Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_{\lambda}(x; q, t) = R_{\lambda}(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$ 

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$
Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\mu) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$  

The normalization is fixed by requiring that the coefficient of $x^{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ [9] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\mu) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of

$$R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t)$$

in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the duality or evaluation symmetry, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n}\tau_1, \ldots, q^{-\mu_1}\tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.$$  

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most $n$ parts, $C_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in C_n$ we define

$$\overline{\beta} := w_\beta(\beta_+),$$

where $w_\beta$ is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\overline{\beta}) = 0 \text{ for } \beta \in C_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$  

The normalization is fixed by requiring that the coefficient of $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax;q,t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x;q,t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (-w_0\beta)$, with $w_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

**Theorem A.** Write $I(\alpha) := \#\{i < j | \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)}W_0H_0G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

**Theorem B.** For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\tilde{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

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**Theorem C.** For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha (x) = \frac{G_\alpha (t^{1-n} a \omega_0 x)}{G_\alpha (a \tau)}.$$ 

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha (x)$ in terms of the $G_\beta (a x)$’s.

## 2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i + 1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \to \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}_n$ and $\mathbb{K}^n$ by $s_i v := (\cdots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \cdots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \mapsto n + 1 - i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i t - k_i (v)}$ with

$$k_i (v) := \# \{ k < i \mid v_k \geq v_i \} + \# \{ k > i \mid v_k > v_i \}.$$ 

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{v} = (q^{v_1 \tau_1}, \ldots, q^{v_n \tau_n})$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_\nu (\overline{v}_+)$ with $w_\nu \in S_n$ the shortest permutation such that $v_\nu := w_\nu^{-1} (v)$ has non-increasing entries, see [4, Section 2]. We write $\overline{v} := -w_0 \nu$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^{\pm 1}] := F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $F[x]$ and $F(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $F$-linear operators

$$H_i = t s_i - \frac{(1 - t) x_i}{x_i - x_{i+1}} (1 - s_i) = t + \frac{x_i - t x_{i+1}}{x_i - x_{i+1}} (s_i - 1).$$
on \( F(x) \) \((1 \leq i < n)\) called Demazure-Lusztig operators, and the automorphism \( \Delta \) of \( F(x) \) defined by

\[
\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).
\]

Note that \( H_i \) \((1 \leq i < n)\) and \( \Delta \) preserve \( F[x^{\pm 1}] \) and \( F[x] \). Cherednik [1, 2] showed that the operators \( H_i \) \((1 \leq i < n)\) and \( \Delta \) satisfy the defining relations of the type A extended affine Hecke algebra,

\[
(H_i - t)(H_i + 1) = 0,
\]

\[
H_iH_j = H_jH_i, \quad |i - j| > 1,
\]

\[
H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1},
\]

\[
\Delta H_{i+1} = H_i \Delta,
\]

\[
\Delta^2 H_1 = H_{n-1} \Delta^2
\]

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For \( w \in S_n \) we write \( H_w := H_{i_1}H_{i_2} \cdots H_{i_\ell} \) with \( w = s_{i_1}s_{i_2} \cdots s_{i_\ell} \) a reduced expression for \( w \in S_n \). It is well defined because of the braid relations for the \( H_i \)'s. Write \( \overline{H}_i := H_i + 1 - t = th_i^{-1} \) and set

\[
\xi_i := t^{1-n}\overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1}H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \tag{1}
\]

The operators \( \xi_i \)'s are pairwise commuting invertible operators, with inverses

\[
\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.
\]

The \( \xi_i^{-1} \) \((1 \leq i \leq n)\) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial \( E_\alpha \in F[x] \) of degree \( \alpha \in C_n \) is the unique polynomial satisfying

\[
\xi_i^{-1}E_\alpha = \overline{u}_iE_\alpha, \quad i = 1, \ldots, n
\]

and normalized such that the coefficient of \( x^\alpha \) in \( E_\alpha \) is 1.

Let \( \iota \) be the field automorphism of \( \mathbb{K} \) inverting \( q, t \) and \( a \). It restricts to a field automorphism of \( F \), inverting \( q \) and \( t \). We extend \( \iota \) to a \( \mathbb{Q} \)-algebra automorphism of \( \mathbb{K}[x] \)
and $\mathbb{F}[x]$ by letting $\iota$ act on the coefficients of the polynomial. Write

$$G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)$$

for $\alpha \in \mathbb{C}_n$. Note that $\overline{\nu}^{-1} = (\iota(\overline{\nu}_1), \ldots, \iota(\overline{\nu}_n))$.

Put $H_i^\circ, H_w^\circ, \overline{H_i}^\circ, \Delta^\circ$ and $\xi_i^\circ$ for the operators $H_i, H_w, \overline{H_i}, \Delta$ and $\xi_i$ with $q, t$ replaced by their inverses. For instance,

$$H_i^\circ = t^{-1}s_i - \frac{(1-t^{-1})x_i}{x_i - x_{i+1}}(1-s_i),$$

$$\Delta^\circ f(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}).$$

We then have $\xi_i^\circ E_\alpha^\circ = \overline{\alpha}_i E_\alpha^\circ$ for $i = 1, \ldots, n$, which characterizes $E_\alpha^\circ$ up to a scalar factor.

**Theorem 1.** For $\alpha \in \mathbb{C}_n$ we have

$$G_\alpha'(x) = t^{(1-n)|\alpha|+I(\alpha)} w_0 H_w^\circ G_\alpha(t^{n-1}x)$$  \hspace{1cm} (2)

with $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$.

**Remark.** Formally set $t = q^r$, replace $x$ by $1 + (q - 1)x$, divide both sides of (2) by $(q - 1)^{|\alpha|}$ and take the limit $q \to 1$. Then

$$G_\alpha'(x; r) = (-1)^{|\alpha|} \sigma(w_0)w_0 G_\alpha(-x - (n - 1)r; r)$$  \hspace{1cm} (3)

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here $\sigma$ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators $H_i$, given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1-s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0\sigma^\circ(w_0)$ with $\sigma^\circ$ the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1-s_i)$$

of $H_i^\circ$. Formula (3) was obtained before in [14, Thm. 1.10].
**Proof.** We show that the right-hand side of (2) satisfies the defining properties of $G_{\alpha'}$. For the vanishing property, note that
\[ t^{n-1} w_0 \tilde{\beta} = \tilde{\beta}^{-1} \] (this is the $q$-analog of [14, Lem. 6.1(2)]); hence,
\[ (w_0 H_0^o G_{\alpha}^o (t^{n-1} x))|_{x=\tilde{\beta}} = (H_0^o G_{\alpha}^o (x))|_{x=\tilde{\beta}^{-1}}. \]
This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{\alpha}^o (w \beta^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that
\[ E_{\alpha} = t^{I(\alpha)} w_0 H_0^o E_{\alpha}^o. \] (5)
Note that $\Psi := w_0 H_0^o$ satisfies the intertwining properties
\[ H_i \Psi = t \Psi H_i^o, \]
\[ \Delta \Psi = t^{n-1} \Psi H_{n-1}^o \cdots H_1^o (\Delta^o)^{-1} H_{n-1}^o \cdots H_1^o \] (6)
for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1} \Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$. Therefore,
\[ E_{\alpha} (x) = c_{\alpha} \Psi E_{\alpha}^o (x) \]
for some constant $c_{\alpha} \in \mathbb{F}$. But the coefficient of $x^\alpha$ in $\Psi x^\alpha$ is $t^{-I(\alpha)}$; hence, $c_{\alpha} = t^{I(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $u_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in C_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_{\alpha} \in \mathbb{F}[x]$ as defined before. Note that
\[ E_{u+(1^n)} = x_1 \cdots x_n E_u (x). \]
It is now easy to check that formula (5) is valid with $\alpha$ replaced by an arbitrary integral vector $u$,

$$E_u = t^{I(u)} w_0 H_{w_0}^\circ E_u$$

(7)

with $E_u^\circ := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector $u$, where $p(x^{-1})$ stands for inverting all the parameters $x_1, \ldots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^\pm 1]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{I(u)} H_{w_0}^\circ E_u(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - qa'(s) + 1}{1 - qa(s) + 1} \right)^{\ell'(s)} \prod_{s \in \alpha} \left( at^{l'(s)} - qa'(s) \right)$$

(8)

was obtained, with $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$a(s) := \alpha_i - j, \quad l(s) := \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \},$$

$$a'(s) := j - 1, \quad l'(s) := \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}. $$

By (8) we have

$$E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n+l'(s)} - qa'(s) + 1}{1 - qa(s) + 1 t^{l(s)} + 1} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in C_n$,

$$\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}.$$
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G^0_{\alpha}(a \tau^{-1}).$$

Proof. Since $t^{n-1}w_0 \tau = \tau^{-1} = \overline{\tau}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a \tau) = t^{(1-n)|\alpha|+I(\alpha)} (H^0_{w_0} G^0_{\alpha})(a \overline{\tau}^{-1}) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G^0_{\alpha}(a \overline{\tau}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^{n} \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a \tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha|-n(\alpha)} q^{n'(\alpha)} G^0_{\alpha}(a^{-1} \tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$. ■

Following [8, (3.9)] we define $\tau_\alpha \in F$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. (10)$$

It only depends on the $S_n$-orbit of $\alpha$. 
Corollary 4. For $\alpha \in \mathbb{C}^n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau^{-1}_\alpha a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). \hfill \blacksquare

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$-valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\natural := (v_2, \ldots, v_n, v_1 + 1)$ and $y^\natural := (y_2, \ldots, y_n, qy_1)$. Denote the inverse of $^\natural$ by $^\flat$, so $v^\flat = (v_n - 1, v_1, \ldots, v_1 - 1)$ and $y^\flat = (y_n/q, y_1, \ldots, y_n - 1)$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(v^\flat) = s_i v$ if $v_i \neq v_{i+1}$.
2. $v_i = t v_{i+1}$ if $v_i = v_{i+1}$.
3. $v^\natural = v^\flat$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^\pm 1])$ generated by the operators $H_i$ ($1 \leq i < n$), $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$-algebra $A$ we write $\mathcal{F}_A$ for the space of $A$-valued functions $f : \mathbb{Z}^n \to A$ on $\mathbb{Z}^n$.

Corollary 6. Let $A$ be a unital $\mathbb{K}$-algebra. Consider the $A$-linear operators $\hat{H}_i$ ($1 \leq i < n$), $\hat{\Delta}$ and $\hat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{align*}
(\hat{H}_i f)(v) &:= tf(v) + \frac{v_i - t v_{i+1}}{v_i - v_{i+1}} (f(s_i v) - f(v)), \\
(\hat{\Delta} f)(v) &:= f(v^\flat), \\
(\hat{x}_j f)(v) &:= a v_j f(v)
\end{align*}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \hat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \hat{\Delta}$ and $x_j \mapsto \hat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \to \text{End}_A(\mathcal{F}_A)$, $X \mapsto \hat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$. 

Proof. Let $\mathcal{O} \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\mathfrak{t}$-invariant subset that contains \{a$\overline{v}$ | $v \in \mathbb{Z}^n$\}. Note that $\mathcal{O}$ is contained in \{y $\in \mathbb{K}^n$ | $y_i \neq y_j$ if $i \neq j$\}. The Demazure–Lusztig operators $H_i$ (1 $\leq$ i $<$ n), $\Delta^\pm$ and the coordinate multiplication operators $x_j$ (1 $\leq$ j $\leq$ n) act $A$-linearly on the space $F^O_A$ of $A$-valued functions on $\mathcal{O}$, and hence turns $F^O_A$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map

$$\text{pr} : F^O_A \rightarrow F_A$$

by $\text{pr}(g)(v) := g(a\overline{v})$ ($v \in \mathbb{Z}^n$).

We claim that Ker($\text{pr}$) is an $\mathbb{H}$-submodule of $F^O_A$. Clearly Ker($\text{pr}$) is $x_j$-invariant for $j = 1, \ldots, n$. Let $g \in$ Ker($\text{pr}$). Part 3 of Lemma 5 implies that $\Delta g \in$ Ker($\text{pr}$). To show that $H_j g \in$ Ker($\text{pr}$) we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \overline{v} = \overline{s_i v}$ by part 1 of Lemma 5. Hence,

$$(H_j g)(a\overline{v}) = t g(a\overline{v}) + \frac{\overline{v_i - t\overline{v}_{i+1}}}{\overline{v_i}} (g(a\overline{s_i v}) - g(a\overline{v})) = 0.$$ 

If $v_i = v_{i+1}$ then $\overline{v}_i = t\overline{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_j g)(\overline{v}) = t g(a\overline{v}) + \frac{\overline{v_i - t\overline{v}_{i+1}}}{\overline{v_i}} (g(a\overline{s_i v}) - g(a\overline{v})) = t g(a\overline{v}) = 0.$$

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F^O_A/\text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i$ (1 $\leq$ i $<$ n), $\Delta$ and $x_j$ (1 $\leq$ j $\leq$ n) on $F_A$ is by the operators $\widehat{H}_i$ (1 $\leq$ i $<$ n), $\widehat{\Delta}$ and $\widehat{x}_j$ (1 $\leq$ j $\leq$ n).

Remark 7. With the notations from (the proof of) Corollary 6, let $\overline{g} \in F^O_A$ and set $g := \text{pr}(\overline{g}) \in F_A$. In other words, $g(v) := \overline{g}(a\overline{v})$ for all $v \in \mathbb{Z}^n$. Then

$$(\widehat{X} g)(v) = (X \overline{g})(a\overline{v}), \quad v \in \mathbb{Z}^n$$

for $X = H_i, \Delta^\pm, x_j$.

Remark 8. Let $F_A^+$ be the space of $A$-valued functions on $C_n$. We sometimes will consider $\widehat{H}_i$ (1 $\leq$ i $<$ n), $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ (1 $\leq$ j $\leq$ n), defined by the formulas (11), as linear operators on $F_A^+$.

Definition 9. We call

$$K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a\tau; q, t)} \in \mathbb{K}[x]$$

(12)

the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 
We frequently use the shorthand notation \( K_\alpha(x) := K_\alpha(x; q, t; a) \). We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that \( a \) cannot be specialized to 1 in (12) since \( G_\alpha(\tau) = G_\alpha(0) = 0 \) if \( \alpha \in C_n \) is nonzero. Note furthermore that

\[
\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}
\]

since \( \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x) \).

Recall from [4] the operator \( \Phi_1 = (x_n - t^{1-n})\Delta \in \mathbb{H} \) and the inhomogeneous Cherednik operators

\[
\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.
\]

The operators \( H_i, \, \Xi_j \) and \( \Phi \) preserve \( \mathbb{K}[x] \) (see [4]); hence, they give rise to \( \mathbb{K} \)-linear operators on \( F_{\mathbb{K}[x]}^+ \) (e.g., \( (H_i f)(\alpha) := H_i(f(\alpha)) \) for \( \alpha \in C_n \)). Note that the operators \( H_i, \, \Xi_j \) and \( \Phi \) on \( F_{\mathbb{K}[x]}^+ \) commute with the hat-operators \( \hat{H}_i, \, \hat{x}_j \) and \( \hat{\Delta}^{-1} \) on \( F_{\mathbb{K}[x]}^+ \) (cf. Remark 8). The same remarks hold true for the space \( F_{\mathbb{K}[\tau]}^+ \) of \( \mathbb{K}[\tau] \)-valued functions on \( \mathbb{Z}^n \) (in fact, in this case the hat-operators define a \( \mathbb{H} \)-action on \( F_{\mathbb{K}[\tau]}^+ \)).

Let \( K \in F_{\mathbb{K}[x]}^+ \) be the map \( \alpha \mapsto K_\alpha(\cdot) \) (\( \alpha \in C_n \)).

**Lemma 10.** For \( 1 \leq i < n \) and \( 1 \leq j \leq n \) we have in \( F_{\mathbb{K}[x]}^+ \),

1. \( H_i K = \hat{H}_i K \).
2. \( \Xi_j K = a \hat{x}_j^{-1} K \).
3. \( \Phi K = t^{1-n}(a^2 \hat{x}_j^{-1} - 1) \hat{\Delta}^{-1} K \).

**Proof.** 1. To derive the formula we need to expand \( H_i K_\alpha \) as a linear combination of the \( K_\beta \)'s. As a first step we expand \( H_i G_\alpha \) as linear combination of the \( G_\beta \)'s.

If \( \alpha \in C_n \) satisfies \( \alpha_i < \alpha_{i+1} \) then

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + G_{\alpha}(x)
\]

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that \( H_i \) satisfies the quadratic relation \( (H_i - t)(H_i + 1) = 0 \), it follows that

\[
H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - t\overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1}\overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2} G_{\alpha}(x)
\]
if \( \alpha \in C_n \) satisfies \( \alpha_i > \alpha_{i+1} \). Finally, \( H_i G_\alpha(x) = tG_\alpha(x) \) if \( \alpha \in C_n \) satisfies \( \alpha_i = \alpha_{i+1} \) by [4, Cor. 3.4].

An explicit expansion of \( H_i K_\alpha \) as linear combination of the \( K_\beta \)'s can now be obtained using the formula

\[
G_\alpha(\alpha \tau) = \frac{\overline{\alpha_{i+1}} - t\overline{\alpha_i}}{\overline{\alpha_{i+1}} - \overline{\alpha_i}} G_{s,\alpha}(\alpha \tau)
\]

for \( \alpha \in C_n \) satisfying \( \alpha_i > \alpha_{i+1} \), cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as \( H_i K = \tilde{H}_i K \).

2. See [4, Thm. 2.6].

3. Let \( \alpha \in C_n \). By [14, Lem. 2.2 (1)],

\[
\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\circ}(x).
\]

By the evaluation formula (8) we have

\[
\frac{G_{\alpha^\circ}(\alpha \tau)}{G_\alpha(\alpha \tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.
\]

Hence,

\[
\Phi K_\alpha(x) = t^{1-n}(a\overline{\alpha}_1 - 1)K_{\alpha^\circ}(x).
\]

\[\boxed{\text{Remark 11.} \quad \text{Note that}}

\[
\Phi K_\alpha(x) = (a\overline{\alpha}_n - t^{1-n})K_{\alpha^\circ}(x)
\]

for \( \alpha \in C_n \) since \( \overline{\alpha}_1^{-1} = t^{n-1} w_0 \overline{\alpha} \).

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials \( G_\alpha(x) \) and \( K_\alpha(x) \) to \( \alpha \in \mathbb{Z}^n \). It will be the unique extension of \( K \in \mathcal{F}^+_{[n]}(x) \) to a map \( K \in \mathcal{F}_{[n]}(x) \) such that Lemma 10 remains valid.

Lemma 12. For \( \alpha \in C_n \) we have

\[
G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},
\]

\[
K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a\overline{\alpha}_i^{-1}}{1 - qt^{n-1} x_i} \right) K_{\alpha+(1^n)}(qx).
\]
Proof. Note that for \( f \in \mathbb{K}[x], \)

\[
\Phi^n f(x) = \left( \prod_{i=1}^{n} (x_i - t^{1-n}) \right) f(q^{-1} x).
\]

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10.

For \( m \in \mathbb{Z}_{\geq 0} \) we define \( A_m(x; v) \in \mathbb{K}(x) \) by

\[
A_m(x; v) := \prod_{i=1}^{n} \left( \frac{(q^{1-m} a v_i^{-1}; q)_m}{(q x_i^{-1}; q)_m} \right), \quad \forall v \in \mathbb{Z}^n,
\]

with \( (y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y) \) the \( q \)-shifted factorial.

Definition 13. Let \( v \in \mathbb{Z}^n \) and write \( |v| := v_1 + \cdots + v_n \). Define \( G_v(x) = G_v(x; q, t) \in \mathbb{F}(x) \) and \( K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x) \) by

\[
G_v(x) := q^{m|v|} x^{-m} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m} t^{1-n} x_i^{-1}; q)_m},
\]

\[
K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),
\]

where \( m \) is a nonnegative integer such that \( v + (m^n) \in C_n \) (note that \( G_v \) and \( K_v \) are well defined by Lemma 12).

Example 14. If \( n = 1 \) then for \( m \in \mathbb{Z}_{\geq 0} \),

\[
K_{-m}(x) = \left( \frac{qa; q}{qx; q} \right)_m, \quad K_m(x) = \left( \frac{x}{a} \right)^m \left( \frac{x^{-1}; q}{a^{-1}; q} \right)_m.
\]

Lemma 15. For all \( v \in \mathbb{Z}^n \),

\[
K_v(x) = \frac{G_v(x)}{G_v(at)}.
\]

Proof. Let \( v \in \mathbb{Z}^n \). Clearly \( G_v(x) \) and \( K_v(x) \) only differ by a multiplicative constant, so it suffices to show that \( K_v(at) = 1 \). Fix \( m \in \mathbb{Z}_{\geq 0} \) such that \( v + (m^n) \in C_n \). Then

\[
K_v(at) = A_m(at; v) K_{v+(m^n)}(q^m a t) = A_m(at; v) \frac{G_{v+(m^n)}(q^m a t)}{G_{v+(m^n)}(at)} = 1,
\]
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map $K : C_n \to \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \to \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in $\mathcal{F}_K(x)$,

1. $H_iK = \hat{H}_iK$.
2. $\Xi_jK = a\hat{x}_j^{-1}K$.
3. $\Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1}K$.

**Proof.** Write $A_m \in \mathcal{F}_K(x)$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_K(x)$ defined by $(A_m f)(v) := A_m(x; v)f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_K(x)$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_K(x)$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \ldots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_K(x)$,

$$((\hat{H}_i \circ A_m)f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1}$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $v_1, \ldots, v_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in C_n$. Since

$$K_v(x) = A_m(x; v)K_{v+(m^n)}(q^mx)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_iK)(v) = (\hat{H}_iK)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\hat{H}_iK)(v) = tK_v$ and $H_iK_{v+(m^n)}(q^mx) = tK_{v+(m^n)}(q^mx)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n}(a\hat{v}_1^{-1} - 1)K_v(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi(q^m),$$

where $\Phi(q^m) := (q^mx_n - t^{1-n})\Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_jK_v(x) = \hat{v}_j^{-1}K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition.

■
6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation \( \tilde{v} = -w_0 v \) for \( v \in \mathbb{Z}^n \).

**Theorem 17.** (Duality). For all \( u, v \in \mathbb{Z}^n \) we have

\[
K_u(a\tilde{v}) = K_v(a\tilde{u}).
\]  

**(17)**

**Example 18.** If \( n = 1 \) and \( m, r \in \mathbb{Z}_{\geq 0} \) then

\[
K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r}.
\]  

**(18)**

by the explicit expression for \( K_m(x) \) from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of \( m \) and \( r \).

**Proof.** We divide the proof of the theorem in several steps.

**Step 1.** If \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( v \in \mathbb{Z}^n \) then \( K_{siu}(a\tilde{v}) = K_v(a\tilde{siu}) \) for \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \).

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

\[
\frac{(t-1)\bar{v}_i}{(\bar{v}_i - \bar{v}_{i+1})} K_u(a\tilde{v}) + \left( \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} \right) K_u(a\tilde{s}_{n-i}v) = \frac{(t-1)\bar{u}_i}{(\bar{u}_i - \bar{u}_{i+1})} K_u(a\tilde{v}) + \left( \frac{\bar{u}_i - t\bar{u}_{i+1}}{\bar{u}_i - \bar{u}_{i+1}} \right) K_{siu}(a\tilde{v}).
\]  

**(19)**

Replacing in (19) the role of \( u \) and \( v \) and replacing \( i \) by \( n - i \) we get

\[
\frac{(t-1)\bar{u}_{n-i}}{(\bar{u}_{n-i} - \bar{u}_{n+1-i})} K_v(a\tilde{u}) + \left( \frac{\bar{u}_{n-i} - t\bar{u}_{n+1-i}}{\bar{u}_{n-i} - \bar{u}_{n+1-i}} \right) K_v(a\tilde{s}_i\tilde{u}) = \frac{(t-1)\bar{v}_{n-i}}{(\bar{v}_{n-i} - \bar{v}_{n+1-i})} K_v(a\tilde{u}) + \left( \frac{\bar{v}_{n-i} - t\bar{v}_{n+1-i}}{\bar{v}_{n-i} - \bar{v}_{n+1-i}} \right) K_{s_{n-i}v}(a\tilde{u}).
\]  

**(20)**

Suppose that \( s_{n-i}v = v \). Then \( \bar{v}_{n-i} = t\bar{v}_{n+1-i} \) by the 2nd part of Lemma 5. Since \( \tilde{v} = t^{1-n}w_0\bar{v}^{-1} \), that is, \( \tilde{v}_i = t^{1-n}\bar{v}_{n+1-i}^{-1} \) we then also have \( \tilde{v}_i = t\tilde{v}_{i+1} \). It then follows by a direct computation that (19) reduces to \( K_{siu}(a\tilde{v}) = K_u(a\tilde{v}) \) and (20) to \( K_v(a\tilde{s}_i\tilde{u}) = K_v(a\tilde{u}) \) if \( s_{n-i}v = v \).
We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v$. We have to show that $K_{s_1u}(a\tilde{v}) = K_v(a\tilde{s}_1u)$ for all $v$. It is trivially true if $s_1u = u$, so we may assume that $s_1u \neq u$. Suppose that $v$ satisfies $s_{n-1}v = v$. Then it follows from the previous paragraph that

$$K_{s_1u}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(a\tilde{s}_1u).$$

If $s_{n-1}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_1u}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-1}v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-1}v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-1}v}(a\tilde{s}_1u)$. Hence, we obtain an explicit expression of $K_{s_1u}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s}_1u)$, which turns out to reduce to $K_{s_1u}(a\tilde{v}) = K_v(a\tilde{s}_1u)$ after a direct computation.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(\alpha\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15. ■

**Step 3.** $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_v(a\tilde{v}) = K_v(\alpha\tilde{y})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\perp \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\perp$. Furthermore, note that we have the formula

$$(a\tilde{v}_1^{-1} - 1)K_u(a\tilde{v}) = (a\tilde{u}_1^{-1} - 1)K_{u^\perp}(a\tilde{v}) \quad (21)$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{v}) = \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{\gamma}_1^{-1} - 1)}K_{\gamma^\perp}(a\tilde{v})$$

$$= \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{\gamma}_1^{-1} - 1)}K_v(\alpha\tilde{\gamma}) = K_v(a\tilde{\gamma}) = K_v(a\tilde{\alpha}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step. ■
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = v - (m^n) \) and \( q^{-m}\tilde{u} = u + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v}) \\
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(v - (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(a(u + (m^n))) \\
= A_m(a\tilde{v}; u)K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n))K_v(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m} a\tilde{u}; v - (m^n)) = 1,
\]

which follows by a straightforward computation using (4). \( \square \)

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}_{\text{Fix}}^+ \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i \overline{K} = \hat{H}_i \overline{K} \).
2. \( \xi_j \overline{K} = \hat{\xi}_j^{-1} \overline{K} \).
3. \( x_n \Delta \overline{K} = t^{1-n} \hat{\Delta}^{-1} \overline{K} \).
Note that

$$(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).$$

Then repeated application of part 3 of Lemma 19 shows that for $\alpha \in C_n$,

$$E_\alpha(x) = \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n},$$

$$\overline{K}_\alpha(x) = q^{\alpha |} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha}_i x_i)^{-1} \right) \overline{K}_{\alpha+(1^n)}(x). \tag{22}$$

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials $E_\nu(x) := E_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ for arbitrary $\nu \in \mathbb{Z}^n$ to those labeled by compositions through the formula

$$E_\nu(x) = \frac{E_{\nu+(m^n)}(x)}{(x_1 \cdots x_n)^m}.$$ 

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees $\nu \in \mathbb{Z}^n$.

**Definition 20.** Let $\nu \in \mathbb{Z}^n$ and $m \in \mathbb{Z}_{\geq 0}$ such that $\nu + (m^n) \in C_n$. Then $\overline{K}_\nu(x) := \overline{K}_\nu(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ is defined by

$$\overline{K}_\nu(x) := q^{\nu |} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-1} \right) \overline{K}_{\nu+(m^n)}(x).$$

Using

$$\lim_{a \to \infty} A_m(ax; \nu) = q^{-m^2 n} t^{(1-n)n} \prod_{i=1}^{n} (\overline{\nu}_i x_i)^{-m}$$

and the definitions of $G_\nu(x)$ and $K_\nu(x)$ it follows that

$$\lim_{a \to \infty} a^{-|\nu|} G_\nu(ax) = E_\nu(x),$$

$$\lim_{a \to \infty} K_\nu(ax) = \overline{K}_\nu(x)$$

for all $\nu \in \mathbb{Z}^n$, so in particular

$$\overline{K}_\nu(x) = \frac{E_\nu(x)}{E_\nu(\nu)} \quad \forall \nu \in \mathbb{Z}^n.$$
Lemma 19 holds true for the extension of \( \mathcal{K} \) to the map \( \mathcal{K} \in \mathcal{F}[x^\pm 1] \) defined by \( \nu \mapsto \mathcal{K}_\nu \) \((\nu \in \mathbb{Z}^n)\). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all \( u, v \in \mathbb{Z}^n \),

\[
\mathcal{K}_u(\tilde{v}) = \mathcal{K}_v(\tilde{u}).
\]

### 7.2 \( O \)-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the \( O \)-polynomials \( O_\alpha \) (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for \( O_\alpha \) in terms of the non-symmetric interpolation Macdonald polynomial \( K_\alpha \).

**Proposition 22.** For all \( \alpha \in C_n \) we have

\[
O_\alpha(x) = K_\alpha(t^{1-n}a\nu_0x).
\]

**Proof.** The polynomial \( \tilde{O}_\alpha(x) := K_\alpha(t^{1-n}a\nu_0x) \) is of degree at most \( |\alpha| \) and

\[
\tilde{O}_\alpha(\beta^{-1}) = K_\alpha(t^{1-n}a\nu_0\beta^{-1}) = K_\alpha(a\tilde{\beta}) = K_\alpha(a\tilde{\alpha})
\]

for all \( \beta \in C_n \) by (4) and Theorem 17. Hence, \( \tilde{O}_\alpha = O_\alpha \). \( \blacksquare \)

### 7.3 Okounkov’s duality

Write \( F[x]^{S_n} \) for the symmetric polynomials in \( x_1, \ldots, x_n \) with coefficients in a field \( F \). Write \( C_+ := \sum_{w \in S_n} H_w \). The symmetric interpolation Macdonald polynomial \( R_\lambda(x) \in F[x]^{S_n} \) is the multiple of \( C_+ G_\lambda \) such that the coefficient of \( x^\lambda \) is one (see, e.g., [13]). We write

\[
K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(\alpha \tau)} \in \mathbb{K}[x]^{S_n}
\]

for the normalized symmetric interpolation Macdonald polynomial. Then

\[
C_+ K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\alpha^+(x)
\]

for \( \alpha \in C_n \). Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a\mu^{-1}) = K_\mu^+(a\lambda^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\widehat{C}_+ = \sum_{w \in S_n} \widehat{H}_w$, with $\widehat{H}_w := \widehat{H}_{i_1} \cdots \widehat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_\lambda$ for the function $f_\mu(u) := K_u(a\mu)(u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\mu) = (C_+ K_\lambda)(a\mu) = (\widehat{C}_+ f_\mu)(\lambda)$$

(24)

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{u}) = (Jw_0 K_\mu(t^{1-n}x))|_{x=a^{-1}\mu}$$

(25)

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_iJ = (H_i^\circ)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^\circ)^{-1}$$

(26)

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x=a^{-1}\lambda}.$$ 

By (23) and (4) this simplifies to

$$(\widehat{C}_+ f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\lambda).$$

Returning to (24) we conclude that $K_\lambda^+(a\mu) = K_\mu^+(a\lambda)$. Since $K_\lambda^+$ is symmetric we obtain from (4) that

$$K_\lambda^+(a\mu^{-1}) = K_\mu^+(a\lambda^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).
Lemma 24. For \( u, v \in \mathbb{Z}^n \) we have

\[
(H_w K_u)(a \tilde{v}) = (H_w K_v)(a \tilde{u}).
\] (27)

Proof. We proceed as in the previous subsection. Set \( f_v(u) := K_u(a \tilde{v}) \) for \( u, v \in \mathbb{Z}^n \). By part 1 of Proposition 16,

\[
(H_w K_u)(a \tilde{v}) = (\hat{H}_w f_v)(u).
\]

Since \( f_v(u) = (Iw_0 K_v)(a^{-1} t^{n-1} \tilde{u}) \) by (4), Remark 7 implies that

\[
(\hat{H}_w f_v)(u) = (H_w Jw_0 K_v)(a^{-1} t^{n-1} \tilde{u}).
\]

Now \( H_w Jw_0 = Jw_0 H_w \) by (26); hence,

\[
(\hat{H}_w f_v)(u) = (Jw_0 H_w K_v)(a^{-1} t^{n-1} \tilde{u}) = (H_w K_v)(a \tilde{u}),
\]

which completes the proof.

Recall from Theorem 1 that

\[
G'(\beta)(x) = t^{(1-n)|\beta|+I(\beta)} \Psi G_\omega(t^{n-1} x)
\]

with \( \Psi := w_0 H_w^\omega \). We define normalized versions by

\[
K'(\beta)(x) := \frac{G'(\beta)(x)}{G'_\omega(a^{-1} \tau)} = t^{\ell(w_0)} \Psi K_\omega(t^{n-1} x), \quad \beta \in \mathcal{C}_n,
\]

with \( K_\omega := \iota(K_\omega) \) for \( \nu \in \mathbb{Z}^n \) (the 2nd formula follows from Lemma 2). More generally, we define for \( \nu \in \mathbb{Z}^n \),

\[
K'_\nu(x) := t^{\ell(w_0)} \Psi K_\omega(t^{n-1} x).
\] (28)

We write \( K' : \mathbb{Z}^n \to \mathbb{K}(x) \) for the map \( \nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n) \). Since \( H_i \Psi = \Psi H_i^\omega \), part 1 of Proposition 16 gives \( H_i K' = \hat{H}_i^\omega K' \). Considering the action of \( ((x_n - 1)\Delta^\circ)^n \) on \( K'_\nu(x) \) we get, using the fact that \( ((x_n - 1)\Delta^\circ)^n \) commutes with \( \Psi \) and part 3 of Proposition 16,

\[
K'_\nu(x) = \left( \prod_{i=1}^{n} \frac{(1 - a^{-1} x_i)}{(1 - q^{-1} x_i)} \right) K'_\nu((1)) (q^{-1} x),
\]
in particular
\[
K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v_i; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v^n(m)}(q^{-m}x).
\]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K'_v(x) \) for \( v \in \mathbb{Z} \); hence,
\[
K'_{-m}(x) = \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m},
\]
\[
K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a^{-1}; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m}
\]
for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have
\[
K'_v(a^{-1}u) = K'_u(a^{-1}v).
\]

**Proof.** Note that
\[
K'_v(a^{-1}u) = t^{\ell(W_0)} \psi K'_{v}(t^{n-1}x)|_{x=a^{-1}u} = t^{\ell(W_0)} (H_{W_0} \circ K_{v})(a^{-1}\tilde{u}^{-1})
\]
by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in \mathbb{C}_n \) the generalized binomial coefficient by
\[
\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} := \frac{G_\beta(\alpha)}{G_\beta(\beta)}.
\]

(29)

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get
\[
\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1},t^{-1}} = \frac{G_\beta(\alpha^{-1})}{G_\beta(\beta^{-1})}.
\]
Theorem 27. For $\alpha, \beta \in C_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in C_n} a^{\mid \beta \mid} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} \frac{G'_{\beta}(x)}{G_{\beta}(ax)}.$$  

(30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in C_n} \tau^{-1}_{\beta} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q^{-1}, t^{-1}} \frac{K'_{\beta}(x)}{K_{\beta}(\alpha x)}$$

$$= \sum_{\beta \in C_n} \frac{K^\circ_{\beta}(\bar{\alpha}^{-1})K'_{\beta}(x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{K^\circ_{\beta}(\bar{\alpha}^{-1})\Psi K^0_{\beta}(t^{n-1}x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}$$

(31)

with $\Psi = w_0H_{w_0}$ (note that the dependence on $\alpha$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^\circ_{\beta}(x)$ and $K_{\beta}(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K^\circ_{\beta}(\bar{\alpha}^{-1})K^0_{\beta}(t^{n-1}w_0x)}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}.$$  

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in C_n} \frac{K^\circ_{\beta}(\bar{\alpha}^{-1})K^0_{\beta}(\bar{\gamma}^{-1})}{\tau_{\beta}K_{\beta}(\bar{\beta}^{-1})}.$$  

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_{\alpha}$ and $G_{\alpha}$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G^o_\beta(\bar{\alpha}^{-1}) \Psi K^o_\beta(t^{n-1}x)}{\tau_\beta G^o_\beta(\bar{\beta}^{-1})},$$

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0Hw_0$ on both sides. Since $w_0Hw_0\Psi = \Id$ we obtain

$$\Psi K^o_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K^o_\alpha(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\bar{\alpha}^{-1})K_\beta(ax). \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}. \quad (34)$$

Since $\left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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