Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$
Then $R_{\lambda}(x) = R_{\lambda}(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_{\lambda}(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$  

The normalization is fixed by requiring that the coefficient of $x^{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_{\lambda}(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_{\lambda}(x)$ is the Macdonald polynomial $P_{\lambda}(x)$ [9] and $R_{\lambda}(x)$ satisfies the extra vanishing property $R_{\lambda}(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_{\lambda}(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_{\lambda}(ax) = R_{\lambda}(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_{\mu}(x; q^{-1}, t^{-1})'$s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_{\lambda}(a\tilde{\mu})}{R_{\lambda}(a\tau)} = \frac{R_{\mu}(a\tilde{\lambda})}{R_{\mu}(a\tau)}.$$  

The interpolation polynomials have natural non-symmetric analogs $G_{\alpha}(x) = G_{\alpha}(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in C_n$ we define

$$\bar{\beta} := w_{\beta}(\beta_+),$$

where $w_{\beta}$ is the shortest permutation such that $\beta_+ = w_{\beta}^{-1}(\beta)$ is a partition. Then $G_{\alpha}(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_{\alpha}(\bar{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$  

The normalization is fixed by requiring that the coefficient of $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_{\alpha}(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_{\lambda}(x)$ admit non-symmetric counterparts for the $G_{\alpha}(x)$. For instance, the top homogeneous part of $G_{\alpha}(x)$
is the non-symmetric Macdonald polynomial \( E_\alpha(x) \) and \( G_\alpha(x) \) satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of \( G_\alpha(ax; q, t) \) in terms of a 2nd family of interpolation polynomials \( G'_\alpha(x) = G'_\alpha(x; q, t) \). These latter polynomials are characterized by having the same top homogeneous part as \( G_\alpha(x) \), namely the non-symmetric polynomial \( E_\alpha(x) \), and the following vanishing conditions at the evaluation points \( \tilde{\beta} := (-w_0\beta) \), with \( w_0 \) the longest element of the symmetric group \( S_n \):

\[
G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.
\]

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials \( G'_\alpha(x) \) in terms of \( G_\alpha(x) \), which involves the symmetric group action on the algebra of polynomials in \( n \) variables over \( \mathbb{F} \) by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators \( H_w \) \((w \in S_n)\) as described in the next section.

**Theorem A.** Write \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \). Then we have

\[
G'_\alpha(t^{n-1} x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)} w_0 H w_0 G_\alpha(x; q, t).
\]

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for \( G_\alpha(x) \), which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions \( \alpha, \beta \in C_n \) we have

\[
\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\bar{\alpha})}{G_\beta(a\tau)}.
\]

This is a special case of Theorem 17 below.

We now recall the interpolation \( O \)-polynomials introduced in [14, Thm. 1.1]. Write \( x^{-1} \) for \((x_1^{-1}, \ldots, x_n^{-1})\). Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial \( O_\alpha(x) = O_\alpha(x; q, t; a) \) of degree at most \( |\alpha| \) with coefficients in the field \( \mathbb{K} \) such that

\[
O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\bar{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.
\]

Our 3rd result is a simple expression for the \( O \)-polynomials in terms of the interpolation polynomials \( G_\alpha(x) \).
Theorem C. For all compositions $\alpha \in C_n$ we have

$$O_{\alpha}(x) = \frac{G_{\alpha}(t^{1-n}a\omega_0x)}{G_{\alpha}(a\tau)}.$$  

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_{\beta}(ax)$’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13], and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i \nu := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots)$ for $\nu = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \mapsto n+1-i$ for $i = 1, \ldots, n$.

For $\nu = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{\nu} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i \tau_i - k_i(\nu)}$ with

$$k_i(\nu) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$  

If $\nu \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{\nu} = (q^{v_1 \tau_1}, \ldots, q^{v_n \tau_n})$. For arbitrary $\nu \in \mathbb{Z}^n$ we have $\overline{\nu} = w_{\nu}(\overline{\nu})$ with $w_{\nu} \in S_n$ the shortest permutation such that $\nu_+ := w_{\nu}^{-1}(\nu)$ has non-increasing entries, see [4, Section 2]. We write $\overline{\nu} := -w_0 \nu$ for $\nu \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in C_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x_{\pm 1}^1 := F[x_1^\pm, \ldots, x_n^\pm]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i-x_{i+1}}(1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)$$
on \(\mathbb{F}(x)\) \((1 \leq i < n)\) called Demazure-Lusztig operators, and the automorphism \(\Delta\) of \(\mathbb{F}(x)\) defined by

\[
\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).
\]

Note that \(H_i\) \((1 \leq i < n)\) and \(\Delta\) preserve \(\mathbb{F}[x^\pm 1]\) and \(\mathbb{F}[x]\). Cherednik \([1, 2]\) showed that the operators \(H_i\) \((1 \leq i < n)\) and \(\Delta\) satisfy the defining relations of the type A extended affine Hecke algebra,

\[
(H_i - t)(H_i + 1) = 0,
\]

\[
H_iH_j = H_jH_i, \quad |i - j| > 1,
\]

\[
H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1},
\]

\[
\Delta H_{i+1} = H_i\Delta,
\]

\[
\Delta^2 H_1 = H_{n-1}\Delta^2
\]

for all the indices such that both sides of the equation make sense (see also \([4, \text{ Section 3}]\)). For \(w \in S_n\) we write \(H_w := H_{s_1s_2\cdots s_i}\) with \(w = s_1s_2\cdots s_i\) a reduced expression for \(w \in S_n\). It is well defined because of the braid relations for the \(H_i\)'s. Write \(H_i := H_i + 1 - t = tH_i^{-1}\) and set

\[
\xi_i := t^{1-n}H_{i-1}\cdots H_1\Delta^{-1}H_{n-1}\cdots H_i, \quad 1 \leq i \leq n. \tag{1}
\]

The operators \(\xi_i\)'s are pairwise commuting invertible operators, with inverses

\[
\xi_i^{-1} = H_i\cdots H_{n-1}\Delta H_1\cdots H_{i-1}.
\]

The \(\xi_i^{-1} (1 \leq i \leq n)\) are the Cherednik operators \([2, 4]\).

The monic non-symmetric Macdonald polynomial \(E_\alpha \in \mathbb{F}[x]\) of degree \(\alpha \in C_n\) is the unique polynomial satisfying

\[
\xi_i^{-1}E_\alpha = \overline{\alpha}_iE_\alpha, \quad i = 1, \ldots, n
\]

and normalized such that the coefficient of \(x^\alpha\) in \(E_\alpha\) is 1.

Let \(\iota\) be the field automorphism of \(\mathbb{K}\) inverting \(q, t\) and \(a\). It restricts to a field automorphism of \(\mathbb{F}\), inverting \(q\) and \(t\). We extend \(\iota\) to a \(\mathbb{Q}\)-algebra automorphism of \(\mathbb{K}[x]\)
and \( F[x] \) by letting \( \iota \) act on the coefficients of the polynomial. Write

\[
G^\circ_\alpha := \iota(G_\alpha), \quad E^\circ_\alpha := \iota(E_\alpha)
\]

for \( \alpha \in C_n \). Note that \( \overline{v}^{-1} = (\iota(\overline{v}_1), \ldots, \iota(\overline{v}_n)) \).

Put \( H^\circ_i, H^\circ_w, \overline{H}^\circ_i, \Delta^\circ \) and \( \xi^\circ_i \) for the operators \( H_i, H_w, \overline{H}_i, \Delta \) and \( \xi_i \) with \( q, t \) replaced by their inverses. For instance,

\[
H^\circ_i x = t^{-1} s_i - \frac{(1 - t^{-1}) x_i}{x_i - x_{i+1}} (1 - s_i),
\]

\[
\Delta^\circ f(x_1, \ldots, x_n) = \overline{f(qx_n, x_1, \ldots, x_{n-1})}.
\]

We then have \( \xi^\circ_i E^\circ_\alpha = \overline{\alpha}_i E^\circ_\alpha \) for \( i = 1, \ldots, n \), which characterizes \( E^\circ_\alpha \) up to a scalar factor.

**Theorem 1.** For \( \alpha \in C_n \) we have

\[
G'_\alpha(x) = t^{(1-n)|\alpha| + I(\alpha)} w_0 H^\circ_w G^\circ_\alpha(t^{n-1} x)
\]

with \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \).

**Remark.** Formally set \( t = q^r \), replace \( x \) by \( 1 + (q - 1)x \), divide both sides of (2) by \( (q - 1)|\alpha| \) and take the limit \( q \to 1 \). Then

\[
G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n-1)r; r)
\]

for the non-symmetric interpolation Jack polynomial \( G_\alpha(\cdot; r) \) and its primed version (see [14]). Here \( \sigma \) denotes the action of the symmetric group with \( \sigma(s_i) \) the rational degeneration of the Demazure-Lusztig operators \( H_i \), given explicitly by

\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}} (1 - s_i),
\]

see [14, Section 1]. To establish the formal limit (3) one uses that \( \sigma(w_0)w_0 = w_0 \sigma^\circ(w_0) \) with \( \sigma^\circ \) the action of the symmetric group defined in terms of the rational degeneration

\[
\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}} (1 - s_i)
\]

of \( H^\circ_i \). Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_\alpha$.

For the vanishing property, note that

\[ t^{n-1}w_0\tilde{\beta} = \beta^{-1} \quad (4) \]

(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,

\[ (w_0H_{w_0}^o G^o_\alpha(t^{n-1}x)|_{x=\beta} = (H_{w_0}^o G^o_\alpha(x))|_{x=\beta^{-1}}. \]

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G^o_\alpha(w\beta^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

\[ E_\alpha = t^{1(\alpha)}w_0H_{w_0}^o E^o_\alpha. \quad (5) \]

Note that $\Psi := w_0H_{w_0}^o$ satisfies the intertwining properties

\[ H_i \Psi = t\Psi H_i^o, \]

\[ \Delta \Psi = t^{n-1} \Psi H_{n-1}^o \cdots H_1^o(\Delta^o)^{-1}H_{n-1}^o \cdots H_1^o \quad (6) \]

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$. Therefore,

\[ E_\alpha(x) = c_\alpha \Psi E^o_\alpha(x) \]

for some constant $c_\alpha \in F$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_\alpha = t^{1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $F[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in F[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\bar{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in F[x]$ as defined before. Note that

\[ E_{u+(1^n)} = x_1 \cdots x_n E_u(x). \]
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{I(u)} w_0 H_{w_0}^\infty E_u^\infty$$

(7)

with $E_u^\infty := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E^{-w_0}u(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E^{-w_0}u(x^{-1}) = t^{I(u)} H_{w_0}^\infty E_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s)+1} t^{1-l'(s)}}{1 - q^{a(s)+1} t^{l'(s)+1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})$$

(8)

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

- $a(s) := \alpha_i - j$, $l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \}$,
- $a'(s) := j - 1$, $l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}$.

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - q^{a'(s)+1} t^{l'(s)+1}}{1 - q^{a(s)+1} t^{l'(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}.$$
**Lemma 2.** For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^0_\alpha(a \tau^{-1}).$$

**Proof.** Since $t^{n-1}w_0 \tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)}(H^0_{w_0} G^0_\alpha)(a \overline{\tau}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)}G^0_\alpha(a \overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. 

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

**Lemma 3.** For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|}t^{(1-n)|\alpha|-n(\alpha)}q^{n'(\alpha)}G^0_\alpha(a^{-1} \tau^{-1}).$$

**Proof.** This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G^0_\alpha$. 

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|}q^{n'(\alpha)}t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in C_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau^{-1}_\alpha a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9).

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\sharp := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\sharp := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^\sharp$ by $^\flat$, so $v^\flat = (v_n - 1, v_1, \ldots, v_{n-1})$ and $y^\flat = (y_n/q, y_1, \ldots, y_{n-1})$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = \overline{s_i}v$ if $v_i \neq v_{i+1}$.
2. $\overline{v}_i = t\overline{v}_{i+1}$ if $v_i = v_{i+1}$.
3. $v^\sharp = \overline{v}^\flat$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^{\pm 1}])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{align*}
(\widehat{H}_i f)(v) &:= tf(v) + \frac{\overline{v}_i - t\overline{v}_{i+1}}{\overline{v}_i - \overline{v}_{i+1}} (f(s_i v) - f(v)), \\
(\widehat{\Delta} f)(v) &:= f(v^\sharp), \\
(\widehat{x}_j f)(v) &:= a\overline{v}_j f(v)
\end{align*}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A)$, $X \mapsto \widehat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 
Remark 8. Let $O \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\tau$-invariant subset that contains $\{a \vec{v} \mid \vec{v} \in \mathbb{Z}^n\}$. Note that $O$ is contained in $\{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\}$. The Demazure–Lusztig operators $H_i$ $(1 \leq i < n)$, $\Delta^{\pm 1}$ and the coordinate multiplication operators $x_j$ $(1 \leq j \leq n)$ act $A$-linearly on the space $F_A^O$ of $A$-valued functions on $O$, and hence turns $F_A^O$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map

$$\text{pr} : F_A^O \to F_A$$

by $\text{pr}(g)(\vec{v}) := g(a \vec{v})$ ($\vec{v} \in \mathbb{Z}^n$).

We claim that Ker$(\text{pr})$ is an $\mathbb{H}$-submodule of $F_A^O$. Clearly Ker$(\text{pr})$ is $x_j$-invariant for $j = 1, \ldots, n$. Let $g \in \text{Ker}(\text{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(\text{pr})$. To show that $H_j g \in \text{Ker}(\text{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \vec{v} = s_i \vec{v}$ by part 1 of Lemma 5. Hence,

$$(H_i g)(a \vec{v}) = tg(a \vec{v}) + \frac{v_i - t v_{i+1}}{v_i - v_{i+1}} (g(a s_i \vec{v}) - g(a \vec{v})) = 0.$$ 

If $v_i = v_{i+1}$ then $\vec{v}_i = t \vec{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_i g)(\vec{v}) = tg(a \vec{v}) + \frac{v_i - t v_{i+1}}{v_i - v_{i+1}} (g(a s_i \vec{v}) - g(a \vec{v})) = tg(a \vec{v}) = 0.$$ 

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F_A^O / \text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i$ $(1 \leq i < n)$, $\Delta$ and $x_j$ $(1 \leq j \leq n)$ on $F_A$ is by the operators $\hat{H}_i$ $(1 \leq i < n)$, $\hat{\Delta}$ and $\hat{x}_j$ $(1 \leq j \leq n)$. ■

Remark 7. With the notations from (the proof of) Corollary 6, let $\tilde{g} \in F_A^O$ and set $g := \text{pr}(\tilde{g}) \in F_A$. In other words, $g(\vec{v}) := \tilde{g}(a \vec{v})$ for all $\vec{v} \in \mathbb{Z}^n$. Then

$$(\hat{X} g)(\vec{v}) = (X \tilde{g})(a \vec{v}), \quad \vec{v} \in \mathbb{Z}^n$$

for $X = H_i, \Delta^{\pm 1}, x_j$.

Remark 8. Let $F_A^+$ be the space of $A$-valued functions on $C_n$. We sometimes will consider $\hat{H}_i$ $(1 \leq i < n)$, $\hat{\Delta}^{-1}$ and $\hat{x}_j$ $(1 \leq j \leq n)$, defined by the formulas (11), as linear operators on $F_A^+$.

Definition 9. We call

$$K_\alpha(x; q, t, a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a \tau; q, t)} \in \mathbb{K}[x] \quad (12)$$

the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 
We frequently use the shorthand notation \( K_\alpha(x) := K_\alpha(x; q, t; a) \). We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that \( a \) cannot be specialized to 1 in (12) since \( G_\alpha(\tau) = G_\alpha(0) = 0 \) if \( \alpha \in C_n \) is nonzero. Note furthermore that
\[
\lim_{a \to \infty} K_\alpha(ax) = E_\alpha(x) E_\alpha(\tau) \quad (13)
\]
since \( \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x) \).

Recall from [4] the operator \( \Phi_1 = \left( x_n - t^{1-n} \right) \Delta \in \mathbb{H} \) and the inhomogeneous Cherednik operators
\[
\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.
\]
The operators \( H_i, \Xi_j \) and \( \Phi \) preserve \( \mathbb{K}[x] \) (see [4]); hence, they give rise to \( \mathbb{K} \)-linear operators on \( \mathcal{F}_\mathbb{K}[x] \) (e.g., \( (H_i f)(\alpha) := H_i(f(\alpha)) \) for \( \alpha \in C_n \)). Note that the operators \( H_i, \Xi_j \) and \( \Phi \) on \( \mathcal{F}_\mathbb{K}[x] \) commute with the hat-operators \( \hat{H}_i, \hat{x}_j \) and \( \hat{\Lambda}^{-1} \) on \( \mathcal{F}_\mathbb{K}[x] \) (cf. Remark 8). The same remarks hold true for the space \( \mathcal{F}_{\mathbb{K}(x)} \) of \( \mathbb{K}(x) \)-valued functions on \( \mathbb{Z}^n \) (in fact, in this case the hat-operators define a \( \mathbb{K} \)-action on \( \mathcal{F}_{\mathbb{K}(x)} \)).

Let \( K \in \mathcal{F}_\mathbb{K}[x] \) be the map \( \alpha \mapsto K_\alpha(\cdot) \) (\( \alpha \in C_n \)).

**Lemma 10.** For \( 1 \leq i < n \) and \( 1 \leq j \leq n \) we have in \( \mathcal{F}_\mathbb{K}[x] \),
\[
\begin{align*}
1. \quad H_i K &= \hat{H}_i K \\
2. \quad \Xi_j K &= a \hat{x}_j^{-1} K \\
3. \quad \Phi K &= t^{1-n}(a^2 \hat{x}_1^{-1} - 1) \hat{\Lambda}^{-1} K.
\end{align*}
\]

**Proof.** 1. To derive the formula we need to expand \( H_i K_\alpha \) as a linear combination of the \( K_\beta \)'s. As a 1st step we expand \( H_i G_\alpha \) as linear combination of the \( G_\beta \)'s.

If \( \alpha \in C_n \) satisfies \( \alpha_i < \alpha_{i+1} \) then
\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)
\]
by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that \( H_i \) satisfies the quadratic relation \( (H_i - t)(H_i + 1) = 0 \), it follows that
\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - t\overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1}\overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{s_i \alpha}(x)
\]
if $\alpha \in C_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = t G_\alpha(x)$ if $\alpha \in C_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the $K_\beta$'s can now be obtained using the formula

$$G_\alpha(a\tau) = \frac{\alpha_{i+1} - t\alpha_i}{\alpha_{i+1} - \alpha_i} G_{s\alpha}(a\tau)$$

for $\alpha \in C_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \tilde{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in C_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-|\alpha|} G_{\alpha^{-}}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^{-}}(a\tau)}{G_\alpha(a\tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.$$ 

Hence,

$$\Phi K_\alpha(x) = t^{1-n}(a\alpha_1^{-1} - 1) K_{\alpha^{-}}(x).$$

Remark 11. Note that

$$\Phi K_\alpha(x) = (a\alpha_n^{-} - t^{1-n}) K_{\alpha^{-}}(x)$$

for $\alpha \in C_n$ since $\alpha^{-1} = t^{n-1} w_0 \alpha$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}_{[\mathbb{C}]}^+$ to a map $K \in \mathcal{F}_{\mathbb{C}}(x)$ such that Lemma 10 remains valid.

Lemma 12. For $\alpha \in C_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left( \prod_{i=1}^n \frac{(1-a\alpha_i^{-1})}{(1-q t^{n-1} x_i)} \right) K_{\alpha+(1^n)}(x).$$
Proof. Note that for $f \in \mathbb{K}[x]$,

$$\Phi^nf(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10.

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m}av_i^{-1}; q)_m}{(qt^{n-1}x_i; q)_m} \quad \forall v \in \mathbb{Z}^n,$$

with $(y; q)_m := \prod_{j=0}^{m-1} (1 - q^jy)$ the $q$-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v|-m^2n} \frac{G_{v+(mn)}(q^{m}x)}{\prod_{i=1}^{n} x_i^m(q^{-m}t^{1-n}x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v)K_{v+(mn)}(q^{m}x),$$

where $m$ is a nonnegative integer such that $v + (mn) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left( \frac{x}{a} \right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.$$

Lemma 15. For all $v \in \mathbb{Z}^n$,

$$K_v(x) = \frac{G_v(x)}{G_v(at)}.$$  

Proof. Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(at) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (mn) \in C_n$. Then

$$K_v(at) = A_m(at; v)K_{v+(mn)}(q^{m}at) = A_m(at; v) \frac{G_{v+(mn)}(q^{m}at)}{G_{v+(mn)}(at)} = 1,$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : \mathcal{C}_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a \hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n} (a^{2} \hat{x}_1^{-1} - 1) \Delta^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $$(A_m f)(v) = A_m(x; v)f(v)$$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$((\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1} \quad (15)$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^*) \in \mathcal{C}_n$. Since

$$K_v(x) = A_m(x; v) K_{v + (m^*)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\hat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_i K)(v) = t K_v$ and $H_i K_{v + (m^*)}(q^m x) = t K_{v + (m^*)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n} (a \bar{v}_1^{-1} - 1) K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m), \quad (16)$$

where $\Phi(q^m) := (q^m x_n - t^{1-n}) \Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \bar{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition. ■
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = -w_0 v$ for $v \in \mathbb{Z}^n$.

**Theorem 17.** (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}). \quad (17)$$

**Example 18.** If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m} (a^{-1}; q)_r}. \quad (18)$$

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of $m$ and $r$.

**Proof.** We divide the proof of the theorem in several steps. \[\Box\]

**Step 1.** If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{siu}(a\tilde{v}) = K_v(a\tilde{s_iu})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

$$\left(\frac{(t-1)\tilde{v}_i}{\tilde{v}_i - \tilde{v}_{i+1}}\right) K_u(a\tilde{v}) + \left(\frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}}\right) K_u(a\tilde{s_n-i}v)$$

$$= \left(\frac{(t-1)\tilde{u}_i}{\tilde{u}_i - \tilde{u}_{i+1}}\right) K_u(a\tilde{v}) + \left(\frac{\tilde{u}_i - t\tilde{u}_{i+1}}{\tilde{u}_i - \tilde{u}_{i+1}}\right) K_{siu}(a\tilde{v}). \quad (19)$$

Replacing in (19) the role of $u$ and $v$ and replacing $i$ by $n-i$ we get

$$\left(\frac{(t-1)\tilde{u}_{n-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}}\right) K_v(a\tilde{u}) + \left(\frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}}\right) K_v(a\tilde{s_{n-i}}u)$$

$$= \left(\frac{(t-1)\tilde{v}_{n-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}}\right) K_v(a\tilde{u}) + \left(\frac{\tilde{v}_{n-i} - t\tilde{v}_{n+1-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}}\right) K_{s_{n-i}v}(a\tilde{u}). \quad (20)$$

Suppose that $s_{n-i}v = v$. Then $\tilde{v}_{n-i} = t\tilde{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n}w_0 v^{-1}$, that is, $\tilde{v}_i = t^{1-n}v^{-1}_{n+1-i}$, we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{s_{i}u}(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_v(a\tilde{s_{i}u}) = K_v(a\tilde{u})$ if $s_{n-i}v = v$. 


We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v$. We have to show that $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s_iu})$ for all $v$. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that $v$ satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(a\tilde{s_iu}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i}v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s_iu})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s_iu})$, which turns out to reduce to $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s_iu})$ after a direct computation.

**Step 2.** $K_0(a\tilde{v}) = 1 = K_v(a\tilde{0})$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(at) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a\tilde{v}) = K_v(a\tilde{\gamma})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := a\tilde{\alpha} \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\circ$. Furthermore, note that we have the formula

$$K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha}) = (a\tilde{v}_1^{-1} - 1)K_u(a\tilde{v}) = (a\tilde{u}_1^{-1} - 1)K_u(a\tilde{v})$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\tilde{v}) = K_v(a\tilde{\gamma}) = K_v(a\tilde{\gamma}^\circ) = K_v(a\tilde{\alpha}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $u, v \in \mathbb{Z}^n$.

Proof of Step 4. Fix $u, v \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $u + (m^n) \in C_n$. Note that $q^m\tilde{v} = v - (m^n)$ and $q^{-m}\tilde{u} = u + (m^n)$. Then

$$K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v})$$
$$= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n)))$$
$$= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n)))$$
$$= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n))K_v(a\tilde{u}),$$

where we used Step 3 in the 3rd equality. The result now follows from the fact that

$$A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n)) = 1,$$

which follows by a straightforward computation using (4).

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial $E_\alpha(x)$ of degree $\alpha$ is the top homogeneous component of $G_\alpha(x)$, i.e.,

$$E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|}G_\alpha(ax), \quad \alpha \in C_n.$$

The normalized non-symmetric Macdonald polynomials are

$$\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.$$

We write $\overline{K} \in \mathcal{F}^{+}_{\text{Fix}}$ for the resulting map $\alpha \mapsto \overline{K}_\alpha$. Taking limits in Lemma 10 we get the following.

**Lemma 19.** We have for $1 \leq i < n$ and $1 \leq j \leq n$,

1. $H_i\overline{K} = \tilde{H}_i\overline{K}$.
2. $\xi_j\overline{K} = \tilde{x}_j^{-1}\overline{K}$.
3. $x_n\Delta\overline{K} = t^{1-n}\tilde{x}_1^{-1}\tilde{\Delta}^{-1}\overline{K}$. 

Note that

\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^n x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),

\[E_\alpha(x) = E_{\alpha+(1^n)}(x) \left/ x_1 \cdots x_n \right.,\]

\[\bar{K}_\alpha(x) = q^{[\alpha]} t^{(1-n)n} \left( \prod_{i=1}^n (\bar{\alpha}_i x_i)^{-1} \right) \bar{K}_{\alpha+(1^n)}(x).\]

(22)

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula

\[E_\nu(x) = E_{\nu+(m^n)}(x) \left/ (x_1 \cdots x_n)^m \right..\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n\).

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n\). Then \(\bar{K}_\nu(x) := \bar{K}_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) is defined by

\[\bar{K}_\nu(x) := q^{[\nu]} t^{(1-n)n} \left( \prod_{i=1}^n (\bar{\nu}_i x_i)^{-1} \right) \bar{K}_{\nu+(m^n)}(x).\]

Using

\[\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2 n} t^{(1-n)n} \prod_{i=1}^n (\bar{\nu}_i x_i)^{-m}\]

and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that

\[\lim_{a \to \infty} a^{-[\nu]} G_\nu(ax) = E_\nu(x),\]

\[\lim_{a \to \infty} K_\nu(ax) = \bar{K}_\nu(x)\]

for all \(\nu \in \mathbb{Z}^n\), so in particular

\[\bar{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $\mathbb{K}$ to the map $\mathbb{K} \in \mathcal{F}[x^{\pm 1}]$ defined by $\nu \mapsto \mathbb{K}_\nu$ ($\nu \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$\mathbb{K}_u(\bar{v}) = \mathbb{K}_v(\bar{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\bar{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\bar{\beta}^{-1}) = K_\alpha(\bar{a}\tilde{\beta}) = K_\beta(\bar{a}\tilde{\alpha})$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$.  

### 7.3 Okounkov’s duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]^{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of $x_\lambda$ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(\alpha \tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{(w)} \right) K_\alpha^+(x)$$

for $\alpha \in C_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in P_\mathbb{N}$ we have

$$K^+_{\lambda}(a\mu^{-1}) = K^+_{\mu}(a\lambda^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\tilde{C}_+ = \sum_{w \in S_n} \tilde{H}_w$, with $\tilde{H}_w := \tilde{H}_{s_{i_1}} \cdots \tilde{H}_{s_{i_r}}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_K$ for the function $f_\mu(u) := K_u(a\tilde{\mu}) (u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_{\lambda}(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\tilde{C}_+ f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{\mu}) = (Jw_0 K_\mu(t^{1-n}x))|_{x=a^{-1}u}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in K(x)$. A direct computation shows that

$$JH_i J = (H_i^o)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^o)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\tilde{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x=a^{-1}\lambda}. $$

By (23) and (4) this simplifies to

$$(\tilde{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_{\mu}(a\tilde{\lambda}).$$

Returning to (24) we conclude that $K^+_{\lambda}(a\tilde{\mu}) = K^+_{\mu}(a\tilde{\lambda})$. Since $K^+_{\lambda}$ is symmetric we obtain from (4) that

$$K^+_{\lambda}(a\mu^{-1}) = K^+_{\mu}(a\lambda^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0}K_u)(a \tilde{v}) = (H_{w_0}K_v)(a \tilde{u}).$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a \tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0}K_u)(a \tilde{v}) = (\hat{H}_{w_0}f_v)(u).$$

Since $f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}u)$ by (4), Remark 7 implies that

$$(\hat{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}u).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$(\hat{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}u) = (H_{w_0}K_v)(a \tilde{u}),$$

which completes the proof.

Recall from Theorem 1 that

$$G'_\beta(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G^\circ_\beta(t^{n-1}x)$$

with $\Psi := w_0H^\circ_{w_0}$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1}\tau)} = t^{\ell(w_0)\Psi K^\circ_\beta(t^{n-1}x)}, \quad \beta \in C_n,$$

with $K^\circ_\nu := \iota(K_\nu)$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K'_\nu(x) := t^{\ell(w_0)\Psi K^\circ_\nu(t^{n-1}x)}.$$  \hspace{1cm} (28)

We write $K' : \mathbb{Z}^n \rightarrow \mathbb{K}(x)$ for the map $\nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n)$. Since $H_i \Psi = \Psi H^\circ_i$, part 1 of Proposition 16 gives $H_i K' = \hat{H}^\circ_i K'$. Considering the action of $((x_{n-1})^n(\Delta^o))^n$ on $K'_\beta(x)$ we get, using the fact that $((x_{n-1})^n(\Delta^o))^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K'_\nu(x) = \left(\prod_{i=1}^{n} \frac{(1-a^{-1}\tilde{v_i})}{(1-q^{-1}x_i)}\right)K'_{\nu + (1^n)}(q^{-1}x),$$
in particular

\[ K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v_i; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+(m)}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K''_v(x) \) for \( v \in \mathbb{Z} \); hence,

\[ K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m}, \]

\[ K'_{m}(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m} \]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_v(a^{-1}u) = K'_u(a^{-1}v). \]

**Proof.** Note that

\[ K'_v(a^{-1}u) = t^{\ell(w_0)} \Psi K^\circ_v(t^{n-1}x) |_{x = a^{-1}u} = t^{\ell(w_0)} (H^\circ_{w_0} K_v)(a^{-1}u^{-1}) \]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \). □

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_{\alpha} \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} := \frac{G_{\beta}(\overline{\alpha})}{G_{\beta}(\overline{\beta})}. \]  

(29)

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get

\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1},t^{-1}} = \frac{G^\circ_{\beta}(\alpha^{-1})}{G^\circ_{\beta}(\beta^{-1})}. \]
Theorem 27. For $\alpha, \beta \in \mathbb{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} a^{\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}. \quad (30)$$

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in \mathbb{C}_n} \tau^{-1}_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} K'_\beta(x)$$

$$= \sum_{\beta \in \mathbb{C}_n} \frac{K^0_\beta(\alpha^{-1}) K'_\beta(x)}{\tau_\beta K^0_\beta(\beta^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in \mathbb{C}_n} \frac{K^0_\beta(\alpha^{-1}) \Psi K^0_\beta(t^{n-1}x)}{\tau_\beta K^0_\beta(\beta^{-1})} \quad (31)$$

with $\Psi = w_0 H^0 w_0$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^0_\beta(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H w_0 \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H w_0 K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K^0_\beta(\alpha^{-1}) K^0_\beta(t^{n-1}w_0x)}{\tau_\beta K^0_\beta(\beta^{-1})}.$$ 

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H w_0 K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in \mathbb{C}_n} \frac{K^0_\beta(\alpha^{-1}) K^0_\beta(\gamma^{-1})}{\tau_\beta K^0_\beta(\beta^{-1})}.$$ 

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all \( \alpha \in C_n \) we have

\[
K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax). \tag{32}
\]

The starting point of the alternative proof of (32) is the binomial formula in the form

\[
K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta(\overline{\alpha}^{-1}) \Psi K_\beta(t^{n-1}x)}{\tau_\beta G_\beta(\overline{\beta}^{-1})},
\]

see (31). Replace \((a, x, q, t)\) by \((a^{-1}, at^{n-1}x, q^{-1}, t^{-1})\) and act by \(w_0Hw_0\) on both sides. Since \(w_0Hw_0\Psi = \text{Id}\) we obtain

\[
\Psi K_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax).
\]

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

\[
\Psi K_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \frac{\tau_\beta K_\beta(\overline{\alpha}) K_\beta(ax)}{K_\beta(\overline{\beta})}. \tag{33}
\]

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

\[
\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} \begin{bmatrix} \beta \\ \gamma \end{bmatrix}_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.
\]

Since \(\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_{q,t} = 0\) unless \(\delta \supseteq \epsilon\), the terms in the sum are zero unless \(\gamma \subseteq \beta \subseteq \alpha\).

**Acknowledgments**

We thank Eric Rains for sharing with us his unpublished results with Alain Lascoux and Ole Warnaar on a one-parameter rational extension of the non-symmetric interpolation Macdonald
polynomials. It leads to a different proof of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem B). We thank an anonymous referee for detailed comments.

**Funding**

This work was partially supported by Simons Foundation [509766 to S.S.].

**References**


