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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials \( R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t) \) form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in \( n \) variables over the field \( \mathbb{F} := \mathbb{Q}(q, t) \). They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most \( n \) parts

\[
P_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.
\]

For a partition \( \mu \in P_n \) we define \( |\mu| = \mu_1 + \cdots + \mu_n \) and write

\[
\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \quad \text{where} \quad \tau := (\tau_1, \ldots, \tau_n) \quad \text{with} \quad \tau_i := t^{1-i}.
\]
Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$  

The normalization is fixed by requiring that the coefficient of $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ [9] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.$$

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := \{\mathbb{Z}_{\geq 0}\}^n$. For a composition $\beta \in C_n$ we define

$$\overline{\beta} := w_\beta(\beta_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\overline{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$  

The normalization is fixed by requiring that the coefficient of $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (w_0 \beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$ 

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \# \{i < j \mid \alpha_i \geq \alpha_j \}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha|-I(\alpha)} w_0 H_{w_0} G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov's duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\bar{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\bar{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$ 

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

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Theorem C. For all compositions \( \alpha \in \mathcal{C}_n \) we have

\[
O_\alpha(x) = \frac{G_\alpha(t^{1-n}a\omega_0 x)}{G_\alpha(a\tau)}.
\]

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials \( G'_\alpha(x) \) in terms of the \( G_\beta(ax) \)'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let \( S_n \) be the symmetric group in \( n \) letters and \( s_i \in S_n \) the permutation that swaps \( i \) and \( i+1 \). The \( s_i \) (\( 1 \leq i < n \)) are Coxeter generators for \( S_n \). Let \( \ell: S_n \to \mathbb{Z}_{\geq 0} \) be the associated length function. Let \( S_n \) act on \( \mathbb{Z}^n \) and \( \mathbb{K}^n \) by \( s_i \nu := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots) \) for \( \nu = (v_1, \ldots, v_n) \). Write \( w_0 \in S_n \) for the longest element, given explicitly by \( i \to n+1-i \) for \( i = 1, \ldots, n \).

For \( \nu = (v_1, \ldots, v_n) \in \mathbb{Z}^n \) define \( \overline{\nu} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n \) by \( \overline{v}_i := q^{v_i}t^{-k_i(\nu)} \) with

\[
k_i(\nu) := \# \{ k < i \mid v_k \geq v_i \} + \# \{ k > i \mid v_k > v_i \}.
\]

If \( \nu \in \mathbb{Z}^n \) has non-increasing entries \( v_1 \geq v_2 \geq \cdots \geq v_n \), then \( \overline{\nu} = (q^{v_1} \tau_1, \ldots, q^{v_n} \tau_n) \). For arbitrary \( \nu \in \mathbb{Z}^n \) we have \( \overline{\nu} = w_\nu(\overline{\nu}^-) \) with \( w_\nu \in S_n \) the shortest permutation such that \( \nu_- := w_\nu^{-1}(\nu) \) has non-increasing entries, see [4, Section 2]. We write \( \overline{\nu} := -w_0 \nu \) for \( \nu \in \mathbb{Z}^n \).

Note that \( \overline{\alpha_n} = t^{1-n} \) if \( \alpha \in \mathcal{C}_n \) with \( \alpha_n = 0 \).

For a field \( F \) we write \( F[x] := F[x_1, \ldots, x_n] \), \( F[x_{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) and \( F(x) \) for the quotient field of \( F[x] \). The symmetric group acts by algebra automorphisms on \( F[x] \) and \( F(x) \), with the action of \( s_i \) by interchanging \( x_i \) and \( x_{i+1} \) for \( 1 \leq i < n \). Consider the \( F \)-linear operators

\[
H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)
\]
on $\mathbb{F}(x)$ ($1 \leq i < n$) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ ($1 \leq i < n$) and $\Delta$ preserve $\mathbb{F}[x^\pm 1]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ ($1 \leq i < n$) and $\Delta$ satisfy the defining relations of the type A extended affine Hecke algebra,

$$(H_i - t)(H_i + 1) = 0,$$

$$H_i H_j = H_j H_i, \quad |i - j| > 1,$$

$$H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},$$

$$\Delta H_{i+1} = H_i \Delta,$$

$$\Delta^2 H_1 = H_{n-1} \Delta^2$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$’s. Write $H_i := H_i + 1 - t = t H_i^{-1}$ and set

$$\xi_i := t^{1-n} H_{i-1} \cdots H_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators $\xi_i$’s are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = H_1 \cdots H_{n-1} \Delta H_1 \cdots H_{i-1}.$$

The $\xi_i^{-1}$ ($1 \leq i \leq n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$\xi_i^{-1} E_\alpha = \overline{\alpha} E_\alpha, \quad i = 1, \ldots, n$$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$
and \( F[x] \) by letting \( \iota \) act on the coefficients of the polynomial. Write

\[
G_\alpha^0 := \iota(G_\alpha), \quad E_\alpha^0 := \iota(E_\alpha)
\]

for \( \alpha \in C_n \). Note that \( \overline{\nu}^{-1} = (\iota(\overline{\nu}_1), \ldots, \iota(\overline{\nu}_n)) \).

Put \( H_i^0, H_w^0, \overline{H}_i, \Delta^0 \) and \( \xi_i^0 \) for the operators \( H_i, H_w, \overline{H}_i, \Delta \) and \( \xi_i \) with \( q, t \) replaced by their inverses. For instance,

\[
H_i^0 = t^{-1} s_i - \frac{(1 - t^{-1}) x_i}{x_i - x_{i+1}} (1 - s_i),
\]

\[
\Delta^0 f(x_1, \ldots, x_n) = f(q x_n, x_1, \ldots, x_{n-1}).
\]

We then have \( \xi_i^0 E_\alpha^0 = \overline{\alpha}_i E_\alpha^0 \) for \( i = 1, \ldots, n \), which characterizes \( E_\alpha^0 \) up to a scalar factor.

**Theorem 1.** For \( \alpha \in C_n \) we have

\[
G_\alpha'(x) = t^{(1-n)|\alpha|+I(\alpha)} w_0 H_w^0 G_\alpha(t^{-1} x)
\]

with \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \).

**Remark.** Formally set \( t = q^r \), replace \( x \) by \( 1 + (q - 1)x \), divide both sides of (2) by \( (q - 1)^{|\alpha|} \) and take the limit \( q \to 1 \). Then

\[
G_\alpha'(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n - 1)r; r)
\]

for the non-symmetric interpolation Jack polynomial \( G_\alpha(\cdot; r) \) and its primed version (see [14]). Here \( \sigma \) denotes the action of the symmetric group with \( \sigma(s_i) \) the rational degeneration of the Demazure-Lusztig operators \( H_i \), given explicitly by

\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}} (1 - s_i),
\]

see [14, Section 1]. To establish the formal limit (3) one uses that \( \sigma(w_0) w_0 = w_0 \sigma^0(w_0) \) with \( \sigma^0 \) the action of the symmetric group defined in terms of the rational degeneration

\[
\sigma^0(s_i) = s_i - \frac{r}{x_i - x_{i+1}} (1 - s_i)
\]

of \( H_i^0 \). Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G_\alpha'$. For the vanishing property, note that

$$t^{n-1} w_0 \tilde{\beta} = \tilde{\beta}^{-1}$$

(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,

$$(w_0 H_{w_0}^o G_\alpha^o (t^{n-1} x))|_{x=\beta} = (H_{w_0}^o G_\alpha^o (x))|_{x=\beta^{-1}}.$$

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_\alpha^o (w \beta^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_\alpha = t^{1(\alpha)} w_0 H_{w_0}^o E_\alpha^o.$$ (5)

Note that $\Psi := w_0 H_{w_0}^o$ satisfies the intertwining properties

$$H_i \Psi = t \Psi H_i^o,$$

$$\Delta \Psi = t^{n-1} \Psi H_{n-1}^o \cdots H_1^o (\Delta^o)^{-1} H_{n-1}^o \cdots H_1^o$$

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1} \Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$. Therefore,

$$E_\alpha (x) = c_\alpha \Psi E_\alpha^o (x)$$

for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_\alpha = t^{1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\overline{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1 \cdots x_n E_u (x).$$
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{I(u)} w_0 H_{w_0}^\circ E_u$$  \hspace{1cm} (7)

with $E_u^\circ := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x \pm 1]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{I(u)} H_{w_0}^\circ E_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

### 3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)} + tl(s)}{1 - q^{a(s)+1} t^{l(s)+1}} \right) \prod_{s \in \alpha} \left( at^{l(s)} - q^{a'(s)} \right)$$  \hspace{1cm} (8)

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \},$$

$$a'(s) := j - 1, \quad l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}.$$ 

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l(s)} - q^{a'(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}.$$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha) - \ell(w_0)} G^0_{\alpha}(a \tau^{-1}).$$

Proof. Since $t^{n-1}w_0 \tau = \tau^{-1} = \bar{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha)} (H^0_{w_0} G^0_{\alpha})(a \bar{\tau}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha) - \ell(w_0)} G^0_{\alpha}(a \bar{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality.

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^{n} (\alpha_i \choose 2)$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha).$$ (9)

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a\tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha| - n(\alpha)} q^{n'(\alpha)} G^0_{\alpha}(a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}.$$ (10)

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in \mathbb{C}_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau^{-1}_\alpha a^{-|\alpha|}G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). □

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\natural := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\natural := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $v^\natural$ by $v^\flat$, so

$$v^\flat = (v_n - 1, v_1, \ldots, v_1 - 1).$$

We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = s_i^\natural v$ if $v_i \neq v_{i+1}$.
2. $v_i = t_i^\natural v_{i+1}$ if $v_i = v_{i+1}$.
3. $v_i^\natural = v_i^\flat$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^\pm 1])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^\pm 1$, and the multiplication operators $x_j^\pm 1$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\left(\widehat{H}_if\right)(v) := tf(v) + \frac{v_i - t_{i+1}v_{i+1}}{v_i - v_{i+1}}(f(s_i v) - f(v)),$$

$$\left(\widehat{\Delta}f\right)(v) := f(v^\natural), \quad \left(\widehat{\Delta}^{-1}f\right)(v) := f(v^\flat),$$

$$(\widehat{x}_j f)(v) := a v_j f(v)$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A), X \mapsto \widehat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 
Proof. Let $O \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\pi$-invariant subset that contains $\{a \overrightarrow{v} \mid v \in \mathbb{Z}^n\}$. Note that $O$ is contained in $\{y \in \mathbb{K}^n \mid y_i \neq y_j$ if $i \neq j\}$. The Demazure–Lusztig operators $H_i$ $(1 \leq i < n)$, $\Delta^{\pm 1}$ and the coordinate multiplication operators $x_j$ $(1 \leq j \leq n)$ act $A$-linearly on the space $F_A^O$ of $A$-valued functions on $O$, and hence turns $F_A^O$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map

$$\text{pr} : F_A^O \rightarrow F_A$$

by $\text{pr}(g)(v) := g(a \overrightarrow{v})$ $(v \in \mathbb{Z}^n)$.

We claim that Ker(pr) is an $\mathbb{H}$-submodule of $F_A^O$. Clearly Ker(pr) is $x_j$-invariant for $j = 1, \dots, n$. Let $g \in \text{Ker}(pr)$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(pr)$. To show that $H_i g \in \text{Ker}(pr)$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \overrightarrow{v} = s_i \overrightarrow{v}$ by part 1 of Lemma 5. Hence,

$$(H_i g)(a \overrightarrow{v}) = tg(a \overrightarrow{v}) + \frac{\overrightarrow{v}_i - t \overrightarrow{v}_{i+1}}{\overrightarrow{v}_i - \overrightarrow{v}_{i+1}} (g(as_i \overrightarrow{v}) - g(a \overrightarrow{v})) = 0.$$

If $v_i = v_{i+1}$ then $\overrightarrow{v}_i = t \overrightarrow{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_i g)(\overrightarrow{v}) = tg(a \overrightarrow{v}) + \frac{\overrightarrow{v}_i - t \overrightarrow{v}_{i+1}}{\overrightarrow{v}_i - \overrightarrow{v}_{i+1}} (g(as_i \overrightarrow{v}) - g(a \overrightarrow{v})) = tg(a \overrightarrow{v}) = 0.$$

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F_A^O/\text{Ker}(pr)$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i$ $(1 \leq i < n)$, $\Delta$ and $x_j$ $(1 \leq j \leq n)$ on $F_A$ is by the operators $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$. 

Remark 7. With the notations from (the proof of) Corollary 6, let $\tilde{g} \in F_A^O$ and set $g := \text{pr}(\tilde{g}) \in F_A$. In other words, $g(v) := \tilde{g}(a \overrightarrow{v})$ for all $v \in \mathbb{Z}^n$. Then

$$(\widehat{X} g)(v) = (X \tilde{g})(a \overrightarrow{v}), \quad v \in \mathbb{Z}^n$$

for $X = H_i, \Delta^{\pm 1}, x_j$.

Remark 8. Let $F_A^+$ be the space of $A$-valued functions on $\mathcal{C}_n$. We sometimes will consider $\widehat{H}_i$ $(1 \leq i < n)$, $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ $(1 \leq j \leq n)$, defined by the formulas (11), as linear operators on $F_A^+$.

Definition 9. We call

$$K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(\alpha \tau; q, t)} \in \mathbb{K}[x]$$

(12)

the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 


We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(0) = 0$ if $\alpha \in \mathbb{C}$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}$$

since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi_1 = (x_n - t^{1-n})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$ 

The operators $H_i$, $\Xi_j$ and $\Phi$ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $\mathcal{F}_+^{\mathbb{K}[x]}$ (e.g., $(H_\alpha f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in \mathbb{C}$). Note that the operators $H_i$, $\Xi_j$ and $\Phi$ on $\mathcal{F}_+^{\mathbb{K}[x]}$ commute with the hat-operators $\widehat{H}_i$, $\widehat{\xi}_j$ and $\widehat{\Delta}^{-1}$ on $\mathcal{F}_+^{\mathbb{K}[x]}$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_+^{\mathbb{K}[x]}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $\mathcal{F}_+^{\mathbb{K}[x]}$).

Let $K \in \mathcal{F}_+^{\mathbb{K}[x]}$ be the map $\alpha \mapsto K_\alpha(\cdot) (\alpha \in \mathbb{C})$.

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $\mathcal{F}_+^{\mathbb{K}[x]}$:

1. $H_i K = \widehat{H}_i K$.
2. $\Xi_j K = a \widehat{\xi}_j K$.
3. $\Phi K = t^{1-n} (a^2 \widehat{\xi}_1^{-1} - 1) \widehat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$'s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$'s.

If $\alpha \in \mathbb{C}$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t - 1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t - 1) \overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - t \overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1} \overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{s_i \alpha}(x)$$
if $\alpha \in C_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = t G_\alpha(x)$ if $\alpha \in C_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the $K_\beta$’s can now be obtained using the formula

$$G_\alpha(at) = \frac{\alpha_i - t \alpha_i}{\alpha_{i+1} - \alpha_i} G_{\alpha_{i+1}}(at)$$

for $\alpha \in C_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \tilde{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in C_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-a_1} G_{\alpha^{-}}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^{-}}(at)}{G_\alpha(at)} = at^{1-n+k_1(\alpha)} - q^{a_1} t^{1-n}.$$ 

Hence,

$$\Phi K_\alpha(x) = t^{1-n}(a_\alpha^{-1} - 1) K_{\alpha^{-}}(x).$$

**Remark 11.** Note that

$$\Phi K_\alpha(x) = (a_\alpha^{-1} - t^{1-n}) K_{\alpha^{-}}(x)$$

for $\alpha \in C_n$ since $a_\alpha^{-1} = t^{n-1} w_0 a_\alpha$.

## 5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}_{\mathbb{Z}[x]}^+$ to a map $K \in \mathcal{F}_{\mathbb{Z}[x]}$ such that Lemma 10 remains valid.

**Lemma 12.** For $\alpha \in C_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a_\alpha^{-1}}{1 - qt^{n-1} x_i} \right) K_{\alpha+(1^n)}(qx).$$
Proof. Note that for $f \in \mathbb{K}[x]$,

$$\Phi^n f(x) = \left(\prod_{i=1}^{n}(x_i - t^{1-n})\right)f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \left(\frac{(q^{1-m}av_i^{-1}; q)_m}{(qt^{n-1}x_i; q)_m}\right), \quad \forall v \in \mathbb{Z}^n,$$  \tag{14}

with $(y; q)_m := \prod_{j=0}^{m-1}(1 - q^jy)$ the $q$-shifted factorial.

**Definition 13.** Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v|-m^2n}\frac{G_v+mn(q^nx)}{\prod_{i=1}^{n}x_i^m(q^{-m}t^{1-n}x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v)K_{v+mn}(q^nx),$$

where $m$ is a nonnegative integer such that $v + (mn) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

**Example 14.** If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left(\frac{x}{a}\right)^{m}\frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.$$  

**Lemma 15.** For all $v \in \mathbb{Z}^n$,

$$K_v(x) = \frac{G_v(x)}{G_v(at)}.$$

**Proof.** Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(at) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (mn) \in C_n$. Then

$$K_v(at) = A_m(at; v)K_{v+mn}(q^na_t) = A_m(at; v)\frac{G_{v+mn}(q^na_t)}{G_{v+mn}(at)} = 1,$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\tilde{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^{2}\tilde{x}_1 - 1)\Delta^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$((\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $\overline{v}_1, \ldots, \overline{v}_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = tK_v$ and $H_i K_{v+(m^n)}(q^m x) = tK_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\overline{v}_1 - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

where $\Phi(q^m) := (q^m x_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi J K_v(x) = \overline{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition. 

\[ \square \]
6  Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{v} = -w_0 v \) for \( v \in \mathbb{Z}^n \).

**Theorem 17.** (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a\tilde{v}) = K_v(a\tilde{u}).
\]

(17)

**Example 18.** If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m(a^{-1}; q)_r}}
\]

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

**Proof.** We divide the proof of the theorem in several steps.

---

**Step 1.** If \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{siu}(a\tilde{v}) = K_{sv}(as_i\tilde{u}) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})}K_u(a\tilde{v}) + \frac{(\tilde{v}_i - t\tilde{v}_{i+1})}{(\tilde{v}_i - \tilde{v}_{i+1})}K_u(as_{n-i}v)
\]

\[
= \frac{(t-1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{i+1})}K_u(a\tilde{v}) + \frac{(\tilde{u}_i - t\tilde{u}_{i+1})}{(\tilde{u}_i - \tilde{u}_{i+1})}K_{siu}(a\tilde{v}).
\]

(19)

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t-1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})}K_v(a\tilde{u}) + \frac{(\tilde{u}_{n-i} - t\tilde{u}_{n+1-i})}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})}K_v(as_i\tilde{u})
\]

\[
= \frac{(t-1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})}K_v(a\tilde{u}) + \frac{(\tilde{v}_{n-i} - t\tilde{v}_{n+1-i})}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})}K_{s_{n-i}v}(a\tilde{u}).
\]

(20)

Suppose that \( s_{n-i}v = v \). Then \( \tilde{v}_{n-i} = t\tilde{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n}w_0v^{-1} \), that is, \( \tilde{v}_i = t^{1-n}\tilde{v}_{n+1-i} \), we then also have \( \tilde{v}_i = t\tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{siu}(a\tilde{v}) = K_u(a\tilde{v}) \) and (20) to \( K_v(as_i\tilde{u}) = K_v(a\tilde{u}) \) if \( s_{n-i}v = v \).
We now use these observations to prove Step 1. Assume that $K_u(a\bar{v}) = K_v(a\bar{u})$ for all $v$. We have to show that $K_{s_iu}(a\bar{v}) = K_v(a\bar{s_iu})$ for all $v$. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that $v$ satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\bar{v}) = K_u(a\bar{v}) = K_v(a\bar{u}) = K_v(a\bar{s_iu}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\bar{v})$ as an explicit linear combination of $K_v(a\bar{u})$ and $K_{s_{n-i}v}(a\bar{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\bar{u})$ as an explicit linear combination of $K_v(a\bar{u})$ and $K_v(a\bar{s_iu})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\bar{v})$ as linear combination of $K_v(a\bar{u})$ and $K_v(a\bar{s_iu})$, which turns out to reduce to $K_{s_iu}(a\bar{v}) = K_v(a\bar{s_iu})$ after a direct computation.

**Step 2.** $K_0(a\bar{v}) = 1 = K_v(a\bar{u})$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\bar{u}) = K_v(a\bar{r}) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_\alpha(a\bar{v}) = K_v(a\bar{u})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma (a\bar{v}) = K_\gamma (a\bar{y})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\bar{v}) = K_v(a\bar{u})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\natural \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\natural$. Furthermore, note that we have the formula

$$(a\bar{v}_{-1} - 1)K_u(a\bar{v}) = (a\bar{u}_{-1} - 1)K_u(a\bar{v})$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\bar{v}) = K_\gamma(a\bar{v}) = \frac{(a\bar{v}_{-1} - 1)}{(a\gamma_{-1} - 1)}K_{\gamma}(a\bar{v}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + m^n \in C_n \). Note that \( q^m \tilde{v} = \tilde{v} - (m^n) \) and \( q^{-m} \tilde{u} = \tilde{u} + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v}) \\
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(\tilde{v} - (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; \tilde{v} - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; \tilde{v} - (m^n)) = 1,
\]

which follows by a straightforward computation using (4).

\[
\square
\]

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|}G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F} \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i \overline{K} = \overline{H}_i \overline{K} \).
2. \( \xi_j \overline{K} = \overline{\xi}_j^{-1} \overline{K} \).
3. \( x_n \Delta \overline{K} = t^{1-n} \hat{\Delta}^{-1} \overline{K} \).
Note that
\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[E_\alpha(x) = \frac{E_{\alpha+1^n}(x)}{x_1 \cdots x_n},\]
\[K_\alpha(x) = q^{\left| \alpha \right|} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{v_i} x_i)^{-1} \right) K_{\alpha+1^n}(x).\]  \(22\)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.\]

The 2nd formula of \(22\) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n\).

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n\). Then \(K_\nu(x) := \overline{K}_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) is defined by
\[K_\nu(x) := q^{\left| \nu \right|} t^{(1-n)n} m \left( \prod_{i=1}^{n} (\overline{v_i} x_i)^{-m} \right) K_{\nu+1^n}(x).\]

Using
\[\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2 n} t^{(1-n)n} m \left( \prod_{i=1}^{n} (\overline{v_i} x_i)^{-m} \right)\]
and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that
\[\lim_{a \to \infty} \frac{a^{-\left| \nu \right|} G_\nu(ax)}{E_\nu(x)} = E_\nu(x),\]
\[\lim_{a \to \infty} \frac{a^{-\left| \nu \right|} K_\nu(ax)}{E_\nu(x)} = \overline{K}_\nu(x)\]
for all \(\nu \in \mathbb{Z}^n\), so in particular
\[\overline{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}[x^{\pm 1}]$ defined by $v \mapsto K_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\tilde{v}) = K_v(\tilde{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in \mathbb{C}_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\tilde{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\tilde{\beta}^{-1}) = K_\alpha(a\tilde{\beta}) = K_\beta(\tilde{a})$$

for all $\beta \in \mathbb{C}_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. ■

### 7.3 Okounkov’s duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]^{S_n}$ is the multiple of $C_+G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(\alpha\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_{\alpha+}(x)$$

for $\alpha \in \mathbb{C}_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in P_n$ we have

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_\mathbb{K}$ for the function $f_\mu(u) := K_u(a\tilde{\mu})$ ($u \in \mathbb{Z}^n$). Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\hat{C}_+ f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{u}) = (Jw_0 K_\mu(t^{1-n}x))|_{x = a^{-1}\overline{u}}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_i J = (H_i^o)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^o)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\hat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x = a^{-1}\overline{\lambda}}.$$  

By (23) and (4) this simplifies to

$$(\hat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\tilde{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\tilde{\mu}) = K_\mu^+(a\tilde{\lambda})$. Since $K_\lambda^+$ is symmetric we obtain from (4) that

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$\left( H_{w_0}K_u \right) (a \tilde{v}) = (H_{w_0}K_v)(a \tilde{u}). \tag{27}$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a \tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$\left( H_{w_0}K_u \right) (a \tilde{v}) = (\mathcal{H}_{w_0}f_v)(u).$$

Since $f_v(u) = (Iw_0K_v)(a^{-1} t^{n-1} \tilde{u})$ by (4), Remark 7 implies that

$$\left( \mathcal{H}_{w_0}f_v \right) (u) = (H_{w_0}Jw_0K_v)(a^{-1} t^{n-1} \tilde{u}).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$\left( \mathcal{H}_{w_0}f_v \right) (u) = (Jw_0H_{w_0}K_v)(a^{-1} t^{n-1} \tilde{u}) = (H_{w_0}K_v)(a \tilde{u}),$$

which completes the proof.

Recall from Theorem 1 that

$$G_{\beta}'(x) = t^{(1-n)|\beta| + I(\beta)} \Psi G_{\beta}^o(t^{n-1}x)$$

with $\Psi := w_0 H_{w_0}^o$. We define normalized versions by

$$K_{\beta}'(x) := \frac{G_{\beta}'(x)}{G_{\beta}(a^{-1} \tau)} = t^{\ell(w_0)} \Psi K_{\beta}^o(t^{n-1}x), \quad \beta \in \mathcal{C}_n,$$

with $K_{\nu}^o := \iota(K_{\nu})$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K_{\nu}'(x) := t^{\ell(w_0)} \Psi K_{\nu}^o(t^{n-1}x). \tag{28}$$

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K_{\nu}'(\nu \in \mathbb{Z}^n)$. Since $H_i \Psi = \Psi H_{i}^o$, part 1 of Proposition 16 gives $H_i K' = \mathcal{H}_i K'$. Considering the action of $((x_{n-1} \Delta^o)^n$ on $K_{\beta}'(x)$ we get, using the fact that $((x_{n-1} \Delta^o)^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K_{\nu}'(x) = \left( \prod_{i=1}^{n} \frac{(1 - a^{-1} \tilde{\nu}_i)}{(1 - q^{-1} x_i)} \right) K_{\nu+(1^n)}(q^{-1}x),$$
in particular

\[ K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1} \bar{v}_i; q)_m}{(q^{-m} x_i; q)_m} \right) K'_{v(m^n)}(q^{-m} x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K_v^0(x) \) for \( v \in \mathbb{Z} \); hence,

\[
K'_{-m}(x) = \frac{(q^{-1} a^{-1}; q^{-1})_m}{(q^{-1} x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m},
\]

\[
K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}
\]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_v(a^{-1} u) = K'_u(a^{-1} v). \]

**Proof.** Note that

\[ K'_v(a^{-1} u) = t^{\ell(w_0)} \Psi K_v^0(t^{n-1} x) \big|_{x = a^{-1} u} = t^{\ell(w_0)} (H_{w_0}^0) K_v^0(a^{-1} \bar{u}^{-1}) \]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} := \frac{G_\beta(\bar{\alpha})}{G_\beta(\bar{\beta})}. \tag{29}
\]

Applying the automorphism \( \iota \) of \( F \) to (29) we get

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} = \frac{G_\beta(\bar{\alpha}^{-1})}{G_\beta(\bar{\beta}^{-1})}.
\]
Theorem 27. For $\alpha, \beta \in C_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in C_n} a^{\beta} \binom{\alpha}{\beta}_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}.$$  \hfill (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in C_n} \tau_{\beta}^{-1} \binom{\alpha}{\beta}_{q^{-1}, t^{-1}} K'_\beta(x)$$

$$= \sum_{\beta \in C_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1})K'_\beta(x)}{\tau_{\beta}K^\circ_\beta(\overline{\beta}^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1})\Psi K^\circ_\beta(t^{n-1}x)}{\tau_{\beta}K^\circ_\beta(\overline{\beta}^{-1})}$$

with $\Psi = w_0H^\circ_{w_0}$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^\circ_\beta(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K^\circ_\beta(\overline{\alpha}^{-1})K^\circ_\beta(t^{n-1}w_0x)}{\tau_{\beta}K^\circ_\beta(\overline{\beta}^{-1})}.$$  

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in C_n} \frac{K^\circ_\beta(\overline{\alpha}^{-1})K^\circ_\beta(\overline{\gamma}^{-1})}{\tau_{\beta}K^\circ_\beta(\overline{\beta}^{-1})}.$$  

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta^{\alpha}(\omega^{-1})K_\beta(t^{n-1}x)}{\tau_\beta G_\beta^{\alpha}(\beta^{-1})},$$

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0Hw_0$ on both sides. Since $w_0Hw_0\Psi = \text{Id}$ we obtain

$$\Psi K_\alpha^{\circ}(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K_\alpha^{\circ}(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \frac{\tau_\beta K_\beta(\alpha)K_\beta(ax)}{K_\beta(\beta)}. \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}. \quad \text{Since } \left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0 \text{ unless } \delta \supseteq \epsilon, \text{ the terms in the sum are zero unless } \gamma \subseteq \beta \subseteq \alpha.$$  

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