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Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \ldots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in $n$ variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most $n$ parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \cdots + \mu_n$ and write

$$\overline{\mu} = (q^{\mu_1} \tau_1, \ldots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \ldots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$
Then \( R_\lambda(x) = R_\lambda(x; q, t) \) is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most \(|\lambda|\) satisfying the vanishing conditions

\[
R_\lambda(\mu) = 0 \quad \text{for} \quad \mu \in \mathcal{P}_n \quad \text{such that} \quad |\mu| \leq |\lambda|, \mu \neq \lambda.
\]

The normalization is fixed by requiring that the coefficient of \( x^{\lambda} := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \) in the monomial expansion of \( R_\lambda(x) \) is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of \( R_\lambda(x) \) is the Macdonald polynomial \( P_\lambda(x) \) and \( R_\lambda(x) \) satisfies the extra vanishing property \( R_\lambda(\mu) = 0 \) unless \( \lambda \subseteq \mu \) as Ferrer diagrams. Other key properties of \( R_\lambda(x) \), which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of \( R_\lambda(ax) = R_\lambda(ax_1, \ldots, ax_n; q, t) \) in terms of the \( R_\mu(x; q^{-1}, t^{-1})'s \) over the field \( K := \mathbb{Q}(q, t, a) \), and the duality or evaluation symmetry, which involves the evaluation points

\[
\tilde{\mu} = (q^{-\mu_n} \tau_1, \ldots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n
\]

and takes the form

\[
\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.
\]

The interpolation polynomials have natural non-symmetric analogs \( G_\alpha(x) = G_\alpha(x; q, t) \), which were also defined in [4, 13]. These are indexed by the set of compositions with at most \( n \) parts, \( \mathcal{C}_n := (\mathbb{Z}_{\geq 0})^n \). For a composition \( \beta \in \mathcal{C}_n \) we define

\[
\overline{\beta} := w_\beta(\overline{\beta_+}),
\]

where \( w_\beta \) is the shortest permutation such that \( \beta_+ = w_\beta^{-1}(\beta) \) is a partition. Then \( G_\alpha(x) \) is, up to normalization, characterized as the unique polynomial of degree at most \(|\alpha| := \alpha_1 + \cdots + \alpha_n \) satisfying the vanishing conditions

\[
G_\alpha(\overline{\beta}) = 0 \quad \text{for} \quad \beta \in \mathcal{C}_n \quad \text{such that} \quad |\beta| \leq |\alpha|, \beta \neq \alpha.
\]

The normalization is fixed by requiring that the coefficient of \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) in the monomial expansion of \( G_\alpha(x) \) is 1.

Many properties of the symmetric interpolation polynomials \( R_\lambda(x) \) admit non-symmetric counterparts for the \( G_\alpha(x) \). For instance, the top homogeneous part of \( G_\alpha(x) \)
is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := (\omega_0 \beta)$, with $\omega_0$ the longest element of the symmetric group $S_n$:

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in $n$ variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators $H_w$ ($w \in S_n$) as described in the next section.

Theorem A. Write $I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \}$. Then we have

$$G'_\alpha (t^{n-1} x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)} \omega_0 H_\omega \omega_0 G_\alpha (x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov’s duality result.

Theorem B. For all compositions $\alpha, \beta \in C_n$ we have

$$\frac{G_\alpha (a\tilde{\beta})}{G_\alpha (a\tau)} = \frac{G_\beta (a\tilde{\alpha})}{G_\beta (a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation $O$-polynomials introduced in [14, Thm. 1.1]. Write $x^{-1}$ for $(x_1^{-1}, \ldots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha (\tilde{\beta}^{-1}) = \frac{G_\beta (a\tilde{\alpha})}{G_\beta (a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the $O$-polynomials in terms of the interpolation polynomials $G_\alpha(x)$. 

Theorem C. For all compositions $\alpha \in C_n$ we have

$$O_{\alpha}(x) = \frac{G_{\alpha}(t^{1-n}aw_0x)}{G_{\alpha}(a\tau)}.$$ 

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov’s [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G_{\alpha}'(x)$ in terms of the $G_{\beta}(ax)$'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let $S_n$ be the symmetric group in $n$ letters and $s_i \in S_n$ the permutation that swaps $i$ and $i+1$. The $s_i$ ($1 \leq i < n$) are Coxeter generators for $S_n$. Let $\ell : S_n \rightarrow \mathbb{Z}_{\geq 0}$ be the associated length function. Let $S_n$ act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots)$ for $v = (v_1, \ldots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \rightarrow n+1-i$ for $i = 1, \ldots, n$.

For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ define $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n) \in \mathbb{F}^n$ by $\overline{v}_i := q^{v_i}t^{-k_i(v)}$ with

$$k_i(v) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$ 

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \cdots \geq v_n$, then $\overline{v} = (q^{v_1}\tau_1, \ldots, q^{v_n}\tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\overline{v} = w_v(\overline{v}^+)$ with $w_v \in S_n$ the shortest permutation such that $\overline{v}^+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\overline{v} := -w_0v$ for $v \in \mathbb{Z}^n$.

Note that $\overline{\alpha} = t^{1-n}$ if $\alpha \in C_n$ with $\alpha_n = 0$.

For a field $F$ we write $F[x] := F[x_1, \ldots, x_n]$, $F[x^\pm 1] := F[x_1^\pm, \ldots, x_n^\pm]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $\mathbb{F}[x]$ and $\mathbb{F}(x)$, with the action of $s_i$ by interchanging $x_i$ and $x_{i+1}$ for $1 \leq i < n$. Consider the $\mathbb{F}$-linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1 - s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1).$$
on $F(x)$ $(1 \leq i < n)$ called Demazure-Lusztig operators, and the automorphism $\Delta$ of $F(x)$ defined by

$$\Delta f(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

Note that $H_i$ $(1 \leq i < n)$ and $\Delta$ preserve $F[x^{\pm 1}]$ and $F[x]$. Cherednik [1, 2] showed that the operators $H_i$ $(1 \leq i < n)$ and $\Delta$ satisfy the defining relations of the type A extended affine Hecke algebra,

$$(H_i - t)(H_i + 1) = 0,$$

$$H_iH_j = H_jH_i, \quad |i - j| > 1,$$

$$H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1},$$

$$\Delta H_{i+1} = H_i\Delta,$$

$$\Delta^2 H_1 = H_{n-1}\Delta^2$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]).

For $w \in S_n$ we write $H_w := H_{i_1}H_{i_2}\cdots H_{i_\ell}$ with $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the $H_i$'s. Write $\overline{H}_i := H_i + 1 - t = tH_i^{-1}$ and set

$$\xi_i := t^{1-n}\overline{H}_{i-1}\cdots\overline{H}_1\Delta^{-1}H_{n-1}\cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators $\xi_i$'s are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i\cdots\overline{H}_{n-1}\Delta H_1\cdots H_{i-1}.$$

The $\xi_i^{-1}$ $(1 \leq i \leq n)$ are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in F[x]$ of degree $\alpha \in C_n$ is the unique polynomial satisfying

$$\xi_i^{-1}E_\alpha = \overline{u}_iE_\alpha, \quad i = 1, \ldots, n$$

and normalized such that the coefficient of $x^\alpha$ in $E_\alpha$ is 1.

Let $\iota$ be the field automorphism of $K$ inverting $q$, $t$ and $a$. It restricts to a field automorphism of $F$, inverting $q$ and $t$. We extend $\iota$ to a $Q$-algebra automorphism of $K[x]$
and \( \mathbb{F}[x] \) by letting \( \iota \) act on the coefficients of the polynomial. Write
\[
G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)
\]
for \( \alpha \in C_n \). Note that \( \overrightarrow{v}^{-1} = (\iota(v_1), \ldots, \iota(v_n)) \).

Put \( H_i^\circ, H_w^\circ, \overrightarrow{H_i}, \Delta^\circ \) and \( \xi_i^\circ \) for the operators \( H_i, H_w, \overrightarrow{H_i}, \Delta \) and \( \xi_i \) with \( q, t \) replaced by their inverses. For instance,
\[
H_i^\circ = t^{-1} s_i - \frac{(1 - t^{-1})x_i}{x_i - x_{i+1}} (1 - s_i),
\]
\[
\Delta^\circ f(x_1, \ldots, x_n) = f(q x_n, x_1, \ldots, x_{n-1}).
\]

We then have \( \xi_i^\circ E_\alpha^\circ = \overrightarrow{E}_\alpha \) for \( i = 1, \ldots, n \), which characterizes \( E_\alpha^\circ \) up to a scalar factor.

**Theorem 1.** For \( \alpha \in C_n \) we have
\[
G'_\alpha(x) = t^{(1 - n)|\alpha| + I(\alpha)} w_0 H_{w_0}^\circ G_\alpha^\circ (t^{n-1} x)
\]
with \( I(\alpha) := \# \{ i < j \mid \alpha_i \geq \alpha_j \} \).

**Remark.** Formally set \( t = q^r \), replace \( x \) by \( 1 + (q - 1)x \), divide both sides of (2) by \( (q - 1)^{|\alpha|} \) and take the limit \( q \to 1 \). Then
\[
G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n - 1)r; r)
\]
for the non-symmetric interpolation Jack polynomial \( G_\alpha(\cdot; r) \) and its primed version (see [14]). Here \( \sigma \) denotes the action of the symmetric group with \( \sigma(s_i) \) the rational degeneration of the Demazure-Lusztig operators \( H_i \), given explicitly by
\[
\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}} (1 - s_i),
\]
see [14, Section 1]. To establish the formal limit (3) one uses that \( \sigma(w_0) w_0 = w_0 \sigma^\circ(w_0) \) with \( \sigma^\circ \) the action of the symmetric group defined in terms of the rational degeneration
\[
\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}} (1 - s_i)
\]
of \( H_i^\circ \). Formula (3) was obtained before in [14, Thm. 1.10].
Proof. We show that the right-hand side of (2) satisfies the defining properties of $G'_{\alpha}$. For the vanishing property, note that

$$t^{n-1}w_0\tilde{\beta} = \tilde{\beta}^{-1}$$

(this is the $q$-analog of [14, Lem. 6.1(2)]); hence,

$$(w_0H_{w_0}^oG_{\alpha}^o(t^{n-1}x))|_{x=\tilde{\beta}} = (H_{w_0}^oG_{\alpha}^o(x))|_{x=\tilde{\beta}^{-1}}.$$ 

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_{\alpha}^o(w\tilde{\beta}^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_{\alpha} = t^{I(\alpha)}w_0H_{w_0}^oE_{\alpha}^o.$$ 

Note that $\Psi := w_0H_{w_0}^o$ satisfies the intertwining properties

$$H_i\Psi = t\Psi H_i^o,$$

$$\Delta \Psi = t^{n-1}\Psi H_{n-1}^o \cdots H_1^o(\Delta^o)^{-1}H_{n-1}^o \cdots H_1^o$$

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi \xi_i^o$ for $i = 1, \ldots, n$. Therefore,

$$E_{\alpha}(x) = c_{\alpha}\Psi E_{\alpha}^o(x)$$

for some constant $c_{\alpha} \in \mathbb{F}$. But the coefficient of $x^{\alpha}$ in $\Psi x^{\alpha}$ is $t^{-I(\alpha)}$; hence, $c_{\alpha} = t^{I(\alpha)}$. ■

Consider the Demazure operators $H_i$ and the Cherednik operators $\xi_j^{-1}$ as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators $\xi_j^{-1}$ with eigenvalues $\bar{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in $E_u$ is 1. For $u = \alpha \in \mathcal{C}_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_{\alpha} \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1 \cdots x_nE_u(x).$$
It is now easy to check that formula (5) is valid with \( \alpha \) replaced by an arbitrary integral vector \( u \),

\[
E_u = t^l(u) w_0 H_{w_0}^\circ E_u
\]

(7)

with \( E_u^\circ := \iota(E_u) \). Furthermore, one can show in the same vein as the proof of (5) that

\[
w_0 E_{-w_0 u}(x^{-1}) = E_u(x)
\]

for an integral vector \( u \), where \( p(x^{-1}) \) stands for inverting all the parameters \( x_1, \ldots, x_n \) in the Laurent polynomial \( p(x) \in F[x^{\pm 1}] \). Combining this equality with (7) yields

\[
E_{-w_0 u}(x^{-1}) = t^l(u) H_{w_0}^\circ E_u(x),
\]

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

\[
G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{1-n} - q^{a'(s) + l'(s)}}{1 - q^{a(s) + l(s) + 1}} \right) \prod_{s \in \alpha} (at^{l'(s)} - q^{a'(s)})
\]

(8)

was obtained, with \( a(s), l(s), a'(s) \) and \( l'(s) \) the arm, leg, coarm and coleg of \( s = (i, j) \in \alpha \), defined by

\[
\begin{align*}
a(s) &:= \alpha_i - j, \\
l(s) &:= \# \{ k > i \mid j \leq \alpha_k \leq \alpha_i \} + \# \{ k < i \mid j \leq \alpha_k + 1 \leq \alpha_i \}, \\
a'(s) &:= j - 1, \\
l'(s) &:= \# \{ k > i \mid \alpha_k > \alpha_i \} + \# \{ k < i \mid \alpha_k \geq \alpha_i \}.
\end{align*}
\]

By (8) we have

\[
E_\alpha(\tau) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(a \tau) = \prod_{s \in \alpha} \left( \frac{t^{l(s) + l'(s)} - q^{a'(s) + l'(s) + 1}}{1 - q^{a(s) + l(s) + 1}} \right),
\]

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for \( \alpha \in C_n \),

\[
\ell(w_0) - I(\alpha) = \# \{ i < j \mid \alpha_i < \alpha_j \}.
\]
Lemma 2. For $\alpha \in C_n$ we have

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha) - \ell(w_0)}G^0_\alpha(a\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = \overline{0}^{-1}$ we have by Theorem 1,

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha)}(H^0_{w_0}G^0_\alpha)(a\overline{0}^{-1})$$

$$= t^{(1-n)|\alpha|+I(\alpha) - \ell(w_0)}G^0_\alpha(a\overline{0}^{-1}),$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality.

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G^0_\alpha(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the $S_n$-orbit of $\alpha$, while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in C_n$ we have

$$G_\alpha(a\tau) = (-a)^{|\alpha|}t^{(1-n)|\alpha|-n(\alpha)}q^{n'(\alpha)}G^0_\alpha(a^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial $G_\alpha$.

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in C_n$) by

$$\tau_\alpha := (-1)^{|\alpha|}q^{n'(\alpha)}t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the $S_n$-orbit of $\alpha$.\]
Corollary 4.  For \( \alpha \in \mathbb{C}^n \) we have

\[
G'_\alpha(a^{-1}\tau) = \tau^{-1}_\alpha a^{-|\alpha|} G_\alpha(\alpha \tau).
\]

Proof. Use Lemmas 2 and 3 and (9). \( \blacksquare \)

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of \( \mathbb{K} \)-valued functions on \( \mathbb{Z}^n \), which is constructed as follows.

For \( v \in \mathbb{Z}^n \) and \( y \in \mathbb{K}^n \) write \( v^\natural := (v_2, \ldots, v_n, v_1 + 1) \) and \( y^\natural := (y_2, \ldots, y_n, qy_1) \).

Denote the inverse of \( \natural \) by \( \sharp \), so \( v^\natural \sharp = (v_n - 1, v_1, \ldots, v_1 - 1) \) and \( y^\natural \sharp = (y_n/q, y_1, \ldots, y_1 - 1) \). We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let \( v \in \mathbb{Z}^n \) and \( 1 \leq i < n \). Then we have

1. \( s_i(v) = s_i^\natural v \) if \( v_i \neq v_{i+1} \).
2. \( \bar{v}_i = t\bar{v}_{i+1} \) if \( v_i = v_{i+1} \).
3. \( \bar{v}^\natural = v^\natural \).

Let \( \mathbb{H} \) be the double affine Hecke algebra over \( \mathbb{K} \). It is isomorphic to the subalgebra of \( \text{End}(\mathbb{K}[x^{\pm 1}]) \) generated by the operators \( H_i \ (1 \leq i < n), \Delta^{\pm 1}, \) and the multiplication operators \( x_j^{\pm 1} \ (1 \leq j \leq n) \).

For a unital \( \mathbb{K} \)-algebra \( A \) we write \( \mathcal{F}_A \) for the space of \( A \)-valued functions \( f: \mathbb{Z}^n \to A \) on \( \mathbb{Z}^n \).

Corollary 6. Let \( A \) be a unital \( \mathbb{K} \)-algebra. Consider the \( A \)-linear operators \( \hat{H}_i \ (1 \leq i < n) \), \( \hat{\Delta} \) and \( \hat{x}_j \ (1 \leq j \leq n) \) on \( \mathcal{F}_A \) defined by

\[
(\hat{H}_i f)(v) := tf(v) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (f(s_i v) - f(v)),
\]

\[
(\hat{\Delta} f)(v) := f(v^\natural), \quad (\hat{\Delta}^{-1} f)(v) := f(v^\sharp),
\]

\[
(\hat{x}_j f)(v) := a\bar{v}_j f(v)
\]  (11)

for \( f \in \mathcal{F}_A \) and \( v \in \mathbb{Z}^n \). Then \( H_i \mapsto \hat{H}_i \ (1 \leq i < n), \Delta \mapsto \hat{\Delta} \) and \( x_j \mapsto \hat{x}_j \ (1 \leq j \leq n) \) defines a representation \( \mathbb{H} \to \text{End}_A(\mathcal{F}_A), X \mapsto \hat{X} \ (X \in \mathbb{H}) \) of the double affine Hecke algebra \( \mathbb{H} \) on \( \mathcal{F}_A \).
**Proof.** Let $O \subset \mathbb{K}^n$ be the smallest $S_n$-invariant and $\mathfrak{z}$-invariant subset that contains \{a\bar{v} \mid v \in \mathbb{Z}^n\}. Note that $O$ is contained in \{y \in \mathbb{K}^n \mid y_i \neq y_j$ if $i \neq j\}. The Demazure–Lusztig operators $H_i (1 \leq i < n)$, $\Delta^{\pm 1}$ and the coordinate multiplication operators $x_j (1 \leq j \leq n)$ act $A$-linearly on the space $F_A^O$ of $A$-valued functions on $O$, and hence turns $F_A^O$ into an $\mathbb{H}$-module. Define the surjective $A$-linear map

$$\text{pr} : F_A^O \rightarrow F_A$$

by $\text{pr}(g)(v) := g(a\bar{v}) (v \in \mathbb{Z}^n)$.

We claim that $\text{Ker}(\text{pr})$ is an $\mathbb{H}$-submodule of $F_A^O$. Clearly $\text{Ker}(\text{pr})$ is $x_j$-invariant for $j = 1, \ldots, n$. Let $g \in \text{Ker}(\text{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(\text{pr})$. To show that $H_j g \in \text{Ker}(\text{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \bar{v} = s_i \bar{v}$ by part 1 of Lemma 5. Hence,

$$(H_i g)(a\bar{v}) = tg(a\bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a\bar{v}_i) - g(a\bar{v})) = 0.$$

If $v_i = v_{i+1}$ then $\bar{v}_i = t\bar{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_i g)(\bar{v}) = tg(a\bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (g(a\bar{v}_i) - g(a\bar{v})) = tg(a\bar{v}) = 0.$$

Hence, $F_A$ inherits the $\mathbb{H}$-module structure of $F_A^O / \text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of $H_i (1 \leq i < n)$, $\Delta$ and $x_j (1 \leq j \leq n)$ on $F_A$ is by the operators $\hat{H}_i (1 \leq i < n)$, $\hat{\Delta}$ and $\hat{x}_j (1 \leq j \leq n)$. ■

**Remark 7.** With the notations from (the proof of) Corollary 6, let $\tilde{g} \in F_A^O$ and set $g := \text{pr}(\tilde{g}) \in F_A$. In other words, $g(v) := \tilde{g}(a\bar{v})$ for all $v \in \mathbb{Z}^n$. Then

$$(\hat{X} g)(v) = (X \tilde{g})(a\bar{v}), \quad v \in \mathbb{Z}^n$$

for $X = H_i, \Delta^{\pm 1}, x_j$.

**Remark 8.** Let $F_A^+$ be the space of $A$-valued functions on $\mathcal{C}_n$. We sometimes will consider $\hat{H}_i (1 \leq i < n)$, $\hat{\Delta}^{-1}$ and $\hat{x}_j (1 \leq j \leq n)$, defined by the formulas (11), as linear operators on $F_A^+$.

**Definition 9.** We call

$$K_\alpha(x; q, t) := \frac{G_\alpha(x; q, t)}{G_\alpha(\alpha \tau; q, t)} \in \mathbb{K}[x]$$

(12)

the normalized non-symmetric interpolation Macdonald polynomial of degree $\alpha$. 

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We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that $a$ cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(0) = 0$ if $\alpha \in C_n$ is nonzero. Note furthermore that

$$\lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{a^{-|\alpha|}E_\alpha(\tau)}$$

since $\lim_{a \to \infty} a^{-|\alpha|}G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi_1 = (x_n - t^{-1})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j}H_j \cdots H_{n-1}\Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$  

The operators $H_i$, $\Xi_j$ and $\Phi$ preserve $K[x]$ (see [4]); hence, they give rise to $\mathbb{K}$-linear operators on $F_{\mathbb{K}[x]}^+$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in C_n$). Note that the operators $H_i$, $\Xi_j$ and $\Phi$ on $F_{\mathbb{K}[x]}^+$ commute with the hat-operators $\hat{H}_i$, $\hat{x}_j$ and $\hat{\Delta}^{-1}$ on $F_{\mathbb{K}[x]}^+$ (cf. Remark 8).

The same remarks hold true for the space $F_{\mathbb{K}(x)}$ of $\mathbb{K}(x)$-valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$-action on $F_{\mathbb{K}(x)}$).

Let $K \in F_{\mathbb{K}[x]}^+$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in C_n$).

**Lemma 10.** For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $F_{\mathbb{K}[x]}^+$,

1. $H_i K = \hat{H}_i K$.
2. $\Xi_j K = a\hat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1} K$.

**Proof.** 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the $K_\beta$'s. As a 1st step we expand $H_i G_\alpha$ as linear combination of the $G_\beta$'s.

If $\alpha \in C_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}G_\alpha(x) + G_{s_i \alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that $H_i$ satisfies the quadratic relation $(H_i - t)H_i + 1 = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t - 1)\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}G_\alpha(x) + \frac{t(\overline{\alpha}_{i+1} - t\overline{\alpha}_i)(\overline{\alpha}_{i+1} - t^{-1}\overline{\alpha}_i)}{(\overline{\alpha}_{i+1} - \overline{\alpha}_i)^2}G_{s_i \alpha}(x)$$
if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = t G_\alpha(x)$ if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the $K_\beta$'s can now be obtained using the formula

$$G_\alpha(\alpha \tau) = \frac{\overline{\alpha}_{i+1} - t\overline{\alpha}_i}{\overline{\alpha}_{i+1} - \overline{\alpha}_i} G_{s\alpha}(\alpha \tau)$$

for $\alpha \in \mathcal{C}_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \tilde{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in \mathcal{C}_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\circ}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^\circ}(\alpha \tau)}{G_\alpha(\alpha \tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.$$

Hence,

$$\Phi K_\alpha(x) = t^{1-n}(a\overline{\alpha}_1 - 1)K_{\alpha^\circ}(x).$$

**Remark 11.** Note that

$$\Phi K_\alpha(x) = (a\overline{\alpha}_n - t^{1-n})K_{\alpha^\circ}(x)$$

for $\alpha \in \mathcal{C}_n$ since $\overline{\alpha}_n = t^{n-1} w_0 \overline{\alpha}$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}_+^{\mathbb{Z}[x]}$ to a map $K \in \mathcal{F}_+^{\mathbb{Z}[x]}$ such that Lemma 10 remains valid.

**Lemma 12.** For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left( \prod_{i=1}^n \frac{1 - a\overline{\alpha}_i^{-1}}{1 - q^{n-1} x_i} \right) K_{\alpha+(1^n)}(qx).$$
Proof. Note that for $f \in \mathbb{K}[x],$

$$\Phi^nf(x) = \left(\prod_{i=1}^{n}(x_i - t^{1-n})\right)f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. 

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^{n} \left(\frac{(q^{1-m}a \tau^{-1}; q)_m}{(qt^{n-1}x_i; q)_m}\right) \forall v \in \mathbb{Z}^n,$$

with $(y; q)_m := \prod_{j=0}^{m-1}(1 - q^jy)$ the $q$-shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \cdots + v_n.$ Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v| - m^2n} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^{n} x_i^m (q^{-m}t^{1-n}x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v)K_{v+(m^n)}(q^m x),$$

where $m$ is a nonnegative integer such that $v + (m^n) \in \mathcal{C}_n$ (note that $G_v$ and $K_v$ are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0},$

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left(\frac{x}{a}\right)^m \left(\frac{x^{-1}; q)_m}{(a^{-1}; q)_m}. $$

Lemma 15. For all $v \in \mathbb{Z}^n,$

$$K_v(x) = \frac{G_v(x)}{G_v(a \tau)}.$$

Proof. Let $v \in \mathbb{Z}^n.$ Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(a \tau) = 1.$ Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n.$ Then

$$K_v(a \tau) = A_m(a \tau; v)K_{v+(m^n)}(q^m a \tau) = A_m(a \tau; v)\frac{G_{v+(m^n)}(q^m a \tau)}{G_{v+(m^n)}(a \tau)} = 1,$$
where the last formula follows from a direct computation using the evaluation formula (8).

We extend the map \( K : \mathcal{C}_n \rightarrow \mathbb{K}[x] \) to a map

\[
K : \mathbb{Z}^n \rightarrow \mathbb{K}(x)
\]

by setting \( v \mapsto K_v(x) \) for all \( v \in \mathbb{Z}^n \). Lemma 10 now extends as follows.

**Proposition 16.** We have, as identities in \( \mathcal{F}_{\mathbb{K}(x)} \),

1. \( H_iK = \hat{H}_iK \).
2. \( \Xi_jK = a\hat{x}_j^{-1}K \).
3. \( \Phi K = t^{1-n}(a^2\hat{x}_1^{-1} - 1)\hat{\Delta}^{-1}K \).

**Proof.** Write \( A_m \in \mathcal{F}_{\mathbb{K}(x)} \) for the map \( v \mapsto A_m(x; v) \) for \( v \in \mathbb{Z}^n \). Consider the linear operator on \( \mathcal{F}_{\mathbb{K}(x)} \) defined by \( (A_m f)(v) := A_m(x; v)f(v) \) for \( v \in \mathbb{Z}^n \) and \( f \in \mathcal{F}_{\mathbb{K}(x)} \). For \( 1 \leq i < n \) we have \([H_i, A_m] = 0\) as linear operators on \( \mathcal{F}_{\mathbb{K}(x)} \), since \( A_m(x; v) \) is a symmetric rational function in \( x_1, \ldots, x_n \). Furthermore, for \( v \in \mathbb{Z}^n \) and \( f \in \mathcal{F}_{\mathbb{K}(x)} \),

\[
((\hat{H}_i \circ A_m f)(v) = ((A_m \circ \hat{H}_i)f)(v) \quad \text{if} \quad v_i \neq v_{i+1} \quad (15)
\]

by part 2 of Lemma 5 and the fact that \( A_m(x; v) \) is symmetric in \( v_1, \ldots, v_n \). Fix \( v \in \mathbb{Z}^n \) and choose \( m \in \mathbb{Z}_{\geq 0} \) such that \( v + (m^n) \in \mathcal{C}_n \). Since

\[
K_v(x) = A_m(x; v)K_{v + (m^n)}(q^m x)
\]

we obtain from \([H_i, A_m] = 0\) and (15) that \((H_iK)(v) = (\hat{H}_iK)(v)\) if \( v_i \neq v_{i+1} \). This also holds true if \( v_i = v_{i+1} \) since then \((\hat{H}_iK)(v) = tK_v \) and \( H_iK_{v + (m^n)}(q^m x) = tK_{v + (m^n)}(q^m x) \). This proves part 1 of the proposition.

Note that \( \Phi K_v(x) = t^{1-n}(a\bar{v}_1^{-1} - 1)K_v(x) \) for arbitrary \( v \in \mathbb{Z}^n \) by Lemma 10 and the commutation relation

\[
\Phi \circ A_m = A_m \circ \Phi(q^m), \quad (16)
\]

where \( \Phi(q^m) := (q^mx_n - t^{1-n})\Delta \). This proves part 3 of the proposition.

Finally we have \( \Xi_jK_v(x) = \bar{v}_j^{-1}K_v(x) \) for all \( v \in \mathbb{Z}^n \) by \([H_i, A_m] = 0\), (16) and Lemma 10. This proves part 2 of the proposition.

\[ \qed \]
6  Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = -w_0 v$ for $v \in \mathbb{Z}^n$.

**Theorem 17.** (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}).$$  \hspace{1cm} (17)

**Example 18.** If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_{m}(a^{-1}; q)_r}$$  \hspace{1cm} (18)

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of $m$ and $r$.

**Proof.** We divide the proof of the theorem in several steps. ■

**Step 1.** If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{siu}(a\tilde{v}) = K_v(as_{i+1}u)$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

**Proof of Step 1.** Writing out the formula from part 1 of Proposition 16 gives

$$\frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{v}) + \frac{(\tilde{v}_i - t\tilde{v}_{i+1})}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{s}_{n-i}v)$$

$$= \frac{(t-1)\tilde{u}_i}{(\tilde{u}_i - \tilde{u}_{n-i})} K_u(a\tilde{v}) + \frac{(\tilde{u}_i - t\tilde{u}_{n-i})}{(\tilde{u}_i - \tilde{u}_{n-i})} K_{siu}(a\tilde{v}).$$  \hspace{1cm} (19)

Replacing in (19) the role of $u$ and $v$ and replacing $i$ by $n - i$ we get

$$\frac{(t-1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a\tilde{u}) + \frac{(\tilde{v}_{n-i} - t\tilde{v}_{n+1-i})}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_{v}(as_{i+1}u)$$

$$= \frac{(t-1)\tilde{v}_{n-i}}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_v(a\tilde{u}) + \frac{(\tilde{v}_{n-i} - t\tilde{v}_{n+1-i})}{(\tilde{v}_{n-i} - \tilde{v}_{n+1-i})} K_{v}(as_{i+1}u).$$  \hspace{1cm} (20)

Suppose that $s_{n-i}v = v$. Then $\tilde{v}_{n-i} = t\tilde{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n}v_0^{-1}$, that is, $\tilde{v}_i = t^{1-n}v_{n+1-i}$, we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{siu}(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_{v}(as_{i+1}u) = K_{v}(a\tilde{u})$ if $s_{n-i}v = v$. 


We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v$. We have to show that $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s_iu})$ for all $v$. It is trivially true if $s_iu = u$, so we may assume that $s_iu \neq u$. Suppose that $v$ satisfies $s_{n-i}v = v$. Then it follows from the previous paragraph that

$$K_{s_iu}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(a\tilde{s_iu}).$$

If $s_{n-i}v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_iu}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i}v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i}v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s_iu})$. Hence, we obtain an explicit expression of $K_{s_iu}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(a\tilde{s_iu})$, which turns out to reduce to $K_{s_iu}(a\tilde{v}) = K_v(a\tilde{s_iu})$ after a direct computation.

**Step 2.** $K_0(a\tilde{v}) = 1 = K_v(a\tilde{0})$ for all $v \in \mathbb{Z}^n$.

**Proof of Step 2.** Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(a\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15.

**Step 3.** $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

**Proof of Step 3.** We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a\tilde{v}) = K_v(a\tilde{\gamma})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha \cdot n > 0$. Then $\gamma := \alpha \tilde{\gamma} \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma \tilde{\gamma}$. Furthermore, note that we have the formula

$$(a\tilde{v}_1^{-1} - 1)K_u(a\tilde{v}) = (a\tilde{u}_1^{-1} - 1)K_u(a\tilde{\gamma})$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$K_\alpha(a\tilde{v}) = K_\gamma(a\tilde{\gamma}) = \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{\gamma}_1^{-1} - 1)} K_\gamma(a\tilde{\gamma}) = \frac{(a\tilde{v}_1^{-1} - 1)}{(a\tilde{\gamma}_1^{-1} - 1)} K_{\alpha \tilde{\gamma}} = K_v(a\tilde{\alpha}),$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step.■
Step 4. \( K_u(a\tilde{v}) = K_v(a\tilde{u}) \) for all \( u, v \in \mathbb{Z}^n \).

Proof of Step 4. Fix \( u, v \in \mathbb{Z}^n \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( u + (m^n) \in C_n \). Note that \( q^m\tilde{v} = \tilde{v} - (m^n) \) and \( q^{-m}\tilde{u} = \tilde{u} + (m^n) \). Then

\[
K_u(a\tilde{v}) = A_m(a\tilde{v}; u)K_{u+(m^n)}(q^m a\tilde{v})
\]
\[
= A_m(a\tilde{v}; u)K_{u+(m^n)}(a(\tilde{v} - (m^n)))
\]
\[
= A_m(a\tilde{v}; u)K_{\nu-(m^n)}(a(u + (m^n)))
\]
\[
= A_m(a\tilde{v}; u)K_{\nu-(m^n)}(q^{-m}a\tilde{u}) = A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; \nu - (m^n))K_\nu(a\tilde{u}),
\]

where we used Step 3 in the 3rd equality. The result now follows from the fact that

\[
A_m(a\tilde{v}; u)A_m(q^{-m}a\tilde{u}; \nu - (m^n)) = 1,
\]

which follows by a straightforward computation using (4). \( \blacksquare \)

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial \( E_\alpha(x) \) of degree \( \alpha \) is the top homogeneous component of \( G_\alpha(x) \), i.e.,

\[
E_\alpha(x) = \lim_{a \to \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in C_n.
\]

The normalized non-symmetric Macdonald polynomials are

\[
\overline{K}_\alpha(x) := \lim_{a \to \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in C_n.
\]

We write \( \overline{K} \in \mathcal{F}_{\text{Fix}}^+ \) for the resulting map \( \alpha \mapsto \overline{K}_\alpha \). Taking limits in Lemma 10 we get the following.

Lemma 19. We have for \( 1 \leq i < n \) and \( 1 \leq j \leq n \),

1. \( H_i\overline{K} = \tilde{H}_i\overline{K} \).
2. \( \xi_j\overline{K} = \tilde{\xi}_j^{-1}\overline{K} \).
3. \( x_n\Delta\overline{K} = t^{1-n}\tilde{x}_1^{-1}\tilde{\Delta}^{-1}\overline{K} \).
Note that
\[(x_n \Delta)^n f(x) = \left( \prod_{i=1}^{n} x_i \right) f(q^{-1} x).\]

Then repeated application of part 3 of Lemma 19 shows that for \(\alpha \in C_n\),
\[
E_\alpha(x) = \frac{E_{\alpha + (1^n)}(x)}{x_1 \cdots x_n},
\]
\[
\overline{K}_\alpha(x) = q^{\alpha |_d} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha_i})^{-1} \right) \overline{K}_{\alpha + (1^n)}(x). \tag{22}
\]

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials \(E_v(x) := E_v(x; q, t) \in \mathbb{F}[x^\pm 1]\) for arbitrary \(v \in \mathbb{Z}^n\) to those labeled by compositions through the formula
\[
E_v(x) = \frac{E_{v + (m^n)}(x)}{(x_1 \cdots x_n)^m}.
\]

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees \(v \in \mathbb{Z}^n\).

**Definition 20.** Let \(v \in \mathbb{Z}^n\) and \(m \in \mathbb{Z}_{\geq 0}\) such that \(v + (m^n) \in C_n\). Then \(\overline{K}_v(x) := \overline{K}_v(x; q, t) \in \mathbb{F}[x^\pm 1]\) is defined by
\[
\overline{K}_v(x) := q^{\frac{v |_d}{t}} t^{(1-n)n} \left( \prod_{i=1}^{n} (\overline{\alpha_i})^{-1} \right) \overline{K}_{v + (m^n)}(x).
\]

Using
\[
\lim_{a \to \infty} A_{m}(ax; v) = q^{-m^2 n t^{(1-n)n} \prod_{i=1}^{n} (\overline{\alpha_i})^{-m}}
\]
and the definitions of \(G_v(x)\) and \(K_v(x)\) it follows that
\[
\lim_{a \to \infty} a^{-v |_d} G_v(ax) = E_v(x),
\]
\[
\lim_{a \to \infty} K_v(ax) = \overline{K}_v(x)
\]
for all \(v \in \mathbb{Z}^n\), so in particular
\[
\overline{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \quad \forall v \in \mathbb{Z}^n.
\]
Lemma 19 holds true for the extension of $K$ to the map $K \in \mathcal{F}[\mathbb{Z}^{\pm 1}]$ defined by $v \mapsto K_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

**Corollary 21.** For all $u, v \in \mathbb{Z}^n$,

$$K_u(\bar{v}) = K_v(\bar{u}).$$

### 7.2 $O$-polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the $O$-polynomials $O_\alpha$ (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for $O_\alpha$ in terms of the non-symmetric interpolation Macdonald polynomial $K_\alpha$.

**Proposition 22.** For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

**Proof.** The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\bar{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\bar{\beta}^{-1}) = K_\alpha(a\bar{\beta}) = K_\beta(a\bar{\alpha})$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$.

### 7.3 Okounkov’s duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in a field $F$. Write $C_\perp := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in F[x]^{S_n}$ is the multiple of $C_\perp G_\lambda$ such that the coefficient of $x^\lambda$ is one (see, e.g., [13]). We write

$$K_\perp^{\lambda}(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_\perp K_\alpha(x) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K_\perp^{\lambda}(x)$$

(23)

for $\alpha \in C_n$. Okounkov’s [10, Section 2] duality result now reads as follows.
Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K^+_\lambda(a\mu^{-1}) = K^+_\mu(a\lambda^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_\mu$ for the function $f_\mu(u) := K_u(a\mu)(u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\lambda(a\mu) = (C_+K_\lambda)(a\mu) = (\widehat{C}_+f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\mu) = (Jw_0K_\mu(t^{1-n}x))|_{x=a^{-1}\mu^{-1}}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_iJ = (H_i^\circ)^{-1}, \quad w_0H_iw_0 = (H_{n-i}^\circ)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+f_\mu)(\lambda) = (Jw_0C_+K_\mu(t^{1-n}x))|_{x=a^{-1}\mu}.\tag{25}$$

By (23) and (4) this simplifies to

$$(\widehat{C}_+f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\mu(a\mu).$$

Returning to (24) we conclude that $K^+_\lambda(a\mu) = K^+_\mu(a\lambda)$. Since $K^+_\lambda$ is symmetric we obtain from (4) that

$$K^+_\lambda(a\mu^{-1}) = K^+_\mu(a\lambda^{-1}),$$

which is Okounkov’s duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).

Let us derive Theorem 23 as consequence of Theorem 17. Write $\hat{C}_+ = \sum_{w \in S_n} \hat{H}_w$, with $\hat{H}_w := \hat{H}_{i_1} \cdots \hat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_\mu$ for the function $f_\mu(u) := K_u(a\mu)(u \in \mathbb{Z}^n)$. Then

$$\left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\lambda(a\mu) = (C_+K_\lambda)(a\mu) = (\widehat{C}_+f_\mu)(\lambda)$$

by part 1 of Proposition 16. The duality (17) of $K_u$ and (4) imply that

$$f_\mu(u) = K_\mu(a\mu) = (Jw_0K_\mu(t^{1-n}x))|_{x=a^{-1}\mu^{-1}}$$

with $(Jf)(x) := f(x_1^{-1}, \ldots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_iJ = (H_i^\circ)^{-1}, \quad w_0H_iw_0 = (H_{n-i}^\circ)^{-1}$$

for $1 \leq i < n$. In particular, $Jw_0C_+ = C_+Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+f_\mu)(\lambda) = (Jw_0C_+K_\mu(t^{1-n}x))|_{x=a^{-1}\mu}.\tag{25}$$

By (23) and (4) this simplifies to

$$(\widehat{C}_+f_\mu)(\lambda) = \left( \sum_{w \in S_n} t^{\ell(w)} \right) K^+_\mu(a\mu).$$

Returning to (24) we conclude that $K^+_\lambda(a\mu) = K^+_\mu(a\lambda)$. Since $K^+_\lambda$ is symmetric we obtain from (4) that

$$K^+_\lambda(a\mu^{-1}) = K^+_\mu(a\lambda^{-1}),$$

which is Okounkov’s duality result.
Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0} K_u)(a \tilde{v}) = (H_{w_0} K_v)(a \tilde{u}).$$

(27)

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a \tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0} K_u)(a \tilde{v}) = (\hat{H}_{w_0} f_v)(u).$$

Since $f_v(u) = (I w_0 K_v)(a^{-1} t^{n-1} \tilde{u})$ by (4), Remark 7 implies that

$$(\hat{H}_{w_0} f_v)(u) = (H_{w_0} J w_0 K_v)(a^{-1} t^{n-1} \tilde{u}).$$

Now $H_{w_0} J w_0 = J w_0 H_{w_0}$ by (26); hence,

$$(\hat{H}_{w_0} f_v)(u) = (J w_0 H_{w_0} K_v)(a^{-1} t^{n-1} \tilde{u}) = (H_{w_0} K_v)(a \tilde{u}),$$

which completes the proof.

Recall from Theorem 1 that

$$G'_\beta(x) = t^{(1-n)|\beta| + I(\beta)} \psi G_\beta^\circ(t^{n-1} x)$$

with $\psi := w_0 H_{w_0}^\circ$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1} \tau)} = t^\ell(w_0) \psi K_\beta^\circ(t^{n-1} x), \quad \beta \in \mathcal{C}_n,$$

with $K_\nu^\circ := \iota(K_\nu)$ for $\nu \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $\nu \in \mathbb{Z}^n$,

$$K'_\nu(x) := t^\ell(w_0) \psi K_\nu^\circ(t^{n-1} x).$$

(28)

We write $K' : \mathbb{Z}^n \to \mathbb{K}(x)$ for the map $\nu \mapsto K'_\nu (\nu \in \mathbb{Z}^n)$. Since $H_i \psi = \psi H_i^\circ$, part 1 of Proposition 16 gives $H_i K' = \hat{H}_i^s K'$. Considering the action of $((x_n - 1) \Delta^\circ)^n$ on $K'_\beta(x)$ we get, using the fact that $((x_n - 1) \Delta^\circ)^n$ commutes with $\psi$ and part 3 of Proposition 16,

$$K'_\nu(x) = \left( \prod_{i=1}^n \frac{1-a^{-1} \tilde{v}_i}{1-q^{-1} x_i} \right) K'_{\nu + (1^n)}(q^{-1} x),$$
in particular

\[ K'_v(x) = \left( \prod_{i=1}^{n} \frac{(a^{-1}v; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+(m^n)}(q^{-m}x). \]

**Example 25.** For \( n = 1 \) we have \( K'_v(x) = K^0_v(x) \) for \( v \in \mathbb{Z} \); hence,

\[ K'_{-m}(x) = (q^{-1}a^{-1}; q^{-1})_m = (ax)^{-m} \frac{(qa; q)_m}{(q^{-1}x; q^{-1})_m}, \]

\[ K'_m(x) = (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a^{-1}; q)_m}. \]

for \( m \in \mathbb{Z}_{\geq 0} \) by Example 14.

**Proposition 26.** For all \( u, v \in \mathbb{Z}^n \) we have

\[ K'_v(a^{-1} \tilde{u}) = K'_u(a^{-1} \tilde{v}). \]

**Proof.** Note that

\[ K'_v(a^{-1} \tilde{u}) = t^{\ell(w_0)} \Psi K^0_v(t^{n-1}x)|_{x=a^{-1} \tilde{u}} = t^{\ell(w_0)} (H_{w_0}^0 K^0_v)(a^{-1} \tilde{u}^{-1}) \]

by (4). By (27) the right-hand side is invariant under the interchange of \( u \) and \( v \).  

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of \( O_\alpha \) was used to prove the following binomial theorem [14, Thm. 1.3]. Define for \( \alpha, \beta \in C_n \) the generalized binomial coefficient by

\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q, t} := \frac{G^0_{\beta}(\overline{\alpha})}{G^0_{\beta}(\overline{\beta})}. \] (29)

Applying the automorphism \( \iota \) of \( \mathbb{F} \) to (29) we get

\[ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} = \frac{G^0_{\beta}(\overline{\alpha}^{-1})}{G^0_{\beta}(\overline{\beta}^{-1})}. \]
Theorem 27. For $\alpha, \beta \in C_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in C_n} a^{|\beta|} \binom{\alpha}{\beta}_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(ax)}.$$  \hspace{1cm} (30)

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$K_\alpha(ax) = \sum_{\beta \in C_n} \tau^{-1}_{\beta} \binom{\alpha}{\beta}_{q^{-1}, t^{-1}} K'_\beta(x)$$

$$= \sum_{\beta \in C_n} \frac{K^0_{\beta}(\overline{\alpha}^{-1})K'_\beta(x)}{\tau_\beta K^0_{\beta}(\overline{\beta}^{-1})}$$

$$= t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{K^0_{\beta}(\overline{\alpha}^{-1})\Psi K^0_{\beta}(t^{n-1}x)}{\tau_\beta K^0_{\beta}(\overline{\beta}^{-1})}$$ \hspace{1cm} (31)

with $\Psi = w_0 H_{w_0}^0$ (note that the dependence on $a$ in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K^0_\beta(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of $K_\alpha$ as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0}K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K^0_{\beta}(\overline{\alpha}^{-1})K^0_{\beta}(t^{n-1}w_0x)}{\tau_\beta K^0_{\beta}(\overline{\beta}^{-1})}.$$ \hspace{1cm} \text{(4)}

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0}K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in C_n} \frac{K^0_{\beta}(\overline{\alpha}^{-1})K^0_{\beta}(\overline{\gamma}^{-1})}{\tau_\beta K^0_{\beta}(\overline{\beta}^{-1})}.$$ \hspace{1cm} \text{(5)}

The right-hand side is manifestly invariant under interchanging $\alpha$ and $\gamma$, which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating $G'_\alpha$ and $G_\alpha$ is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).
The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

**Theorem 29.** For all $\alpha \in C_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$  \hfill (32)

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta^\alpha(\bar{\alpha}^{-1}) \Psi K_\beta^\alpha(t^{n-1}x)}{\tau_\beta G_\beta^\alpha(\bar{\beta}^{-1})},$$

see (31). Replace $(a, x, q, t)$ by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0 H w_0$ on both sides. Since $w_0 H w_0 \Psi = \text{Id}$ we obtain

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

**Remark 30.** It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta K_\beta(\bar{\alpha} \bar{K}_\beta(ax)) \frac{K_\beta(ax)}{K_\beta(\bar{\beta})}. \hfill (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_{q,t} \left[ \begin{array}{c} \beta \\ \gamma \end{array} \right]_{q^{-1},t^{-1}} = \delta_{\alpha,\gamma}.$$ 

Since $\left[ \begin{array}{c} \delta \\ \epsilon \end{array} \right]_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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