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**DOI**
10.1017/9780511777165.002
**Publication date**
2020
**Document Version**
Author accepted manuscript
**Published in**
Encyclopedia of special functions : the Askey-Bateman project. - Volume 2

# 1

# General overview of multivariable special functions

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This is a preliminary version of Chapter 1 in the book *Encyclopedia of special functions: The Askey–Bateman project, Vol. 2: Multivariable special functions*, T. H. Koornwinder and J. V. Stokman (eds.), Cambridge University Press, 2021.

## 1.1 Introduction

The theory of one-variable (ordinary) hypergeometric and basic hypergeometric series goes back to work of Euler, Gauss and Jacobi. The theory of elliptic hypergeometric series is of a much more recent vintage [20]. The three theories deal with the study of series $\sum_{k\geq 0} c_k$ with $f(k) := c_{k+1}/c_k$ a rational function in $k$ (*hypergeometric* theory), a rational function in $q^k$ (*basic hypergeometric* theory), or a doubly periodic meromorphic function in $k$ (*elliptic hypergeometric* theory, see [21, Ch. 11] for an overview).

Examples of elementary functions admitting hypergeometric and basic hypergeometric series representations are

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, z^k, \qquad \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k \tag{1.1.1}$$

for $|z| < 1$ and $\alpha, a \in \mathbb{C}$, with $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k \in \mathbb{Z}_{\geq 0}$ the *shifted factroial* (or *Pochhammer symbol*), $(a;q)_k := (1-a)(1-qa)\cdots(1-q^{k-1}a)$ for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the *q-shifted factorial*. Here, and throughout the entire chapter, we assume for convenience that $0 < q < 1$. Note that the series in the second identity, with $a = q^\alpha$, tends to the series in the first identity as $q \uparrow 1$, at least formally, and that the identities (1.1.1) reduce to polynomial identities when $\alpha \in \mathbb{Z}_{\leq 0}$. Also note that the series in (1.1.1) are indeed hypergeometric and basic hypergeometric series, respectively, since $f(k) = \frac{k+\alpha}{k+1}\, z$ and $f(k) = \frac{1-q^k a}{1-q^{k+1}}\, z$ for the first and second series in (1.1.1). These are the well known Newton (generalized) *binomial theorem* and its $q$-analogue [21, §1.3]. They form, apart from the ($q$)-exponential series, the simplest nontrivial examples of an impressive scheme of hypergeometric and basic hypergeometric summation identities [21], with the members in the scheme related by limit transitions.

The summands of elliptic, basic and classical hypergeometric series are expressible in terms of products and quotients of elliptic, basic and classical shifted factorials. The basic and classical ones are the ($q$-)shifted factorials as defined in the previous paragraph. The *elliptic* (or *theta*) *shifted factorial* is given by $(z;q,p)_k := \prod_{i=0}^{k-1} \theta(zq^i;p)$ for $k \in \mathbb{Z}_{\geq 0}$ and $0 < p < 1$, with $\theta(z;p) := \prod_{i=0}^{\infty}(1-p^i z)(1-p^{i+1}/z)$ the *modified theta function*. These shifted factorials can be expressed as $\Gamma(q^k z)/\Gamma(z)$ (or, in the classical case, $\Gamma(z+k)/\Gamma(z)$) with $\Gamma(z)$ an appropriate analogue of the classical Gamma function. For the elliptic hypergeometric case this is

Ruijsenaars' *elliptic Gamma function* [66]

$$\prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \qquad 0<p,q<1,$$

for the basic hypergeometric case the (modified) *q-Gamma function* $(z;q)_\infty^{-1}$, and for the classical hypergeometric case the classical *Gamma function*.

There is no "simple" elliptic analogue of (1.1.1). In fact, the first elliptic hypergeometric summation formula that was found [20] generalizes the *top level* terminating (basic) hypergeometric summation identity! This is a general pattern for the elliptic hypergeometric theory: the top levels of the (basic) hypergeometric theory admit elliptic versions, and there is little room for degenerations without falling outside the realm of elliptic hypergeometric series. Possibly this is one of the reasons for the late discovery of elliptic hypergeometric series.

Parallel to the theory of hypergeometric series there is a theory of hypergeometric integrals, see §1.2.3 and, in Chapters 5 and 6, the sections 5.3 and 6.2. Such integrals can often be identified with hypergeometric series. But, certainly in the elliptic case, there are many instances where the hypergeometric integral is convergent while a possible corresponding hypergeometric series diverges [65, §2.10]. Hypergeometric integrals naturally appear as coordinates of vector-valued solutions of Knizhnik–Zamolodchikov (*KZ*) and Knizhnik–Zamolodchikov–Bernard (*KZB*) equations and their $q$-analogues, see Chapter 11. The elliptic case, corresponding to solutions of *qKZB* equations, already appeared in 1996 in [17, §7] (yet formally) and soon afterwards rigorously in [18, §6].

This volume deals with multivariable generalizations of ordinary, basic and elliptic hypergeometric series and integrals. This includes various multivariable extensions of classical (bi)orthogonal polynomials and functions, which form an important subclass of hypergeometric series within the one-variable theory.

Various multivariable theories have emerged, each with its own characteristic features depending on the particular motivation for, and context behind, its multivariable extension. For instance, there are important multivariable theories motivated by special function theory itself (see Chapters 2–6), by representation theory and Lie theory (see Chapters 7–9 and 12), by combinatorics (see Chapter 10) and by theoretical physics (see Chapters 8–9 and 11–12).

In the remainder of this introductory chapter we give a short discussion of each type of multivariable special functions treated in this volume, and we highlight their interrelations and differences. In §1.2 we first discuss the multivariable series which may be seen as extensions of the three types of hypergeometric series. The different classes of multivariable extensions of classical (bi)orthogonal functions will be discussed in §1.3.

We hope that this short impression of the various classes of multivariable special functions and their interrelations helps the reader to oversee the chapters in this volume, and how they are related.

## 1.2 Multivariable classical, basic and elliptic hypergeometric series

### *1.2.1 Appell and Lauricella hypergeometric series*

*Gauss' hypergeometric series* is given by

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}\, x^k, \tag{1.2.1}$$

which absolutely converges for $|x| < 1$. One of the oldest generalizations of the Gauss hypergeometric series to several variables was given by Appell, who introduced the four *Appell hypergeometric series* in two variables [1], [16, §5.7], denoted by $F_1, F_2, F_3, F_4$. For instance,

$$F_2(a,b_1,b_2,c_1,c_2;x,y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n\, m!\, n!}\, x^m y^n, \quad (x,y)\in\mathbb{C}^2,\ |x|+|y|<1. \tag{1.2.2}$$

The series (1.2.2) are double series $\sum_{m,n=0}^{\infty} c_{m,n}/(m!\,n!)$ with $c_{m+1,n}/c_{m,n}$ and $c_{m,n+1}/c_{m,n}$ of the form $p_1(m,n)/r_1(m,n)$ and $p_2(m,n)/r_2(m,n)$ for suitable relative prime polynomials $p_i$ and $r_i$ in two variables ($i = 1,2$). This extends the property characterizing hypergeometric series in one variable, and such series are therefore also called *hypergeometric*. The highest degree of the four polynomials $p_1, r_1, p_2, r_2$ is called the *order* of the hypergeometric series in two variables. The Appell hypergeometric series have order two. Horn classified all hypergeometric series of order two, see the list of 34 series in [16, §5.7.1]. Lauricella defined $n$-variable analogues $F_A, F_B, F_C, F_D$ of $F_2, F_3, F_4, F_1$, respectively. The Appell and Lauricella hypergeometric series are discussed in Chapter 3.

Many properties and formulas for Gauss hypergeometric series generalize to Appell and Lauricella hypergeometric series, but, not surprisingly, one has to deal with interesting complications concerning, for instance, the integral representations, systems of partial differential equations, and monodromy, see Chapter 3. Furthermore, solutions of the system of partial differential equations for these series form a much richer collection than in the case of the Gauss hypergeometric series, where all local solutions at regular singularities are expressed in terms of series of the same type. For instance, for $F_2$ six different types of series occur as local solutions, including some which are hypergeometric series of order higher than two, or even not hypergeometric series at all. See Olsson [54]. Gel'fand's $A$-hypergeometric functions (see Chapter 4) offer a fruitful point of view for the study of Appell and Lauricella hypergeometric series. This can also give inspiration for a study of $q$-analogues, see [51], where also a connection is made with quantum groups. In [21, Chapter 10] there is a chapter on $q$-series in two or more variables.

Appell and Lauricella hypergeometric series have several interrelations with other special functions in several variables. The first example (which may have been the motivating example for Appell) are the biorthogonal polynomials on the simplex and ball, see Chapter 2. Further examples deal with Heckman–Opdam hypergeometric functions (see Chapter 8). In the case of root system A these functions can be identified for certain degenerate parameter values with a special Lauricella $F_D$ (or, in two variables, with Appell $F_1$), see [72]. A special case of Heckman–Opdam hypergeometric functions for root system $BC_2$ can be written as a sum of two Appell $F_4$ functions (see [3, Theorems 3.3 and 2.3]). In the polynomial case one of the $F_4$ terms vanishes, so that special $BC_2$ Jacobi polyomials can be written as a terminating $F_4$, see [39, (7.15)]. Another special case of the Heckman–Opdam functions for $BC_n$, now for general $n$, can be expressed as $n$-variable analogues [3, (5.1) and Theorem 5.4] of *Kampé de Fériet hypergeometric series* (certain hypergeometric series in two variables of order three).

### 1.2.2 *A*-hypergeometric functions

The $A$-hypergeometric (or GKZ hypergeometric) functions were introduced by Gel'fand, Zelevinsky & Kapranov [23] in 1989, but there have been analogous approaches before. In particular, W. Miller Jr. [48] described in 1973 a new approach to the *hypergeometric differential equation*

$$z(1-z)f''(z) + (c-(a+b+1)z)f'(z) - abf(z) = 0, \tag{1.2.3}$$

of which the Gauss hypergeometric series (1.2.1) is a solution. He observed that, if the parameters $a, b, c$ in (1.2.3) are replaced by $s\partial_s, u\partial_u, t\partial_t$, then the resulting system of PDEs

$$QF = 0, \quad s\partial_s F = aF, \quad u\partial_u F = bF, \quad t\partial_t F = cF \tag{1.2.4}$$

with

$$Q := z(1-z)\partial_{zz} + t\partial_{tz} - z(s\partial_{sz} + u\partial_{uz} + \partial_z) - su\partial_{su}$$

has a solution

$$F(s,u,t,z) = s^a u^b t^c \ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right). \tag{1.2.5}$$

Miller defines the *dynamical symmetry algebra* $\mathcal{G}$ of $Q$ as the set of all first order PDEs $L$ such that $QLf = 0$ whenever $Qf = 0$. It is a Lie algebra which has a basis of operators acting on solutions of the form (1.2.5) (so-called *contiguity relations*). Then $\mathcal{G}$ is seen to be isomorphic to sl(4). Miller [48] pointed out that a similar approach works for generalized hypergeometric series ${}_{r+1}F_r$ and for Appell and Lauricella hypergeometric series. This was elaborated by him in several papers in 1972, 1973.

In 1980 Kalnins, Manocha & Miller [32] transformed systems like (1.2.4), in the case of Appell's and Horn's hypergeometric series in two variables, into so-called *canonical systems*. These systems coincide with special cases of the later introduced $A$-hypergeometric systems [23]. M. Saito [68, 69] recognized the relevance of [32] for the GKZ theory. He also worked with a symmetry algebra for operators $Q$ which no longer requires that the operators in the algebra are first order.

A change of variables turns the system (1.2.4) of PDEs in the following canonical (or $A$-hypergeometric) form,

$$(\partial_{xy} - \partial_{zw})f = 0, \quad (x\partial_x - y\partial_y)f = (1-c)f, \quad (x\partial_x + z\partial_z)f = -af, \quad (x\partial_x + w\partial_w)f = -bf \tag{1.2.6}$$

with corresponding solution

$$f(x,y,z,w) = y^{c-1} z^{-a} w^{-b}\,{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; \frac{xy}{zw}\right), \tag{1.2.7}$$

see for instance [75, §2.6, §3.2.1]. Note that the change of variables has transformed the second order partial differential operator $Q$ to $\partial_{xy} - \partial_{zw}$, which is essentially the 4D Laplace operator. This makes it manifest that the dynamical symmetry algebra of $Q$ is $\mathfrak{sl}(4)$, see also [10].

The general $A$-hypergeometric system in $n$ variables $x = (x_1, \ldots, x_n)$ depends on a $d \times n$ matrix $A = (a_{ij}) = (a_1 \ldots a_n)$ with integer column vectors $a_j \in \mathbb{Z}^d$ (from which the $A$ in $A$-hypergeometric) such that the $\mathbb{Z}$-span of the $a_j$ equals $\mathbb{Z}^d$. The $A$-hypergeometric system, depending on parameters $\beta_1, \ldots, \beta_d$, is given by

$$\Big(\prod_{u_i>0} \partial_{x_i}^{u_i}\Big) f = \Big(\prod_{u_i<0} \partial_{x_i}^{-u_i}\Big) f \quad (u \in L\backslash\{0\}), \qquad \Big(\sum_{j=1}^{n} a_{ij} x_j \partial_{x_j}\Big) f = \beta_i f \quad (i = 1, \ldots, d) \tag{1.2.8}$$

with $L := \{u \in \mathbb{Z}^n \mid Au = 0\}$. It can be seen to have the system (1.2.6) as the special case

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta = (1-c, -a, -b)^t. \tag{1.2.9}$$

For each $\nu \in \mathbb{C}^n$ such that $A\nu = \beta$ we have a formal solution of the ($\nu$-independent) differential equations (1.2.8) given by the series

$$\sum_{u \in L} \prod_{j=1}^{n} \frac{x_j^{\nu_j + u_j}}{\Gamma(\nu_j + u_j + 1)}\,. \tag{1.2.10}$$

called *A-hypergeometric series in Gamma function form*. With the choice (1.2.9) of $A, \beta$ and with $\nu := (0, c-1, -a, -b)^t$, $u := k(1,1,-1,-1)^t$ $(k \in \mathbb{Z})$ the series (1.2.10) becomes

$$\sum_{k=-\infty}^{\infty} \frac{x_1^k x_2^{c-1+k} x_3^{-a-k} x_4^{-b-k}}{\Gamma(k+1)\,\Gamma(c+k)\,\Gamma(-a-k+1)\,\Gamma(-b-k+1)} = \frac{x_2^{c-1} x_3^{-a} x_4^{-b}}{\Gamma(c)\,\Gamma(1-a)\,\Gamma(1-b)} \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} \left(\frac{x_1 x_2}{x_3 x_4}\right)^k,$$

which is (1.2.7) apart from the Gamma factors in the denominator in front of the summation.

Choices for $A$ and $\nu$ in (1.2.10) can be made such that the resulting series involves ${}_{r+1}F_r(z)$ or an Appell or Lauricella hypergeometric series. For instance, for Appell's $F_2$ one can take

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \qquad \begin{aligned} &\beta = (-a, -b_1, -b_2, c_1 - 1, c_2 - 1)^t, \\ &u = m(-1,-1,0,1,0,1,0) + n(-1,0,-1,0,1,0,1)^t, \\ &\nu = (-a, -b_1, -b_2, c_1 - 1, c_2 - 1, 0, 0)^t. \end{aligned}$$

Then the first part of the system (1.2.8) is generated by $\partial_1\partial_2 f = \partial_4\partial_6 f$, $\partial_1\partial_3 f = \partial_5\partial_7 f$, and (1.2.10) becomes

$$\sum_{m,n=-\infty}^{\infty} \frac{x_1^{-a-m-n} x_2^{-b_1-m} x_3^{-b_2-n} x_4^{c_1+m-1} x_5^{c_2+n-1} x_6^m x_7^n}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(c_1+m)\,\Gamma(c_2+n)\,\Gamma(1-a-m-n)\,\Gamma(1-b_1-m)\,\Gamma(1-b_2-n)}$$
$$= \frac{x_1^{-a} x_2^{-b_1} x_3^{-b_2} x_4^{c_1-1} x_5^{c_2-1}}{\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(1-a)\,\Gamma(1-b_1)\,\Gamma(1-b_2)}\, F_2\left(a, b_1, b_2, c_1, c_2; \frac{x_4 x_6}{x_1 x_2}, \frac{x_5 x_7}{x_1 x_3}\right).$$

The GKZ theory, of which Chapter 4 gives a survey, not only unifies the study of many classes of multivariable special functions, but also exploits methods from algebra, geometry, $D$-module theory and combinatorics, far beyond the methods used in classical approaches.

### *1.2.3 Classical, basic and elliptic hypergeometric series and integrals associated with root systems*

*Hypergeometric integrals* of classical, basic and elliptic type are integrals with integrand expressed in terms of products and quotients of Gamma factors $\Gamma(ax)$ (in the classical case, $\Gamma(a+x)$), with $\Gamma(x)$ the Gamma function of the appropriate type. In the classical case integrands involving products of the form $(1-x)^a$ are also considered to be hypergeometric ($(1-x)^a$ is formally the $q \to 1$ limit of the quotient $(q^x;q)_\infty/(q^{a+x};q)_\infty$ of $q$-Gamma functions). The singular set of the integrand of a hypergeometric integral is a union of geometric (in the classical case, arithmetic) progressions. Hypergeometric series naturally arise as the sum of residues of the integrand over such pole progressions.

*Multidimensional* hypergeometric integrals typically arise in contexts involving representation theory of algebraic and Lie groups. For instance, in harmonic analysis on compact symmetric spaces, the zonal spherical functions give rise to a family of multivariable orthogonal polynomials with respect to a measure on a compact torus that is absolutely continuous with respect to the Haar measure. The associated weight function admits a natural factorization in terms of the root system underlying the symmetric space. Such multivariable integrals often admit generalizations beyond the representation theoretic context. They provide the prototypical examples of *hypergeometric integrals associated with root systems*.

Let us focus now more closely on the structure of such integrals. Suppose $R$ is an irreducible root system in $\mathbb{R}^n$, and fix a choice $R^+$ of positive roots. The co-weight lattice $P^\vee$ of $R$ is the lattice in $\mathbb{R}^n$ dual to the $\mathbb{Z}$-span of $R$. For the classical root systems we take the usual realization of $R = R^+ \cup (-R^+)$ in $\mathbb{R}^n$ with respect to the standard orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. Concretely, $R^+ = \{e_i - e_j\}_{1\le i<j\le n}$ for type $\mathrm{A}_{n-1}$, $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n}$ for type $\mathrm{D}_n$, $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n} \cup \{2e_i\}_{i=1}^n$ for type $\mathrm{C}_n$, and $R^+ = \{e_i \pm e_j\}_{1\le i<j\le n} \cup \{e_i, 2e_i\}_{i=1}^n$ for type $\mathrm{BC}_n$.

Let $k_\alpha \in \mathbb{C}$ be parameters that only depend on the Weyl group orbit of the root $\alpha \in R$ (equivalently, $k_\alpha$ only depends on the root length $\|\alpha\|$ of the root $\alpha \in R$). The prototypical example of a classical hypergeometric integral associated with $R$ is

$$\int_{A_R} w_{\boldsymbol{k}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad w_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{\alpha\in R} \left(1 - e^{2\pi i(\alpha,\boldsymbol{x})}\right)^{k_\alpha} \tag{1.2.11}$$

with $A_R \subset \mathbb{R}^n$ a fundamental domain for the translation action of $P^\vee$ on $\mathbb{R}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1 \ldots \mathrm{d}x_n$ and $\boldsymbol{k} := \{k_\alpha\}_{\alpha \in R}$ the collection of the parameters $k_\alpha$ (here the $k_\alpha$ should satisfy appropriate conditions to ensure convergence of the integral). Remarkably the integral (1.2.11) admits an explicit evaluation as a product of Gamma functions. The resulting identity is known as the *Macdonald constant term identity* (see Theorem 8.4.2(i)). It gives the volume of the orthogonality measure of root system generalizations of the Jacobi polynomials, also known nowadays as *Heckman–Opdam polynomials*, see Chapter 8 for a detailed discussion.

Of particular interest is the special case that the root system $R$ is of type $\mathrm{BC}_n$. In that case the Macdonald constant term identity reduces after the change of variables $z_j = \sin^2(\pi x_j)$ to the well known *Selberg integral* [71]

$$\int_{[0,1]^n} \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1 \le i < j \le n} |z_i - z_j|^{2\gamma}\, \mathrm{d}\boldsymbol{z} = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(\alpha + \beta + (n + j - 1)\gamma)\Gamma(1+\gamma)}$$

with parameters $\alpha = k_{\epsilon_1} + k_{2\epsilon_1} + \frac{1}{2}, \beta = k_{2\epsilon_1} + \frac{1}{2}$ and $\gamma = k_{\epsilon_1 - \epsilon_2}$, which in turn is a multidimensional generalization of the beta integral. There are many applications of the Selberg integral, for instance in the theory of integrable systems (Chapters 8 and 9), in conformal field theory (Chapter 11) and in random matrix theory; see the overview article [19].

For basic hypergeometric integrals associated with root systems a similar story applies. The role of Lie groups and root systems are taken over by quantum groups and affine root systems, although this time the representation theoretic context came later. The *affine root system* associated to an irreducible reduced root system $R$ is denoted by $R^{(1)}$ and consists of the collection of affine linear functionals $a\colon \mathbb{R}^n \to \mathbb{R}$ of the form $a(\boldsymbol{x}) = (\alpha, \boldsymbol{x}) + m$ ($\alpha \in R$ and $m \in \mathbb{Z}$). The role of $w_{\boldsymbol{k}}(\boldsymbol{x})$ is now taken over by

$$w_{\boldsymbol{k},q}(\boldsymbol{x}) = \prod_{a \in R^{(1)};\, a(0) \ge 0} \left( \frac{1 - q^{a(\boldsymbol{x})}}{1 - q^{k_a + a(\boldsymbol{x})}} \right) = \prod_{\alpha \in R} \frac{(q^{(\alpha,\boldsymbol{x})}; q_\alpha)_\infty}{(q^{k_\alpha + (\alpha,\boldsymbol{x})}; q_\alpha)_\infty},$$

where $k_a = k_\alpha$ if $\alpha$ is the gradient of $a \in R^{(1)}$. Macdonald [43] conjectured an explicit evaluation for the *basic hypergeometric integral*

$$\int_{A_R} w_{\boldsymbol{k},q}(\boldsymbol{x}/\tau)\, \mathrm{d}\boldsymbol{x}, \qquad q = \exp(2\pi i \tau) \tag{1.2.12}$$

associated with $R$, which was proved in full generality by Cherednik [7] using the theory of double affine Hecke algebras. The evaluation formula gives the volume of the orthogonality measure of the Macdonald polynomials, see Chapter 9. The integral (1.2.12) and its evaluation generalize to arbitrary (possibly non-reduced) irreducible affine root systems and with milder equivariance conditions on $\boldsymbol{k} = \{k_a\}_{a \in R^{(1)}}$. In case of the non-reduced affine root system of type $\mathrm{C}^\vee\mathrm{C}_n$ this leads to Gustafson's [25] multivariable analogue of the Askey–Wilson integral which depends, apart from $q$, on five additional parameters. It gives the volume of the orthogonality measure of the Koornwinder polynomials, see Chapter 9.

A very general elliptic analogue of the Selberg integral and of Gustafson's multivariable analogue of the Askey–Wilson integral was conjectured by van Diejen & Spiridonov [11, Theorem 4.2] and proven by Rains [60, Theorem 6.1],

$$\frac{1}{(2\pi i)^n}\int_{T^n}\prod_{1\le i<j\le n}\frac{\Gamma(tz_iz_j,tz_iz_j^{-1},tz_i^{-1}z_j,tz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j,z_iz_j^{-1},z_i^{-1}z_j,z_i^{-1}z_j^{-1})}\prod_{k=1}^n\frac{\prod_{i=1}^6\Gamma(t_iz_k,t_iz_k^{-1})}{\Gamma(z_k^2,z_k^{-2})}\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}$$
$$=\frac{2^nn!}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{m=1}^n\left(\frac{\Gamma(t^m)}{\Gamma(t)}\prod_{1\le i<j\le 6}\Gamma(t^{m-1}t_it_j)\right),\qquad(1.2.13)$$

with $T$ the positively oriented unit circle in the complex plane, $\Gamma(x_1,\ldots,x_r):=\Gamma(x_1)\cdots\Gamma(x_r)$ a product of elliptic Gamma functions, and parameters $t,t_i\in\mathbb{C}$ satisfying $|t|,|t_i|<1$ and $t^{2n-2}t_1\cdots t_6=pq$. The integral (1.2.13) is an example of an *elliptic hypergeometric integral* associated with the root system of type $\mathrm{C}_n$. For $n=1$ it reduces to Spiridonov's elliptic beta integral [74]. It is a special case of a family of transformation formulas that relate elliptic hypergeometric integrals associated with type C root systems of different ranks [5]. The basic analogue of (1.2.13) is a multivariable analogue of the *Nassrallah–Rahman integral* [50],

$$\frac{1}{2\pi i}\int_T\frac{(z^2,z^{-2},Az,Az^{-1};q)_\infty}{\prod_{j=1}^5(t_jz,t_jz^{-1};q)_\infty}\frac{\mathrm{d}z}{z}=\frac{2\prod_{j=1}^5(At_j^{-1};q)_\infty}{(q;q)_\infty\prod_{1\le j<k\le 5}(t_jt_k;q)_\infty},\qquad|t_j|<1\qquad(1.2.14)$$

where $A:=t_1t_2t_3t_4t_5$ and $(a_1,\ldots,a_r;q)_\infty:=\prod_{i=1}^r(a_i;q)_\infty$. Just as (1.2.14) gives the Askey–Wilson integral for $t_5=0$, its multivariable analogue yields Gustafson's integral [25] by the same substitution. The identity (1.2.13) and some of its degenerations give the volumes of (bi)orthogonality measures for important families of multivariable (bi)orthogonal functions, see §1.4.

The multivariable elliptic integrals appearing in [17, 18] as coordinates of vector-valued solutions of *qKZB* equations are associated with the root system of type $\mathrm{A}_n$. Their semiclassical limits, which provide solutions of the *KZB* equation, as well as their degenerations to the basic and classical hypergeometric level, are discussed in Chapter 11.

A further rough division of hypergeometric integrals associated with root systems involves the notion of types. Multidimensional integrals are said to be *type II basic* (resp. *elliptic*) *hypergeometric integrals* associated with the root system $R$ if the integrand contains a factor of the form $\prod_{\alpha\in R}\left(\Gamma(q^{(\alpha,\boldsymbol{x})})/\Gamma(q^{k_\alpha+(\alpha,\boldsymbol{x})})\right)$ with $\Gamma(x)$ the basic (resp. elliptic) Gamma function. It is called *type I* if it contains a factor of the form $\prod_{\alpha\in R}\Gamma(q^{(\alpha,\boldsymbol{x})})^{-1}$. Similarly, a multidimensional integral is said to be a *type II classical hypergeometric integral* associated with the root system $R$ if the integrand contains a factor of the form $\Delta_{\boldsymbol{k}}(\boldsymbol{x})$ or $\prod_{\alpha\in R}\left(\Gamma(\alpha,\boldsymbol{x})/\Gamma(k_\alpha+(\alpha,\boldsymbol{x}))\right)$, with $\Gamma(x)$ the classical Gamma function (and a similar adjustment for type I). The examples of multidimensional integral evaluations highlighted so far, are type II. In Chapter 5 and Chapter 6 many examples of type I and type II multidimensional integral evaluations and transformations are discussed. Note that there are also hypergeometric integrals of mixed type, see (6.2.3) for an example.

Next we turn our attention to multivariable hypergeometric series. For a given root system

$R$, we can define a *Weyl type denominator* by

$$\Delta(\boldsymbol{x}) := \prod_{\alpha\in R^+} h((\alpha,\boldsymbol{x})) \quad \text{with} \quad h(z) = \begin{cases} z & \text{(classical hypergeometric type)},\\ 1-q^z & \text{(basic hypergeometric type)},\\ \theta(z;p) & \text{(elliptic hypergeometric type)}.\end{cases}$$

In the basic hypergeometric case $\Delta(\boldsymbol{x})$ is the Weyl denominator of the semisimple Lie algebra associated with $R$, while in the elliptic case $\Delta(\boldsymbol{x})$ is closely related to the Weyl denominator of the affine Lie algebra associated with $R^{(1)}$, see §6.1.2.

*Multivariable classical, basic and elliptic hypergeometric series* $\sum_{\boldsymbol{k}\in D} f(\boldsymbol{k})$ ($D \subseteq \mathbb{Z}^n$) are said to be *associated with the classical root system* $R$ if $f(\boldsymbol{k})$ contains the factor $\Delta(\boldsymbol{y}+\boldsymbol{k})$ for some fixed $\boldsymbol{y}\in\mathbb{C}^n$ in a nontrivial way. First examples of multivariable classical hypergeometric series identities appeared in the work of Holman, Biedenharn and Louck [28] on $6j$-symbols for SU($n$) (the associated root system is of type A). An important nontrivial example of a multivariable basic hypergeometric series identity is Milne's [49] fundamental theorem

$$\sum_{\boldsymbol{k}\in D_N} \Delta(\boldsymbol{y}+\boldsymbol{k}) \prod_{\ell=1}^{n} q^{(\ell-1)k_\ell} \prod_{i,j=1}^{n} \frac{(q^{\beta_i+y_j-y_i};q)_{k_j}}{(q^{1+y_j-y_i};q)_{k_j}} = \frac{(q^{\beta_1+\cdots+\beta_n};q)_N}{(q;q)_N}\Delta(\boldsymbol{y})$$

with $D_N := \{\boldsymbol{k}\in\mathbb{Z}^n_{\geq 0} \mid k_1+\cdots+k_n = N\}$ and $\Delta(\boldsymbol{x}) = \prod_{1\leq i<j\leq n}(1-q^{x_i-x_j})$ the Weyl type denominator for the root system $R$ of type $\mathrm{A}_{n-1}$. An elliptic generalization is the elliptic Jackson summation formula (6.3.1a) due to Rosengren [64, Theorem 5.1].

For classical root systems, identities and transformations for multivariable hypergeometric series naturally arise from related multidimensional hypergeometric integral identities and transformations through residue calculus. In this process, the Weyl denominator $\Delta(\boldsymbol{k})$ arises from the integrands of the multidimensional hypergeometric integrals through the formula (6.1.3). The residue calculus typically involves iterated small contour deformations per coordinate, avoiding at each step the poles of the factors of the integrand that do not depend on a single coordinate $x_j$. This technique was developed in [76] where it was applied to type II basic hypergeometric integrals associated with Koornwinder polynomials. When applied to the elliptic Selberg integral (1.2.13) one obtains a type C elliptic hypergeometric series identity (see (6.3.6)) that reduces for $n=1$ to the *Frenkel–Turaev elliptic summation formula [20]*

$$\begin{aligned}\sum_{m=0}^{N} \frac{\theta(aq^{2m};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-N};p,q)_m}{(q,aq/b,aq/c,aq/d,aq/e,aq^{1+N};p,q)_m} q^m \\ = \frac{(aq,aq/bc.aq/bd.aq/cd;p,q)_N}{(aq/b,aq/c,aq/d,aq/bcd;p,q)_N} \qquad (1.2.15)\end{aligned}$$

for $bcde = a^2q^{N+1}$, where $(x_1,\ldots,x_r;p,q)_m = \prod_{i=1}^r (x_i;p,q)_m$. In this way, many of the hypergeometric series identities and transformations associated with classical root systems as discussed in Chapter 5 (classical and basic hypergeometric) and in Chapter 6 (elliptic hypergeometric) can be viewed as discrete versions of multidimensional hypergeometric integrals.

## 1.3 Multivariable (bi)orthogonal polynomials and functions

### *1.3.1 One-variable cases*

The class of one-variable (bi)orthogonal polynomials and functions splits up into various natural subclasses, each subclass having its distinct features that are vital for the construction and study of its multivariable generalization.

**(a)** General theory of orthogonal polynomials [78].
**(b)** Classical orthogonal polynomials.
**(c)** Classical biorthogonal rational functions.
**(d)** Bessel functions [55, §10.22(v)] and Jacobi functions [36].

By *classical orthogonal polynomials* we mean (more generally than in Chapter 2) the one-variable orthogonal polynomials belonging to the Askey or $q$-Askey scheme [34, Ch. 9,14]. They are characterized as the orthogonal polynomials that are joint eigenfunctions of a suitable type of second order differential or ($q$-)difference operator. The corresponding classification results are called (generalized) Bochner theorems [24, 31, 80]). Prominent members are the Jacobi polynomials [78, Ch. IV] and their top level $q$-analogues, the Askey–Wilson polynomials [2]. By *classical biorthogonal rational functions* we refer to the generalizations of classical orthogonal polynomials due to Rahman [58, 59] and Wilson [81], and their elliptic analogues due to Spiridonov and Zhedanov [74].

Classical orthogonal polynomials and biorthogonal rational functions are expressible as ordinary, basic and elliptic hypergeometric series. The various classes admit (bi)orthogonality relations with respect to explicit measures whose total masses are the outcome of important integral evaluation formulas. For example, for the classical Jacobi polynomials the integral evaluation is the beta integral

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},\qquad \alpha,\beta>-1,$$

with $\Gamma(x)$ the classical Gamma function, for Rahman's [58] biorthogonal basic hypergeometric rational functions it is the Nassrallah–Rahman integral (1.2.14), and for Spiridonov–Zhedanov's [74] elliptic biorthogonal rational functions it is Spiridonov's [73] elliptic beta integral (the $n=1$ case of (1.2.13)).

### *1.3.2 Multivariable generalizations*

The subclasses (a)–(d) of one-variable (bi)orthogonal polynomials and functions generalize to the multivariable case as follows. Note that the one-variable subclass (b) generalizes to two subclasses (b1) and (b2).

**(a)** General theory of multivariable orthogonal polynomials [15] with respect to orthogonality measures on $\mathbb{R}^d$. This is discussed in Chapter 2.
**(b1)** Multivariable orthogonal polynomials expressible as (non-straightforward) products of one-variable classical orthogonal polynomials and elementary polynomials, see Tratnik [79] and Gasper & Rahman [22] for the continuous cases. Examples are in Chapter 2.

**(b2)** Root system generalizations of classical orthogonal polynomials. Prominent examples are the BC-type Heckman–Opdam polynomials [26] (see Chapter 8) and Koornwinder polynomials [37] (see Chapter 9), which provide multivariable generalizations of the Jacobi polynomials and their top-level $q$-analogues in the $q$-Askey scheme (Askey–Wilson polynomials), respectively.

**(c)** By the extension of the class of classical orthogonal polynomials to biorthogonal rational functions, one arrives at a class of one-variable special functions that generalizes to the elliptic hypergeometric level as well as to the multivariable level. In the one-variable setup these are the biorthogonal elliptic rational functions from Spiridonov & Zhedanov [74]. The multivariable generalization is due to Rains [60]. This is discussed in Chapter 6. The Macdonald–Koornwinder polynomials associated with classical root systems are limit cases of Rains' elliptic biorthogonal rational functions.

**(d)** Root system generalizations of Bessel functions and the associated Fourier transforms [14] are discussed in Chapter 7. Root system generalizations of Jacobi functions and the associated harmonic analysis [57] are discussed in Chapter 8. These functions can be thought of as non-polynomial generalizations of the Heckman–Opdam polynomials so that they may be called *Heckman–Opdam functions*. Their basic hypergeometric analogues (Cherednik [8], Stokman [77]) are not discussed in this volume.

In the next section we briefly discuss for these five subclasses how these multivariable extensions came about, and what techniques are used to study them.

## 1.4 Multivariable (bi)orthogonal polynomials and functions, some details

### *Class (a): General theory of multivariable orthogonal polynomials*

Important topics are the development of multivariable versions of Gram–Schmidt orthogonalization and of three-term recurrence relations (Favard's theorem). Much of the general theory deals with the space $\mathcal{V}_n$ of *orthogonal polynomials of degree n in d variables* $x_1, \ldots, x_d$, defined as the space of all polynomials of total degree $\leq n$ which are orthogonal to all polynomials of total degree $\leq n-1$ with respect to a fixed inner product on the space of polynomials in $x_1, \ldots, x_d$. The subspace $\mathcal{V}_n$ does not have a canonical orthogonal basis. In particular, the *monomial basis* of $\mathcal{V}_n$, consisting of the polynomials in $\mathcal{V}_n$ of the form $x^\alpha + Q_\alpha(x)$ ($\alpha \in \mathbb{Z}^d_{\geq 0}$ multi-index of degree $n$) with $Q_\alpha(x)$ of total degree $\leq n-1$, is usually not an orthogonal basis of $\mathcal{V}_n$. Still any basis of $\mathcal{V}_n$, in particular the monomial basis, admits another basis which is biorthogonal to the first basis. If the subspaces $\mathcal{V}_n$ are the eigenspaces of some second order partial differential operator $L$ then the orthogonal polynomials are said to be *classical*. For two-variable examples, see §2.3.4. Remarkably, while the factorizable multivariable orthogonal polynomials (class (b1)) are usually classical, the root system generalizations of the classical orthogonal polynomials (class (b2)) are usually not. In both cases, for $q = 1$, there is a se cond order differential operator around, but in the second case its eigenspaces are usually not the full spaces $\mathcal{V}_n$.

### *Class (b1): Factorizable multivariable orthogonal polynomials*

For two-dimensional regions like the disk and the triangle with obvious weight functions generalizing the ultraspherical or Jacobi weight function, the monomial basis and the basis biorthogonal to it were already explicitly given by Appell in the late nineteenth century. These bases were expressed in terms of Appell hypergeometric functions. Explicit orthogonal bases for these cases were given much later, although they have a very simple factorized form. See a survey in [35, §3.4]. These bases usually came up in close connection with various kinds of applications. Very noteworthy are the *disk polynomials* of Zernike and Brinkman [82], motivated by optics and still having important applications there.

The factorized orthogonal polynomials were extended to higher dimensions $d$ (ball and simplex). In addition to the second order operator $L$ mentioned under class (a), there are $d-1$ further partial differential operator generating together with $L$ a commutative algebra of which the orthogonal polynomials are the joint eigenfunctions.
These orthogonal polynomials also occur [35, §3.5], [12], [33] in connection with spherical harmonics, see also Chapter 2. Furthermore, these polynomials naturally arise as coupling coefficients for tensor product representations of $\mathrm{SL}_2(\mathbb{R})$, while multivariable basic hypergeometric orthogonal polynomials arise as coupling coefficients for tensor products of the associated quantum group [62, 63].

### *Class (b2): Root system generalizations of classical orthogonal polynomials.*

Restrictions of zonal spherical functions on compact symmetric spaces to the torus provide multivariable root system generalizations of Jacobi polynomials depending on special discrete parameter values, which come from the root multiplicities of the symmetric space [27, Ch. V, §4]. Heckman–Opdam polynomials [26] provide generalizations without restrictions on the parameters. The techniques from geometric group theory now fail. Initially the only alternative was laborious analytic work, but the early nineties brought the great insight that there is another Lie type setting for these polynomials, namely representation theory of degenerate affine Hecke algebras [57] in terms of Heckman–Dunkl differential-reflection operators [26], which generalize the Dunkl operators [13] treated in Chapter 7 (see also Cherednik's [6, pp. 429–430] slight variants of the Heckman–Dunkl operators). As mentioned already in §1.2.3, the total mass of the orthogonality measure of the Heckman–Opdam polynomials is evaluated by the Macdonald constant term identity. A significant difference with multivariable orthogonal polynomials in class (b1) is the fact that explicit hypergeometric series expressions of the Heckman–Opdam polynomials are not available. However, there is a binomial formula in the $\mathrm{BC}_n$ case, see [38, (9.1), (11.5)], as we will discuss for the elliptic case in §1.4, Class (c).

All key properties, including orthogonality and norm formulas, need representation theoretic tools involving the Dunkl type operators and related operators, such as intertwining and shift operators. The Heckman–Opdam polynomials are joint eigenfunctions of a commutative algebra generated by $d$ commuting partial differential operators, where $d$ is the rank of the associated root system. It includes the explicit second order differential operator $L$ arising as the radial part of the Laplace–Beltrami operator with the root multiplicities taken continuously, see Chapter 8. An important application and source of further applications is the fact that they produce the eigenstates for an important class of integrable one-dimensional quantum many-body systems in theoretical physics called quantum Calogero-Moser systems, see [56].

In the $q$-case the Macdonald [46] and Koornwinder [37] polynomials provide multivariable root system generalizations of the Askey–Wilson polynomials (the top level in the $q$-Askey scheme) and its subclasses of continuous $q$-Jacobi and continuous $q$-ultraspherical polynomials (only for root system of type BC all three classes generalize). Just as for $q = 1$, they relate for special parameter values to harmonic analysis on quantum compact symmetric spaces [52, 42] (although in this case it was not the original motivation for introducing these polynomials). Many properties of the Heckman–Opdam polynomials as described above generalize to Macdonald and Koornwinder polynomials, with the role of differential operators now taken over by $q$-difference operators. The deeper study of these polynomials involves Cherednik's theory on (double) affine Hecke algebras [9, 67, 47], see Chapter 9.

Prior to the Macdonald polynomials for arbitrary root systems, Macdonald [44], [45, Ch. VI] introduced a version of these polynomials which is related to the general linear group (they relate to the Macdonald polynomials for root system of type $A$ as, for $q = 1$, the Jack polynomials relate to the A-type Heckman–Opdam polynomials). These $\mathrm{GL}_d$ Macdonald polynomials are homogeneous symmetric polynomials in $d$ (or countably many) variables depending on two parameters $q$ and $t$, which generalize various important classes of symmetric functions such as Jack polynomials, Hall–Littlewood polynomials, and Schur functions. Their important role in modern algebraic combinatorics is discussed in Chapter 10.

### *Class (c): Biorthogonal rational functions*

From the $q$-Askey scheme of classical basic hypergeometric orthogonal polynomials and their biorthogonal rational extensions, only Rahman's [58] biorthogonal rational functions have been generalized to the elliptic regime (Spiridonov and Zhedanov [74]). Multivariable generalizations in the class (b2) were introduced by Rains [60]. The associated explicit evaluation formula of the total mass of the biorthogonality measure is the type II elliptic hypergeometric integral associated to the root system of type C given by (1.2.13). The elliptic variant of the theory of interpolation functions, which goes back to work of Kostant and Sahi [40, 41], has been particularly useful in the development of Rains' multivariable elliptic biorthogonal rational functions. These are classes of multivariable polynomials defined by explicit vanishing properties, and serve as a kind of monomial type basis within the theory. In particular, one can write down explicit series expansions of the elliptic biorthogonal rational functions in terms of interpolation functions. These so-called *binomial formulas* (earlier given for Koornwinder polynomials by Okounkov [53]) are for the moment the closest to explicit elliptic or basic hypergeometric series expressions one can get. See also §6.4.

The elliptic generalization of the double affine Hecke algebra is currently in development [61] from the point of view of algebraic geometry.

#### *Class (d): Root system generalizations of Bessel and Jacobi functions*

The discussion about class (b2) largely applies here too. For special parmeter values Jacobi functions relate to zonal spherical functions on *noncompact* Riemannian symmetric spaces and multivariable Bessel functions to spherical functions for Cartan motion groups. The nonsymmetric versions of multivariable Bessel functions are called *Dunkl kernels*, and it is there that the Hecke algebraic approach, mentioned before in the discussion of class (b2), first arose with Dunkl's invention of a commuting family of differential-difference operators serving as deformations of directional derivatives. These operators are nowadays known as *Dunkl operators*, see Chapter 7.

It remains puzzling that nonsymmetric Jacobi functions and nonsymmetric Jacobi polynomials associated with root systems do not seem to live, for special parameter values, on Riemannian symmetric spaces, as symmetric Jacobi functions and polynomials do. However, partial interpretations of nonsymmetric special functions are known in connection with representations of affine Lie algebras and p-adic groups, see [70, 29, 30, 4].

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