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Sahi, S.; Stokman, J.

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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

Siddhartha Sahi\textsuperscript{1} and Jasper Stokman\textsuperscript{2,}\textsuperscript{*}

\textsuperscript{1}Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854-8019, USA and \textsuperscript{2}KdV Institute for Mathematics, University of Amsterdam, Science Park 105-107, 1098 XG Amsterdam, The Netherlands

*Correspondence to be sent to: e-mail: j.v.stokman@uva.nl

We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$ 

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write $\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n)$ where $\tau := (\tau_1, \ldots, \tau_n)$ with $\tau_i := t^{1-i}$.

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Then $R_{\lambda}(x) = R_{\lambda}(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_{\lambda}(\mu) = 0 \text{ for } \mu \in P_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$ 

The normalization is fixed by requiring that the coefficient of $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_{\lambda}(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_{\lambda}(x)$ is the Macdonald polynomial $P_{\lambda}(x)$ [9] and $R_{\lambda}(x)$ satisfies the extra vanishing property $R_{\lambda}(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_{\lambda}(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_{\lambda}(ax) = R_{\lambda}(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_{\mu}(x; q^{-1}, t^{-1})$’s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in P_n$$

and takes the form

$$\frac{R_{\lambda}(a \tilde{\mu})}{R_{\lambda}(a \tau)} = \frac{R_{\mu}(a \tilde{\lambda})}{R_{\mu}(a \tau)}.$$ 

The interpolation polynomials have natural non-symmetric analogs $G_{\alpha}(x) = G_{\alpha}(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in C_n$ we define

$$\bar{\beta} := w_{\beta}(\beta_+),$$

where $w_{\beta}$ is the shortest permutation such that $\beta_+ = w_{\beta}^{-1}(\beta)$ is a partition. Then $G_{\alpha}(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_{\alpha}(\bar{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$ 

The normalization is fixed by requiring that the coefficient of $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_{\alpha}(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_{\lambda}(x)$ admit non-symmetric counterparts for the $G_{\alpha}(x)$. For instance, the top homogeneous part of $G_{\alpha}(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha|-I(\alpha)}w_0 H_{w_0} G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 
Theorem C. For all compositions $\alpha \in C_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n}a\omega_0 x)}{G_\alpha(\omega_0 x)}.$$  

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \rightarrow n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i} t^{-k_i(v)}$ with

$$k_i(v) := \# \{ k < i | v_k \geq v_i \} + \# \{ k > i | v_k > v_i \}.$$  

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{v} = (q^{v_1} \tau_1, \ldots, q^{v_n} \tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_v(\overline{v}_+)$ with $w_v \in S_n$ the shortest permutation such that $v_+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\tilde{v} := -w_0 v$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in C_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)$$

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on $\mathbb{F}(x)$ \((1 \leq i < n)\) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

\[
\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).
\]

Note that $H_i$ \((1 \leq i < n)\) and $\Delta$ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ \((1 \leq i < n)\) and $\Delta$ satisfy the defining relations of the type $A$ extended affine Hecke algebra,

\[
(H_i - t)(H_i + 1) = 0,
\]

\[
H_i H_j = H_j H_i, \quad |i - j| > 1,
\]

\[
H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1},
\]

\[
\Delta H_{i+1} = H_i \Delta,
\]

\[
\Delta^2 H_1 = H_{n-1} \Delta^2
\]

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = t H_i^{-1}$ and set

\[
\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \tag{1}
\]

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

\[
\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.
\]

The $\xi_i^{-1}$ \((1 \leq i \leq n)\) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in \mathbb{C}^n$ is the unique polynomial satisfying

\[
\xi_i^{-1} E_\alpha = \overline{\alpha} E_\alpha, \quad i = 1, \ldots, n
\]

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$
and \(\mathbb{F}[x]\) by letting \(\iota\) act on the coefficients of the polynomial. Write

\[
G^\circ_\alpha := \iota(G_\alpha), \quad E^\circ_\alpha := \iota(E_\alpha)
\]

for \(\alpha \in \mathbb{C}_n\). Note that \(\overline{v}^{-1} = (\iota(v_1), \ldots, \iota(v_n))\).

Put \(H^\circ_i, H^\circ_w, \overline{H}_i, \Delta^\circ\) and \(\xi^\circ_i\) for the operators \(H_i, H_w, \overline{H}_i, \Delta\) and \(\xi_i\) with \(q, t\) replaced by their inverses. For instance,

\[
H^\circ_i = t^{-1} s_i - \frac{(1 - t^{-1}) x_i}{x_i - x_{i+1}} (1 - s_i),
\]

\[
\Delta^\circ f(x_1, \ldots, x_n) = f(q x_n, x_1, \ldots, x_{n-1}).
\]

We then have \(\xi^\circ_i E^\circ_\alpha = \overline{e}_i E^\circ_\alpha\) for \(i = 1, \ldots, n\), which characterizes \(E^\circ_\alpha\) up to a scalar factor.

**Theorem 1.** For \(\alpha \in \mathbb{C}_n\) we have

\[
G'_\alpha(x) = t^{(1-n)|\alpha|+1(\alpha)} w_0 H^\circ_w G^\circ_\alpha(t^{-1} x)
\]

with \(I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}\).

**Remark.** Formally set \(t = q^r\), replace \(x\) by \(1 + (q - 1)x\), divide both sides of (2) by \((q - 1)^{|\alpha|}\) and take the limit \(q \to 1\). Then

\[
G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n - 1)r; r)
\]

for the non-symmetric interpolation Jack polynomial \(G_\alpha(\cdot; r)\) and its primed version (see [14]). Here \(\sigma\) denotes the action of the symmetric group with \(\sigma(s_i)\) the rational degeneration of the Demazure-Lusztig operators \(H_i\), given explicitly by

\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}} (1 - s_i),
\]

see [14, Section 1]. To establish the formal limit (3) one uses that \(\sigma(w_0) w_0 = w_0 \sigma^\circ(w_0)\) with \(\sigma^\circ\) the action of the symmetric group defined in terms of the rational degeneration

\[
\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}} (1 - s_i)
\]

of \(H^\circ_i\). Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_\alpha$.

For the vanishing property, note that
\[ t^{n-1} w_0 \beta = \tilde{\beta}^{-1} \]  
(this is the $q$-analog of \[14\, \text{Lem. 6.1(2)}\]); hence,
\[ (w_0 H_{w_0}^o G_{w_0}^o (t^{n-1} x))|_{x=\beta} = (H_{w_0}^o G_{w_0}^o (x))|_{x=\tilde{\beta}^{-1}}. \]

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{w_0}^o (w_0 H_{w_0}^o \alpha)$ (use e.g., \[2\, \text{Prop. 3.2.2}\]). It follows that $\xi_i^{-1} \Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$.

Therefore, \( E_\alpha (x) = c_\alpha \Psi E_\alpha^o (x) \)
for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_\alpha = t^{-1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $u_j$ (1 $\leq$ $j$ $\leq$ $n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that
\[ E_{u+(1^n)} = x_1 \cdots x_n E_u (x). \]
It is now easy to check that formula (5) is valid with \( \alpha \) replaced by an arbitrary integral vector \( u \),

\[
E_u = t^{l(u)} w_0 H^\circ w_0 E_u^\circ
\]

with \( E_u^\circ := \iota(E_u) \). Furthermore, one can show in the same vein as the proof of (5) that

\[
w_0 E_{-w_0 u}(x^{-1}) = E_u(x)
\]

for an integral vector \( u \), where \( \rho(x^{-1}) \) stands for inverting all the parameters \( x_1, \ldots, x_n \) in the Laurent polynomial \( p(x) \in F[x^\pm 1] \). Combining this equality with (7) yields

\[
E_{-w_0 u}(x^{-1}) = t^{l(u)} H_{w_0} \circ E_{w_0}(x),
\]

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

### 3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

\[
G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)} + t^{1-l'(s)}}{1 - q^{a(s)} + t^{l'(s)+1}} \right) \prod_{s \in \alpha} \left( at^{l'(s)} - q^{a'(s)} \right)
\]

was obtained, with \( a(s), l(s), a'(s) \) and \( l'(s) \) the arm, leg, coarm and coleg of \( s = (i, j) \in \alpha \), defined by

\[
a(s) := \alpha_i - j, \quad l(s) := \#\{k > i \mid j \leq \alpha_k \leq \alpha_i\} + \#\{k < i \mid j \leq \alpha_k + 1 \leq \alpha_i\},
\]

\[
a'(s) := j - 1, \quad l'(s) := \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \geq \alpha_i\}.
\]

By (8) we have

\[
E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n + l'(s)} - q^{a'(s)} + 1}{1 - q^{a(s)} + t^{l'(s)+1}} \right),
\]

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for \( \alpha \in C_n \),

\[
\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.
\]
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha| + I(\alpha) - \ell(w_0)} G^0_\alpha(a \tau^{-1}).$$

Proof. Since $t^{n-1}w_0 \tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha| + I(\alpha)} (H^{0}_{w_0} G^0_\alpha)(a \overline{\tau}^{-1})$$

$$= t^{(1-n)|\alpha| + I(\alpha) - \ell(w_0)} G^0_\alpha(a \overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha| - n(\alpha)} q^{n'(\alpha)} G^0_\alpha(a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in \mathbb{C}^n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau^{-1}_a a^{-|\alpha|}G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\natural := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\natural := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^\natural$ by $^\flat$, so $v^\flat = (v_n - 1, v_1, \ldots, v_1)$ and $y^\flat = (y_n/q, y_1, \ldots, y_1)$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = s_i^\natural v$ if $v_i \neq v_{i+1}$.
2. $\overline{v}_i = t\overline{v}_{i+1}$ if $v_i = v_{i+1}$.
3. $v^\natural = v^\flat$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^{\pm 1}])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $F_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$) on $F_A$ defined by

\[
\begin{align*}
(\widehat{H}_if)(v) &:= tf(v) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}}(f(s_i v) - f(v)), \\
(\widehat{\Delta}f)(v) &:= f(v^\natural), \quad (\widehat{\Delta}^{-1}f)(v) := f(v^\flat), \\
(\widehat{x}_jf)(v) &:= a\overline{v}_j f(v)
\end{align*}
\]

for $f \in F_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(F_A)$, $X \mapsto \widehat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $F_A$. 
Proof. Let \( \mathcal{O} \subset \mathbb{K}^n \) be the smallest \( S_n \)-invariant and \( \mathfrak{z} \)-invariant subset that contains \( \{a \bar{v} \mid v \in \mathbb{Z}^n\} \). Note that \( \mathcal{O} \) is contained in \( \{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\} \). The Demazure–Lusztig operators \( H_i \) \((1 \leq i < n)\), \( \Delta^{\pm 1} \) and the coordinate multiplication operators \( x_j \) \((1 \leq j \leq n)\) act \( A \)-linearly on the space \( F^O_A \) of \( A \)-valued functions on \( \mathcal{O} \), and hence turns \( F^O_A \) into an \( \mathbb{H} \)-module. Define the surjective \( A \)-linear map

\[
\text{pr} : F^O_A \to \mathcal{F}_A
\]

by \( \text{pr}(g)(v) := g(a \bar{v}) \) \((v \in \mathbb{Z}^n)\).

We claim that \( \ker(\text{pr}) \) is an \( \mathbb{H} \)-submodule of \( F^O_A \). Clearly \( \ker(\text{pr}) \) is \( x_j \)-invariant for \( j = 1, \ldots, n \). Let \( g \in \ker(\text{pr}) \). Part 3 of Lemma 5 implies that \( \Delta g \in \ker(\text{pr}) \). To show that \( H_i g \in \ker(\text{pr}) \) we consider two cases. If \( v_i \neq v_{i+1} \) then \( s_i \bar{v} = \bar{s}_i \bar{v} \) by part 1 of Lemma 5. Hence,

\[
(H_i g)(a \bar{v}) = t g(a \bar{v}) + \frac{\bar{v}_i - t \bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a \bar{s}_i \bar{v}) - g(a \bar{v})) = 0.
\]

If \( v_i = v_{i+1} \) then \( \bar{v}_i = t \bar{v}_{i+1} \) by part 2 of Lemma 5. Hence,

\[
(H_i g)(\bar{v}) = t g(a \bar{v}) + \frac{\bar{v}_i - t \bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a \bar{s}_i \bar{v}) - g(a \bar{v})) = t g(a \bar{v}) = 0.
\]

Hence, \( \mathcal{F}_A \) inherits the \( \mathbb{H} \)-module structure of \( F^O_A / \ker(\text{pr}) \). It is a straightforward computation, using Lemma 5 again, to show that the resulting action of \( H_i \) \((1 \leq i < n)\), \( \Delta \) and \( x_j \) \((1 \leq j \leq n)\) on \( \mathcal{F}_A \) is by the operators \( \hat{H}_i \) \((1 \leq i < n)\), \( \hat{\Delta} \) and \( \hat{x}_j \) \((1 \leq j \leq n)\). \( \blacksquare \)

Remark 7. With the notations from (the proof of) Corollary 6, let \( \bar{g} \in F^O_A \) and set \( g := \text{pr}(\bar{g}) \in \mathcal{F}_A \). In other words, \( g(v) := \bar{g}(a \bar{v}) \) for all \( v \in \mathbb{Z}^n \). Then

\[
(\hat{X} g)(v) = (X \bar{g})(a \bar{v}), \quad v \in \mathbb{Z}^n
\]

for \( X = H_i, \Delta^{\pm 1}, x_j \).

Remark 8. Let \( \mathcal{F}^+_A \) be the space of \( A \)-valued functions on \( \mathcal{C}_n \). We sometimes will consider \( \hat{H}_i \) \((1 \leq i < n)\), \( \hat{\Delta}^{-1} \) and \( \hat{x}_j \) \((1 \leq j \leq n)\), defined by the formulas (11), as linear operators on \( \mathcal{F}^+_A \).

Definition 9. We call

\[
K_{\alpha}(x; q, t; a) := \frac{G_{\alpha}(x; q, t)}{G_{\alpha}(a \tau; q, t)} \in \mathbb{K}[x] \quad (12)
\]

the normalized non-symmetric interpolation Macdonald polynomial of degree \( \alpha \).
We frequently use the shorthand notation \( K_\alpha(x) := K_\alpha(x; q, t; a) \). We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that \( a \) cannot be specialized to 1 in (12) since \( G_\alpha(\tau) = G_\alpha(0) = 0 \) if \( \alpha \in C_n \) is nonzero. Note furthermore that

\[
\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}
\]

since \( \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x) \).

Recall from [4] the operator \( /\Phi_1 = \left( x_n - t^{1-n} \right) \Delta \in \mathbb{H} \) and the inhomogeneous Cherednik operators

\[
/\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.
\]

The operators \( H_i, /\Xi_j \) and \( \Phi \) preserve \( \mathbb{K}[x] \) (see [4]); hence, they give rise to \( \mathbb{K} \)-linear operators on \( \mathcal{F}_+^{+}[x] \) (e.g., \( (H_i f)(\alpha) := H_i(f(\alpha)) \) for \( \alpha \in C_n \)). Note that the operators \( H_i, /\Xi_j \) and \( \Phi \) on \( \mathcal{F}_+^{+}[x] \) commute with the hat-operators \( \hat{H}_i, \hat{x}_j \) and \( \hat{\Lambda}^{-1} \) on \( \mathcal{F}_+^{+}[x] \) (cf. Remark 8).

The same remarks hold true for the space \( \mathcal{F}_+^{\mathbb{K}(x)} \) of \( \mathbb{K}(x) \)-valued functions on \( \mathbb{Z}^n \) (in fact, in this case the hat-operators define a \( \mathbb{H} \)-action on \( \mathcal{F}_+^{\mathbb{K}(x)} \)).

Let \( K \in \mathcal{F}_+^{\mathbb{K}(x)} \) be the map \( \alpha \mapsto K_\alpha(\cdot) (\alpha \in C_n) \).

**Lemma 10.** For \( 1 \leq i < n \) and \( 1 \leq j \leq n \) we have in \( \mathcal{F}_+^{\mathbb{K}[x]} \):

1. \( H_i K = \hat{H}_i K \).
2. \( /\Xi_j K = a \hat{x}_j^{-1} K \).
3. \( \Phi K = t^{1-n}(a^2 \hat{x}_1^{-1} - 1) \hat{\Lambda}^{-1} K \).

**Proof.** 1. To derive the formula we need to expand \( H_i K_\alpha \) as a linear combination of the \( K_\beta \)'s. As a 1st step we expand \( H_i G_\alpha \) as linear combination of the \( G_\beta \)'s.

If \( \alpha \in C_n \) satisfies \( \alpha_i < \alpha_{i+1} \) then

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{a}_i}{\overline{a}_i - \overline{a}_{i+1}} G_\alpha(x) + G_{s_{i\alpha}}(x)
\]

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that \( H_i \) satisfies the quadratic relation \( (H_i - t)(H_i + 1) = 0 \), it follows that

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{a}_i}{\overline{a}_i - \overline{a}_{i+1}} G_\alpha(x) + \frac{t(\overline{a}_{i+1} - t\overline{a}_i)(\overline{a}_{i+1} - t^{-1}\overline{a}_i)}{(\overline{a}_{i+1} - \overline{a}_i)^2} G_{s_{i\alpha}}(x)
\]
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_i G_\alpha(x) = t G_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_i K_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(\alpha \tau) = \frac{\alpha_{i+1} - t \alpha_i}{\alpha_{i+1} - \alpha_i} G_{\alpha_i}(\alpha \tau)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_i K = \tilde{H}_i K \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\circ}(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_{\alpha^\circ}(\alpha \tau)}{G_\alpha(\alpha \tau)} = a t^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n}(a \alpha_{n} - 1) K_{\alpha^\circ}(x).
\]

Remark 11. Note that

\[
\Phi K_\alpha(x) = (a \alpha_{n} - t^{1-n}) K_{\alpha^\circ}(x)
\]

for \( \alpha \in C_n \) since \( \tilde{\alpha}^{-1} = t^{n-1} w_0 \alpha \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in \mathcal{F}^+_{[x]} \) to a map \( K \in \mathcal{F}_{[x]} \) such that Lemma 10 remains valid.

Lemma 12. For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-\vert \alpha \vert} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^{n}(qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^{n} \frac{1 - a \alpha_i^{-1}}{1 - qa^{n-1} x_i} \right) K_{\alpha+(1^n)}(qx).
\]
Proof. Note that for $f \in \mathbb{K}[x],$

$$\Phi^n f(x) = \left(\prod_{i=1}^{n} (x_i - t^{1-n})\right) f(q^{-1} x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10.

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m} a v - 1; q)_m}{(q t^{n-1} x_i; q)_m}, \quad \forall v \in \mathbb{Z}^n,$$

with $(y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y)$ the $q$-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n.$ Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t, a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v|-m^2} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m} t^{1-n} x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),$$

where $m$ is a nonnegative integer such that $v + (m^n) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0},$

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left(\frac{x}{a}\right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.$$

Lemma 15. For all $v \in \mathbb{Z}^n,$

$$K_v(x) = \frac{G_v(x)}{G_v(at)}.$$

Proof. Let $v \in \mathbb{Z}^n.$ Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(at) = 1.$ Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n.$ Then

$$K_v(at) = A_m(at; v) K_{v+(m^n)}(q^m at) = A_m(at; v) \frac{G_{v+(m^n)}(q^m at)}{G_{v+(m^n)}(at)} = 1.$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : \mathbb{C}^n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a \hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2 \hat{x}_1^{-1} - 1) \hat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $(A_m f)(v) = A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$(\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

(15)

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathbb{C}_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = tK_v$ and $H_i K_{v+(m^n)}(q^m x) = tK_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a \overline{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

(16)

where $\Phi(q^m) = (q^m x_n - t^{1-n}) \Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \overline{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition. ■
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{v} = w_0 v \) for \( v \in \mathbb{Z}^n \).

**Theorem 17.** (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a \tilde{v}) = K_v(a \tilde{u}).
\]  

(17)

**Example 18.** If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r}
\]  

(18)

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

**Proof.** We divide the proof of the theorem in several steps. ■

**Step 1.** If \( K_u(a \tilde{v}) = K_v(a \tilde{u}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{siu}(a \tilde{v}) = K_{v}(a \tilde{siu}) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a \tilde{v}) + \frac{(\tilde{v}_i - t\tilde{v}_{i+1})}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a \tilde{v}_i \tilde{u}) \]

(19)

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t-1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a \tilde{u}) + \frac{(\tilde{u}_{n-i} - t\tilde{u}_{n+1-i})}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a \tilde{u}_i \tilde{u}) \]

(20)

Suppose that \( s_{n-i} v = v \). Then \( \tilde{v}_{n-i} = t \tilde{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n} w_0^{n} v^{-1} \), that is, \( \tilde{v}_i = t^{1-n} v_{n+1-i}^{-1} \), we then also have \( \tilde{v}_i = t \tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{siu}(a \tilde{v}) = K_u(a \tilde{v}) \) and (20) to \( K_v(a \tilde{siu}) = K_v(a \tilde{u}) \) if \( s_{n-i} v = v \).
We now use these observations to prove Step 1. Assume that $K_u(a\nu) = K_v(a\nu)$ for all $v$. We have to show that $K_{s_i u}(a\nu) = K_v(a\nu)$ for all $v$. It is trivially true if $s_i u = u$, so we may assume that $s_i u \neq u$. Suppose that $v$ satisfies $s_{n-i} v = v$. Then it follows from the previous paragraph that

$$K_{s_i u}(a\nu) = K_u(a\nu) = K_v(a\nu) = K_v(a\nu).$$

If $s_{n-i} v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_i u}(a\nu)$ as an explicit linear combination of $K_v(a\nu)$ and $K_{s_{n-i} v}(a\nu)$. Then (20) can be used to rewrite the term involving $K_{s_{n-i} v}(a\nu)$ as an explicit linear combination of $K_v(a\nu)$ and $K_v(a\nu s_i u)$. Hence, we obtain an explicit expression of $K_{s_i u}(a\nu)$ as linear combination of $K_v(a\nu)$ and $K_v(a\nu s_i u)$, which turns out to reduce to $K_{s_i u}(a\nu) = K_v(a\nu s_i u)$ after a direct computation.

**Step 2.** $K_0(a\nu) = 1 = K_v(a\nu)$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\nu) = K_v(a\nu) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_\alpha(a\nu) = K_v(a\nu)$ for $v \in \mathbb{Z}^n$ and $\alpha \in C_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_v(a\nu) = K_v(a\nu)$ for $v \in \mathbb{Z}^n$ and $\gamma \in C_n$ with $|\gamma| < m$. Let $\alpha \in C_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\nu) = K_v(a\nu)$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\vee \in C_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\vee$. Furthermore, note that we have the formula

$$(a\nu^{-1}_1 - 1)K_u(a\nu^\vee) = (a\nu_1^{-1} - 1)K_u(a\nu^\vee)$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\nu) = K_v(a\nu) = \frac{(a\nu_1^{-1} - 1)}{(a\gamma_1^{-1} - 1)}K_v(a\nu^\vee) = \frac{(a\nu_1^{-1} - 1)}{(a\gamma_1^{-1} - 1)}K_v(a\nu^\vee) = K_v(a\nu^\vee) = K_v(a\nu),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4.  \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = \tilde{v} - (m^n) \) and \( q^{-m}\tilde{u} = u + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v})
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n)))
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n)))
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m}a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n)) = 1,
\]

which follows by a straightforward computation using (4).

\[
\square
\]

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_{\alpha}(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_{\alpha}(x) \), i.e.,

\[
E_{\alpha}(x) = \lim_{a \to \infty} a^{-|\alpha|} G_{\alpha}(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_{\alpha}(x) := \lim_{a \to \infty} K_{a}(ax) = \frac{E_{\alpha}(x)}{E_{\alpha}(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}_{\text{pfix}}^+ \) for the resulting map \( \alpha \mapsto \overline{K}_{\alpha} \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\overline{K} = \overline{H}_i\overline{K} \).
2. \( \xi_j\overline{K} = \overline{\xi}_j^{-1}\overline{K} \).
3. \( x_n\Delta\overline{K} = t_1^{1-n}\hat{\Delta}^{-1}\overline{K} \).
Note that
\[(x_n\Delta)^nf(x) = \left(\prod_{i=1}^{n} x_i\right)f(q^{-1}x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},\]
\[K_\alpha(x) = q^{||\alpha||t(1-n)n} \left(\prod_{i=1}^{n} (\overline{\alpha}_ix_i)^{-1}\right)K_{\alpha+(1^n)}(x).\] (22)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_v(x) := E_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) for arbitrary \(v \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[E_v(x) = \frac{E_{v+(m^n)}(x)}{(x_1 \cdots x_n)^m}.\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(v \in \mathbb{Z}^n\).

**Definition 20.** Let \(v \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(v + (m^n) \in C_n\). Then \(K_v(x) := K_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]\) is defined by
\[K_v(x) := q^{m||v||t(1-n)n} \left(\prod_{i=1}^{n} (\overline{v}_ix_i)^{-m}\right)K_{v+(m^n)}(x).\]

Using
\[\lim_{a \to \infty} A_m(ax; v) = q^{-m^2nt(1-n)n} \prod_{i=1}^{n} (\overline{v}_ix_i)^{-m}\]
and the definitions of \(G_v(x)\) and \(K_v(x)\) it follows that
\[\lim_{a \to \infty} a^{-|v|}G_v(ax) = E_v(x),\]
\[\lim_{a \to \infty} K_v(ax) = \overline{K}_v(x)\]
for all \(v \in \mathbb{Z}^n\), so in particular
\[\overline{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \quad \forall v \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $\overline{K}$ to the map $\overline{K} \in \mathcal{F}[x^{\pm 1}]$ defined by $\nu \mapsto \overline{K}_\nu$ ($\nu \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$\overline{K}_u(\overline{v}) = \overline{K}_v(\overline{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\beta^-) = K_\alpha(t^{1-n}aw_0\beta^-) = K_\alpha(a\tilde{\beta}) = K_\beta(a\tilde{\alpha})$$

for all $\beta \in \mathcal{C}_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. $\blacksquare$

### 7.3 Okounkov’s duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]^{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K^+_\lambda(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{i(w)} \right) K^+_\alpha(x)$$

for $\alpha \in \mathcal{C}_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K^+_\lambda(a\tilde{\mu}^{-1}) = K^+_\mu(a\tilde{\lambda}^{-1}).$$

Let us derive Theorem 23 as a consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_K$ for the function $f_\mu(u) := K_u(a\tilde{\mu}) (u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\lambda(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\hat{C}_+ f_\mu)(\lambda) \quad (24)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{u}) = (Jw_0 K_\mu(t^{1-n}x))_{|x = a^{-1}\tilde{u}} \quad (25)$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_iJ = (H^o_i)^{-1}, \quad w_0 H_i w_0 = (H^o_{n-i})^{-1} \quad (26)$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\hat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))_{|x = a^{-1}\tilde{x}}.$$\text{By (23) and (4) this simplifies to}

$$(\hat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\mu(a\tilde{\lambda}).$$

Returning to (24) we conclude that $K^+_\lambda(a\tilde{\mu}) = K^+_\mu(a\tilde{\lambda})$. Since $K^+_\lambda$ is symmetric we obtain from (4) that

$$K^+_\lambda(a\tilde{\mu}^{-1}) = K^+_\mu(a\tilde{\lambda}^{-1}),$$

which is Okounkov's duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For \( u, v \in \mathbb{Z}^n \) we have
\[
(H_{w_0} K_u)(a \tilde{v}) = (H_{w_0} K_v)(a \tilde{u}).
\] (27)

Proof. We proceed as in the previous subsection. Set \( f_v(u) := K_u(a \tilde{v}) \) for \( u, v \in \mathbb{Z}^n \). By part 1 of Proposition 16,
\[
(H_{w_0} K_u)(a \tilde{v}) = (\hat{H}_{w_0} f_v)(u).
\]

Since \( f_v(u) = (Iw_0 K_v)(a^{-1} t^{n-1} \tilde{u}) \) by (4), Remark 7 implies that
\[
(\hat{H}_{w_0} f_v)(u) = (H_{w_0} Jw_0 K_v)(a^{-1} t^{n-1} \tilde{u}).
\]

Now \( H_{w_0} Jw_0 = Jw_0 H_{w_0} \) by (26); hence,
\[
(\hat{H}_{w_0} f_v)(u) = (Jw_0 H_{w_0} K_v)(a^{-1} t^{n-1} \tilde{u}) = (H_{w_0} K_v)(a \tilde{u}),
\]
which completes the proof. \( \blacksquare \)

Recall from Theorem 1 that
\[
G'_\beta(x) = t^{(1-n)|\beta| + I(\beta)} \Psi G_\beta^o(t^{n-1}x)
\]
with \( \Psi := w_0 H_{w_0}^o \). We define normalized versions by
\[
K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1} \tau)} = t^{\ell(w_0)} \Psi K_\beta^o(t^{n-1}x), \quad \beta \in C_n,
\]
with \( K_\nu^o := \iota(K_\nu) \) for \( \nu \in \mathbb{Z}^n \) (the 2nd formula follows from Lemma 2). More generally, we define for \( \nu \in \mathbb{Z}^n \),
\[
K'_\nu(x) := t^{\ell(w_0)} \Psi K_\nu^o(t^{n-1}x).
\] (28)

We write \( K' : \mathbb{Z}^n \to \mathbb{K}(x) \) for the map \( \nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n) \). Since \( H_i \Psi = \Psi H_i^o \), part 1 of Proposition 16 gives \( H_i K' = \hat{H}_{w_0}^i K' \). Considering the action of \((x_n - 1) \Delta^o)^n\) on \( K'_\beta(x) \) we get, using the fact that \((x_n - 1) \Delta^o)^n\) commutes with \( \Psi \) and part 3 of Proposition 16,
\[
K'_\nu(x) = \left( \prod_{i=1}^n \frac{1 - a^{-1} \nu_i}{1 - q^{-1} x_i} \right) K'_{\nu+(1^n)}(q^{-1}x),
\]
in particular

\[ K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}q^{-1}; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+m}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K'^0_v(x) \) for \( v \in \mathbb{Z} \); hence,

\[
K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m}, \\
K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}
\]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_v(a^{-1}u) = K'_v(a^{-1}v). \]

**Proof.** Note that

\[ K'_v(a^{-1}u) = t_{\ell(w_0)} \Psi K'^0_v(t^{\ell-1}x)|_{x=a^{-1}u} = t_{\ell(w_0)} (H_w^o K_v^o) (a^{-1} \tilde{u}^{-1}) \]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q, t} := \frac{G_\beta(\overline{\alpha})}{G_\beta(\overline{\beta})}. 
\]

(29)

Applying the automorphism \( \iota \) of \( F \) to (29) we get

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} = \frac{G_\beta(\overline{\alpha}^{-1})}{G_\beta(\overline{\beta}^{-1})}. 
\]
Theorem 27. For $\alpha, \beta \in \mathbb{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} a^{\beta} \binom{\alpha}{\beta} \frac{G'_\beta(x)}{G_\beta(ax)}.$$  \hspace{1cm} (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} \tau_{\beta}^{-1} \binom{\alpha}{\beta} K'_\beta(x)$$

$$= \sum_{\beta \in \mathbb{C}_n} \frac{K^o_{\beta}(\alpha^{-1})K'_\beta(x)}{\tau_{\beta}K^o_{\beta}(\beta^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K^o_{\beta}(\alpha^{-1})\Psi K^o_{\beta}(t^{n-1}x)}{\tau_{\beta}K^o_{\beta}(\beta^{-1})}.$$  \hspace{1cm} (31)

with $\Psi = w_0 H^o_{w_0}$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^o_{\beta}(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K^o_{\beta}(\alpha^{-1})K^o_{\beta}(t^{n-1}w_0x)}{\tau_{\beta}K^o_{\beta}(\beta^{-1})}.$$  

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in \mathbb{C}_n} \frac{K^o_{\beta}(\alpha^{-1})K^o_{\beta}(\gamma^{-1})}{\tau_{\beta}K^o_{\beta}(\beta^{-1})}.$$  

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K_\alpha'(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta^\circ(\overline{\alpha})\Psi K_\beta^\circ(t^{n-1}x)}{\tau_\beta G_\beta^\circ(\overline{\beta})}, \quad \text{see (31).}$$

Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0Hw_0$ on both sides. Since $w_0Hw_0\Psi = \text{Id}$ we obtain

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\overline{\alpha})K_\beta(ax) \frac{K_\beta(\overline{\beta})}{K_\beta(\overline{\beta})}. \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}. \quad \text{Since} \quad \left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0 \text{ unless } \delta \supseteq \epsilon, \text{ the terms in the sum are zero unless } \gamma \subseteq \beta \subseteq \alpha.$$

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