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DOI
10.1093/imrn/rnz229

Publication date
2021

Document Version
Final published version

Published in
International Mathematics Research Notices: IMRN

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Citation for published version (APA):
Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials
\[ R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t) \]
form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in \( n \) variables over the field \( \mathbb{F} := \mathbb{Q}(q, t) \). They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most \( n \) parts
\[ \mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}. \]

For a partition \( \mu \in \mathcal{P}_n \) we define \( |\mu| = \mu_1 + \cdots + \mu_n \) and write
\[ \overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \]
where \( \tau := (\tau_1, \ldots, \tau_n) \) with \( \tau_i := t^{1-i} \).
Then $R_{\lambda}(x) = R_{\lambda}(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_{\lambda}(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$ 

The normalization is fixed by requiring that the coefficient of $x_{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_{\lambda}(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_{\lambda}(x)$ is the Macdonald polynomial $P_{\lambda}(x)$ [9] and $R_{\lambda}(x)$ satisfies the extra vanishing property $R_{\lambda}(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_{\lambda}(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_{\lambda}(ax) = R_{\lambda}(ax_1, \ldots, ax_n; q, t)$ in terms of the $R_{\mu}(x; q^{-1}, t^{-1})$’s over the field $K := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_{\lambda}(a\tilde{\mu})}{R_{\lambda}(a\tau)} = \frac{R_{\mu}(a\tilde{\lambda})}{R_{\mu}(a\tau)}.$$

The interpolation polynomials have natural non-symmetric analogs $G_{\alpha}(x) = G_{\alpha}(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := \{\mathbb{Z}_{\geq 0}\}^n$. For a composition $\beta \in C_n$ we define

$$\bar{\beta} := w_\beta(\bar{\beta}_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_{\alpha}(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_{\alpha}(\bar{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$ 

The normalization is fixed by requiring that the coefficient of $x_{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_{\alpha}(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_{\lambda}(x)$ admit non-symmetric counterparts for the $G_{\alpha}(x)$. For instance, the top homogeneous part of $G_{\alpha}(x)$...
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w \ (w \in S_n)$ as described in the next section.

**Theorem A.** Write $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha|-I(\alpha)}H_w_0 G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 


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Theorem C. For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n}aW_0x)}{G_\alpha(a\tau)}.$$  

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$’s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i + 1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $K^n$ by $s_i \nu := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots)$ for $\nu = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \mapsto n+1-i$ for $i = 1, \ldots, n$.

For $\nu = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\bar{\nu} = (\bar{v}_1, \ldots, \bar{v}_n) \in \mathbb{F}^n$ by $\bar{v}_i := q^{v_i}t^{-k_i(\nu)}$ with

$$k_i(\nu) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$  

If $\nu \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\bar{\nu} = (q^{v_1} \tau_1, \ldots, q^{v_n} \tau_n)$. For arbitrary $\nu \in \mathbb{Z}^n$ we have $\bar{\nu} = w_\nu(\bar{\nu}^+) + w_\nu \in S_n$ the shortest permutation such that $\nu_+ := w_\nu^{-1}(\nu)$ has non-increasing entries, see [4, Section 2]. We write $\bar{\nu} := -w_0 \nu$ for $\nu \in \mathbb{Z}^n$.

Note that $\bar{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^\pm 1] := F[x_1^\pm, \ldots, x_n^\pm]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1).$$
on \( \mathbb{F}(x) \) \( (1 \leq i < n) \) called Demazure-Lusztig operators, and the automorphism \( \Delta \) of \( \mathbb{F}(x) \) defined by

\[
\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).
\]

Note that \( H_i \) \( (1 \leq i < n) \) and \( \Delta \) preserve \( \mathbb{F}[x^{\pm 1}] \) and \( \mathbb{F}[x] \). Cherednik [1, 2] showed that the operators \( H_i \) \( (1 \leq i < n) \) and \( \Delta \) satisfy the defining relations of the type A extended affine Hecke algebra,

\[
(H_i - t)(H_i + 1) = 0, \\
H_i H_j = H_j H_i, \quad |i - j| > 1, \\
H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}, \\
\Delta H_{i+1} = H_i \Delta, \\
\Delta^2 H_1 = H_{n-1} \Delta^2
\]

for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For \( w \in S_n \) we write \( H_w := H_{i_1} H_{i_2} \cdots H_{i_l} \) with \( w = s_{i_1} s_{i_2} \cdots s_{i_l} \) a reduced expression for \( w \in S_n \). It is well defined because of the braid relations for the \( H_i \)'s. Write \( \overline{H}_i := H_i + 1 - t = tH_i^{-1} \) and set

\[
\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)
\]

The operators \( \xi_i \)'s are pairwise commuting invertible operators, with inverses

\[
\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.
\]

The \( \xi_i^{-1} \) \( (1 \leq i \leq n) \) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial \( E_\alpha \in \mathbb{F}[x] \) of degree \( \alpha \in \mathbb{C}_n \) is the unique polynomial satisfying

\[
\xi_i^{-1} E_\alpha = \overline{u}_i E_\alpha, \quad i = 1, \ldots, n
\]

and normalized such that the coefficient of \( x^\alpha \) in \( E_\alpha \) is 1.

Let \( \iota \) be the field automorphism of \( \mathbb{K} \) inverting \( q, t \) and \( a \). It restricts to a field automorphism of \( \mathbb{F} \), inverting \( q \) and \( t \). We extend \( \iota \) to a \( \mathbb{Q} \)-algebra automorphism of \( \mathbb{K}[x] \).
and $\mathbb{F}[x]$ by letting $\iota$ act on the coefficients of the polynomial. Write

$$G^\circ_\alpha := \iota(G_\alpha), \quad E^\circ_\alpha := \iota(E_\alpha)$$

for $\alpha \in C_n$. Note that $\nabla^{-1} = (\iota(\nabla_1), \ldots, \iota(\nabla_n))$.

Put $H^\circ_\iota, \ H^\circ_w, \ \overline{H}_i, \ \Delta^\circ$ and $\xi^\circ_i$ for the operators $H_\iota, \ H_w, \ \overline{H}_i, \ \Delta$ and $\xi_i$ with $q, t$ replaced by their inverses. For instance,

$$H^\circ_i = t^{-1}s_i - \frac{(1 - t^{-1})x_i}{x_i - x_{i+1}}(1 - s_i),$$

$$\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi^\circ_i E^\circ_\alpha = \overline{\alpha}_i E^\circ_\alpha$ for $i = 1, \ldots, n$, which characterizes $E^\circ_\alpha$ up to a scalar factor.

**Theorem 1.** For $\alpha \in C_n$ we have

$$G'_\alpha(x) = t^{(1-n)|\alpha|+I(\alpha)}w_0H^\circ_wG^\circ_\alpha(t^{n-1}x) \quad (2)$$

with $I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \}$.

**Remark.** Formally set $t = qr$, replace $x$ by $1 + (q - 1)x$, divide both sides of (2) by $(q - 1)^{|\alpha|}$ and take the limit $q \to 1$. Then

$$G'_\alpha(x; r) = (-1)^{|\alpha|}\sigma(w_0)w_0G_\alpha(-x - (n - 1)r; r) \quad (3)$$

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here $\sigma$ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators $H_i$, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0\sigma^\circ(w_0)$ with $\sigma^\circ$ the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)$$

of $H^\circ_i$. Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_\alpha$.

For the vanishing property, note that
\[ t^{n-1}w_0\beta = \beta^{-1} \]  
\[ (w_0H_{w_0}^0 G_\alpha^0(t^{n-1}x))|_{x=\beta} = (H_{w_0}^0 G_\alpha^0(x))|_{x=\beta^{-1}}. \]

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_\alpha^0(w\beta^{-1}) (w \in S_n)$ by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that
\[ E_\alpha = t^{1(\alpha)}w_0H_{w_0}^0 E_\alpha^{\circ}. \]  

Note that $\Psi := w_0H_{w_0}^0$ satisfies the intertwining properties
\[ H_i\Psi = t\Psi H_i^0, \]
\[ \Delta \Psi = t^{n-1}\Psi H_{n-1}^0 \cdots H_1^0(\Delta^0)^{-1}H_{n-1}^0 \cdots H_1^0 \]
for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi \xi_i^0$ for $i = 1, \ldots, n$. Therefore,
\[ E_\alpha(x) = c_\alpha \Psi E_\alpha^{\circ}(x) \]
for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-1(\alpha)}$; hence, $c_\alpha = t^{1(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $u_j (1 \leq j \leq n)$, normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that
\[ E_{u+(1^n)} = x_1 \cdots x_n E_u(x). \]
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{l(u)} w_0 H^\circ_{w_0} E^\circ_u$$

(7)

with $E^\circ_u := t(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u} (x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0 u} (x^{-1}) = t^{l(u)} H^\circ_{w_0} E^\circ_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - qa^\prime(s) + t^{1-l'(s)}}{1 - q^{a(s) + 1} t^{l(s) + 1}} \right) \prod_{s \in \alpha} (at^{l(s)} - q^{a'(s)})$$

(8)

was obtained, with $a(s), l(s), a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \},$$

$$a'(s) := j - 1, \quad l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}.$$  

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - qa' \cdot (s) + t^{1-l'(s) + 1}}{1 - q^{a(s) + 1} t^{l(s) + 1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}.$$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(\mathbf{a}\tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G_\alpha^0(\mathbf{a}\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = \overline{0}^{-1}$ we have by Theorem 1,

$$G'_\alpha(\mathbf{a}\tau) = t^{(1-n)|\alpha|+I(\alpha)}(H^0_{w_0}G_\alpha^0)(\mathbf{a}\overline{0}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G_\alpha^0(\mathbf{a}\overline{0}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G_\alpha^0(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(\mathbf{a}\tau) = (-\mathbf{a})^{n(\alpha)} t^{n(\alpha)-n(\phi)} q^{n'(\alpha)} G_\alpha^0(\mathbf{a}^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{l(\alpha)} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in \mathbb{C}^n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau_a^{-1}a^{-|\alpha|}G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). $\blacksquare$

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^i := (v_2, \ldots, v_n, v_1 + 1)$ and $y^i := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^i$ by $^\natural$, so $v^\natural = (v_n - 1, v_1, \ldots, v_1)$ and $y^\natural = (y_n/q, y_1, \ldots, y_n)$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v) = s_i^\natural v$ if $v_i \neq v_{i+1}$.
2. $v_i = t^{i+1}_v(v_{i+1})$ if $v_i = v_{i+1}$.
3. $v^\natural = v^\natural$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^{\pm 1}])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\hat{H}_i$ ($1 \leq i < n$), $\hat{\Delta}$ and $\hat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{align*}
(\hat{H}_if)(v) &:= tf(v) + \frac{v_i - t^{i+1}_v(v_{i+1})}{v_i - v_{i+1}}(f(s_i v) - f(v)), \\
(\hat{\Delta}f)(v) &:= f(v^\natural), \quad (\hat{\Delta}^{-1}f)(v) := f(v^\natural), \\
(\hat{x}_jf)(v) &:= a v_j f(v)
\end{align*}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \hat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \hat{\Delta}$ and $x_j \mapsto \hat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A)$, $X \mapsto \hat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 

Remark 8. Let \( O \subset \mathbb{K}^n \) be the smallest \( S_n \)-invariant and \( t \)-invariant subset that contains \( \{a \bar{v} \mid v \in \mathbb{Z}^n\} \). Note that \( O \) is contained in \( \{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\} \). The Demazure–Lusztig operators \( H_i \) (\( 1 \leq i < n \)), \( \Delta^{\pm 1} \) and the coordinate multiplication operators \( x_j \) (\( 1 \leq j \leq n \)) act \( A \)-linearly on the space \( F_A^O \) of \( A \)-valued functions on \( O \), and hence turns \( F_A^O \) into an \( \mathbb{H} \)-module. Define the surjective \( A \)-linear map

\[
pr : F_A^O \to F_A
\]

by \( pr(g)(v) := g(a \bar{v}) \) (\( v \in \mathbb{Z}^n \)).

We claim that \( Ker(pr) \) is an \( \mathbb{H} \)-submodule of \( F_A^O \). Clearly \( Ker(pr) \) is \( x_j \)-invariant for \( j = 1, \ldots, n \). Let \( g \in Ker(pr) \). Part 3 of Lemma 5 implies that \( \Delta g \in Ker(pr) \). To show that \( H_i g \in Ker(pr) \) we consider two cases. If \( v_i \neq v_{i+1} \) then \( s_i \bar{v} = \bar{s}_i \bar{v} \) by part 1 of Lemma 5. Hence,

\[
(H_i g)(a \bar{v}) = tg(a \bar{v}) + \frac{\bar{v}_i - t \bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a s_i \bar{v}) - g(a \bar{v})) = 0.
\]

If \( v_i = v_{i+1} \) then \( \bar{v}_i = t \bar{v}_{i+1} \) by part 2 of Lemma 5. Hence,

\[
(H_i g)(\bar{v}) = tg(a \bar{v}) + \frac{\bar{v}_i - t \bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a s_i \bar{v}) - g(a \bar{v})) = tg(a \bar{v}) = 0.
\]

Hence, \( F_A \) inherits the \( \mathbb{H} \)-module structure of \( F_A^O / Ker(pr) \). It is a straightforward computation, using Lemma 5 again, to show that the resulting action of \( H_i \) (\( 1 \leq i < n \)), \( \Delta \) and \( x_j \) (\( 1 \leq j \leq n \)) on \( F_A \) is by the operators \( \hat{H}_i \) (\( 1 \leq i < n \)), \( \widehat{\Delta} \) and \( \widehat{x}_j \) (\( 1 \leq j \leq n \)).

Remark 7. With the notations from (the proof of) Corollary 6, let \( \tilde{g} \in F_A^O \) and set \( g := pr(\tilde{g}) \in F_A \). In other words, \( g(v) := \tilde{g}(a \bar{v}) \) for all \( v \in \mathbb{Z}^n \). Then

\[
(\hat{X} g)(v) = (X \tilde{g})(a \bar{v}), \quad v \in \mathbb{Z}^n
\]

for \( X = H_i, \Delta^{\pm 1}, x_j \).

Remark 8. Let \( F_A^+ \) be the space of \( A \)-valued functions on \( \mathcal{C}_n \). We sometimes will consider \( \hat{H}_i \) (\( 1 \leq i < n \)), \( \widehat{\Delta}^{-1} \) and \( \widehat{x}_j \) (\( 1 \leq j \leq n \)), defined by the formulas (11), as linear operators on \( F_A^+ \).

Definition 9. We call

\[
K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a \tau; q, t)} \in \mathbb{K}[x]
\]

the normalized non-symmetric interpolation Macdonald polynomial of degree \( \alpha \).
We frequently use the shorthand notation \( K_\alpha(x) := K_\alpha(x; q, t; a) \). We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that \( a \) cannot be specialized to 1 in (12) since \( G_\alpha(\tau) = G_\alpha(0) = 0 \) if \( \alpha \in C_n \) is nonzero. Note furthermore that

\[
\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}
\]

since \( \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x) \).

Recall from [4] the operator \( \Phi_1 = (x_n - t^{1-n}) \Delta \in \mathbb{H} \) and the inhomogeneous Cherednik operators

\[
\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.
\]

The operators \( H_i, \Xi_j \) and \( \Phi \) preserve \( K[x] \) (see [4]); hence, they give rise to \( \mathbb{K}\)-linear operators on \( F^+_{K[x]} \) (e.g., \( (H_i f)(\alpha) := H_i(f(\alpha)) \) for \( \alpha \in C_n \)). Note that the operators \( H_i, \Xi_j \) and \( \Phi \) on \( F^+_{K[x]} \) commute with the hat-operators \( \tilde{H}_i, \tilde{x}_j \) and \( \tilde{\Delta}^{-1} \) on \( F^+_{K[x]} \) (cf. Remark 8). The same remarks hold true for the space \( F^+_\mathbb{K}[x) \) of \( \mathbb{K}(x) \)-valued functions on \( \mathbb{Z}^n \) (in fact, in this case the hat-operators define a \( \mathbb{H} \)-action on \( F^+_\mathbb{K}[x) \)).

Let \( K \in F^+_\mathbb{K}[x] \) be the map \( \alpha \mapsto K_\alpha(\cdot) (\alpha \in C_n) \).

**Lemma 10.** For \( 1 \leq i < n \) and \( 1 \leq j \leq n \) we have in \( F^+_\mathbb{K}[x] \):

1. \( H_i K = \tilde{H}_i K \).
2. \( \Xi_j K = a \tilde{x}_j^{-1} K \).
3. \( \Phi K = t^{1-n}(a^2 \tilde{x}_j^{-1} - 1) \tilde{\Delta}^{-1} K \).

**Proof.**

1. To derive the formula we need to expand \( H_i K_\alpha \) as a linear combination of the \( K_\beta \)'s. As a 1st step we expand \( H_i G_\alpha \) as linear combination of the \( G_\beta \)'s.

If \( \alpha \in C_n \) satisfies \( \alpha_i < \alpha_{i+1} \) then

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)
\]

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that \( H_i \) satisfies the quadratic relation \( (H_i - t)(H_i + 1) = 0 \), it follows that

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - \overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1} \overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{s_i \alpha}(x)
\]
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_i G_\alpha(x) = tG_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_iK_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(at) = \frac{\overline{\alpha}_{i+1} - t\overline{\alpha}_i}{\overline{\alpha}_{i+1} - \overline{\alpha}_i} G_{\overline{\alpha}_i}(at)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_iK = \widetilde{H}_iK \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-a_1} G_{\alpha^2}(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_{\alpha^2}(at)}{G_{\alpha}(at)} = at^{1-n+k_1(a)} - q^{a_1} t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n} (a\overline{a}^{-1} - 1) K_{\alpha^2}(x).
\]

Remark 11. Note that

\[
\Phi K_\alpha(x) = (a\overline{a}_n - t^{1-n}) K_{\alpha^2}(x)
\]

for \( \alpha \in C_n \) since \( \overline{a}^{-1} = t^{n-1}w_0\overline{a} \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in F_+^{[\mathbb{K}[x]]} \) to a map \( K \in F_+^{[\mathbb{K}[x]]} \) such that Lemma 10 remains valid.

Lemma 12. For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n(qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^n \frac{(1 - a\overline{a}_i^{-1})}{(1 - q^{n-1}x_i)} \right) K_{\alpha+(1^n)}(qx).
\]
Proof. Note that for $f \in \mathbb{K}[x]$, 
\[ \Phi^n f(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1} x). \]

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10.

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by
\[ A_m(x; v) := \prod_{i=1}^{n} \frac{(q^{1-m}a \nu_i^{-1}; q)_m}{(qt^{n-1}x_i; q)_m} \quad \forall v \in \mathbb{Z}^n, \quad (14) \]
with $(y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y)$ the $q$-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by
\[ G_v(x) := q^{-m|v|-m} \frac{G_{v+(m^n)}(x^{m}x)}{\prod_{i=1}^{n} x_i^m(q^{-m}t^{1-n}x_i^{-1}; q)_m}, \]
\[ K_v(x) := A_m(x; v)K_{v+(m^n)}(x^{m}x), \]
where $m$ is a nonnegative integer such that $v + (m^n) \in C_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,
\[ K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}. \]

Lemma 15. For all $v \in \mathbb{Z}^n$,
\[ K_v(x) = \frac{G_v(x)}{G_v(at)}. \]

Proof. Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(at) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Then
\[ K_v(at) = A_m(at; v)K_{v+(m^n)}(q^m at) = A_m(at; v) \frac{G_{v+(m^n)}(q^m at)}{G_{v+(m^n)}(at)} = 1, \]
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_i K = \widehat{H}_i K$.
2. $\Xi_j K = a\widehat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2\widehat{x}_1^{-1} - 1)\widehat{\Delta}^{-1} K$.

**Proof.** Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $(A_m f)(v) = A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$((\widehat{H}_i \circ A_m)f)(v) = ((A_m \circ \widehat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

(15)

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\widehat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\widehat{H}_i K)(v) = tK_v$ and $H_i K_{v+(m^n)}(q^m x) = tK_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\widehat{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

(16)

where $\Phi(q^m) := (q^m x_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \widehat{v}_j^{-1}K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{\nu} = -w_0\nu \) for \( \nu \in \mathbb{Z}^n \).

**Theorem 17.** (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a\tilde{\nu}) = K_v(a\tilde{\nu}). \quad (17)
\]

**Example 18.** If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(aq^{-r}) = q^{-mr}(a^{-1};q)_{m+r}(a^{-1};q)_m(q^{-mr}(a^{-1};q)_r) \quad (18)
\]

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

**Proof.** We divide the proof of the theorem in several steps. ■

**Step 1.** If \( K_u(a\tilde{\nu}) = K_v(a\tilde{\nu}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{siu}(a\tilde{\nu}) = K_v(a\tilde{s}_i\tilde{u}) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{\nu}) + \left( \frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}} \right) K_u(a\tilde{s}_{n-i}v) = \frac{(t-1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{i+1})} K_v(a\tilde{\nu}) + \left( \frac{\tilde{u}_i - t\tilde{u}_{i+1}}{\tilde{u}_i - \tilde{u}_{i+1}} \right) K_{siu}(a\tilde{\nu}). \quad (19)
\]

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t-1)\tilde{u}_{n-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} K_v(a\tilde{u}) + \left( \frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} \right) K_v(a\tilde{s}_i\tilde{u}) = \frac{(t-1)\tilde{v}_{n-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} K_v(a\tilde{\nu}) + \left( \frac{\tilde{v}_{n-i} - t\tilde{v}_{n+1-i}}{\tilde{v}_{n-i} - \tilde{v}_{n+1-i}} \right) K_{s_{n-i}v}(a\tilde{u}). \quad (20)
\]

Suppose that \( s_{n-i}v = v \). Then \( \tilde{v}_{n-i} = t\tilde{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n}w_0\tilde{v}^{-1} \), that is, \( \tilde{v}_i = t^{-n}\tilde{v}_{n+1-i} \), we then also have \( \tilde{v}_i = t\tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{s_{n-i}u}(a\tilde{\nu}) = K_u(a\tilde{\nu}) \) and (20) to \( K_v(a\tilde{s}_i\tilde{u}) = K_v(a\tilde{u}) \) if \( s_{n-i}v = v \).
We now use these observations to prove Step 1. Assume that \( K_u(a \tilde{v}) = K_v(a \tilde{u}) \) for all \( v \). We have to show that \( K_{s_1 u}(a \tilde{v}) = K_v(a \tilde{s}_1 \tilde{u}) \) for all \( v \). It is trivially true if \( s_1 u = u \), so we may assume that \( s_1 u \neq u \). Suppose that \( v \) satisfies \( s_{n-1} v = v \). Then it follows from the previous paragraph that

\[
K_{s_1 u}(a \tilde{v}) = K_u(a \tilde{v}) = K_v(a \tilde{u}) = K_v(a \tilde{s}_1 \tilde{u}).
\]

If \( s_{n-1} v \neq v \) then (19) and the induction hypothesis can be used to write \( K_{s_1 u}(a \tilde{v}) \) as an explicit linear combination of \( K_v(a \tilde{u}) \) and \( K_{s_1 v}(a \tilde{u}) \). Then (20) can be used to rewrite the term involving \( K_{s_{n-1} v}(a \tilde{u}) \) as an explicit linear combination of \( K_v(a \tilde{u}) \) and \( K_v(a \tilde{s}_1 \tilde{u}) \). Hence, we obtain an explicit expression of \( K_{s_1 u}(a \tilde{v}) \) as linear combination of \( K_v(a \tilde{u}) \) and \( K_v(a \tilde{s}_1 \tilde{u}) \), which turns out to reduce to \( K_{s_1 u}(a \tilde{v}) = K_v(a \tilde{s}_1 \tilde{u}) \) after a direct computation.

**Step 2.** \( K_0(a \tilde{v}) = 1 = K_v(a \tilde{0}) \) for all \( v \in \mathbb{Z}^n \).

**Proof of Step 2.** Clearly \( K_0(x) = 1 \) and \( K_v(a \tilde{0}) = K_v(\alpha \tilde{r}) = 1 \) for \( v \in \mathbb{Z}^n \) by Lemma 15.

**Step 3.** \( K_\alpha(a \tilde{v}) = K_v(a \tilde{\alpha}) \) for \( v \in \mathbb{Z}^n \) and \( \alpha \in \mathcal{C}_n \).

**Proof of Step 3.** We prove it by induction. It is true for \( \alpha = 0 \) by Step 2. Let \( m \in \mathbb{Z}_{>0} \) and suppose that \( K_\gamma(a \tilde{v}) = K_v(a \tilde{\gamma}) \) for \( v \in \mathbb{Z}^n \) and \( \gamma \in \mathcal{C}_n \) with \( |\gamma| < m \). Let \( \alpha \in \mathcal{C}_n \) with \( |\alpha| = m \).

We need to show that \( K_\alpha(a \tilde{v}) = K_v(a \tilde{\alpha}) \) for all \( v \in \mathbb{Z}^n \). By Step 1 we may assume without loss of generality that \( \alpha_n > 0 \). Then \( \gamma := \alpha^\flat \in \mathcal{C}_n \) satisfies \( |\gamma| = m - 1 \), and \( \alpha = \gamma^\sharp \). Furthermore, note that we have the formula

\[
(a \tilde{v}_1^{-1} - 1) K_u(a \tilde{v}) = (a \tilde{u}_1^{-1} - 1) K_{u^\flat}(a \tilde{v})
\]

for all \( u, v \in \mathbb{Z}^n \), which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

\[
K_\alpha(a \tilde{v}) = K_{\gamma \flat}(a \tilde{v}) = \frac{(a \tilde{v}_1^{-1} - 1)}{(a \tilde{\gamma}_1^{-1} - 1)} K_{\gamma}(a \tilde{\gamma}) = (a \tilde{v}_1^{-1} - 1) K_{u^\flat}(a \tilde{v}) = (a \tilde{u}_1^{-1} - 1) K_{u^\flat}(a \tilde{v}) = K_{u^\flat}(a \tilde{v}) = K_v(a \tilde{\alpha}),
\]

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.
Step 4. \( K_u(a \tilde{v}) = K_v(a \tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in \mathcal{C}_n \). Note that \( q^m \tilde{v} = \tilde{v} - (m^n) \) and \( q^{-m} \tilde{u} = \tilde{u} + (m^n) \). Then

\[
K_u(a \tilde{v}) = A_m(a \tilde{v}; u)K_{u+(m^n)}(q^m a \tilde{v})
\]

\[
= A_m(a \tilde{v}; u)K_{u+(m^n)}(a(\tilde{v} - (m^n)))
\]

\[
= A_m(a \tilde{v}; u)K_{v-(m^n)}(a(u + (m^n)))
\]

\[
= A_m(a \tilde{v}; u)K_{v-(m^n)}(q^{-m} a \tilde{u}) = A_m(a \tilde{v}; u)A_m(q^{-m} a \tilde{u}; \tilde{v} - (m^n))K_v(a \tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a \tilde{v}; u)A_m(q^{-m} a \tilde{u}; \tilde{v} - (m^n)) = 1,
\]

which follows by a straightforward computation using (4).

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_{\alpha}(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_{\alpha}(x) \), i.e.,

\[
E_{\alpha}(x) = \lim_{a \to \infty} a^{-|\alpha|} G_{\alpha}(ax), \quad \alpha \in \mathcal{C}_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_{\alpha}(x) := \lim_{a \to \infty} K_{\alpha}(ax) = \frac{E_{\alpha}(x)}{E_{\alpha}(\tau)}, \quad \alpha \in \mathcal{C}_n.
\]

We write \( \overline{K} \in \mathcal{J}^+_\text{fix} \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( \overline{H}_i \overline{K} = \overline{H}_i \overline{K} \).
2. \( \xi_j \overline{K} = \hat{\xi}_j^{-1} \overline{K} \).
3. \( x_n \Delta \overline{K} = t^{1-n} \hat{\Delta}^{-1} \overline{K} \).
Note that
\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},\]
\[\overline{K}_\alpha(x) = q^{\lvert \alpha \rvert} t^{(1-n)\lvert \alpha \rvert} \left( \prod_{i=1}^{n} (\overline{\alpha}_i x_i)^{-1} \right) \overline{K}_{\alpha+(1^n)}(x).\]

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) for arbitrary \(\nu \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(\nu \in \mathbb{Z}^n\).

**Definition 20.** Let \(\nu \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(\nu + (m^n) \in C_n\). Then \(\overline{K}_\nu(x) := \overline{K}_\nu(x; q, t) \in \mathbb{F}[x^\pm 1]\) is defined by
\[\overline{K}_\nu(x) := q^{\lvert \nu \rvert} t^{(1-n)\lvert \nu \rvert} m \left( \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-1} \right) \overline{K}_{\nu+(m^n)}(x).\]

Using
\[\lim_{a \to \infty} A_m(ax ; \nu) = q^{-m^2 n t^{(1-n)nm}} \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-m}\]
and the definitions of \(G_\nu(x)\) and \(K_\nu(x)\) it follows that
\[\lim_{a \to \infty} a^{-\lvert \nu \rvert} G_\nu(ax) = E_\nu(x),\]
\[\lim_{a \to \infty} K_\nu(ax) = \overline{K}_\nu(x)\]
for all \(\nu \in \mathbb{Z}^n\), so in particular
\[\overline{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\tau)} \quad \forall \nu \in \mathbb{Z}^n.\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}_{[x^1]}$ defined by $v \mapsto K_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\tilde{v}) = K_v(\tilde{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\beta^{-1}) = K_\alpha(t^{1-n}aw_0\beta^{-1}) = K_\alpha(a\tilde{\beta}) = K_\beta(a\tilde{\alpha})$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. 

### 7.3 Okounkov's duality

Write $F[x]_{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]_{S_n}$ is the multiple of $C_+ G_\lambda$ such that the coefficient of $x_\lambda$ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(\alpha\tau)} \in \mathbb{K}[x]_{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\alpha^+(x) \tag{23}$$

for $\alpha \in C_n$. Okounkov's [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a \mu^{-1}) = K_\mu^+(a \lambda^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\widehat{C}_+ = \sum_{w \in S_n} \widehat{H}_w$, with $\widehat{H}_w := \widehat{H}_{i_1} \cdots \widehat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in F_\mathbb{K}$ for the function $f_\mu(u) := K_u(a \mu)$ $(u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a \mu) = (C_+ K_\lambda)(a \mu) = (\widehat{C}_+ f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a \mu) = (Jw_0 K_\mu(t^{1-n}x))|_{x=a^{-1} \mu}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_i J = (H_i^o)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^o)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x=a^{-1} \lambda}.$$ 

By (23) and (4) this simplifies to

$$(\widehat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a \lambda).$$

Returning to (24) we conclude that $K_\lambda^+(a \mu^{-1}) = K_\mu^+(a \lambda^{-1})$. Since $K_\lambda^+$ is symmetric we obtain from (4) that

$$K_\lambda^+(a \mu^{-1}) = K_\mu^+(a \lambda^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0}K_u)(a\tilde{v}) = (H_{w_0}K_v)(a\tilde{u}). \quad (27)$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a\tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0}K_u)(a\tilde{v}) = (\hat{H}_{w_0}f_v)(u).$$

Since $f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}\tilde{u})$ by (4), Remark 7 implies that

$$(\hat{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}\tilde{u}).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$(\hat{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}\tilde{u}) = (H_{w_0}K_v)(a\tilde{u}),$$

which completes the proof. ■

Recall from Theorem 1 that

$$G_\beta'(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G_\beta^o(t^{n-1}x)$$

with $\Psi := w_0H_{w_0}^o$. We define normalized versions by

$$K_\beta'(x) := \frac{G_\beta'(x)}{G_\beta'(a^{-1}\tau)} = t^{\ell(w_0)}\Psi K_\beta^o(t^{n-1}x), \quad \beta \in C_n,$$

with $K_\nu^o := \iota(K_\nu)$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K_\nu'(x) := t^{\ell(w_0)}\Psi K_\nu^o(t^{n-1}x). \quad (28)$$

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K_\nu'(\nu \in \mathbb{Z}^n)$. Since $H_i\Psi = \Psi H_i^o$, part 1 of Proposition 16 gives $H_iK' = \hat{H}_i^oK'$. Considering the action of $((x_n - 1)\Delta^x)^n$ on $K_\beta'(x)$ we get, using the fact that $((x_n - 1)\Delta^x)^n$ commutes with $\Psi$ and part 3 of Proposition 16,

$$K_\nu'(x) = \left(\prod_{i=1}^n \frac{1-a^{-1}\tilde{\nu}_i}{1-q^{-1}x_i}\right)K_{\nu+(1^n)}(q^{-1}x).$$
in particular

$$K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v_i; q)_m}{(q^{-m}x_i; q)_m} \right)K'_{v+(m^n)}(q^{-m}x).$$

**Example 25.** For $n = 1$ we have $K'_v(x) = K^*_v(x)$ for $v \in \mathbb{Z}$; hence,

$$K'_{m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = \frac{(a^m x^{-1})_m}{(a^{-1}; q)_m},$$

$$K'_{m}(x) = \frac{(a^m x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}$$

for $m \in \mathbb{Z}_{\geq 0}$ by Example 14.

**Proposition 26.** For all $u, v \in \mathbb{Z}^n$ we have

$$K'_v(a^{-1}u) = K'_u(a^{-1}v).$$

**Proof.** Note that

$$K'_v(a^{-1}u) = t^{\ell(w_0)}t^{-1} \Psi K'_{v}(q^{-m^{-1}x})|_{x=a^{-1}u} = t^{\ell(w_0)}(H^\circ_{w_0}K'_{v})(a^{-1}u^{-1})$$

by (4). By (27) the right-hand side is invariant under the interchange of $u$ and $v$. □

### 7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of $O_\alpha$ was used to prove the following binomial theorem [14, Thm. 1.3]. Define for $\alpha, \beta \in C_n$ the generalized binomial coefficient by

$$\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} := \frac{G_{\beta}(\overline{\alpha})}{G_{\beta}(\overline{\beta})}. \quad (29)$$

Applying the automorphism $\iota$ of $\mathbb{F}$ to (29) we get

$$\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1},t^{-1}} = \frac{G^\circ_{\beta}(\overline{\alpha}^{-1})}{G^\circ_{\beta}(\overline{\beta}^{-1})}.$$
Theorem 27. For $\alpha, \beta \in C_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in C_n} a^{\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \frac{G_\beta'(x)}{G_\beta(ax)}. \quad (30)$$

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in C_n} \tau_\beta^{-1} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \frac{K_\beta'(x)}{K_\beta(1/x)} \quad (31)$$

with $w_0 H_0$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K_\beta(x)$ and $K_\beta'(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_0 \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_0 K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{K_\beta'(1/x)K_\beta'(t^{-1}w_0 x)}{\tau_\beta K_\beta(1/x)}.$$

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_0 K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in C_n} \frac{K_\beta'(1/x)K_\beta'(t^{-1}w_0 x)}{\tau_\beta K_\beta(1/x)}.$$

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G_\alpha'$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all \( \alpha \in C_n \) we have

\[
K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax). \tag{32}
\]

The starting point of the alternative proof of (32) is the binomial formula in the form

\[
K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta(\overline{a}) \Psi K_\beta(t^{n-1}x)}{\tau_\beta G_\beta(\overline{\beta}^{-1})},
\]

see (31). Replace \((a, x, q, t)\) by \((a^{-1}, at^{n-1}x, q^{-1}, t^{-1})\) and act by \(w_0 H w_0\) on both sides. Since \(w_0 H w_0 \Psi = \text{Id}\) we obtain

\[
\Psi K_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).
\]

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

\[
\Psi K'_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\overline{a}) K_\beta(ax).
\]

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

\[
\sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.
\]

Since \( \left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0 \) unless \( \delta \supseteq \epsilon \), the terms in the sum are zero unless \( \gamma \subseteq \beta \subseteq \alpha \).

**Acknowledgments**

We thank Eric Rains for sharing with us his unpublished results with Alain Lascoux and Ole Warnaar on a one-parameter rational extension of the non-symmetric interpolation Macdonald
polynomials. It leads to a different proof of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem B). We thank an anonymous referee for detailed comments.

Funding

This work was partially supported by Simons Foundation [509766 to S.S.].

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