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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n)$$

where $\tau := (\tau_1, \ldots, \tau_n)$ with $\tau_i := t^{1-i}$.

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Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\mu) = 0 \text{ for } \mu \in P_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$ 

The normalization is fixed by requiring that the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in \[4, 13\], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ \[9\] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov \[10\], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n, q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in P_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.$$

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in \[4, 13\]. These are indexed by the set of compositions with at most $n$ parts, $C_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in C_n$ we define

$$\overline{\beta} := w_\beta(\beta_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\overline{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$ 

The normalization is fixed by requiring that the coefficient of $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax;q,t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x;q,t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)}w_0Hw_0G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}$$

for all $\beta$.

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

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Theorem C. For all compositions $\alpha \in C_n$ we have

$$O_\alpha (x) = \frac{G_\alpha (t^{1-n}a \omega_0 x)}{G_\alpha (at)}.$$

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha (x)$ in terms of the $G_\beta (ax)$'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \rightarrow \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i \nu := (\cdots, \nu_{i-1}, \nu_{i+1}, \nu_i, \nu_{i+2}, \cdots)$ for $\nu = (\nu_1, \ldots, \nu_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \mapsto n+1-i$ for $i = 1, \ldots, n$.

For $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ define $\overline{\nu} = (\overline{\nu}_1, \ldots, \overline{\nu}_n) \in \mathbb{F}^n$ by $\overline{\nu}_i := q^{\nu_i} t^{-k_i(\nu)}$ with

$$k_i(\nu) := \# \{ k < i \mid \nu_k \geq \nu_i \} + \# \{ k > i \mid \nu_k > \nu_i \}.$$

If $\nu \in \mathbb{Z}^n$ has non-increasing entries $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, then $\overline{\nu} = (q^{\nu_1} \tau_1, \ldots, q^{\nu_n} \tau_n)$. For arbitrary $\nu \in \mathbb{Z}^n$ we have $\overline{\nu} = w_\nu(\overline{\nu}_+) \nu \in S_n$ the shortest permutation such that $\nu_+ := w^{-1}_\nu(\nu)$ has non-increasing entries, see [4, Section 2]. We write $\overline{\nu} := -w_0 \nu$ for $\nu \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in C_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x_{\pm 1}^1] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_i+1$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = ts_i - (1-t)x_i (1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)$$
on $\mathbb{F}(x)$ ($1 \leq i < n$) called Demazure-Lusztig operators, and the automorphism $\Delta$ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ ($1 \leq i < n$) and $\Delta$ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators $H_i$ ($1 \leq i < n$) and $\Delta$ satisfy the defining relations of the type $A$ extended affine Hecke algebra,

$$ (H_i - t)(H_i + 1) = 0, $$

$$ H_i H_j = H_j H_i, \quad |i - j| > 1, $$

$$ H_i H_{i+1} H_i = H_{i+1} H_i H_i, $$

$$ \Delta H_{i+1} = H_i \Delta, $$

$$ \Delta^2 H_1 = H_{n-1} \Delta^2 $$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_{\ell}}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell}}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = tH_i^{-1}$ and set

$$ \xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1) $$

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

$$ \xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}. $$

The $\xi_i^{-1}$ ($1 \leq i \leq n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$ \xi_i^{-1} E_\alpha = \overline{a}_i E_\alpha, \quad i = 1, \ldots, n $$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $\mathbb{K}$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $\mathbb{F}$, inverting $q$ and $t$. We extend $\iota$ to a $\mathbb{Q}$-algebra automorphism of $\mathbb{K}[x]$
and \( \mathbb{F}[x] \) by letting \( \iota \) act on the coefficients of the polynomial. Write

\[
G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)
\]

for \( \alpha \in C_n \). Note that \( \overline{\nu}^{-1} = (\iota(\overline{\nu}_1), \ldots, \iota(\overline{\nu}_n)) \).

Put \( H_i^\circ, H_w^\circ, \overline{\Delta_i}^\circ, \Delta^\circ \) and \( \xi_i^\circ \) for the operators \( H_i, H_w, \overline{\Delta_i}, \Delta \) and \( \xi_i \) with \( q, t \) replaced by their inverses. For instance,

\[
H_i^\circ = t^{-1}s_i - \frac{(1 - t^{-1})x_i}{x_i - x_{i+1}}(1 - s_i),
\]

\[
\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).
\]

We then have \( \xi_i^\circ E_\alpha^\circ = \overline{\alpha}_i E_\alpha^\circ \) for \( i = 1, \ldots, n \), which characterizes \( E_\alpha^\circ \) up to a scalar factor.

**Theorem 1.** For \( \alpha \in C_n \) we have

\[
G_\alpha'(x) = t^{(1-n)|\alpha|} w_0 H_w^\circ G_\alpha^0 (t^{n-1} x)
\]

with \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \).

**Remark.** Formally set \( t = q^r \), replace \( x \) by \( 1 + (q - 1)x \), divide both sides of (2) by \( (q - 1)^{|\alpha|} \) and take the limit \( q \to 1 \). Then

\[
G_\alpha'(x; r) = (-1)^{\alpha} \sigma(w_0) w_0 G_\alpha(-x - (n - 1)r; r)
\]

for the non-symmetric interpolation Jack polynomial \( G_\alpha(\cdot; r) \) and its primed version (see [14]). Here \( \sigma \) denotes the action of the symmetric group with \( \sigma(s_i) \) the rational degeneration of the Demazure-Lusztig operators \( H_i \), given explicitly by

\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),
\]

see [14, Section 1]. To establish the formal limit (3) one uses that \( \sigma(w_0) w_0 = w_0 \sigma^\circ(w_0) \) with \( \sigma^\circ \) the action of the symmetric group defined in terms of the rational degeneration

\[
\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)
\]

of \( H_i^\circ \). Formula (3) was obtained before in [14, Thm. 1.10].
**Proof.** We show that the right-hand side of (2) satisfies the defining properties of $G_{\alpha}'$.

For the vanishing property, note that
\[ t^{n-1}w_0\tilde{\beta} = \tilde{\beta}^{-1} \]
(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,
\[ (w_0H_{w_0}^o G_{\alpha}^o(t^{n-1}x))|_{x=\tilde{\beta}} = (H_{w_0}^o G_{\alpha}^o(x))|_{x=\tilde{\beta}^{-1}}. \]

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{\alpha}^o(w\beta^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that
\[ E_{\alpha} = t^{1(\alpha)}w_0H_{w_0}^o E_{\alpha}^o. \]

Note that $\Psi := w_0H_{w_0}^o$ satisfies the intertwining properties
\[ H_i\Psi = t\Psi H_i^o, \]
\[ \Delta \Psi = t^{n-1}\Psi H_{n-1}^o \cdots H_1^o (\Delta^o)^{-1}H_{n-1}^o \cdots H_1^o \]
for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$.

Therefore,
\[ E_{\alpha}(x) = c_{\alpha} \Psi E_{\alpha}^o(x) \]
for some constant $c_{\alpha} \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_{\alpha} = t^{1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\overline{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_{\alpha} \in \mathbb{F}[x]$ as defined before. Note that
\[ E_{u+(1^n)} = x_1 \cdots x_n E_u(x). \]
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{l(u)} w_0 H_{w_0}^c E_u^c$$ (7)

with $E_u^c := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{-1}]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{l(u)} H_{w_0}^c E_u^c(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)+1} t^{1-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})$$ (8)

was obtained, with $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i,j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \#\{k > i \mid j \leq \alpha_k \leq \alpha_i\} + \#\{k < i \mid j \leq \alpha_k + 1 \leq \alpha_i\},$$

$$a'(s) := j - 1, \quad l'(s) := \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \geq \alpha_i\}.$$

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - q^{a'(s)+1} t^{l'(s)+1}}{1 - q^{a(s)+1} t^{l(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.$$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G_\alpha^0(a \tau^{-1}).$$

Proof. Since $t^{n-1}w_0 \tau = \tau^{-1} = \bar{\omega}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)}(H^\circ_{w_0}G^\circ_\alpha)(a \bar{\omega}^{-1}) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^\circ_\alpha(a \bar{\omega}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^\circ_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha| - n(\alpha)} q^{n'(\alpha)} G^\circ_\alpha(a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For \( \alpha \in C_n \) we have

\[
G'_\alpha(a^{-1}\tau) = \tau^{-1}_\alpha a^{-|\alpha|}G_\alpha(a\tau).
\]

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of \( \mathbb{K} \)-valued functions on \( \mathbb{Z}^n \), which is constructed as follows.

For \( \nu \in \mathbb{Z}^n \) and \( y \in \mathbb{K}^n \) write \( \nu^\natural := (\nu_2, \ldots, \nu_n, \nu_1 + 1) \) and \( y^\natural := (y_2, \ldots, y_n, qy_1) \). Denote the inverse of \( ^\natural \) by \( ^\# \), so \( \nu^\# = (\nu_n - 1, \nu_1, \ldots, \nu_1 - 1) \) and \( y^\# = (y_n/q, y_1, \ldots, y_{n-1}) \). We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let \( \nu \in \mathbb{Z}^n \) and \( 1 \leq i < n \). Then we have

1. \( s_i(\nu) = s_i^{\#} \nu \) if \( \nu_i \neq \nu_{i+1} \).
2. \( \nu_i = t^\natural \nu_{i+1} \) if \( \nu_i = \nu_{i+1} \).
3. \( \nu^\# = \nu^\natural \).

Let \( \mathbb{H} \) be the double affine Hecke algebra over \( \mathbb{K} \). It is isomorphic to the subalgebra of \( \text{End}(\mathbb{K}[x^{\pm 1}]) \) generated by the operators \( H_i \) (\( 1 \leq i < n \)), \( \Delta^{\pm 1} \), and the multiplication operators \( x_j^{\pm 1} \) (\( 1 \leq j \leq n \)).

For a unital \( \mathbb{K} \)-algebra \( A \) we write \( \mathcal{F}_A \) for the space of \( A \)-valued functions \( f : \mathbb{Z}^n \to A \) on \( \mathbb{Z}^n \).

Corollary 6. Let \( A \) be a unital \( \mathbb{K} \)-algebra. Consider the \( A \)-linear operators \( \hat{H}_i \) (\( 1 \leq i < n \)), \( \hat{\Delta} \) and \( \hat{x}_j \) (\( 1 \leq j \leq n \)) on \( \mathcal{F}_A \) defined by

\[
(\hat{H}_if)(\nu) := tf(\nu) + \frac{\nu_i - t\nu_{i+1}}{\nu_i - \nu_{i+1}}(f(s_i\nu) - f(\nu)),
\]

\[
(\hat{\Delta}f)(\nu) := f(\nu^\natural), \quad (\hat{\Delta}^{-1}f)(\nu) := f(\nu^\natural),
\]

\[
(\hat{x}_jf)(\nu) := a\nu_jf(\nu)
\]

for \( f \in \mathcal{F}_A \) and \( \nu \in \mathbb{Z}^n \). Then \( H_i \mapsto \hat{H}_i \) (\( 1 \leq i < n \)), \( \Delta \mapsto \hat{\Delta} \) and \( x_j \mapsto \hat{x}_j \) (\( 1 \leq j \leq n \)) defines a representation \( \mathbb{H} \to \text{End}_A(\mathcal{F}_A) \), \( X \mapsto \hat{X} \) (\( X \in \mathbb{H} \)) of the double affine Hecke algebra \( \mathbb{H} \) on \( \mathcal{F}_A \).
Remark 8. Let \( O \subset \mathbb{K}^n \) be the smallest \( S_n \)-invariant and \( z \)-invariant subset that contains \( \{a\overline{v} \mid v \in \mathbb{Z}^n \} \). Note that \( O \) is contained in \( \{ y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j \} \). The Demazure–Lusztig operators \( H_i (1 \leq i < n), \Delta_{\pm 1} \) and the coordinate multiplication operators \( x_j (1 \leq j \leq n) \) act \( A \)-linearly on the space \( F_A^O \) of \( A \)-valued functions on \( O \), and hence turns \( F_A^O \) into an \( \mathbb{H} \)-module. Define the surjective \( A \)-linear map

\[
pr : F_A^O \to F_A
\]

by \( pr(g)(v) := g(a\overline{v}) (v \in \mathbb{Z}^n) \).

We claim that Ker\(\( pr \) is an \( \mathbb{H} \)-submodule of \( F_A^O \). Clearly Ker\(\( pr \) is \( x_j \)-invariant for \( j = 1, \ldots, n \). Let \( g \in \text{Ker}(pr) \). Part 3 of Lemma 5 implies that \( \Delta g \in \text{Ker}(pr) \). To show that \( H_i g \in \text{Ker}(pr) \) we consider two cases. If \( v_i \neq v_{i+1} \) then \( s_i \overline{v} = \overline{s_i v} \) by part 1 of Lemma 5. Hence,

\[
(H_i g)(a\overline{v}) = tg(a\overline{v}) + \frac{\overline{v_i} - t\overline{v}_{i+1}}{v_i - \overline{v}_{i+1}} (g(a\overline{s_i v}) - g(a\overline{v})) = 0.
\]

If \( v_i = v_{i+1} \) then \( \overline{v_i} = t\overline{v}_{i+1} \) by part 2 of Lemma 5. Hence,

\[
(H_i g)(\overline{v}) = tg(a\overline{v}) + \frac{\overline{v_i} - t\overline{v}_{i+1}}{v_i - \overline{v}_{i+1}} (g(a\overline{s_i v}) - g(a\overline{v})) = tg(a\overline{v}) = 0.
\]

Hence, \( F_A \) inherits the \( \mathbb{H} \)-module structure of \( F_A^O / \text{Ker}(pr) \). It is a straightforward computation, using Lemma 5 again, to show that the resulting action of \( H_i (1 \leq i < n), \Delta \) and \( x_j (1 \leq j \leq n) \) on \( F_A \) is by the operators \( \tilde{H}_i (1 \leq i < n), \tilde{\Delta} \) and \( \tilde{x}_j (1 \leq j \leq n) \).

Remark 7. With the notations from (the proof of) Corollary 6, let \( \tilde{g} \in F_A^O \) and set \( g := pr(\tilde{g}) \in F_A \). In other words, \( g(v) := \tilde{g}(a\overline{v}) \) for all \( v \in \mathbb{Z}^n \). Then

\[
(\tilde{X}g)(v) = (X\tilde{g})(a\overline{v}), \quad v \in \mathbb{Z}^n
\]

for \( X = H_i, \Delta_{\pm 1}, x_j \).

Remark 8. Let \( F_A^+ \) be the space of \( A \)-valued functions on \( C_n \). We sometimes will consider \( \tilde{H}_i (1 \leq i < n), \tilde{\Delta}^{-1} \) and \( \tilde{x}_j (1 \leq j \leq n) \), defined by the formulas (11), as linear operators on \( F_A^+ \).

Definition 9. We call

\[
K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(\alpha; q, t)} \in \mathbb{K}[x]
\]

(12)

the normalized non-symmetric interpolation Macdonald polynomial of degree \( \alpha \).
We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(0) = 0$ if $\alpha \in C_n$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}$$

since $\lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi_1 = (x_n - t^{1-n})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$H_i, \quad \Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_n \Phi H_1 \cdots H_{n-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$ 

The operators $H_i$, $\Xi_j$ and $\Phi$ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $\mathcal{F}_{\mathbb{K}[x]}^+$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in C_n$). Note that the operators $H_i, \Xi_j$ and $\Phi$ on $\mathcal{F}_{\mathbb{K}[x]}^+$ commute with the hat-operators $\hat{H}_i, \hat{x}_j$ and $\hat{\Delta}^{-1}$ on $\mathcal{F}_{\mathbb{K}[x]}^+$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_{\mathbb{K}[x]}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $\mathcal{F}_{\mathbb{K}[x]}$).

Let $K \in \mathcal{F}_{\mathbb{K}[x]}^+$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in C_n$).

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $\mathcal{F}_{\mathbb{K}[x]}^+$

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a \hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2 \hat{x}_1^{-1} - 1) \hat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$’s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$’s.

If $\alpha \in C_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t - 1) \bar{\alpha}_i}{\bar{\alpha}_i - \bar{\alpha}_{i+1}} G_\alpha(x) + G_{s_\alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t - 1) \bar{\alpha}_i}{\bar{\alpha}_i - \bar{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\bar{\alpha}_{i+1} - t \bar{\alpha}_i)(\bar{\alpha}_{i+1} - t^{-1} \bar{\alpha}_i)}{(\bar{\alpha}_{i+1} - \bar{\alpha}_i)^2} G_{s_\alpha}(x)$$
if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_iG_\alpha(x) = tG_\alpha(x)$ if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_iK_\alpha$ as linear combination of the $K_\beta$'s can now be obtained using the formula

$$G_\alpha(at) = \frac{\alpha_{i+1} - t\alpha_i}{\alpha_{i+1} - \alpha_i} G_{\alpha}^{s_\alpha}(at)$$

for $\alpha \in \mathcal{C}_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_iK = \tilde{H}_iK$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in \mathcal{C}_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-a_1} G_{\alpha^\circ}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^\circ}(at)}{G_\alpha(at)} = at^{1-n+k_1(\alpha)} - q^{a_1} t^{1-n}.$$ 

Hence,

$$\Phi K_\alpha(x) = t^{1-n} (a\alpha_1^{-1} - 1) K_{\alpha^\circ}(x).$$

Remark 11. Note that

$$\Phi K_\alpha(x) = (a\alpha_1 - t^{1-n}) K_{\alpha^\circ}(x)$$

for $\alpha \in \mathcal{C}_n$ since $\alpha_1^{-1} = t^{n-1}w_0\alpha$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}^+_{[k]}$ to a map $K \in \mathcal{F}^+_{[k]}(x)$ such that Lemma 10 remains valid.

Lemma 12. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a_\alpha_i^{-1}}{1 - q^{t^{n-1}x_i}} \right) K_{\alpha+(1^n)}(qx).$$
Proof. Note that for \( f \in \mathbb{K}[x], \)

\[
\Phi^n f(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1}x).
\]

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For \( m \in \mathbb{Z}_{\geq 0} \) we define \( A_m(x; v) \in \mathbb{K}(x) \) by

\[
A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m}a \bar{v}_i^{-1}; q)_m}{(qt^{n-1}x_i; q)_m}, \quad \forall v \in \mathbb{Z}^n,
\]

with \((y; q)_m := \prod_{j=0}^{m-1} (1 - q^jy)\) the \( q \)-shifted factorial.

Definition 13. Let \( v \in \mathbb{Z}^n \) and write \(|v| := v_1 + \cdots + v_n\). Define \( G_v(x) = G_v(x; q, t) \in \mathbb{F}(x) \) and \( K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x) \) by

\[
G_v(x) := q^{-m|v|-m^2n} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m(q^{-m}t^{1-n}x_i^{-1}; q)_m},
\]

\[
K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),
\]

where \( m \) is a nonnegative integer such that \( v + (m^n) \in C_n \) (note that \( G_v \) and \( K_v \) are well defined by Lemma 12).

Example 14. If \( n = 1 \) then for \( m \in \mathbb{Z}_{\geq 0} \),

\[
K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left( \frac{x}{a} \right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.
\]

Lemma 15. For all \( v \in \mathbb{Z}^n \),

\[
K_v(x) = \frac{G_v(x)}{G_v(at)}.
\]

Proof. Let \( v \in \mathbb{Z}^n \). Clearly \( G_v(x) \) and \( K_v(x) \) only differ by a multiplicative constant, so it suffices to show that \( K_v(at) = 1 \). Fix \( m \in \mathbb{Z}_{\geq 0} \) such that \( v + (m^n) \in C_n \). Then

\[
K_v(at) = A_m(at; v) K_{v+(m^n)}(q^m at) = A_m(at; v) \frac{G_{v+(m^n)}(q^m at)}{G_{v+(m^n)}(at)} = 1,
\]
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathbb{F}K(x)$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in \mathbb{F}K(x)$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathbb{F}K(x)$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathbb{F}K(x)$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathbb{F}K(x)$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathbb{F}K(x)$,

$$(\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $\overline{v}_1, \ldots, \overline{v}_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^1) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{\overline{v}+(m^1)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = tK_v$ and $H_i K_{\overline{v}+(m^1)}(q^m x) = tK_{\overline{v}+(m^1)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\overline{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

where $\Phi(q^m) := (q^m x_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \overline{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.
6  Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = -w_0 v$ for $v \in \mathbb{Z}^n$.

**Theorem 17.** (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}).$$  \hspace{1cm} (17)

**Example 18.** If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr}(a^{-1}; q)_{m+r} \frac{(a^{-1}; q)_{m}(a^{-1}; q)_r}{(a^{-1}; q)_{m+r}}$$  \hspace{1cm} (18)

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of $m$ and $r$.

**Proof.** We divide the proof of the theorem in several steps.  \hspace{1cm} ■

**Step 1.** If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{su}(a\tilde{v}) = K_v(a\tilde{s_i}\tilde{u})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

$$\frac{(t - 1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})}K_u(a\tilde{v}) + \frac{(\tilde{v}_i - t\tilde{v}_{i+1})}{(\tilde{v}_i - \tilde{v}_{i+1})}K_u(a\tilde{s_{i}}\tilde{u})$$

$$= \frac{(t - 1)\tilde{v}_i}{(\tilde{u}_i - \tilde{u}_{i+1})}K_u(a\tilde{v}) + \frac{(\tilde{u}_i - t\tilde{u}_{i+1})}{(\tilde{u}_i - \tilde{u}_{i+1})}K_{su}(a\tilde{v}).$$  \hspace{1cm} (19)

Replacing in (19) the role of $u$ and $v$ and replacing $i$ by $n - i$ we get

$$\frac{(t - 1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})}K_v(a\tilde{u}) + \frac{(\tilde{u}_{n-i} - t\tilde{u}_{n+1-i})}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})}K_v(a\tilde{s_i}\tilde{u})$$

$$= \frac{(t - 1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})}K_v(a\tilde{u}) + \frac{(\tilde{v}_{n-i} - t\tilde{v}_{n+1-i})}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})}K_{sv}(a\tilde{v}).$$  \hspace{1cm} (20)

Suppose that $s_{n-i}v = v$. Then $\tilde{v}_{n-i} = t\tilde{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n}w_0 \tilde{v}^{-1}$, that is, $\tilde{v}_i = t^{1-n}\tilde{v}_{n+1-i}$ we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{su}(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_v(a\tilde{s_i}\tilde{u}) = K_v(a\tilde{u})$ if $s_{n-i}v = v$. 
Hence, we obtain
\[ K_{s_i u}(a v) = K_u(a v) = K_v(a u) = K_v(a s_i u). \]
If \( s_{n-i}v \neq v \) then (19) and the induction hypothesis can be used to write \( K_{s_i u}(a v) \) as an explicit linear combination of \( K_v(a u) \) and \( K_{s_{n-i}v}(a u) \). Then (20) can be used to rewrite the term involving \( K_{s_{n-i}v}(a u) \) as an explicit linear combination of \( K_v(a u) \) and \( K_v(a s_i u) \). Hence, we obtain an explicit expression of \( K_{s_i u}(a v) \) as linear combination of \( K_v(a u) \) and \( K_v(a s_i u) \), which turns out to reduce to \( K_{s_i u}(a v) = K_v(a s_i u) \) after a direct computation.

**Step 2.** \( K_0(a v) = 1 = K_v(a 0) \) for all \( v \in \mathbb{Z}^n \).

**Proof of Step 2.** Clearly \( K_0(x) = 1 \) and \( K_v(a 0) = K_v(a \tau) = 1 \) for \( v \in \mathbb{Z}^n \) by Lemma 15.

**Step 3.** \( K_\alpha(a v) = K_v(a \alpha) \) for \( v \in \mathbb{Z}^n \) and \( \alpha \in C_n \).

**Proof of Step 3.** We prove it by induction. It is true for \( \alpha = 0 \) by Step 2. Let \( m \in \mathbb{Z}_{>0} \) and suppose that \( K_\gamma(a v) = K_v(a \gamma) \) for \( v \in \mathbb{Z}^n \) and \( \gamma \in C_n \) with \( |\gamma| < m \). Let \( \alpha \in C_n \) with \( |\alpha| = m \).

We need to show that \( K_\alpha(a v) = K_v(a \alpha) \) for all \( v \in \mathbb{Z}^n \). By Step 1 we may assume without loss of generality that \( \alpha_n > 0 \). Then \( \gamma := \alpha_n \in C_n \) satisfies \( |\gamma| = m - 1 \), and \( \alpha = \gamma + \alpha_n \). Furthermore, note that we have the formula
\[
(a v_{1}^{-1} - 1)K_u(a v_{1}) = (a u_{1}^{-1} - 1)K_{u_{1}}(a v_{1})
\]
for all \( u, v \in \mathbb{Z}^n \), which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain
\[
K_\alpha(a v) = K_\gamma(a v) = \frac{(a v_{1}^{-1} - 1)K_v(a v_{1})}{(a \gamma_{1}^{-1} - 1)}
\]
\[
= \frac{(a v_{1}^{-1} - 1)}{(a \gamma_{1}^{-1} - 1)}K_v(a \gamma) = K_v(a \gamma) = K_v(a \alpha),
\]
where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = v - (m^n) \) and \( q^{-m}\tilde{u} = u + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v})
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n)))
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n)))
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m}a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; v - (m^n)) = 1,
\]

which follows by a straightforward computation using (4). 

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}_{\text{Fix}}^+ \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\overline{K} = \overline{H}_i\overline{K} \).
2. \( \xi_j\overline{K} = \hat{\xi}_j^{-1}\overline{K} \).
3. \( x_n\Delta\overline{K} = t^{1-n}\hat{\Delta}^{-1}\overline{K} \).
Note that
\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1}x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},\]
\[\bar{K}_\alpha(x) = q^{\alpha|}\, t^{(1-n)n} \left( \prod_{i=1}^{n} (\bar{\alpha}_i x_i)^{-1} \right) \bar{K}_{\alpha+(1^n)}(x).\]  

(22)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_v(x) := E_v(x; q, t) \in \mathbb{F}[x_{\pm 1}]\) for arbitrary \(v \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[E_v(x) = \frac{E_{v+(m^n)}(x)}{(x_1 \cdots x_n)^m}.\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(v \in \mathbb{Z}^n\).

**Definition 20.** Let \(v \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(v + (m^n) \in C_n\). Then \(\bar{K}_v(x) := \bar{K}_v(x; q, t) \in \mathbb{F}[x_{\pm 1}]\) is defined by
\[\bar{K}_v(x) := q^{m|v|}\, t^{(1-n)n m} \left( \prod_{i=1}^{n} (\bar{\alpha}_i x_i)^{-m} \right) \bar{K}_{v+(m^n)}(x).\]

Using
\[\lim_{a \to \infty} A_m(ax; v) = q^{-m^2n} t^{(1-n)n m} \left( \prod_{i=1}^{n} (\bar{\alpha}_i x_i)^{-m} \right)\]
and the definitions of \(G_v(x)\) and \(K_v(x)\) it follows that
\[\lim_{a \to \infty} a^{-|v|} G_v(ax) = E_v(x),\]
\[\lim_{a \to \infty} K_v(ax) = \bar{K}_v(x)\]
for all \(v \in \mathbb{Z}^n\), so in particular
\[\bar{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \quad \forall v \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $\overline{K}$ to the map $\overline{K} \in F[x^{\pm 1}]$ defined by $\nu \mapsto \overline{K}_\nu$ ($\nu \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$\overline{K}_u(\overline{v}) = \overline{K}_v(\overline{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\widetilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\widetilde{O}_\alpha(\beta^{-1}) = K_\alpha(t^{1-n}aw_0\beta^{-1}) = K_\alpha(a\overline{\beta}) = K_{\beta}(a\overline{\alpha})$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\widetilde{O}_\alpha = O_\alpha$. ■

### 7.3 Okounkov’s duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]^{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in K[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\alpha^+(x)$$

(23)

for $\alpha \in C_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in P_n$ we have

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_\mathbb{K}$ for the function $f_\mu(u) := K_u(a\overline{\mu})(u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\overline{\mu}) = (C_+ K_\lambda)(a\overline{\mu}) = (\hat{C}_+ f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\overline{\mu}) = (Jw_0 K_\mu(t_{1-n}x))|_{x=a^{-1}\overline{u}}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_i J = (H_i^o)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^o)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\hat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t_{1-n}x))|_{x=a^{-1}\overline{\lambda}}.$$ 

By (23) and (4) this simplifies to

$$(\hat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\overline{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\overline{\mu}) = K_\mu^+(a\overline{\lambda})$. Since $K_\lambda^+$ is symmetric we obtain from (4) that

$$K_\lambda^+(a\overline{\mu}^{-1}) = K_\mu^+(a\overline{\lambda}^{-1}),$$

which is Okounkov's duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For \( u, v \in \mathbb{Z}^n \) we have
\[
(H_{w_0}K_u)(a\tilde{v}) = (H_{w_0}K_v)(a\tilde{u}).
\] (27)

Proof. We proceed as in the previous subsection. Set \( f_v(u) := K_u(a\tilde{v}) \) for \( u, v \in \mathbb{Z}^n \). By part 1 of Proposition 16,
\[
(H_{w_0}K_u)(a\tilde{v}) = (\hat{H}_{w_0}f_v)(u).
\]
Since \( f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}\tilde{u}) \) by (4), Remark 7 implies that
\[
(\hat{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}\tilde{u}).
\]

Now \( H_{w_0}Jw_0 = Jw_0H_{w_0} \) by (26); hence,
\[
(\hat{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}\tilde{u}) = (H_{w_0}K_v)(a\tilde{u}),
\]
which completes the proof. \[\blacksquare\]

Recall from Theorem 1 that
\[
G'_\beta(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G^o_{\beta}(t^{n-1}x)
\]
with \( \Psi := w_0H^o_{w_0} \). We define normalized versions by
\[
K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1}x)} = t^{\ell(w_0)}\Psi K^o_{\beta}(t^{n-1}x), \quad \beta \in C_n,
\]
with \( K^o_{\nu} := \iota(K_{\nu}) \) for \( \nu \in \mathbb{Z}^n \) (the 2nd formula follows from Lemma 2). More generally, we define for \( \nu \in \mathbb{Z}^n \),
\[
K'_\nu(x) := t^{\ell(w_0)}\Psi K^o_{\nu}(t^{n-1}x).
\] (28)

We write \( K' : \mathbb{Z}^n \to \mathbb{K}(x) \) for the map \( \nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n) \). Since \( H_i\Psi = \Psi H^o_i \), part 1 of Proposition 16 gives \( H_i K' = \hat{H}^{o}_i K' \). Considering the action of \((x_n - 1)^{\Delta^o})^n\) on \( K'_\beta(x) \) we get, using the fact that \((x_n - 1)^{\Delta^o})^n\) commutes with \( \Psi \) and part 3 of Proposition 16,
\[
K'_\nu(x) = \left( \prod_{i=1}^n \frac{(1-a^{-1}\tilde{v}_i)}{(1-q^{-1}x_i)} \right) K'_\nu+(1^n)(q^{-1}x),
\]
in particular

$$K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v_i; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+(m^n)}(q^{-m}x).$$

**Example 25.** For $n = 1$ we have $K'_v(x) = K'_v(x)$ for $v \in \mathbb{Z}$; hence,

$$K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m},$$

$$K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}$$

for $m \in \mathbb{Z}_{\geq 0}$ by Example 14.

**Proposition 26.** For all $u, v \in \mathbb{Z}^n$ we have

$$K'_v(a^{-1}u) = K'_u(a^{-1}v).$$

**Proof.** Note that

$$K'_v(a^{-1}u) = t^{(w_0)} \psi K'_{v+u}(t^{u^{-1}}x)|_{x=a^{-1}u} = t^{(w_0)} H_{w_0}^{u} K'_{v}(a^{-1}u^{-1})$$

by (4). By (27) the right-hand side is invariant under the interchange of $u$ and $v$. ■

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of $O_\alpha$ was used to prove the following binomial theorem [14, Thm. 1.3]. Define for $\alpha, \beta \in C_n$ the generalized binomial coefficient by

$$\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} := \frac{G_\alpha(\overline{\alpha})}{G_\beta(\overline{\beta})}.$$ (29)

Applying the automorphism $\iota$ of $\mathbb{F}$ to (29) we get

$$\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1},t^{-1}} = \frac{G_\alpha(\overline{\alpha}^{-1})}{G_\beta(\overline{\beta}^{-1})}.$$
Theorem 27. For $\alpha, \beta \in C_n$ we have the binomial formula
\[ K_\alpha(ax) = \sum_{\beta \in C_n} a^{\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}. \] (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as
\[
K_\alpha(ax) = \sum_{\beta \in C_n} \tau^{-1}_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} K'_\beta(x)
= \sum_{\beta \in C_n} \frac{K_\beta^0(\overline{\alpha}^{-1})K'_\beta(x)}{\tau_\beta K_\beta^0(\overline{\beta}^{-1})}
= t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{K_\beta^0(\overline{\alpha}^{-1}) \Psi K_\beta^0(t^{n-1}x)}{\tau_\beta K_\beta^0(\overline{\beta}^{-1})}
\] with $\Psi = w_0 H_0(\overline{w_0})$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K_\beta^0(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion
\[
(H_{w_0} K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K_\beta^0(\overline{\alpha}^{-1})K_\beta^0(t^{n-1}w_0x)}{\tau_\beta K_\beta^0(\overline{\beta}^{-1})}.
\]
Substituting $x = \tilde{\gamma}$ and using (4) we obtain
\[
(H_{w_0} K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in C_n} \frac{K_\beta^0(\overline{\alpha}^{-1})K_\beta^0(\overline{\gamma}^{-1})}{\tau_\beta K_\beta^0(\overline{\beta}^{-1})}.
\]
The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G^\circ_\beta(\overline{\alpha}) \Psi K^\circ_\beta(t^{n-1}x)}{\tau_\beta G^\circ_\beta(\overline{\beta})},$$

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, a t^{n-1} x, q^{-1}, t^{-1})$ and act by $w_0 H w_0$ on both sides. Since $w_0 H w_0 \Psi = \text{Id}$ we obtain

$$\Psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K^\circ_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \frac{\tau_\beta \Psi K^\circ_\beta(\overline{\alpha}) K_\beta(ax)}{K^\circ_\beta(\overline{\beta})}. \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.$$ 

Since $\left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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**References**


