the operators $H_i, H_w, \overline{H}_i, \Delta$ and $\xi_i$ with $q, t$ replaced by their inverses. For instance,

$$H_i^\circ = t^{-1}s_i - \frac{(1-t^{-1})x_i}{x_i - x_{i+1}}(1-s_i),$$

$$\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi_i^\circ E_\alpha^\circ = \overline{\alpha}_i E_\alpha^\circ$ for $i = 1, \ldots, n$, which characterizes $E_\alpha^\circ$ up to a scalar factor.

**Theorem 1.** For $\alpha \in \mathbb{C}_n$ we have

$$G_\alpha'(x) = t^{(1-n)|\alpha|+1(\alpha)}w_0H_{w_0}^\circ G_\alpha(t^{n-1}x)$$

(2)

with $I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \}$.

**Remark.** Formally set $t = q^r$, replace $x$ by $1 + (q - 1)x$, divide both sides of (2) by $(q - 1)^{|\alpha|}$ and take the limit $q \to 1$. Then

$$G_\alpha'(x; r) = (-1)^{|\alpha|} \sigma(w_0)w_0G_\alpha(-x - (n - 1)r; r)$$

(3)

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here $\sigma$ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators $H_i$, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1-s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0\sigma^\circ(w_0)$ with $\sigma^\circ$ the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1-s_i)$$

of $H_i^\circ$. Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_\alpha$. For the vanishing property, note that
\[
t^{n-1}w_0\tilde{\beta} = \tilde{\beta}^{-1}
\] (this is the $q$-analog of [14, Lem. 6.1(2)]); hence,
\[
(w_0H^o_{w_0}G^o_\alpha(t^{n-1}x))|_{x=\tilde{\beta}} = (H^o_{w_0}G^o_\alpha(x))|_{x=\tilde{\beta}^{-1}}.
\]
This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G^o_\alpha(w\tilde{\beta}^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that
\[
E_\alpha = t^{I(\alpha)}w_0H^o_{w_0}E^o_\alpha.
\]
Note that $\Psi := w_0H^o_{w_0}$ satisfies the intertwining properties
\[
H^o_i \Psi = t\Psi H^o_i,
\]
\[
\Delta \Psi = t^{n-1} \Psi H^o_{n-1} \cdots H^o_1 (\Delta^o)^{-1} H^o_{n-1} \cdots H^o_1
\]
for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi^{-1}_i \Psi = \Psi \xi^o_i$ for $i = 1, \ldots, n$.

Therefore,
\[
E_\alpha(x) = c_\alpha \Psi E^o_\alpha(x)
\]
for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-I(\alpha)}$; hence, $c_\alpha = t^{I(\alpha)}$. □

Consider the Demazure operators $H^o_i$ and the Cherednik operators $\xi^{-1}_j$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi^{-1}_j$ with eigenvalues $\overline{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that
\[
E_{u+(1^n)} = x_1 \cdots x_n E_u(x).
\]
It is now easy to check that formula (5) is valid with \( \alpha \) replaced by an arbitrary integral vector \( u \),

\[
E_u = t^{l(u)} w_0 H_{w_0}^\alpha E^\circ_u
\]

with \( E^\circ_u := \iota(E_u) \). Furthermore, one can show in the same vein as the proof of (5) that

\[
w_0 E_{-w_0}u(x^{-1}) = E_u(x)
\]

for an integral vector \( u \), where \( p(x^{-1}) \) stands for inverting all the parameters \( x_1, \ldots, x_n \) in the Laurent polynomial \( p(x) \in \mathbb{F}[x^{\pm 1}] \). Combining this equality with (7) yields

\[
E_{-w_0}u(x^{-1}) = t^{l(u)} H_{w_0}^\circ E^\circ_u(x),
\]

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

### 3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

\[
G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)} + 1 t^{1-l'(s)}}{1 - q^{a(s)} + 1 t^{l(s)+1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})
\]

was obtained, with \( a(s), l(s), a'(s) \) and \( l'(s) \) the arm, leg, coarm and coleg of \( s = (i, j) \in \alpha \), defined by

\[
a(s) := \alpha_i - j, \quad l(s) := \#\{k > i \mid j \leq \alpha_k \leq \alpha_i\} + \#\{k < i \mid j \leq \alpha_k + 1 \leq \alpha_i\},
\]

\[
a'(s) := j - 1, \quad l'(s) := \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \geq \alpha_i\}.
\]

By (8) we have

\[
E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - q^{a'(s)} + 1 t^{l'(s)+1}}{1 - q^{a(s)} + 1 t^{l(s)+1}} \right),
\]

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for \( \alpha \in C_n \),

\[
\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.
\]
Lemma 2. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^o_\alpha(a\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = 0^{-1}$ we have by Theorem 1,

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha)}H_{w_0}^oG^o_\alpha(a0^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^o_\alpha(a0^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^o_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^{n} \left( \frac{\alpha_i}{2} \right)$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(a\tau) = (-a)^{|\alpha|}t^{(1-n)|\alpha|-n(\alpha)}q^{n'(\alpha)}G^o_\alpha(a^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in \mathcal{C}_n$) by

$$\tau_\alpha := (-1)^{|\alpha|}q^{n'(\alpha)}t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in C_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). ■

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $K$-valued functions on $Z^n$, which is constructed as follows.

For $v \in Z^n$ and $y \in \mathbb{K}^n$ write $\nu^i := (v_2, \ldots, v_n, v_1 + 1)$ and $\nu^i := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $\nu^i$ by $\nu^i$, so $\nu^i = (v_n - 1, v_1, \ldots, v_{n-1})$ and $\nu^i = (y_n/q, y_1, \ldots, y_{n-1})$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in Z^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = s_i \nu^i$ if $v_i \neq v_{i+1}$.
2. $\nu_i = t \nu_{i+1}$ if $v_i = v_{i+1}$.
3. $\nu^i = \nu^i$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^\pm 1])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^\pm 1$, and the multiplication operators $x_j^\pm 1$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : Z^n \to A$ on $Z^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\hat{H}_i$ ($1 \leq i < n$), $\hat{\Delta}$ and $\hat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{align*}
(\hat{H}_i f)(v) &:= tf(v) + \frac{\nu_i - t \nu_{i+1}}{\nu_i - \nu_{i+1}} (f(s_i v) - f(v)), \\
(\hat{\Delta} f)(v) &:= f(\nu^i), \quad (\hat{\Delta}^{-1} f)(v) := f(\nu^i), \\
(\hat{x}_j f)(v) &:= a \nu_j f(v)
\end{align*}$$

(11)

for $f \in \mathcal{F}_A$ and $v \in Z^n$. Then $H_i \mapsto \hat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \hat{\Delta}$ and $x_j \mapsto \hat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A)$, $X \mapsto \hat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 
**Remark 8.** Let \( \mathcal{O} \subset \mathbb{K}^n \) be the smallest \( S_n \)-invariant and \( \mathfrak{t} \)-invariant subset that contains \( \{a \bar{v} \mid v \in \mathbb{Z}^n\} \). Note that \( \mathcal{O} \) is contained in \( \{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\} \). The Demazure–Lusztig operators \( H_i(1 \leq i < n) \), \( \Delta^{\pm 1} \) and the coordinate multiplication operators \( x_j(1 \leq j \leq n) \) act \( A \)-linearly on the space \( F_A^\mathcal{O} \) of \( A \)-valued functions on \( \mathcal{O} \), and hence turns \( F_A^\mathcal{O} \) into an \( \mathbb{H} \)-module. Define the surjective \( A \)-linear map
\[
pr : F_A^\mathcal{O} \to \mathcal{F}_A
\]
by \( pr(g)(v) := g(a \bar{v}) (v \in \mathbb{Z}^n) \).

We claim that Ker(pr) is an \( \mathbb{H} \)-submodule of \( F_A^\mathcal{O} \). Clearly Ker(pr) is \( x_j \)-invariant for \( j = 1, \ldots, n \). Let \( g \in \text{Ker}(pr) \). Part 3 of Lemma 5 implies that \( \Delta g \in \text{Ker}(pr) \). To show that \( H_j g \in \text{Ker}(pr) \) we consider two cases. If \( v_i \neq v_{i+1} \) then \( s_i \bar{v} = \bar{s}_i \bar{v} \) by part 1 of Lemma 5. Hence,
\[
(H_j g)(a \bar{v}) = tg(a \bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a \bar{v}_i) - g(a \bar{v})) = 0.
\]
If \( v_i = v_{i+1} \) then \( \bar{v}_i = t\bar{v}_{i+1} \) by part 2 of Lemma 5. Hence,
\[
(H_j g)(a \bar{v}) = tg(a \bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a \bar{v}_i) - g(a \bar{v})) = t g(a \bar{v}) = 0.
\]

Hence, \( \mathcal{F}_A \) inherits the \( \mathbb{H} \)-module structure of \( F_A^\mathcal{O} / \text{Ker}(pr) \). It is a straightforward computation, using Lemma 5 again, to show that the resulting action of \( H_i(1 \leq i < n) \), \( \Delta \) and \( x_j(1 \leq j \leq n) \) on \( \mathcal{F}_A \) is by the operators \( \tilde{H}_i(1 \leq i < n) \), \( \tilde{\Delta} \) and \( \tilde{x}_j(1 \leq j \leq n) \).

**Remark 7.** With the notations from (the proof of) Corollary 6, let \( \tilde{g} \in F_A^\mathcal{O} \) and set \( g := pr(\tilde{g}) \in \mathcal{F}_A \). In other words, \( g(v) := \tilde{g}(a \bar{v}) \) for all \( \bar{v} \in \mathbb{Z}^n \). Then
\[
(\tilde{X} g)(v) = (X \tilde{g})(a \bar{v}), \quad v \in \mathbb{Z}^n
\]
for \( X = H_i, \Delta^{\pm 1}, x_j \).

**Remark 8.** Let \( \mathcal{F}_A^+ \) be the space of \( A \)-valued functions on \( \mathcal{C}_n \). We sometimes will consider \( \tilde{H}_i(1 \leq i < n) \), \( \tilde{\Delta}^{-1} \) and \( \tilde{x}_j(1 \leq j \leq n) \), defined by the formulas (11), as linear operators on \( \mathcal{F}_A^+ \).

**Definition 9.** We call
\[
K_\alpha(x; q, t; \bar{a}) := \frac{G_\alpha(x; q, t)}{G_\alpha(\alpha \tau; q, t)} \in \mathbb{K}[x]
\]
(12)
the normalized non-symmetric interpolation Macdonald polynomial of degree \( \alpha \).
We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q; t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(0) = 0$ if $\alpha \in C_n$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}$$

since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $/Phi_1 = (x_n - t^{1-n})/Delta_1 \in H$ and the inhomogeneous Cherednik operators

$$/Xi_j = 1 + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$ 

The operators $H_i$, $/Xi_j$ and $/Phi_1$ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in C_n$). Note that the operators $H_i$, $/Xi_j$ and $/Phi_1$ on $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$ commute with the hat-operators $\widehat{H_i}$, $\widehat{x_j}$ and $\widehat{\Delta}^{-1}$ on $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$ of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$).

Let $K \in \mathcal{F}_{+}^{+}(\mathbb{K}[x])$ be the map $\alpha \mapsto K_\alpha(\cdot) (\alpha \in C_n)$.

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $\mathcal{F}_{+}^{+}(\mathbb{K}[x])$,

1. $H_i K = \widehat{H_i} K$.
2. $/Xi_j K = a/\hat{x}_j^{-1} K$.
3. $/Phi_1 K = t^{1-n}(a^2/\hat{x}_1^{-1} - 1)/\widehat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$’s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$’s.

If $\alpha \in C_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t-1)\overline{a}_i}{\overline{a}_i - \overline{a}_{i+1}} G_\alpha(x) + G_{s_\alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t-1)\overline{a}_i}{\overline{a}_i - \overline{a}_{i+1}} G_\alpha(x) + \frac{t(\overline{a}_{i+1} - t\overline{a}_i)(\overline{a}_{i+1} - t^{-1}\overline{a}_i)}{(\overline{a}_{i+1} - \overline{a}_i)^2} G_{s_\alpha}(x)$$
if $\alpha \in C_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = t G_\alpha(x)$ if $\alpha \in C_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the $K_\beta$’s can now be obtained using the formula

$$G_\alpha(at) = \frac{\bar{\alpha}_{i+1} - t\bar{\alpha}_i}{\bar{\alpha}_{i+1} - \bar{\alpha}_i} G_{s_{i+1}}(at)$$

for $\alpha \in C_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \hat{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in C_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-|\alpha|} G_{\bar{\alpha}}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\bar{\alpha}}(at)}{G_\alpha(at)} = at^{1-n+k_1(a)} - q^{|\alpha|} t^{1-n}.$$ 

Hence,

$$\Phi K_\alpha(x) = t^{1-n}(a\bar{\alpha}^{-1} - 1) K_{\bar{\alpha}}(x).$$

Remark 11. Note that

$$\Phi K_\alpha(x) = (a\bar{\alpha}^{-1} - t^{1-n}) K_{\bar{\alpha}}(x)$$

for $\alpha \in C_n$ since $\bar{\alpha}^{-1} = t^{n-1} w_0 \bar{\alpha}$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}^+_{[\bar{k}]}$ to a map $K \in \mathcal{F}_{[\bar{k}]}(x)$ such that Lemma 10 remains valid.

Lemma 12. For $\alpha \in C_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left( \prod_{i=1}^n \frac{1-a\bar{\alpha}_i^{-1}}{1-qt^{n-1}x_i} \right) K_{\alpha+(1^n)}(qx).$$
Proof. Note that for $f \in \mathbb{K}[x]$, 

$$\Phi^n f(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m}a v^{-1}; q)_m}{(q t^{n-1} x_i; q)_m},$$

with $(y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y)$ the $q$-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v| - m^2 n} \sum \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m} t^{1-n} x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),$$

where $m$ is a nonnegative integer such that $v + (m^n) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \frac{(x/a)^m (x^{-1}; q)_m}{(a^{-1}; q)_m}.$$

Lemma 15. For all $v \in \mathbb{Z}^n$,

$$K_v(x) = \frac{G_v(x)}{G_v(\alpha \tau)}.$$

Proof. Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(\alpha \tau) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Then

$$K_v(\alpha \tau) = A_m(\alpha \tau; v) K_{v+(m^n)}(q^m \alpha \tau) = A_m(\alpha \tau; v) \frac{G_{v+(m^n)}(q^m \alpha \tau)}{G_{v+(m^n)}(\alpha \tau)} = 1,$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map \( K : C_n \to \mathbb{K}[x] \) to a map

\[ K : \mathbb{Z}^n \to \mathbb{K}(x) \]

by setting \( \nu \mapsto K_\nu(x) \) for all \( \nu \in \mathbb{Z}^n \). Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in \( \mathcal{F}_{\mathbb{K}(x)} \),

1. \( H_i K = \hat{H}_i K \).
2. \( \Xi_j K = a \hat{x}_j^{-1} K \).
3. \( \Phi K = t^{1-n}(a^2 \hat{x}_1^{-1} - 1) \hat{\Delta}^{-1} K \).

**Proof.** Write \( A_m \in \mathcal{F}_{\mathbb{K}(x)} \) for the map \( \nu \mapsto A_m(x; \nu) \) for \( \nu \in \mathbb{Z}^n \). Consider the linear operator on \( \mathcal{F}_{\mathbb{K}(x)} \) defined by \( (A_m f)(\nu) = A_m(x; \nu) f(\nu) \) for \( \nu \in \mathbb{Z}^n \) and \( f \in \mathcal{F}_{\mathbb{K}(x)} \). For \( 1 \leq i < n \) we have \([H_i, A_m] = 0\) as linear operators on \( \mathcal{F}_{\mathbb{K}(x)} \), since \( A_m(x; \nu) \) is a symmetric rational function in \( x_1, \ldots, x_n \). Furthermore, for \( \nu \in \mathbb{Z}^n \) and \( f \in \mathcal{F}_{\mathbb{K}(x)} \),

\[
(\hat{H}_i \circ A_m f)(\nu) = ((A_m \circ \hat{H}_i) f)(\nu) \quad \text{if} \quad \nu_i \neq \nu_{i+1} \tag{15}
\]

by part 2 of Lemma 5 and the fact that \( A_m(x; \nu) \) is symmetric in \( \nu_1, \ldots, \nu_n \). Fix \( \nu \in \mathbb{Z}^n \) and choose \( m \in \mathbb{Z}_{\geq 0} \) such that \( \nu + (m^n) \in C_n \). Since

\[ K_\nu(x) = A_m(x; \nu) K_{\nu + (m^n)}(q^m x) \]

we obtain from \([H_i, A_m] = 0\) and (15) that \((H_i K)(\nu) = (\hat{H}_i K)(\nu)\) if \( \nu_i \neq \nu_{i+1} \). This also holds true if \( \nu_i = \nu_{i+1} \) since then \((\hat{H}_i K)(\nu) = tK_\nu \) and \( H_i K_{\nu + (m^n)}(q^m x) = tK_{\nu + (m^n)}(q^m x) \). This proves part 1 of the proposition.

Note that \( \Phi K_\nu(x) = t^{1-n}(a \nu_1^{-1} - 1) K_{\nu+c}(x) \) for arbitrary \( \nu \in \mathbb{Z}^n \) by Lemma 10 and the commutation relation

\[
\Phi \circ A_m = A_m \circ \Phi(q^m), \tag{16}
\]

where \( \Phi(q^m) := (q^m x_n - t^{1-n}) \Delta \). This proves part 3 of the proposition.

Finally we have \( \Xi_j K_\nu(x) = \nu_j^{-1} K_\nu(x) \) for all \( \nu \in \mathbb{Z}^n \) by \([H_i, A_m] = 0\), (16) and Lemma 10. This proves part 2 of the proposition. ■
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = -w_0v$ for $v \in \mathbb{Z}^n$.

**Theorem 17.** (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}). \quad (17)$$

**Example 18.** If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r} \quad (18)$$

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of $m$ and $r$.

**Proof.** We divide the proof of the theorem in several steps.

**Step 1.** If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{_{si}}u(a\tilde{v}) = K_v(a\tilde{s}_i\tilde{u})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

$$\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{v}) + \frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}} K_u(a\tilde{v}_{_{si}})$$

$$= \frac{(t-1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{i+1})} K_u(a\tilde{v}) + \frac{\tilde{u}_i - t\tilde{u}_{i+1}}{\tilde{u}_i - \tilde{u}_{i+1}} K_{_{si}}u(a\tilde{v}). \quad (19)$$

Replacing in (19) the role of $u$ and $v$ and replacing $i$ by $n - i$ we get

$$\frac{(t-1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a\tilde{u}) + \frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} K_v(a\tilde{s}_i\tilde{u})$$

$$= \frac{(t-1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a\tilde{v}) + \frac{\tilde{v}_{n-i} - t\tilde{v}_{n+1-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} K_{s_{n-i}}v(a\tilde{u}). \quad (20)$$

Suppose that $s_{n-i}v = v$. Then $\tilde{v}_{n-i} = t\tilde{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n}w_0\tilde{v}^{-1}$, that is, $\tilde{v}_i = t^{1-n}\tilde{v}_{n+1-i}$, we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{_{si}}u(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_v(a\tilde{s}_i\tilde{u}) = K_v(a\tilde{u})$ if $s_{n-i}v = v$. 


We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v$. We have to show that $K_{s_iu}(a\tilde{v}) = K_v(as_i\tilde{u})$ for all $v$. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that $v$ satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(as_i\tilde{u}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i}v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i}v}(as_i\tilde{u})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(as_i\tilde{u})$, which turns out to reduce to $K_{s_iu}(a\tilde{v}) = K_v(as_i\tilde{u})$ after a direct computation.

**Step 2.** $K_0(a\tilde{v}) = 1 = K_v(a\tilde{0})$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(\alpha\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a\tilde{v}) = K_v(a\tilde{\gamma})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^{\sharp} \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\sharp$. Furthermore, note that we have the formula

$$(a\tilde{v}^{-1}_1 - 1)K_u(a\tilde{v}) = (a\tilde{u}^{-1}_1 - 1)K_{u\gamma}(a\tilde{v})$$

(21)

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{v}) = \frac{(a\tilde{v}^{-1}_1 - 1)}{(a\tilde{\gamma}^{-1}_1 - 1)}K_\gamma(a\tilde{\gamma})$$

$$= \frac{(a\tilde{v}^{-1}_1 - 1)}{(a\tilde{\gamma}^{-1}_1 - 1)}K_{\gamma^\sharp}(a\tilde{\gamma}) = K_v(a\tilde{\gamma}) = K_v(a\tilde{\alpha}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $u, v \in \mathbb{Z}^n$.

Proof of Step 4. Fix $u, v \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $u + (m^n) \in C_n$. Note that $q^m\tilde{v} = v - (m^n)$ and $q^{-m}\tilde{u} = u + (m^n)$. Then

$$
K_u(a\tilde{v}) = A_m(a\tilde{v}; u) K_{u+(m^n)}(q^m a\tilde{v})
= A_m(a\tilde{v}; u) K_{u+(m^n)}(a(v - (m^n)))
= A_m(a\tilde{v}; u) K_{v-(m^n)}(a(u + (m^n)))
= A_m(a\tilde{v}; u) K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u) A_m(q^{-m} a\tilde{u}; v - (m^n)) K_v(a\tilde{u}),
$$

where we used Step 3 in the 3rd equality. The result now follows from the fact that

$$
A_m(a\tilde{v}; u) A_m(q^{-m} a\tilde{u}; v - (m^n)) = 1,
$$

which follows by a straightforward computation using (4). □

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial $E_\alpha(x)$ of degree $\alpha$ is the top homogeneous component of $G_\alpha(x)$, i.e.,

$$
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in C_n.
$$

The normalized non-symmetric Macdonald polynomials are

$$
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
$$

We write $\overline{K} \in \mathcal{F}_{\text{Fix}}^+$ for the resulting map $\alpha \mapsto \overline{K}_\alpha$. Taking limits in Lemma 10 we get the following.

Lemma 19. We have for $1 \leq i < n$ and $1 \leq j \leq n$,

1. $H_i \overline{K} = \overline{H}_i \overline{K}$.
2. $\xi_j \overline{K} = \overline{\xi}_j^{-1} \overline{K}$.
3. $x_n \Delta \overline{K} = t^{1-n} \overline{\Delta}^{-1} \overline{K}$. 

Note that
\[(x_n \Delta)^n f(x) = \left(\prod_{i=1}^{n} x_i\right)f(q^{-1}x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n,\)
\[E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},\]
\[K_\alpha(x) = q^{\alpha|t(1-n)n} \left(\prod_{i=1}^{n} (\overline{x_i} - 1)x_i\right)^{-1} K_{\alpha+(1^n)}(x).\] (22)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n.\)

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n.\) Then \(K_\nu(x) := K_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) is defined by
\[K_\nu(x) := q^{m|\nu|t(1-n)n} \left(\prod_{i=1}^{n} (\overline{x_i} - 1)x_i\right)^{-m} K_{\nu+(m^n)}(x).\]

Using
\[\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2n} t^{1-n} \prod_{i=1}^{n} (\overline{x_i} - 1)x_i^{-m}\]
and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that
\[\lim_{a \to \infty} a^{-|\nu|} G_\nu(ax) = E_\nu(x),\]
\[\lim_{a \to \infty} K_\nu(ax) = K_\nu(x)\]
for all \(\nu \in \mathbb{Z}^n,\) so in particular
\[K_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}[x^{\pm 1}]$ defined by $v \mapsto K_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\bar{v}) = K_v(\bar{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in \mathbb{C}^n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\beta^{-1}) = K_\alpha(t^{1-n}aw_0\beta^{-1}) = K_\alpha(a\beta) = K_\beta(a\tilde{\alpha})$$

for all $\beta \in \mathbb{C}^n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. ■

### 7.3 Okounkov’s duality

Write $\mathbb{F}[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $\mathbb{F}$. Write $C_\perp := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in \mathbb{F}[x]^{S_n}$ is the multiple of $C_\perp G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_\perp^+(x) := \frac{R_\lambda(x)}{R_\lambda(at)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_\perp K_\alpha(x) = \left( \sum_{w \in S_n} t^{(w)} \right) K_\perp^+(x)$$

(23)

for $\alpha \in \mathbb{C}^n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_K$ for the function $f_\mu(u) := K_u(a\bar{\mu}) (u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\bar{\mu}) = (C_+ K_\lambda)(a\bar{\mu}) = (\hat{C}_+ f_\mu)(\lambda) \quad (24)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\bar{\mu}) = (Jw_0 K_\mu(t^{1-n} x))|_{x=a^{-1} \bar{\mu}} \quad (25)$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in K(x)$. A direct computation shows that

$$JH_i J = (H_i^\circ)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^\circ)^{-1} \quad (26)$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\hat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n} x))|_{x=a^{-1} \bar{\mu}}.$$

By (23) and (4) this simplifies to

$$(\hat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\bar{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\bar{\mu}) = K_\mu^+(a\bar{\lambda})$. Since $K_\lambda^+$ is symmetric we obtain from (4) that

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0} K_u)(a \bar{v}) = (H_{w_0} K_v)(a \bar{u}).$$  \hfill (27)

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a \bar{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0} K_u)(a \bar{v}) = (\hat{H}_{w_0} f_v)(u).$$

Since $f_v(u) = (\hat{H}_{w_0} f_v)(u)$ by (4), Remark 7 implies that

$$(\hat{H}_{w_0} f_v)(u) = (H_{w_0} J_{w_0} K_v)(a^{-1} t^{n-1} \bar{u}).$$

Now $H_{w_0} J_{w_0} = J_{w_0} H_{w_0}$ by (26); hence,

$$(\hat{H}_{w_0} f_v)(u) = (J_{w_0} H_{w_0} K_v)(a^{-1} t^{n-1} \bar{u}) = (H_{w_0} K_v)(a \bar{u}),$$

which completes the proof. \hfill \blacksquare

Recall from Theorem 1 that

$$G'(\beta)(x) = t^{(1-n)|\beta| + I(\beta)} \Psi G_\beta(t^n x)$$

with $\Psi := w_0 H_{w_0}$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1} \tau)} = t^{\ell(w_0)} \Psi K_\beta(t^{n-1} x), \quad \beta \in \mathbb{C}_n,$$

with $K_\nu := \iota(K_\nu)$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K'_\nu(x) := t^{\ell(w_0)} \Psi K_\nu(t^{n-1} x).$$  \hfill (28)

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n)$. Since $H_i \Psi = \Psi H_i^\circ$, part 1 of Proposition 16 gives $H_i K' = \hat{H}_i^\circ K'$. Considering the action of $((x_n - 1) \Delta^\circ)^n$ on $K'_\beta(x)$ we get, using the fact that $((x_n - 1) \Delta^\circ)^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K'_\nu(x) = \left( \prod_{i=1}^n \frac{(1 - a^{-1} \bar{v}_i)}{(1 - q^{-1} x_i)} \right) K'_{\nu + (1^n)}(q^{-1} x),$$
in particular

\[ K'_\nu(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}\nu_i; q)_m}{(q^{-m}x_i; q)_m} \right)K'_{\nu+(m\nu)}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_\nu(x) = K^\circ_\nu(x) \) for \( \nu \in \mathbb{Z} \); hence,

\[
K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m},
K'_{m}(x) = (ax)^{m} \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}
\]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_\nu(a^{-1}u) = K'_u(a^{-1}v). \]

**Proof.** Note that

\[
K'_\nu(a^{-1}u) = t^{\ell(w_0)} \Psi_\nu K^\circ_\nu(t^{m-1}x)|_{x=a^{-1}u} = t^{\ell(w_0)} (H_{w_0}^\circ K^\circ_\nu)(a^{-1}u^{-1})
\]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q, t} := \frac{G^\circ_\beta(\alpha)}{G^\circ_\beta(\beta)}. \tag{29}
\]

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} = \frac{G^\circ_\beta(\alpha^{-1})}{G^\circ_\beta(\beta^{-1})}.
\]
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\textbf{Theorem 27.} For $\alpha, \beta \in \mathbb{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} a^{\beta|\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}.$$  \hfill (30)

\textbf{Remark 28.} 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} \tau_\beta^{-1} a^{\beta|\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} G'_\beta(x)$$

$$= \sum_{\beta \in \mathbb{C}_n} \frac{K_\alpha(x)}{\tau_\beta K_\beta(x)}$$

$$= t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K_\alpha(x)}{\tau_\beta K_\beta(x)} \left( n^{-1} x \right)$$

with $\Psi = w_0 H_{w_0}$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K_\alpha(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_w \Psi = w$ the binomial formula (31) implies the finite expansion

$$(H_{w_0} K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K_\alpha(x)}{\tau_\beta K_\beta(x)} \left( n^{-1} w_0 x \right).$$

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0} K_\alpha)(a \tilde{\gamma}) = \sum_{\beta \in \mathbb{C}_n} \frac{K_\alpha(x)}{\tau_\beta K_\beta(x)} \left( n^{-1} \tilde{\gamma} \right).$$

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula \[8, \text{Thm. 4.4}\] in our notations reads as follows.

**Theorem 29.** For all \(\alpha \in C_n\) we have

\[
K'_{\alpha}(x) = \sum_{\beta \in C_n} \tau_{\beta} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_{\beta}(ax). \tag{32}
\]

The starting point of the alternative proof of (32) is the binomial formula in the form

\[
K_{\alpha}(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_{\beta}^{\omega}(\alpha^{-1})\Psi K_{\beta}(t^{n-1}x)}{\tau_{\beta} G_{\beta}^{\omega}(\beta^{-1})},
\]

see (31). Replace \((a, x, q, t)\) by \((a^{-1}, at^{n-1}x, q^{-1}, t^{-1})\) and act by \(w_0Hw_0\) on both sides. Since \(w_0Hw_0 \Psi = Id\) we obtain

\[
\Psi K_{\alpha}^{\omega}(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_{\beta} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_{\beta}(ax).
\]

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

\[
\Psi K_{\alpha}^{\omega}(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_{\beta} K_{\beta}(\alpha) K_{\beta}(ax) K_{\beta}(\beta). \tag{33}
\]

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

\[
\sum_{\beta \in C_n} \frac{\tau_{\beta}}{\tau_{\alpha}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} \begin{bmatrix} \beta \\ \gamma \end{bmatrix}_{q^{-1}, t^{-1}} = \delta_{\alpha, \gamma}.
\]

Since \(\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_{q,t} = 0\) unless \(\delta \supseteq \epsilon\), the terms in the sum are zero unless \(\gamma \subseteq \beta \subseteq \alpha\).

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