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MACDONALD-KOORNWINDER POLYNOMIALS

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ABSTRACT. An overview of the basic results on Macdonald-Koornwinder polynomials and double affine Hecke algebras is given. We develop the theory in such a way that it naturally encompasses all known cases as well as a new rank two case. Among the basic properties of the Macdonald-Koornwinder polynomials we treat are the quadratic norm formulas, duality and the evaluation formulas. This text is a provisional version of a chapter on Macdonald-Koornwinder polynomials for volume 5 of the Askey-Bateman project, entitled "Multivariable special functions".

CONTENTS

1. Introduction

This text is a provisional version of a chapter on Macdonald-Koornwinder polynomials for volume 5 of the Askey-Bateman project, entitled "Multivariable special functions". The aim is to introduce nonsymmetric and symmetric Macdonald-Koornwinder polynomials and to present their basic properties: (bi-)orthogonality, norm formulas, q-difference(-reflection) equations, duality and evaluation formulas.

Symmetric GL type Macdonald polynomials were introduced by Macdonald [58] as a two-parameter family of orthogonal multivariate polynomials interpolating between the Jack polynomials and the Hall-Littlewood polynomials. Root system generalizations were subsequently defined by Macdonald in [60]. They were labelled by so called admissible pairs of root systems. Recasting these data in terms of affine root systems (cf. [15, 61]) it is natural to speak of an untwisted theory and a twisted theory of Macdonald polynomials associated to root systems. An important further extension for nonreduced root systems was constructed by Koornwinder [51]. In [61, 32] important steps were undertaken to develop a general theory covering all these cases. In this text we pursue this further. We call the corresponding polynomials Macdonald-Koornwinder polynomials.

The symmetric Macdonald-Koornwinder polynomials are root system generalizations of classical one-variable q-orthogonal polynomials from the q-Askey scheme [48]. Before giving the contents of this chapter in more detail, we first illustrate this point of view for the symmetric GL Macdonald polynomials and the symmetric Koornwinder polynomials.

1.1. **Symmetric GL type Macdonald polynomials.** The symmetric GL_{n+1} type Macdonald [58] polynomials form a linear basis of the space of symmetric Laurent polynomials in $n+1$ variables $t_1,\ldots,t_{n+1}$, depending on two parameters $0<q,\kappa<1$. They form an orthogonal basis of the Hilbert space $L^2(T_u, v_+(t)dt)$, where dt is the normalized Haar measure on $T_u=\{t=(t_1,\ldots,t_{n+1})\in\mathbb{C}^{n+1}\mid |t_i|=1\}$ and with weight function $v_+(t)$

explicitly given by

$$v_+(t) = \prod_{1\le i\ne j\le n+1} \frac{\big(t_i/t_j;q\big)_\infty}{\big(\kappa^2 t_i/t_j;q\big)_\infty}$$

(the q-shifted factorial is defined by (3.11)). In addition, the symmetric GL_{n+1} type Macdonald polynomials are common eigenfunctions of the commuting trigonometric Ruijsenaars-Macdonald [73, 58] q-difference operators

$$(D_j f)(t) := \sum_{\substack{I\subseteq\{1,\ldots,n+1\}\\ \#I=j}} \left(\prod_{r\in I, s\notin I} \frac{\kappa^{-1}t_r - \kappa t_s}{t_r - t_s}\right) f(q^{-\sum_{r\in I}\epsilon_r} t) \qquad 1\le j\le n+1,$$

where $q^{-\sum_{r\in I}\epsilon_r}$ is the $(n+1)$-vector with q^{-1}'s at the entries labelled by I and ones elsewhere. The eigenvalue equation for D_{n+1} is equivalent to a homogeneity condition for the symmetric Macdonald polynomials. Up to a monomial factor the symmetric GL_{n+1} Macdonald polynomials thus only depend on the n variables $t_1/t_2,\ldots,t_n/t_{n+1}$. For $n=1$ the Macdonald polynomials are essentially the continuous q-ultraspherical polynomials in the variable t_1/t_2. The above results then reduce to the orthogonality relations and the second order q-difference equation satisfied by the continuous q-ultraspherical polynomials, see Subsection 3.7 for further details.

1.2. **Symmetric Koornwinder polynomials.** The symmetric Koornwinder [51] polynomials form a linear basis of the space of Laurent polynomials in n variables $t_1,\ldots,t_n$ which are invariant under the hyperoctahedral group (acting by permutations and inversions of the variables), depending on six parameters $0<q,a,b,c,d,k<1$. They form an orthogonal basis of the Hilbert space $L^2(T_u, v_+(t)dt)$ with $T_u=\{t\in\mathbb{C}^n \,|\, |t_i|=1\}$, Haar measure dt on T_u, and weight function

$$v_+(t) = \prod_{i=1}^n \frac{\big(t_i^{\pm 2};q\big)_\infty}{\big(at_i^{\pm1};q\big)_\infty\big(bt_i^{\pm1};q\big)_\infty\big(ct_i^{\pm1};q\big)_\infty\big(dt_i^{\pm1};q\big)_\infty} \prod_{1\le r\ne s\le n} \frac{\big(t_t t_s^{\pm1};q\big)_\infty}{\big(kt_r t_s^{\pm1};q\big)_\infty},$$

where $\big(uz^{\pm1};q\big)_\infty := \big(uz;q\big)_\infty\big(uz^{-1};q\big)_\infty$. The symmetric Koornwinder polynomials are eigenfunctions of Koornwinder's [51] multivariable extension of the Askey-Wilson [1] second order q-difference operator

$$\begin{aligned}(Df)(t) :=& \sum_{i=1}^n \sum_{\xi\in\{\pm1\}} A_i^\xi(t)\big(f(q^{\xi\epsilon_i}t) - f(t)\big),\\ A_i^\xi(t) :=& \frac{(1-at_i^\xi)(1-bt_i^\xi)(1-ct_i^\xi)(1-dt_i^\xi)}{(1-t_i^{2\xi})(1-qt_i^{2\xi})} \prod_{j\ne i} \frac{(1-kt_i^\xi t_j)(1-kt_i^\xi t_j^{-1})}{(1-t_i^\xi t_j)(1-t_i^\xi t_j^{-1})}.\end{aligned}$$

For $n=1$ the k-dependence drops out and the symmetric Koornwinder polynomials are the Askey-Wilson polynomials from [1]. See Subsection 3.8 for further details.

1.3. **General description of the contents.** Most literature on Macdonald-Koornwinder polynomials deals with one out of the above mentioned four cases of Macdonald-Koornwinder type polynomials (the GL-case, the untwisted case, the twisted case and the Koornwinder case). Cherednik [15] treats the first three cases separately. Macdonald's [61] exposition covers the last three cases, but various steps still need case by case analysis. We develop the theory in such a way that it naturally unifies the above four cases, but in addition contains a new class of rank two Macdonald-Koornwinder type polynomials (see Subsection 3.9). The setup will be close to Haiman's [32] approach, which allowed him to give a uniform proof of the duality of the Macdonald-Koornwinder polynomials.

It is tempting to believe that the initial data for the Macdonald-Koornwinder polynomials should be similarity classes of irreducible affine root systems R together with a choice of a deformation parameter q and a multiplicity function (playing the role of the free parameters in the theory). In such a parametrization the untwisted and twisted cases should relate to the similarity classes of the irreducible reduced affine root systems of untwisted and twisted type respectively, the GL case to the irreducible reduced affine root system of type A with a "reductive" extension of the affine Weyl group, and the Koornwinder case with the nonreduced irreducible affine root system of type $C^\vee C$ (we refer here to the classification of affine root systems from [56], see also the appendix). It turns out though that a more subtle labelling is needed to capture all fundamental properties of the Macdonald-Koornwinder polynomials.

We take as initial data quintuples $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ with R_0 a finite reduced irreducible root system, Δ_0 an ordered basis of R_0, $\bullet \in \{u, t\}$ (u stands for untwisted and t stands for twisted), and Λ, Λ^d two lattices satisfying appropriate compatibility conditions with respect to the (co)root lattice of R_0 (see (2.2)). We build from D an irreducible affine root system R and an extended affine Weyl group W. The extended affine Weyl group W is simply the semi-direct product group $W_0 \ltimes \Lambda^d$ with W_0 the Weyl group of R_0. The affine root system R is constructed as follows. We associate to R_0 and $\bullet$ the reduced irreducible affine root system $R^\bullet$ of type $\bullet$ with gradient root system R_0. Then R is an irreducible affine root system obtained from $R^\bullet$ by adding $2a$ if $a \in R^\bullet$ has the property that the pairings of the associated coroot $a^\vee$ to elements of Λ take value in $2\mathbb{Z}$ (see (2.5)). The Macdonald-Koornwinder polynomials associated to D are the ones naturally related to R in the labelling proposed in the previous paragraph.

In the literature the terminology Macdonald polynomials is used when the underlying affine root system R is reduced, while the terminology Koornwinder polynomials or Macdonald-Koornwinder polynomials is used when R is nonreduced and of type $C^\vee C$. In this text we will use the terminology Macdonald-Koornwinder polynomials when dealing with arbitrary initial data D. We will speak of Macdonald polynomials if the underlying affine root system R is reduced and of Koornwinder polynomials if R is of type $C^\vee C$.

Duality is related to a simple involution on initial data, $D \mapsto D^d = (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda)$ with R_0^d the coroot system $R_0^\vee$ if $\bullet = u$ and the root system R_0 if $\bullet = t$. This duality is subtle on the level of affine root systems (it can for instance happen that the affine root system R^d associated to D^d is reduced while R is nonreduced). The reason that the present choice of initial data is convenient is the fact that the duality map $D \mapsto D^d$ on initial data naturally lifts to the duality antiisomorphism of the associated double affine braid group (see [32] and Section 4). This duality antiisomorphism is the key tool to prove duality, evaluation formulas and norm formulas for the Macdonald-Koornwinder polynomials.

1.4. **Topics that are not discussed.** We do not discuss the shift operators for the Macdonald-Koornwinder polynomials (see, e.g., [9, 79]), leading to the explicit evaluations of the constant terms (generalized Selberg integrals). Other important developments involving Macdonald-Koornwinder polynomials that are not discussed in this chapter, are

- connections to combinatorics. This is discussed in a separate chapter of the fifth volume of the Askey-Bateman project,
- connections to algebraic geometry, see, e.g., [30, 31, 76],
- connections to representation theory, see [15, 16, 17, 41, 38, 39] and references therein.
- applications to harmonic analysis on quantum groups, see, e.g., [62, 66, 64, 53, 22, 24, 25, 67],
- limit cases of symmetric Macdonald-Koornwinder polynomials, see, e.g., [58, 60, 15, 33, 34, 18, 78, 2] and the separate chapter of the fifth volume of the Askey-Bateman project on Heckman-Opdam polynomials,
- the theory of Gaussians, Macdonald-Mehta type integrals, and basic hypergeometric functions associated to root systems, see, e.g., [14, 80, 18, 82],
- interpolation Macdonald-Koornwinder polynomials, see, e.g., [46, 47, 74, 69, 68, 52],
- the relation to quantum integrable systems, see, e.g., [7, 8, 73, 77, 5, 6, 70, 44, 45, 27, 81],
- special parameter values (e.g. roots of unity), see, e.g., [15, 12, 19, 43, 41, 42, 26, 5, 6],
- affine and elliptic generalizations, see, e.g., [23, 10, 73] and [49, 71, 72, 20] respectively.

See the introductory chapter of Cherednik's [15] book for an extensive discussion on the ramifications of Macdonald polynomials.

1.5. **Detailed description of the contents.** Precise references to the literature are given in the main text.

In Section 2 we give the definition of the affine braid group, affine Weyl group and affine Hecke algebra. We determine an explicit realization of the affine Hecke algebra, which will serve as starting point of the Cherednik-Macdonald theory on Macdonald-Koornwinder polynomials in the next section. In addition we introduce the initial data. We introduce the space of multiplicity functions associated to the fixed initial data D. We extend the duality on initial data to an isomorphism of the associated spaces of multiplicity functions. We give the basic representation of the extended affine Hecke algebra associated to D, using the explicit realization of the affine Hecke algebra.

In Section 3 we define and study the nonsymmetric and symmetric monic Macdonald-Koornwinder polynomials associated to the initial data D. We first focus on the nonsymmetric monic Macdonald-Koornwinder polynomials. We characterize them as common eigenfunctions of a family of commuting q-difference reflection operators. These operators are obtained as the images under the basic representation of the elements of the Bernstein-Zelevinsky abelian subalgebra of the extended affine Hecke algebra. We determine the biorthogonality relations of the nonsymmetric monic Macdonald-Koornwinder polynomials and use the finite Hecke symmetrizer to obtain the symmetric monic Macdonald-Koornwinder polynomials. We give the orthogonality relations of the symmetric monic Macdonald-Koornwinder polynomials and show that they are common eigenfunctions of the commuting Macdonald q-difference operators (also known as Ruijsenaars operators in the GL case). We finish this section by describing three cases in detail: the GL case, the $C^\vee C$ case, and an exceptional nonreduced rank two case not covered in Cherednik's [15] and Macdonald's [61] treatments.

In Section 4 we introduce the double affine braid group and the double affine Hecke algebra associated to D and (D, κ) respectively, where κ is a choice of a multiplicity function on R. We lift the duality on initial data to a duality antiisomorphism on the level of the associated double affine braid groups. We show how it descends to the level of double affine Hecke algebras and how it leads to an explicit evaluation formula for the monic Macdonald-Koornwinder polynomials. We proceed by defining the associated normalized nonsymmetric and symmetric Macdonald-Koornwinder polynomials and deriving their duality and quadratic norms.

In Section 5 we give the norm and evaluation formulas in terms of q-shifted factorials for the GL case, the $C^\vee C$ case, and for the exceptional nonreduced rank two case.

In the appendix we give a short introduction to (the classification of) affine root systems, following closely [56] but with some adjustments. We close the appendix with a list of all the affine Dynkin diagrams.

2. The basic representation of the extended affine Hecke algebra

We first introduce the affine Hecke algebra and the appropriate initial data for the Cherednik-Macdonald theory on Macdonald-Koornwinder polynomials. Then we introduce the basic representation of the extended affine Hecke algebra, which is fundamental in the development of the Cherednik-Macdonald theory.

2.1. **Affine Hecke algebras.**

A convenient reference for this subsection is [36]. For unexplained notations and terminology regarding affine Weyl groups we refer to the appendix.

For a generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq r}$ let $M = (m_{ij})_{1 \leq i,j \leq r}$ be the matrix with $m_{ii} = 1$ and for $i \neq j$, $m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$ respectively.

Definition 2.1. *Let $A = (a_{ij})_{1 \leq i,j \leq r}$ be a generalized Cartan matrix.*

(1) *The braid group $\mathcal{B}(A)$ is the group generated by T_i ($1 \leq i \leq r$) with defining relations $T_iT_jT_i \cdots = T_jT_iT_j \cdots$ (m_{ij} factors on each side) if $1 \leq i \neq j \leq r$ (which should be interpreted as no relation if $m_{ij} = \infty$).*

(2) *The Coxeter group $\mathcal{W}(A)$ associated to A is the quotient of $\mathcal{B}(A)$ by the normal subgroup generated by T_i^2 ($1 \leq i \leq r$).*

It is convenient to denote by s_i the element in $\mathcal{W}(A)$ corresponding to T_i for $1 \leq i \leq r$. They are the Coxeter generators of the Coxeter group $\mathcal{W}(A)$.

Let $A = (a_{ij})_{1\leq i,j\leq r}$ be a generalized Cartan matrix. Suppose that k_i $(1 \leq i \leq r)$ are nonzero complex numbers such that $k_i = k_j$ if s_i is conjugate to s_j in $\mathcal{W}(A)$. We write k for the collection $\{k_i\}_i$. Let $\mathbb{C}[\mathcal{B}(A)]$ be the complex group algebra of the braid group $\mathcal{B}(A)$.

Definition 2.2. *The Hecke algebra $H(A, \kappa)$ is the complex unital associative algebra given by $\mathbb{C}[\mathcal{B}(A)]/I_k$, where I_k is the two-sided ideal of $\mathbb{C}[\mathcal{B}(A)]$ generated by $(T_i - k_i)(T_i + k_i^{-1})$ for $1 \leq i \leq r$.*

If $k_i = 1$ for all $1 \leq i \leq r$ then the associated affine Hecke algebra is the complex group algebra of $\mathcal{W}(A)$.

If $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression in $\mathcal{W}(A)$, i.e. a shortest expression of w as product of Coxeter generators, then

$$T_w := T_{i_1}T_{i_2}\cdots T_{i_r} \in H(A, k)$$

is well defined (already in the braid group $\mathcal{B}(A)$), and the T_w $(w \in \mathcal{W}(A))$ form a complex linear basis of $H(A, k)$.

Suppose $R_0 \subset V$ is a finite crystallographic root system with ordered basis $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ and write A_0 for the associated Cartan matrix. Then $W_0 \simeq \mathcal{W}(A_0)$ by mapping the simple reflections $s_{\alpha_i} \in W_0$ to the Coxeter generators s_i of $\mathcal{W}(A_0)$ for $1 \leq i \leq n$. The associated finite Hecke algebra $H(A_0, k)$ depends only on k and W_0 (as Coxeter group). We will sometimes denote it by $H(W_0, k)$.

Similarly, if R is an irreducible affine root system with ordered basis $\Delta = (a_0, \ldots, a_n)$ and if A is the associated affine Cartan matrix, then $\mathcal{W}(A) \simeq W(R)$ by $s_i \mapsto s_{a_i}$ $(0 \leq i \leq n)$. Again we write $H(W(R), k)$ for the associated affine Hecke algebra $H(A, k)$.

2.2. **Realizations of the affine Hecke algebra.** We use the notations on affine root systems as introduced in the appendix. The construction is motivated by Cherednik's polynomial representation [15, Thm. 3.2.1] of the affine Hecke algebra and its extension to the nonreduced case by Noumi [63].

Let $R \subset \widehat{E}$ be an irreducible affine root system on the affine Euclidean space E (possibly nonreduced) with affine Weyl group W, and fix an ordered basis $\Delta = (a_0, a_1, \ldots, a_n)$ of R. Write $A = A(R, \Delta)$ for the associated affine Cartan matrix.

Consider the lattice $\mathbb{Z}R$ in $\widehat{E}$. It is a full W-stable lattice with $\mathbb{Z}$-basis the simple affine roots. Denote by F the quotient field $\mathrm{Quot}\big(\mathbb{C}[\mathbb{Z}R]\big)$ of the complex group algebra $\mathbb{C}[\mathbb{Z}R]$ of $\mathbb{Z}R$. It is convenient to write e^λ $(\lambda \in \mathbb{Z}R)$ for the natural complex linear basis of $\mathbb{C}[\mathbb{Z}R]$. The multiplicative structure of F is determined by

$$e^0 = 1, \quad e^{\lambda+\mu} = e^\lambda e^\mu, \qquad \lambda, \mu \in \mathbb{Z}R.$$

The affine Weyl group W canonically acts by field automorphisms on F. On the basis elements e^λ the W-action reads $w(e^\lambda) = e^{w\lambda}$ $(w \in W,\ \lambda \in \mathbb{Z}R)$. Since W acts by algebra automorphisms on F, we can form the semidirect product algebra $W \ltimes F$.

Let $k : R \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $a \mapsto k_a$ be a W-equivariant map, i.e. $k_{wa} = k_a$ for all $w \in W$ and $a \in R$. If $a \in R$ but $2a \notin R$ then we set $k_{2a} := k_a$. Note that $k^{ind} := k|_{R^{ind}}$ is a W-equivariant map on R^{ind}, which is determined by its values $k_i := k_{a_i}$ $(0 \leq i \leq n)$ on the simple affine roots. It satisfies $k_i = k_j$ if s_i is conjugate to s_j in W. Hence we can form the associated affine Hecke algebra $H(W(R), k^{ind})$.

Theorem 2.3. *With the above notations and conventions, there exists a unique algebra monomorphism*

$$\beta = \beta_{R,\Delta,k} : H(W(R), k^{ind}) \hookrightarrow W \ltimes F$$

satisfying

$$\beta(T_i) = k_i s_i + \frac{k_i - k_i^{-1} + (k_{2a_i} - k_{2a_i}^{-1})e^{a_i}}{1 - e^{2a_i}}(1 - s_i) \tag{2.1}$$

for $0 \leq i \leq n$.

The proof uses the Bernstein-Zelevinsky presentation of the affine Hecke algebra, which we present in a slightly more general context in Subsection 3.1.

Remark 2.4. If $2a_i \notin R$ then (2.1) simplifies to

$$\beta(T_i) = k_i s_i + \frac{k_i - k_i^{-1}}{1 - e^{a_i}}(1 - s_i).$$

The notion of similarity of pairs (R, Δ) (see the appendix) can be extended to triples (R, Δ, k) in the obvious way. The algebra homomorphisms β that are associated to different representatives of the similarity class of (R, Δ, k) are then equivalent in a natural sense. Starting from the next subsection we therefore will focus on the explicit representatives of the similarity classes as described in the appendix in Subsection 6.2.

Recall from the appendix that the classification of irreducible affine root systems leads to a subdivision of irreducible affine root systems in three types, namely untwisted, twisted and mixed type. It is easy to show that the algebra map $\beta_{R,\Delta,k}$ in case R is of mixed type (possibly nonreduced) can alternatively be written as $\beta_{(R',\Delta',k')}$ with appropriately chosen triple (R', Δ', k') and with R' of untwisted or of twisted type. In the Cherednik-Macdonald theory, the mixed type can therefore safely be ignored, as we in fact will do.

2.3. **Initial data.** It turns out that labelling Macdonald-Koornwinder polynomials by irreducible affine root systems is not the most convenient way to proceed. In this subsection we give more convenient initial data and we explain how it relates to affine root systems. For basic notations and facts on affine root systems we refer to the appendix.

Definition 2.5. *The set $\mathcal{D}$ of initial data consists of quintuples*

$$D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$$

with

(1) *R_0 is a finite set of nonzero vectors in an Euclidean space Z forming a finite, irreducible, reduced crystallographic root system within the real span V of R_0,*
(2) *$\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0,*

(3) $\bullet = u$ *or* $\bullet = t$,
(4) Λ *and* Λ^d *are full lattices in* Z, *satisfying*

$$\begin{aligned} &\mathbb{Z}R_0 \subseteq \Lambda, \qquad \big(\Lambda, \mathbb{Z}R_0^\vee\big) \subseteq \mathbb{Z},\\ &\mathbb{Z}R_0^d \subseteq \Lambda^d, \qquad \big(\Lambda^d, \mathbb{Z}R_0^{d\vee}\big) \subseteq \mathbb{Z} \end{aligned} \tag{2.2}$$

where $R_0^d = \{\alpha^d := \mu_\alpha^\bullet \alpha^\vee\}_{\alpha \in R_0}$ *and*

$$\begin{aligned} \mu_\alpha^u &:= 1, & \alpha \in R_0,\\ \mu_\alpha^t &:= |\alpha|^2/2, & \alpha \in R_0. \end{aligned}$$

Note that $R_0^d = R_0^\vee$ if $\bullet = u$ and $= R_0$ if $\bullet = t$.

We view the vector space $\widehat{V}$ of real valued affine linear functions on V as the subspace of $\widehat{Z}$ consisting of affine linear functions on Z which are constant on the orthocomplement $V^\perp$ of V in Z. We write c for the constant function one on V as well as on Z. In a similar fashion we view the orthogonal group $O(V)$ as subgroup of $O(Z)$ and $O_c(\widehat{V})$ as subgroup of $O_c(\widehat{Z})$, where $O_c(\widehat{V})$ is the subgroup of linear automorphisms of $\widehat{V}$ preserving c and preserving the natural semi-positive definite form on $\widehat{V}$ (see the appendix for further details).

Fix $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. We associate to D a triple (R, Δ, W) of an affine root system $R = R(D)$, an ordered basis $\Delta = \Delta(D)$ of R and an extended affine Weyl group $W = W(D)$ as follows. We first define a reduced irreducible affine root system $R^\bullet \subset \widehat{V}$ to D as

$$R^\bullet = \{m\mu_\alpha^\bullet c + \alpha\}_{m \in \mathbb{Z}, \alpha \in R_0} \tag{2.3}$$

for $\bullet \in \{u, t\}$. In other words, $R^u := \mathcal{S}(R_0)$ and $R^t := \mathcal{S}(R_0^\vee)^\vee$ in the notations of the appendix (see (6.2)). Let $\varphi \in R_0$ (respectively $\theta \in R_0$) be the highest root (respectively highest short root) of R_0 with respect to the ordered basis Δ_0 of R_0. Then the ordered basis $\Delta = \Delta(D)$ of $R^\bullet$ is set to be

$$\Delta := (a_0, a_1, \ldots, a_n)$$

with $a_i := \alpha_i$ for $1 \leq i \leq n$ and

$$a_0 := \begin{cases} c - \varphi & \text{if} \quad \bullet = u,\\ \frac{|\theta|^2}{2}c - \theta & \text{if} \quad \bullet = t. \end{cases} \tag{2.4}$$

Remark 2.6. Suppose that (R', Δ') is a pair consisting of a reduced irreducible affine root system R' and an ordered basis Δ' of R'. If R' is similar to $R^\bullet$ then there exists a similarity transformation realizing $R' \simeq R^\bullet$ and mapping Δ' to $\Delta^\bullet$ as unordered sets.

The affine root system R is the following extension of $R^\bullet$. Define the subset $S = S(D)$ by

$$S := \{i \in \{0, \ldots, n\} \mid \big(\Lambda, a_i^\vee\big) = 2\mathbb{Z}\}. \tag{2.5}$$

Let $W^\bullet$ be the affine Weyl group of $R^\bullet$. Then we set

$$R = R(D) := R^\bullet \cup \bigcup_{i \in S} W^\bullet(2a_i), \tag{2.6}$$

which is an irreducible affine root system since $\mathbb{Z}R_0 \subseteq \Lambda$. Note that Δ is also an ordered basis of R.

Remark 2.7. Note that R is an irreducible affine root system of untwisted or twisted type, but never of mixed type (see the appendix for the terminology). But the irreducible affine root systems of mixed type are affine root subsystems of the affine root system of type $C^\vee C$, which is the nonreduced extension of the affine root system R^t with R_0 of type B (see Subsection 3.8 for a detailed description of the affine root system of type $C^\vee C$). Accordingly, special cases of the Koornwinder polynomials are naturally attached to affine root systems of mixed type, see Subsection 2.2 and Remark 3.28.

Remark 2.8. The nonreduced extension of the affine root system R^u with R_0 of type B_2 is not an affine root subsystem of the affine root system of type $C^\vee C_2$. It can actually be better viewed as the rank two case of the family R^u with R_0 of type C_n since, in the corresponding affine Dynkin diagram (see Subsection 6.4), the vertex labelled by the affine simple root a_0 is double bonded with the finite Dynkin diagram of R_0. The nonreduced extension of R^u with R_0 of type C_2 was missing in Macdonald's [56] classification list. It was added in [61, (1.3.17)], but the associated theory of Macdonald-Koornwinder polynomials was not developed. In the present setup it is a special case of the general theory. We will describe this particular case in detail in Subsection 3.9.

Finally we define the extended affine Weyl group $W = W(D)$ (please consult the appendix for the notations). Write $s_i := s_{a_i}$ $(0 \leq i \leq n)$ for the simple reflections of $W^\bullet$. Note that $s_i = s_{\alpha_i}$ $(1 \leq i \leq n)$ are the simple reflections of the finite Weyl group W_0 of R_0. Furthermore, $s_0 = \tau(\varphi^\vee)s_\varphi$ if $\bullet = u$ and $s_0 = \tau(\theta)s_\theta$ if $\bullet = t$, where $\tau(v)$ stands for the translation by v (see the appendix). Consequently

$$W^\bullet \simeq W_0 \ltimes \tau(\mathbb{Z}R_0^d).$$

We omit τ from the notations if no confusion can arise. The extended affine Weyl group $W = W(D)$ is now defined as

$$W := W_0 \ltimes \Lambda^d.$$

It contains the affine Weyl group $W^\bullet$ of $R^\bullet$ as normal subgroup, and $W/W^\bullet \simeq \Lambda^d/\mathbb{Z}R_0^d$.

The affine root system $R^\bullet \subset \widehat{Z}$ is W-stable since

$$\tau(\xi)(m\mu_\beta^\bullet c + \beta) = \big(m - \big(\xi, \beta^{d\vee}\big)\big)\mu_\beta^\bullet c + \beta, \qquad m \in \mathbb{Z},\, \beta \in R_0 \tag{2.7}$$

and $(\xi, \beta^{d\vee}) \in \mathbb{Z}$ for $\xi \in \Lambda^d$ and $\beta \in R_0$. Moreover, the affine root system R is W-invariant.

We now proceed by giving key examples of initial data. Recall that for a finite root system $R_0 \subset V$,

$$P(R_0) := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall\, \alpha \in R_0\}$$

is the weight lattice of R_0. If $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0 then we write $\varpi_i \in P(R_0)$ $(1 \leq i \leq n)$ for the corresponding fundamental weights, which are characterized by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$.

Example 2.9. **(i)** *Take an arbitrary finite reduced irreducible root system R_0 in $V = Z$ with ordered basis Δ_0. Choose $\bullet \in \{u, t\}$ and let Λ, Λ^d be lattices in V satisfying*

$$\mathbb{Z}R_0 \subseteq \Lambda \subseteq P(R_0), \qquad \mathbb{Z}R_0^d \subseteq \Lambda^d \subseteq P(R_0^d).$$

Then $(R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. Note that if $\Lambda = P(R_0)$ then $S = \emptyset$ hence $R = R^\bullet$ is reduced (Cherednik's [15] *theory corresponds to the special case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$).*
(ii) *Take $Z = \mathbb{R}^{n+1}$ with standard orthonormal basis $\{\epsilon_i\}_{i=1}^{n+1}$ and $R_0 = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq n+1}$ for the realization of the finite root system of type n in Z. Then $V = (\epsilon_1 + \cdots + \epsilon_{n+1})^\perp$. As ordered basis take*

$$\Delta_0 = (\alpha_1, \ldots, \alpha_n) = (\epsilon_1 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n+1}).$$

Then $\big(R_0, \Delta_0, u, \mathbb{Z}^{n+1}, \mathbb{Z}^{n+1}\big) \in \mathcal{D}$. Note that $\theta = \varphi = \epsilon_1 - \epsilon_{n+1}$, hence the simple affine root a_0 of R is $a_0 = c - \epsilon_1 + \epsilon_{n+1}$. This example is naturally related to the GL_{n+1} type Macdonald polynomials, see Subsection 3.7.

Given a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ we have the dual root system R_0^d with dual ordered basis $\Delta_0^d := (\alpha_1^d, \ldots, \alpha_n^d)$. This extends to an involution $D \mapsto D^d$ on $\mathcal{D}$ with

$$D^d := (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda) \tag{2.8}$$

for $D = (R_0, \Delta, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. We call D^d the initial data dual to D.

We write $\mu = \mu(D)$ and $\mu^d = \mu(D^d)$ for the function $\mu^\bullet$ on R_0 and R_0^d respectively. Let $\varpi_i^d \in P(R_0^d)$ $(1 \leq i \leq n)$ be the fundamental weights with respect to Δ_0^d.

For a given $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ we thus have a dual triple $(R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda) \in \mathcal{D}$ and hence an associated triple (R^d, Δ^d, W^d). Concretely, the highest root φ^d and the highest short root θ^d of R_0^d with respect to Δ_0^d are given by

$$\varphi^d = \begin{cases} \theta^\vee & \text{if } \bullet = u, \\ \varphi & \text{if } \bullet = t, \end{cases}$$

and

$$\theta^d = \begin{cases} \varphi^\vee & \text{if } \bullet = u, \\ \theta & \text{if } \bullet = t. \end{cases}$$

Hence

$$\Delta^d = (a_0^d, a_1^d, \ldots, a_n^d)$$

with $a_i^d = \alpha_i^d$ $(1 \leq i \leq n)$ and $a_0^d = \mu_\theta(c - \theta^\vee)$. The dual affine root system $R^d = R(D^d)$ is

$$R^d = R^{d\bullet} \cup \bigcup_{i \in S^d} W^{d\bullet}(2a_i^d)$$

with $S^d = \{i \in \{0, \ldots, n\} \,|\, (\Lambda^d, a_i^{d\vee}) = 2\mathbb{Z}\}$, with $R^{d\bullet} = \{m\mu_{\alpha^d} + \alpha^d\}_{m \in \mathbb{Z}, \alpha \in R_0}$ and with $W^{d\bullet} \simeq W_0 \ltimes \tau(\mathbb{Z}R_0)$ the affine Weyl group of $R^{d\bullet}$. The dual extended affine Weyl group is $W^d = W_0 \ltimes \Lambda$. The simple reflections $s_i^d := s_{a_i^d} \in W^{d\bullet}$ $(0 \leq i \leq n)$ are $s_i^d = s_i$ for $1 \leq i \leq n$ and $s_0^d = \tau(\theta)s_\theta$.

Example 2.10. *The correpondence $R \leftrightarrow R^d$ can turn nonreduced affine root systems into reduced ones. We give here an example of untwisted type. An example for twisted type will be given in Example 3.27.*

Take $n \geq 3$ and $R_0 \subseteq V = Z := \mathbb{R}^n$ of type B_n, realized as $R_0 = \{\pm\epsilon_i\} \cup \{\pm\epsilon \pm \epsilon_j\}_{i<j}$ (all sign combinations possible), with $\{\epsilon_i\}$ the standard orthonormal basis of V. Take as ordered basis of R_0,

$$\Delta_0 = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n).$$

The highest root is $\varphi = \epsilon_1 + \epsilon_2 \in R_0$. We then have

$$\mathbb{Z}R_0^\vee \subset P(R_0^\vee) = \mathbb{Z}^n = \mathbb{Z}R_0 \subset P(R_0)$$

with both sublattices $\mathbb{Z}R_0^\vee \subset \mathbb{Z}^n$ and $\mathbb{Z}^n \subset P(R_0)$ of index two. Taking $\Lambda = \mathbb{Z}^n = \Lambda^d$ we get the initial data $D = (R_0, \Delta_0, u, \mathbb{Z}^n, \mathbb{Z}^n) \in \mathcal{D}$. Then $S = S(D) = \{n\}$ and $R = R(D)$ is given by

$$R = \{\pm\epsilon_i + mc\}_{1\leq i\leq n, m\in\mathbb{Z}} \cup \{\pm\epsilon_i \pm \epsilon_j + mc\}_{1\leq i<j\leq n, m\in\mathbb{Z}} \cup \{\pm 2\epsilon_i + 2mc\}_{1\leq i\leq n, m\in\mathbb{Z}}.$$

It is nonreduced and of untwisted type B_n. We have written it here as the disjoint union of the three $W = W_0 \ltimes \mathbb{Z}^n$-orbits of R. Note that a_0 lies in the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}_{1\leq i<j\leq n, m\in\mathbb{Z}}$.

Dually, $R^d = R(D^d)$ is the reduced affine root system of untwisted type C_n. Concretely,

$$R^d = \{\pm\epsilon_i \pm \epsilon_j + mc\}_{1\leq i<j\leq n, m\in\mathbb{Z}} \cup \{\pm 2\epsilon_i + 2mc\}_{1\leq i\leq n, m\in\mathbb{Z}} \cup \{\pm 2\epsilon_i + (2m+1)c\}_{1\leq i\leq n, m\in\mathbb{Z}},$$

written here as the disjoint union of the three $W^d = W_0 \ltimes \mathbb{Z}^n$-orbits of R^d.

This example shows that basic features of the affine root system can alter under dualization. It turns out though that the number of orbits with respect to the action of the extended affine Weyl group is unaltered. To establish this fact it is convenient to use the concept of a multiplicity function on R.

Definition 2.11. *Set $\mathcal{M} = \mathcal{M}(D)$ for the complex algebraic group of W-invariant functions $\kappa : R \to \mathbb{C}^*$, and $\nu = \nu(D)$ for the complex dimension of $\mathcal{M}$.*

Note that $\nu = \nu(D)$ equals the number of W-orbits of R. The value of $\kappa \in \mathcal{M}$ at an affine root $a \in R$ is denoted by κ_a. We call $\kappa \in \mathcal{M}$ a multiplicity function. We write $\kappa^\bullet := \kappa|_{R^\bullet}$ for its restriction to $R^\bullet$. For a multiplicity function $\kappa \in \mathcal{M}$ we set $\kappa_{2a} := \kappa_a$ if $a \in R$ is unmultiplyable (i.e. $2a \notin R$).

First we need a more precise description of the sets $S = S(D)$ and $S^d = S(D^d)$, see (2.5). It is obtained using the classification of affine root systems (see Subsection 6.2 and [56]).

Lemma 2.12. *Let $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$. Set $S_0 = S \cap \{1, \ldots, n\}$.*

- **(a)** *If $\bullet = u$ then $\#S \leq 2$. If $\#S = 2$ then R_0 is of type A_1. If $\#S = 1$ then R_0 is of type B_n $(n \geq 2)$ and $S = S_0 = \{j\}$ with $\alpha_j \in \Delta_0$ the unique simple short root.*

(b) *If $\bullet = t$ then $\#S = 0$ or $\#S = 2$. If $\#S = 2$ then R_0 is either of type A_1 or of type B_n ($n \geq 2$) and $S = \{0, j\}$ with $\alpha_j \in \Delta_0$ the unique short simple root.*

Note that in both the untwisted and the twisted case, $\alpha_j^d \in W_0(D(a_0^d))$ if $S_0 = \{j\}$.

The following lemma should be compared with [32, §5.7].

Lemma 2.13. *Let $D \in \mathcal{D}$ and $\kappa \in \mathcal{M}(D)$. Let $\alpha_j \in \Delta_0$ (respectively $\alpha_{j^d}^d \in \Delta_0^d$) be a simple short root. The assignments*

$$\begin{aligned} \kappa^d_{a_0^d} &:= \kappa_{2\alpha_j}, \\ \kappa^d_{\alpha_i^d} &:= \kappa_{\alpha_i} \qquad \text{if } \ i \in \{1, \ldots, n\}, \\ \kappa^d_{2a_0^d} &:= \kappa_{2a_0} \qquad \text{if } \ 0 \in S^d, \\ \kappa^d_{2\alpha^d_{j^d}} &:= \kappa_{a_0} \qquad \text{if } \ j^d \in S^d \end{aligned}$$

uniquely extend to a multiplicity function $\kappa^d \in \mathcal{M}^d := \mathcal{M}(D^d)$. For fixed $D \in \mathcal{D}$ the map $\kappa \mapsto \kappa^d$ defines an isomorphism $\phi_D : \mathcal{M}(D) \xrightarrow{\sim} \mathcal{M}(D^d)$ of complex tori, with inverse ϕ_{D^d}.

Remark 2.14. Let $\kappa \in \mathcal{M}(D)$. Recall the convention that $\kappa_{2a} = \kappa_a$ for $a \in R$ such that $2a \notin R$. Then for all $\alpha \in R_0$,

$$\kappa_{2\alpha} = \kappa^d_{\mu_{\alpha^d} c + \alpha^d}.$$

Corollary 2.15. *Let $D \in \mathcal{D}$. The number ν of W-orbits of R is equal to the number ν^d of W^d-orbits of R^d.*

Remark 2.16. Returning to Example 2.10, the correspondence from Lemma 2.13 links the orbit $\{\pm\epsilon_i + mc\}$ of R to the orbit $\{2\epsilon_i + 2mc\}$ of R^d, the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}$ of R to the orbit $\{\pm\epsilon_i \pm \epsilon_j + mc\}$ of R^d and the orbit $\{\pm 2\epsilon_i + 2mc\}$ of R to the orbit $\{\pm 2\epsilon_i + (2m+1)c\}$ of R^d.

2.4. **The basic representation.** We fix throughout this subsection a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ of initial data. Recall that it gives rise to a triple (R, Δ, W) of an irreducible affine root system R containing $R^\bullet$, an ordered basis Δ of R as well as of $R^\bullet$, and an extended affine Weyl group $W = W_0 \ltimes \Lambda^d$. In addition we fix a multiplicity function $\kappa \in \mathcal{M}(D)$ and we write $\kappa^\bullet := \kappa|_{R^\bullet}$. It is a W-equivariant map $R^\bullet \to \mathbb{C}^*$. Write $\kappa_i := \kappa^\bullet_{a_i}$ for $0 \leq i \leq n$. Note that $\kappa_i = \kappa_j$ if s_i is conjugate to s_j in $W = \Omega \ltimes W^\bullet$.

We write $R^\pm$ and $R^{\bullet\pm}$ for the positive respectively negative affine roots of R and $R^\bullet$ with respect to Δ. Since the affine root system $R^\bullet$ is W-stable, we can define the length $l(w) = l_D(w)$ of $w \in W$ by

$$l(w) := \#\big(R^{\bullet+} \cap w^{-1} R^{\bullet-}\big).$$

If $w \in W^\bullet = W(R^\bullet)$ then $l(w)$ equals the number of simple reflections s_i ($0 \leq i \leq n$) in a reduced expression of w. We have $W = \Omega \ltimes W^\bullet$ with $\Omega = \Omega(D)$ the subgroup

$$\Omega := \{w \in W \mid l(w) = 0\}$$

of W. Then $\Omega \simeq \Lambda^d / \mathbb{Z}R_0^d$. The abelian group Ω permutes the simple affine roots a_i ($0 \leq i \leq n$), which thus gives rise to an action of Ω on the index set $\{0, \ldots, n\}$. Consequently the action of Ω on $W^\bullet$ by conjugation permutes the set $\{s_i\}_{i=0}^n$ of simple reflections, $ws_iw^{-1} = s_{w(i)}$ for $w \in \Omega$ and $0 \leq i \leq n$ (cf., e.g., [61, §2.5]). A detailed description of the group Ω in terms of a complete set of representatives of $\Lambda^d/\mathbb{Z}R_0^d$ will be given in Subsection 3.4.

Extended versions of the affine braid group and of the affine Hecke algebra are defined as follows. Let $A = A(D) := A(R^\bullet, \Delta)$ be the affine Cartan matrix associated to $(R^\bullet, \Delta)$. Recall that the affine braid group $\mathcal{B}^\bullet := \mathcal{B}(A)$ is isomorphic to the abstract group generated by T_w ($w \in W^\bullet$) with defining relations $T_vT_w = T_{vw}$ for all $v, w \in W^\bullet$ satisfying $l(vw) = l(v) + l(w)$.

Definition 2.17. **(i)** *The extended affine braid group $\mathcal{B} = \mathcal{B}(D)$ is the group generated by T_w ($w \in W$) with defining relations $T_vT_w = T_{vw}$ for all $v, w \in W$ satisfying $l(vw) = l(v) + l(w)$.*
(ii) *The extended affine Hecke algebra $H(\kappa^\bullet) = H(D, \kappa^\bullet)$ is the quotient of $\mathbb{C}[\mathcal{B}]$ by the two-sided ideal generated by $(T_i - \kappa_i)(T_i + \kappa_i^{-1})$ ($0 \leq i \leq n$).*

Similarly to the semidirect product decomposition $W \simeq \Omega \ltimes W^\bullet$ we have $\mathcal{B} \simeq \Omega \ltimes \mathcal{B}^\bullet$ and

$$H(\kappa^\bullet) \simeq \Omega \ltimes H(W^\bullet, \kappa^\bullet),$$

where the action of Ω on $\mathcal{B}$ by group automorphisms (respectively on $H(W^\bullet, \kappa^\bullet)$ by algebra automorphisms) is determined by $w \cdot T_i = T_{w(i)}$ for $w \in \Omega$ and $0 \leq i \leq n$. For $\omega \in \Omega$ we will denote the element T_ω in the extended affine Hecke algebra $H(\kappa^\bullet)$ simply by ω.

The algebra homomorphism

$$\beta_{(R,\Delta,\kappa)} : H(W^\bullet, \kappa^\bullet) \hookrightarrow W^\bullet \ltimes F \subseteq W \ltimes F$$

from Subsection 2.2 extends to an injective algebra map

$$\beta_{D,\kappa} : H(\kappa^\bullet) \hookrightarrow W \ltimes F$$

by $\beta_{D,\kappa}(T_\omega) = \omega$ ($\omega \in \Omega$). We will now show that it gives rise to an action of $H(\kappa^\bullet)$ as q-difference reflection operators on a complex torus T_Λ. It is called the basic representation of $H(\kappa^\bullet)$. It is fundamental for the development of the Cherednik-Macdonald theory.

The complex torus T_Λ (of rank $\dim_{\mathbb{R}}(Z)$) is the algebraic group of complex characters of the lattice Λ,

$$T_\Lambda := \mathrm{Hom}_{\mathbb{Z}}\big(\Lambda, \mathbb{C}^*\big).$$

The algebra $\mathbb{C}[T_\Lambda]$ of regular functions on T_Λ is isomorphic to the group algebra $\mathbb{C}[\Lambda]$, where the standard basis element e^λ ($\lambda \in \Lambda$) of $\mathbb{C}[\Lambda]$ is viewed as the regular function $t \mapsto t(\lambda)$ on T_Λ. We write t^λ for the value of e^λ at $t \in T_\Lambda$. Since Λ is W_0-stable, W_0 acts on T_Λ, giving in turn rise to an action of W_0 on $\mathbb{C}[T_\Lambda]$ by algebra automorphisms. Then $w(e^\lambda) = e^{w\lambda}$ ($w \in W_0$ and $\lambda \in \Lambda$). We now first extend it to an action of the extended affine Weyl group W on $\mathbb{C}[T_\Lambda]$ depending on a fixed parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$.

Set

$$q_\alpha := q^{\mu_\alpha}, \qquad \alpha \in R_0$$

and define $q^{\xi} \in T_{\Lambda}$ ($\xi \in \Lambda^d$) to be the character $\lambda \mapsto q^{(\lambda,\xi)}$ of Λ. The action of W_0 on T_Λ extends to a left W-action $(w,t) \mapsto w_q t$ on T_Λ by

$$\tau(\xi)_q t := q^{\xi} t, \qquad \xi \in \Lambda^d, \ t \in T_\Lambda.$$

Then $(w_q p)(t) := p(w_q^{-1} t)$ ($w \in W$, $p \in \mathbb{C}[T_\Lambda]$) is a W-action by algebra automorphisms on $\mathbb{C}[T_\Lambda]$. In particular,

$$\tau(\xi)_q (e^{\lambda}) = q^{-(\lambda,\xi)} e^{\lambda}, \qquad \xi \in \Lambda^d, \ \lambda \in \Lambda.$$

It extends to a W-action by field automorphisms on the quotient field $\mathbb{C}(T_\Lambda)$ of $\mathbb{C}[T_\Lambda]$. It is useful to introduce the notation

$$t_q^{rc+\lambda} := q^r t^{\lambda}, \qquad r \in \mathbb{R}, \, \lambda \in \Lambda$$

for $t \in T_\Lambda$. Then $(w_q^{-1} t)_q^{rc+\lambda} = t_q^{w(rc+\lambda)}$ for $w \in W$, $t \in T_\Lambda$, $r \in \mathbb{R}$ and $\lambda \in \Lambda$.

We write $W \ltimes_q \mathbb{C}(T_\Lambda)$ for the resulting semidirect product algebra. It canonically acts on $\mathbb{C}(T_\Lambda)$ by q-difference reflection operators. We thus have a sequence of algebra maps

$$H(\kappa^{\bullet}) \to W \ltimes F \to W \ltimes_q \mathbb{C}(T_\Lambda) \to \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T_\Lambda)\big),$$

where the first map is $\beta_{D,\kappa}$ and the second map sends $e^{\mu_\alpha mc+\alpha}$ to $q_\alpha^m e^{\alpha}$ for $m\mu_\alpha c + \alpha \in R^{\bullet}$. It gives the following result, which is closely related to [32, 5.13] in the present generality.

Theorem 2.18. *Let $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$ and $\kappa \in \mathcal{M}(D)$. For $a \in R$ set $\kappa_{2a} := \kappa_a$ if $2a \notin R$. There exists a unique algebra monomorphism*

$$\pi_{\kappa,q} = \pi_{D;\kappa,q} : H(D, \kappa^{\bullet}) \hookrightarrow \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_\Lambda]\big)$$

satisfying

$$\begin{aligned}
\big(\pi_{\kappa,q}(T_i)p\big)(t) &= \kappa_{a_i}(s_{i,q}p)(t) + \left(\frac{\kappa_{a_i} - \kappa_{a_i}^{-1} + (\kappa_{2a_i} - \kappa_{2a_i}^{-1}) t_q^{a_i}}{1 - t_q^{2a_i}}\right)(p(t) - (s_{i,q}p)(t)),\\
\big(\pi_{\kappa,q}(\omega)p\big)(t) &= (\omega_q p)(t)
\end{aligned}$$

for $0 \le i \le n$, $\omega \in \Omega$, $p \in \mathbb{C}[T_\Lambda]$ and $t \in T_\Lambda$.

If $2a_i \notin R$ then, by the convention $\kappa_{2a_i} = \kappa_{a_i}$, the first formula reduces to

$$\big(\pi_{\kappa,q}(T_i)p)\big)(t) = \kappa_{a_i}(s_{i,q}p)(t) + \Big(\frac{\kappa_{a_i} - \kappa_{a_i}^{-1}}{1 - t_q^{a_i}}\Big)(p(t) - (s_{i,q}p)(t)).$$

The theorem is due to Cherednik (see [15] and references therein) in the GL_{n+1} case (see Example 2.9**(ii)**) and when $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ with R_0 an arbitrary reduced irreducible root system. The theorem is due to Noumi [63] for $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \ge 2$). This case is special due to its large degree of freedom ($\nu(D) = 4$ if $n = 1$ and $\nu(D) = 5$ if $n \ge 2$). We will describe this case in detail in Subsection 3.8.

Remark 2.19. **(i)** From Theorem 2.3 one first obtains $\pi_{\kappa,q}$ as an algebra map from $H(\kappa^\bullet)$ to $\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(T_\Lambda)\big)$. The image is contained in the subalgebra of endomorphisms preserving $\mathbb{C}[T_\Lambda]$ since, for $\lambda \in \Lambda$, we have $(\lambda, a_i^\vee) \in \mathbb{Z}$ if $2a_i \notin R$ and $(\lambda, a_i^\vee) \in 2\mathbb{Z}$ if $2a_i \in R$.
(ii) Let $s = s(D)$ be the number of W-orbits of $R \setminus R^\bullet$. Extending a W-equivariant map $\kappa^\bullet : R^\bullet \to \mathbb{C}^*$ to a multiplicity function $\kappa \in \mathcal{M}$ on R amounts to choosing s nonzero complex parameters. Hence, the maps $\pi_{\kappa,q}$ define a family of algebra monomorphisms of the extended affine Hecke algebra $H(\kappa^\bullet)$ into $\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_\Lambda]\big)$, parametrized by $s+1$ parameters $\kappa|_{R\setminus R^\bullet}$ and q.

3. Monic Macdonald-Koornwinder polynomials

In this section we introduce the monic nonsymmetric and symmetric Macdonald-Koornwinder polynomials. The terminology Macdonald polynomials is employed in the literature for the cases that R is reduced (i.e. the cases $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ and the GL_{n+1} case). The Koornwinder polynomials correspond to the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 of of type B_n $(n \geq 2)$, in which case $R = R(D)$ is nonreduced and of type $C^\vee C_n$. To have uniform terminology we will speak of Macdonald-Koornwinder polynomials when discussing the theory for arbitrary initial data.

The monic nonsymmetric Macdonald-Koornwinder polynomials will be introduced as the common eigenfunctions in $\mathbb{C}[T_\Lambda]$ of a family of commuting q-difference reflection operators. The operators are obtained as images under the basic representation $\pi_{\kappa,q}$ of elements from a large commutative subalgebra of the extended affine Hecke algebra $H(\kappa^\bullet)$ constructed by Bernstein and Zelevinsky. A Hecke algebra symmetrizer turns the monic nonsymmetric Macdonald-Koornwinder polynomials into monic symmetric Macdonald-Koornwinder polynomials, which are W_0-invariant regular functions on T_Λ solving a suitable spectral problem of a commuting family of q-difference operators, called Macdonald operators. We in addition determine in this section the (bi)orthogonality relations of the polynomials.

Throughout this section we fix

1. a quintuple $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ of initial data,
2. a deformation parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$,
3. a multiplicity function $\kappa \in \mathcal{M}(D)$

and we freely use the resulting notations from the previous section.

3.1. **Bernstein-Zelevinsky presentation.** An expression $w = \omega s_{i_1} s_{i_2} \cdots s_{i_r} \in W$ with $\omega \in \Omega$ and $0 \leq i_j \leq n$ is called a reduced expression if $r = l(w)$.

For a given reduced expression $w = \omega s_{i_1} s_{i_2} \cdots s_{i_r}$,

$$T_w := \omega T_{i_1} T_{i_2} \cdots T_{i_r} \in H(\kappa^\bullet)$$

is well defined, and $\{T_w\}_{w \in W}$ is a complex linear basis of $H(\kappa^\bullet)$.

The cones

$$\Lambda^{d\pm} := \{\xi \in \Lambda^d \mid \big(\xi, \alpha^{d\vee}\big) \geq 0 \quad \forall\, \alpha \in R_0^\pm\}$$

form fundamental domains for the W_0-action on Λ^d. Any $\xi \in \Lambda^d$ can be written as $\xi = \mu - \nu$ with $\mu, \nu \in \Lambda^{d+}$ (and similarly for Λ^{d-}). Furthermore, if $\xi, \xi' \in \Lambda^{d+}$ then

$l(\tau(\xi+\xi')) = l(\tau(\xi)) + l(\tau(\xi'))$. It follows that there exists a unique group homomorphism $\Lambda^d \to \mathcal{B}$, denoted by $\xi \mapsto Y^\xi$, such that

$$Y^\xi = T_{\tau(\xi)} \qquad \forall\, \xi \in \Lambda^{d+}.$$

On the level of the extended affine Hecke algebra it gives rise to an algebra homomorphism

$$\mathbb{C}[T_{\Lambda^d}] \to H(\kappa^\bullet) \tag{3.1}$$

which we denote by $p \mapsto p(Y)$, where T_{Λ^d} is the complex torus $\mathrm{Hom}(\Lambda^d, \mathbb{C}^*)$, and $p(Y) = \sum_\xi c_\xi Y^\xi$ if $p(t) = \sum_\xi c_\xi t^\xi$. The image of the map (3.1) is denoted by $\mathbb{C}_Y[T_{\Lambda^d}]$.

As in Subsection 2.4, the lattice Λ^d is W_0-stable, giving rise to a W_0-action on T_{Λ^d}, hence a W_0-action on $\mathbb{C}[T_{\Lambda^d}]$ by algebra automorphisms.

Denote by H_0 the subalgebra of $H(\kappa^\bullet)$ generated by T_i ($1 \leq i \leq n$). We have a natural surjective algebra map $H(W_0, \kappa|_{R_0}) \to H_0$, sending the algebraic generator T_i of $H(W_0, \kappa|_{R_0})$ to $T_i \in H_0$.

The analog of the semidirect product decomposition $W = W_0 \ltimes \Lambda^d$ for the extended affine Hecke algebra $H(\kappa^\bullet)$ is the following Bernstein-Zelevinsky presentation of $H(\kappa^\bullet)$ (see [54]).

Theorem 3.1. (1) *The algebra maps $\mathbb{C}[T_{\Lambda^d}] \to \mathbb{C}_Y[T_{\Lambda^d}]$ and $H(W_0, \kappa|_{R_0}) \to H_0$ are isomorphisms,*

(2) *multiplication defines a linear isomorphism $H_0 \otimes \mathbb{C}_Y[T_{\Lambda^d}] \simeq H(\kappa^\bullet)$,*

(3) *for $i \in \{1, \ldots, n\}$ such that $\left(\Lambda^d, \alpha_i^{d\vee}\right) = \mathbb{Z}$ we have*

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa_i - \kappa_i^{-1}}{1 - Y^{-\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)) \tag{3.2}$$

in $H(\kappa^\bullet)$ for all $p \in \mathbb{C}[T_{\Lambda^d}]$,

(4) *for $i \in \{1, \ldots, n\}$ such that $\left(\Lambda^d, \alpha_i^{d\vee}\right) = 2\mathbb{Z}$ we have*

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa_i - \kappa_i^{-1} + (\kappa_0 - \kappa_0^{-1})Y^{-\alpha_i^d}}{1 - Y^{-2\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)) \tag{3.3}$$

in $H(\kappa^\bullet)$ for all $p \in \mathbb{C}[T_{\Lambda^d}]$.

These properties characterize $H(\kappa^\bullet)$ as a unital complex associative algebra.

With the notion of the dual multiplicity parameter κ^d (see Lemma 2.13), the cross relations (3.2) and (3.3) in the affine Hecke algebra $H(\kappa^\bullet)$ can be uniformly written as

$$p(Y)T_i - T_i(s_ip)(Y) = \left(\frac{\kappa^d_{\alpha_i^d} - (\kappa^d_{\alpha_i^d})^{-1} + (\kappa^d_{2\alpha_i^d} - (\kappa^d_{2\alpha_i^d})^{-1})Y^{-\alpha_i^d}}{1 - Y^{-2\alpha_i^d}}\right)(p(Y) - (s_ip)(Y)) \tag{3.4}$$

for $p \in \mathbb{C}[T_{\Lambda^d}]$ and $1 \leq i \leq n$.

It follows from the theorem that the center $Z(H(\kappa^\bullet))$ of the extended affine Hecke algebra $H(\kappa^\bullet)$ equals $\mathbb{C}_Y[T_{\Lambda^d}]^{W_0}$.

3.2. **Monic nonsymmetric Macdonald-Koornwinder polynomials.** The results in this section are from [59, 12, 75, 32]. For detailed proofs see, e.g., [61, §2.8, §4.6, §5.2].

We put in this subsection the following conditions on q and $\kappa \in \mathcal{M}(D)$,

$$0 < q < 1, \qquad 0 < \kappa_a < 1 \;\; \forall a \in R, \tag{3.5}$$

or

$$q > 1, \qquad \kappa_a > 1 \;\; \forall a \in R. \tag{3.6}$$

Set

$$\eta(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0 \end{cases} \tag{3.7}$$

and define a W_0-equivariant map $\upsilon : R_0^d \to \mathbb{R}_{>0}$ (depending on $\kappa^\bullet$) by

$$\upsilon_{\alpha^d} := \kappa_\alpha^{\frac{1}{2}} \kappa_{\mu_\alpha c+\alpha}^{\frac{1}{2}}, \qquad \alpha \in R_0. \tag{3.8}$$

Definition 3.2. *For $\lambda \in \Lambda$ define $\gamma_{\lambda,q} = \gamma_{\lambda,q}(D;\kappa^\bullet) \in T_{\Lambda^d}$ by*

$$\gamma_{\lambda,q} := q^\lambda \prod_{\alpha \in R_0^+} \upsilon_{\alpha^d}^{\eta((\lambda,\alpha^\vee))\alpha^{d\vee}}.$$

In other words, for all $\xi \in \Lambda^d$,

$$\gamma_{\lambda,q}^{\xi} = q^{(\lambda,\xi)} \prod_{\alpha \in R_0^+} \upsilon_{\alpha^d}^{\eta((\lambda,\alpha^\vee))(\xi,\alpha^{d\vee})}.$$

As a special case,

$$\gamma_{\lambda,q} = q^\lambda \prod_{\alpha \in R_0^+} \upsilon_{\alpha^d}^{-\alpha^{d\vee}} \qquad \forall\, \lambda \in \Lambda^-,$$

where $\Lambda^\pm = \{\lambda \in \Lambda \mid \big(\lambda, \alpha^\vee\big) \geq 0 \quad \forall\, \alpha \in R_0^\pm\}$.

Write $l^d = l_{D^d}$ for the length function on the dual extended affine Weyl group W^d and $\Omega^d = \Omega(D^d)$ for the subgroup of elements of W^d of length zero with respect to l^d. We have a q-dependent W^d-action on T_{Λ^d} extending the W_0-action by $\tau(\lambda)_q\gamma = q^\lambda\gamma$ for all $\lambda \in \Lambda$ and $\gamma \in T_{\Lambda^d}$. Then $\gamma_{\lambda,q} = \tau(\lambda)_q\gamma_{0,q}$ in T_{Λ^d} if $\lambda \in \Lambda^-$. This generalizes as follows.

Lemma 3.3. *We have for $\lambda \in \Lambda$,*

$$\gamma_{\lambda,q} = u^d(\lambda)_q \gamma_{0,q}$$

in T_{Λ^d}, where $u^d(\lambda) \in W^d$ is the element of minimal length with respect to l^d in the coset $\tau(\lambda)W_0$.

The conditions (3.5) or (3.6) on the parameters, together with Lemma 3.3, imply

Lemma 3.4. *The map $\Lambda \to T_{\Lambda^d}$, defined by $\lambda \mapsto \gamma_{\lambda,q}$, is injective.*

For later purposes it is convenient to record the following compatibility between the q-dependent W^d-action on $\gamma_{\lambda,q} \in T_{\Lambda^d}$ and the W^d-action $(w\tau(\lambda), \lambda') \mapsto w(\lambda + \lambda')$ on Λ ($w \in W_0$ and $\lambda, \lambda' \in \Lambda$).

Proposition 3.5. *Let $\lambda \in \Lambda$. Then*

(a) *If $\omega \in \Omega^d$ then $\omega_q \gamma_{\lambda,q} = \gamma_{\omega\lambda,q}$.*
(b) *If $0 \leq i \leq n$ and $s_i^d \lambda \neq \lambda$ then $s_{i,q}^d \gamma_{\lambda,q} = \gamma_{s_i^d \lambda,q}$.*
(c) *If $0 \leq i \leq n$ and $s_i^d \lambda = \lambda$ then $s_{i,q}^d \gamma_{\lambda,q} = \gamma_{\lambda,q} \upsilon_{D(a_i^d)}^{2D(a_i^d)^\vee}$.*

Warning: $D(a_i^d) = (Da_i)^d$ holds true for $1 \leq i \leq n$ and for $i = 0$ if $\bullet = t$ (then both sides equal $-\theta$). It is not correct when $i = 0$, $\bullet = u$ and R_0 has two root lengths, since then $(Da_0)^d = -\varphi^\vee$ and $D(a_0^d) = -\theta^\vee$.

For $\lambda, \mu \in \Lambda^+$ we write $\lambda \leq \mu$ if $\mu - \lambda$ can be written as a sum of positive roots $\alpha \in R_0^+$. We also write $\leq$ for the Bruhat order of W_0 with respect to the Coxeter generators s_i $(1 \leq i \leq n)$. For $\lambda \in \Lambda$ let $\lambda_\pm$ be the unique element in $\Lambda^\pm \cap W_0\lambda$ and write $v(\lambda) \in W_0$ for the element of shortest length such that $v(\lambda)\lambda = \lambda_-$. Then $\tau(\lambda) = u^d(\lambda)v(\lambda)$ in W^d for $\lambda \in \Lambda$.

Definition 3.6. *Let $\lambda, \mu \in \Lambda$. We write $\lambda \preceq \mu$ if $\lambda_+ < \mu_+$ or if $\lambda_+ = \mu_+$ and $v(\lambda) \geq v(\mu)$.*

Note that $\preceq$ is a partial order on Λ. Furthermore, if $\lambda \in \Lambda^-$ then $\mu \preceq \lambda$ for all $\mu \in W_0\lambda$. For each $\lambda \in \Lambda$ the set of elements $\mu \in \Lambda$ satisfying $\mu \preceq \lambda$ thus is contained in the finite set

$$\{\mu \in \Lambda \,|\, \mu \preceq \lambda_-\} = \bigcup_{\mu_+ \in \Lambda^+ : \mu_+ \leq \lambda_+} W_0 \mu_+,$$

which is the smallest saturated subset $\mathrm{Sat}(\lambda_+)$ of Λ containing λ_+ (a subset $X \subseteq \Lambda$ is saturated if for each $\alpha \in R_0^+$ and $\lambda \in \Lambda$ we have $\lambda - r\alpha \in X$ for all integers r between zero and $(\lambda, \alpha^\vee)$, including both zero and $(\lambda, \alpha^\vee)$).

We write $p = d_\lambda e^\lambda +$ l.o.t. for an element $p = \sum_{\mu \in \Lambda} d_\mu e^\mu \in \mathbb{C}[T_\Lambda]$ satisfying $d_\mu = 0$ if $\mu \not\preceq \lambda$. If in addition $d_\lambda \neq 0$ then we call p of degree λ.

Proposition 3.7. *For $r \in \mathbb{C}[T_{\Lambda^d}]$ and $\lambda \in \Lambda$ we have*

$$\pi_{\kappa,q}(r(Y))e^\lambda = r(\gamma_{\lambda,q}^{-1})e^\lambda + \text{ l.o.t.}$$

in $\mathbb{C}[T_\Lambda]$.

Corollary 3.8. *For each $\lambda \in \Lambda$ there exists a unique $P_\lambda = P_\lambda(D; \kappa, q) \in \mathbb{C}[T_\Lambda]$ satisfying*

$$\pi_{\kappa,q}(r(Y))P_\lambda = r(\gamma_{\lambda,q}^{-1})P_\lambda, \qquad \forall\, r \in \mathbb{C}[T_{\Lambda^d}]$$

and satisfying $P_\lambda = e^\lambda + \mathrm{l.o.t.}$

Definition 3.9. *$P_\lambda = P_\lambda(D; \kappa, q) \in \mathbb{C}[T_\Lambda]$ is called the monic nonsymmetric Macdonald-Koornwinder polynomial of degree $\lambda \in \Lambda$.*

For $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ the definition of the nonsymmetric Macdonald-Koornwinder polynomial is due to Macdonald [59] in the untwisted case ($\bullet = u$) and due to Cherednik [12] in the general case. For $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$) the nonsymmetric Macdonald-Koornwinder polynomials are Sahi's [75] nonsymmetric Koornwinder polynomials. In the present generality (with more flexible choices of lattices Λ and Λ^d) the definition is close to Haiman's [32, §6] definition. The same references apply for the biorthogonality relations of the nonsymmetric Macdonald-Koornwinder polynomials discussed in the next subsection.

The GL_{n+1} nonsymmetric Macdonald polynomials (corresponding to Example 2.9**(ii)**) are often studied separately, see, e.g., [46, 29].

3.3. **Biorthogonality.** We assume in this subsection that $\kappa \in \mathcal{M}$ and q satisfy

$$\begin{aligned} &0 < q < 1, \\ &0 < \kappa_a < 1 \qquad \forall\, a \in R, \\ &0 < \kappa_a \kappa_{2a}^{\pm 1} \leq 1 \qquad \forall\, a \in R^\bullet \end{aligned} \tag{3.9}$$

and write for $\lambda \in \Lambda$,

$$P_\lambda := P_\lambda(D, \kappa, q), \qquad P_\lambda^\circ := P_\lambda(D, \kappa^{-1}, q^{-1}),$$

where $\kappa^{-1} \in \mathcal{M}(D)$ is the multiplicity function $a \mapsto \kappa_a^{-1}$.

For $a \in R^\bullet$ define $c_a = c_a^{\kappa,q}(\cdot; D) \in \mathbb{C}(T_\Lambda)$ by

$$c_a(t) = \frac{(1 - \kappa_a \kappa_{2a} t_q^a)(1 + \kappa_a \kappa_{2a}^{-1} t_q^a)}{(1 - t_q^{2a})}. \tag{3.10}$$

Then for $w \in W$ and $a \in R^\bullet$,

$$c_a(w_q^{-1} t) = c_{wa}(t).$$

In addition,

$$\pi_{\kappa,q}(T_i) = \kappa_i + \kappa_i^{-1} c_{a_i}(s_{i,q} - 1)$$

for $0 \leq i \leq n$.

Since $0 < q < 1$, the infinite product

$$v := \prod_{a \in R^{\bullet +}} \frac{1}{c_a}$$

defines a meromorphic function on T_Λ. In terms of products

$$\big(x_1, \ldots, x_m; q\big)_r := \prod_{i=1}^{m} \prod_{j=0}^{r-1} (1 - q^j x_i), \qquad r \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \tag{3.11}$$

of q-shifted factorials it becomes

$$\begin{aligned} v(t) = &\prod_{\alpha \in R_0^+} \frac{1 - t^{2\alpha}}{(1 - \kappa_\alpha \kappa_{2\alpha} t^\alpha)(1 + \kappa_\alpha \kappa_{2\alpha}^{-1} t^\alpha)} \\ &\times \prod_{\beta \in R_0} \frac{\big(q_\beta^2 t^{2\beta}; q_\beta^2\big)_\infty}{\big(q_\beta^2 \kappa_\beta \kappa_{2\beta} t^\beta, -q_\beta^2 \kappa_\beta \kappa_{2\beta}^{-1} t^\beta, q_\beta \kappa_{\mu_\beta c + \beta} \kappa_{2\mu_\beta c + 2\beta} t^\beta, -q_\beta \kappa_{\mu_\beta c + \beta} \kappa_{2\mu_\beta c + 2\beta}^{-1} t^\beta; q_\beta^2\big)_\infty}. \end{aligned}$$

Remark 3.10. If $\beta \in R_0$ satisfies $\big(\Lambda^d, \beta^{d\vee}\big) = \mathbb{Z}$ then $\kappa_{\mu_\beta c+\beta} = \kappa_\beta$ and $\kappa_{2\mu_\beta c+2\beta} = \kappa_{2\beta}$. In that case the β-factor in the second line of the above formula of v simplifies to

$$\frac{\big(q_\beta^2 t^{2\beta}; q_\beta^2\big)_\infty}{\big(q_\beta \kappa_\beta \kappa_{2\beta} t^\beta, -q_\beta \kappa_\beta \kappa_{2\beta}^{-1} t^\beta; q_\beta\big)_\infty}.$$

If in addition $\kappa_{2\beta} = \kappa_\beta$ (for instance, if $2\beta \notin R$) then the β-factor simplifies further to

$$\frac{\big(q_\beta t^\beta; q_\beta\big)_\infty}{\big(q_\beta \kappa_\beta^2 t^\beta; q_\beta\big)_\infty}.$$

By the conditions (3.9) on the parameters, v is a continuous function on the compact torus $T_\Lambda^u = \mathrm{Hom}(\Lambda, S^1) \subset T_\Lambda$, where $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Write dt for the normalized Haar measure on T_Λ^u.

Definition 3.11. *Define a sesquilinear form* $\langle \cdot, \cdot \rangle : \mathbb{C}[T_\Lambda] \times \mathbb{C}[T_\Lambda] \to \mathbb{C}$ *by*

$$\langle p, r \rangle := \int_{T_\Lambda^u} p(t)\overline{r(t)}v(t)dt.$$

Proposition 3.12. *Let* $p, r \in \mathbb{C}[T_\Lambda]$ *and* $w \in W$. *Then* $\langle \pi_{\kappa,q}(T_w)p, r\rangle = \langle p, \pi_{\kappa^{-1},q^{-1}}(T_w^{-1})r\rangle$.

The biorthogonality of the nonsymmetric Macdonald-Koornwinder polynomials readily follows from Proposition 3.12.

Theorem 3.13. *If* $\lambda, \mu \in \Lambda$ *and* $\lambda \neq \mu$ *then* $\langle P_\lambda, P_\mu^\circ \rangle = 0$.

3.4. **Macdonald operators.** Special cases of what now are known as the Macdonald q-difference operators were explicitly written down by Macdonald [60] when introducing the symmetric Macdonald polynomials. Earlier Ruijsenaars [73] had introduced these commuting q-difference operators for R_0 of type A as the quantum Hamiltonian of a relativistic version of the quantum trigonometric Calogero-Moser system (see Subsection 3.7). Koornwinder [51] introduced a multivariable extension of the second-order Askey-Wilson q-difference operator to define the symmetric Koornwinder polynomials. This case corresponds to $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n with $n \geq 2$ (see Subsection 3.8).

The construction of the whole family of Macdonald q-difference operators using affine Hecke algebras is due to Cherednik [9] in case $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ and due to Noumi [63] in case $D = (R_0, \Delta_0, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$). We explain this construction here, see [61, §4.4] for a treatment close to the present one.

In this subsection we assume that $\kappa \in \mathcal{M}$ and q satisfy (3.5) or (3.6). Consider the linear map

$$\mathrm{Res}_q : W \ltimes_q \mathbb{C}(T_\Lambda) \to \tau(\Lambda^d) \ltimes_q \mathbb{C}(T_\Lambda)$$

defined by

$$\mathrm{Res}_q\big(\sum_{w\in W_0} D_w w\big) := \sum_{w \in W_0} D_w,$$

where $D_w \in \tau(\Lambda^d) \ltimes_q \mathbb{C}(T_\Lambda)$ $(w \in W_0)$. Note that

$$L(p) = \big(\mathrm{Res}_q(L)\big)(p) \qquad \forall\, p \in \mathbb{C}(T_\Lambda)^{W_0}. \tag{3.12}$$

Lemma 3.14. *Let $\beta_{\kappa,q}(H_0)'$ be the commutant of the subalgebra $\beta_{\kappa,q}(H_0)$ in $W \ltimes_q \mathbb{C}(T_\Lambda)$. Then Res_q restricts to an algebra homomorphism*

$$\mathrm{Res}_q : \beta_{\kappa,q}(H_0)' \to (\tau(\Lambda^d) \ltimes_q \mathbb{C}(T_\Lambda))^{W_0}$$

where $(\tau(\Lambda^d) \ltimes_q \mathbb{C}(T_\Lambda))^{W_0}$ is the subalgebra of $W \ltimes_q \mathbb{C}(T_\Lambda)$ consisting of W_0-invariant q-difference operators.

Since $Z(H(\kappa^\bullet)) = \mathbb{C}_Y[T_{\Lambda^d}]^{W_0}$, the lemma implies that the W_0-invariant q-difference operators

$$D_p := \mathrm{Res}_q\big(\beta_{\kappa,q}(p(Y))\big) \in \big(\tau(\Lambda^d) \ltimes_q \mathbb{C}(T_\Lambda)\big)^{W_0}, \qquad p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$$

pairwise commute. The operator D_p is called the Macdonald q-difference operator associated to $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$.

Define the orbit sums $m_\xi^d \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$ $(\xi \in \Lambda^d)$ by

$$m_\xi^d(t) := \sum_{\eta \in W_0\xi} t^\eta.$$

Then $\{m_\xi^d\}_{\xi \in \Lambda^{d-}}$ is a linear basis of $\mathbb{C}[T_{\Lambda^d}]^{W_0}$. We write D_ξ for $D_{m_\xi^d}$.

Set, for $w \in W$,

$$c_w := \prod_{a \in R^{\bullet +} \cap w^{-1} R^{\bullet -}} c_a \in \mathbb{C}(T_\Lambda)$$

and $\kappa_w := \prod_{a \in R^{\bullet +} \cap w^{-1} R^{\bullet -}} \kappa_a$. Write $W_{0,\xi}$ for the stabilizer subgroup of ξ in W_0.

Remark 3.15. If $\xi \in \Lambda^{d-}$ and $w \in W_{0,\xi}$ then $w(c_{\tau(-\xi)}) = c_{\tau(-\xi)}$ in $\mathbb{C}(T_\Lambda)$.

Proposition 3.16. *Let $\xi \in \Lambda^{d-}$. Then*

$$D_\xi = \kappa_{\tau(-\xi)}^{-1} \sum_{w \in W_0/W_{0,\xi}} w(c_{\tau(-\xi)})\tau(w\xi)_q + \sum_{\eta \in \mathrm{Sat}(\xi_+) \setminus W_0\xi} g_\eta \tau(\eta)_q$$

for certain $g_\eta \in \mathbb{C}(T_\Lambda)$ satisfying $g_{w\eta} = w(g_\eta)$ for all $w \in W_0$ and $\eta \in \Lambda^d$.

The set of dominant miniscule weights in Λ^d is defined by

$$\Lambda^{d+}_{min} := \{\xi \in \Lambda^d \mid \big(\xi, \alpha^{d\vee}\big) \in \{0,1\} \quad \forall \alpha \in R_0^+\}.$$

Set $\Lambda_0^d := \Lambda^d \cap V^\perp$. We now first give an explicit description of the dominant miniscule weights.

Recall that $Da_0 = -\varphi$ if $\bullet = u$ and $Da_0 = -\theta$ if $\bullet = t$. Hence $-(Da_0)^{d\vee}$ is the highest root of $R_0^{d\vee}$. Consider the expansion of $-(Da_0)^{d\vee} \in R_0^{d\vee}$ with respect to the ordered basis $\Delta_0^{d\vee} = (\alpha_1^{d\vee}, \ldots, \alpha_n^{d\vee})$ of $R_0^{d\vee}$,

$$-(Da_0)^{d\vee} = \sum_{i=1}^n m_i \alpha_i^{d\vee},$$

then $m_i \in \mathbb{Z}_{\geq 1}$ for all i. Set

$$J^+_{\Lambda^d} := \{i \in \{1,\ldots,n\} \mid m_i = 1 \;\&\; (\varpi_i^d + V^\perp) \cap \Lambda^d \neq \emptyset\},$$

where $V^\perp$ is the orthocomplement of V in Z.

Proposition 3.17. **(i)** *Λ^{d+}_{min} is a complete set of representatives of $\Lambda^d/\mathbb{Z}R_0^d$.*
(ii) *For $j \in J^+_{\Lambda^d}$ choose an element $\widetilde{\varpi}_j^d \in (\varpi_j^d + V^\perp) \cap \Lambda^d$. Then*

$$\Lambda^{d+}_{min} = \Lambda_0^d \cup \bigcup_{j \in J^+_{\Lambda^d}} \left(\widetilde{\varpi}_j^d + \Lambda_0^d\right)$$

(disjoint union).
(iii) *For $\eta \in \Lambda^d$ let $u(\eta) \in W$ be the unique element of minimal length (with respect to l) in the coset $\tau(\eta)W_0$. Then $\Omega = \{u(\xi) \mid \xi \in \Lambda^{d+}_{min}\}$.*

Since $-(Da_0)^{d\vee} \in R_0^{d\vee+}$ is the highest root, $-(Da_0)^d \in R_0^{d+}$ is quasi-miniscule, i.e. $(-(Da_0)^d, \alpha^{d\vee}) \in \{0,1\}$ for all $\alpha^d \in R_0^{d+} \setminus \{-(Da_0)^d\}$.

Let $w_0 \in W_0$ be the longest Weyl group element.

Corollary 3.18. **(i)** *For $j \in J^+_{\Lambda^d}$ we have*

$$D_{w_0\widetilde{\varpi}_j^d} = \kappa^{-1}_{\tau(-w_0\widetilde{\varpi}_j^d)} \sum_{w \in W_0/W_{0,w_0\widetilde{\varpi}_j^d}} w(c_{\tau(-w_0\widetilde{\varpi}_j^d)})\tau(ww_0\widetilde{\varpi}_j^d)_q.$$

(ii) *We have*

$$D_{(Da_0)^d} = \kappa^{-1}_{\tau(-(Da_0)^d)} \sum_{w \in W_0/W_{0,(Da_0)^d}} w(c_{\tau(-(Da_0)^d)})\bigl(\tau(w(Da_0)^d)_q - 1\bigr) + m^d_{(Da_0)^d}(\gamma_{0,q}^{-1}).$$

3.5. **Monic symmetric Macdonald-Koornwinder polynomials.** Historically the symmetric Macdonald-Koornwinder polynomials preceed the nonsymmetric Macdonald-Koornwinder polynomials. The monic symmetric Macdonald polynomials associated to initial data of the form $D = (R_0, \Delta_0, \bullet, P(R_0), P(R_0^d))$ were defined in Macdonald's handwritten preprint from 1987. It appeared in print in 2000, see [60] (see [58] for the GL_n case). Macdonald used the fact that for a fixed $\xi \in \{w_0\widetilde{\varpi}_j^d\}_{j \in J^+_{\Lambda^d}} \cup \{(Da_0)^d\}$, the explicit Macdonald q-difference operators D_ξ (see Corollary 3.18) is a linear operator on $\mathbb{C}[T_\Lambda]^{W_0}$ which is triangular with respect to the suitable partially ordered basis of orbit sums and has (generically) simple spectrum.

This approach was extended by Koornwinder [51] to the case corresponding to the initial data $D = (R_0, \Delta, t, \mathbb{Z}R_0, \mathbb{Z}R_0)$ with R_0 of type A_1 or of type B_n ($n \geq 2$), in which case $D_{-\theta}$ is Koornwinder's multivariable extension of the Askey-Wilson second-order q-difference operator (see Subsection 3.8). The corresponding symmetric Macdonald-Koornwinder polynomials are the Askey-Wilson [1] polynomials if R_0 is of rank one and the symmetric Koornwinder [51] polynomials if R_0 is of higher rank.

In this subsection we introduce the monic symmetric Macdonald-Koornwinder polynomials by symmetrizing the nonsymmetric ones, cf. [9, §4]. We assume throughout this subsection that $\kappa \in \mathcal{M}$ and q satisfy (3.5) or (3.6). Recall the notation $\kappa_w := \prod_{a \in R^{\bullet +} \cap w^{-1} R^{\bullet -}} \kappa_a$. It only depends on $\kappa^\bullet = \kappa|_{R^\bullet}$. It satisfies $\kappa_v \kappa_w = \kappa_{vw}$ if $l(vw) = l(v) + l(w)$. Hence there exists a unique linear character $\chi_+ : H(\kappa^\bullet) \to \mathbb{C}$ satisfying $\chi_+(T_w) = \kappa_w$ for all $w \in W$, the trivial linear character of $H(\kappa^\bullet)$.

Define $C_+ \in H_0(\kappa^\bullet|_{R_0}) \subset H(\kappa^\bullet)$ by

$$C_+ = \frac{1}{\sum_{w \in W_0} \kappa_w^2} \sum_{w \in W_0} \kappa_w T_w. \tag{3.13}$$

The normalization is such that $\chi_+(C_+) = 1$. Then $T_i C_+ = \kappa_i C_+ = C_+ T_i$ for $1 \leq i \leq n$ and $C_+^2 = C_+$. The following lemma follows from the explicit expression of $\pi_{\kappa,q}(T_i)$ $(1 \leq i \leq n)$.

Lemma 3.19. *The linear endomorphism $\pi_{\kappa,q}(C_+)$ of $\mathbb{C}[T_\Lambda]$ is an idempotent with image $\mathbb{C}[T_\Lambda]^{W_0}$.*

Consider the linear basis $\{m_\lambda\}_{\lambda \in \Lambda^+}$ of $\mathbb{C}[T_\Lambda]^{W_0}$ given by orbit sums, $m_\lambda(t) = \sum_{\mu \in W_0 \lambda} t^\mu$. Recall that $P_\lambda = P_\lambda(D; \kappa, q)$ denotes the monic nonsymmetric Macdonald-Koornwinder polynomial of degree $\lambda \in \Lambda$.

Lemma 3.20. *If $\lambda \in \Lambda^+$ then $\pi_{\kappa,q}(C_+) P_\lambda = \sum_{\mu \in \Lambda^+ : \mu \leq \lambda} c_{\lambda,\mu} m_\mu$ for certain $c_{\lambda,\mu} \in \mathbb{C}$ with $c_{\lambda,\lambda} \neq 0$.*

Definition 3.21. *The monic symmetric Macdonald-Koornwinder polynomial $P_\lambda^+ = P_\lambda^+(D; \kappa, q)$ of degree $\lambda \in \Lambda^+$ is defined by*

$$P_\lambda^+ := c_{\lambda,\lambda}^{-1} \pi_{\kappa,q}(C_+) P_\lambda \in \mathbb{C}[T_\Lambda]^{W_0}.$$

Theorem 3.22. *The monic symmetric Macdonald-Koornwinder polynomial P_λ^+ of degree $\lambda \in \Lambda^+$ is the unique element in $\mathbb{C}[T_\Lambda]^{W_0}$ satisfying*

- **(i)** $P_\lambda^+ = \sum_{\mu \in \Lambda^+ : \mu \leq \lambda} d_{\lambda,\mu} m_\mu$ *with* $d_{\lambda,\mu} \in \mathbb{C}$ *and* $d_{\lambda,\lambda} = 1$,
- **(ii)** $D_p P_\lambda^+ = p\big(q^{-\lambda} \prod_{\alpha \in R_0^+} \upsilon_\alpha^{-\alpha^{d\vee}}\big) P_\lambda^+$ *for all* $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$.

(Note that for $p \in \mathbb{C}[T_{\Lambda^d}]^{W_0}$ and $\lambda \in \Lambda^+$ we have $p\big(q^{-\lambda} \prod_{\alpha \in R_0^+} \upsilon_\alpha^{-\alpha^{d\vee}}\big) = p(\gamma_{w_0\lambda,q}^{-1})$).

Explicit expressions of the monic symmetric Macdonald-Koornwinder polynomials when R_0 is of rank one are given in Subsection 3.7 and Subsection 3.8.

3.6. **Orthogonality.** In this subsection we assume that $\kappa \in \mathcal{M}$ and q satisfy the conditions (3.9). We thus have the monic symmetric Macdonald-Koornwinder polynomials $\{P_\lambda^+\}_{\lambda \in \Lambda^+}$ with respect to the parameters (κ, q), as well as the monic symmetric Macdonald-Koornwinder polynomials with respect to the parameters (κ^{-1}, q^{-1}), in which case we denote them by $\{P_\lambda^{\circ +}\}_{\lambda \in \Lambda^+}$.

Define the W_0-invariant meromorphic function

$$v_+ := \prod_{a \in R^\bullet : a(0) \geq 0} \frac{1}{c_a}$$

on T_Λ. It relates to the weight function v by

$$v=\mathcal{C}v_+,\qquad \mathcal{C}=\mathcal{C}(\cdot;D;\kappa,q):=\prod_{\alpha\in R_0^-}c_\alpha. \tag{3.14}$$

One recovers v_+ from v up to a multiplicative constant by symmetrization in view of the following property of the rational function $\mathcal{C}\in\mathbb{C}(T_\Lambda)$ (cf. [55, (2.8)& (2.8 nr)]).

Lemma 3.23. *We have*

$$\sum_{w\in W_0}w\mathcal{C}=\mathcal{C}(\gamma_{0,q}^d) \tag{3.15}$$

as identity in $\mathbb{C}(T_\Lambda)$, where $\gamma_{\xi,q}^d:=\gamma_{\xi,q}(D^d;\kappa^{d\bullet})\in T_\Lambda$ ($\xi\in\Lambda^d$), cf. Definition 3.2. More generally, if $\xi\in\Lambda^{d-}$ and if (κ,q) is generic then

$$\sum_{w\in W_0}w\mathcal{C}=\mathcal{C}(\gamma_{\xi,q}^d)\sum_{\eta\in W_0\xi}\prod_{\alpha\in R_0^+\cap v(\eta)R_0^-}\frac{c_\alpha(\gamma_{\xi,q}^d)}{c_{-\alpha}(\gamma_{\xi,q}^d)}$$

as identity in $\mathbb{C}(T_\Lambda)$, where (recall) $v(\eta)\in W_0$ is the element of shortest length such that $v(\eta)\eta=\eta_-$.

The meromorphic function v_+ on T_Λ reads in terms of q-shifted factorials,

$$v_+(t)=\prod_{\beta\in R_0}\frac{\big(t^{2\beta};q_\beta^2\big)_\infty}{\big(\kappa_\beta\kappa_{2\beta}t^\beta,-\kappa_\beta\kappa_{2\beta}^{-1}t^\beta,q_\beta\kappa_{\mu_\beta c+\beta}\kappa_{2\mu_\beta c+2\beta}t^\beta,-q_\beta\kappa_{\mu_\beta c+\beta}\kappa_{2\mu_\beta c+2\beta}^{-1}t^\beta;q_\beta^2\big)_\infty}. \tag{3.16}$$

It is a nonnegative real-valued continuous function on T_Λ^u (it is nonnegative since it can be written on T_Λ^u as $v_+(t)=|\delta(t)|^2$ with $\delta(t)$ the expression (3.16) with product taken only over the set R_0^+ of positive roots).

Corollary 3.24. *$\langle\cdot,\cdot\rangle$ restricts to a positive definite, sesquilinear form on $\mathbb{C}[T_\Lambda]^{W_0}$. In fact, for $p,r\in\mathbb{C}[T_\Lambda]^{W_0}$,*

$$\langle p,r\rangle=\frac{\mathcal{C}(\gamma_{0,q}^d)}{\#W_0}\int_{T_\Lambda^u}p(t)\overline{r(t)}v_+(t)dt.$$

Symmetrizing the results on the monic nonsymmetric Macdonald-Koornwinder polynomials using the idempotent $C_+\in H_0$ gives the following properties of the monic symmetric Macdonald-Koornwinder polynomials.

Theorem 3.25. *Let $\lambda\in\Lambda^+$.*
(a) *The symmetric Macdonald-Koornwinder polynomial $P_\lambda^+\in\mathbb{C}[T_\Lambda]^{W_0}$ satisfies the following characterizing properties,*
- **(i)** $P_\lambda^+=\sum_{\mu\in\Lambda^+:\mu\leq\lambda}d_{\lambda,\mu}m_\mu$ *with* $d_{\lambda,\lambda}=1$,
- **(ii)** $\langle P_\lambda^+,m_\mu\rangle=0$ *if* $\mu\in\Lambda^+$ *and* $\mu<\lambda$.

(b) $P_\lambda^{\circ+}=P_\lambda^+$.
(c) $\langle P_\lambda^+,P_\mu^+\rangle=0$ *if* $\mu\in\Lambda^+$ *and* $\mu\neq\lambda$.

3.7. **$\mathbf{GL}_{n+1}$ Macdonald polynomials.** Take $n\geq 1$ and

$$V:=(\epsilon_1+\cdots+\epsilon_{n+1})^{\perp}\subset\mathbb{R}^{n+1}=:Z$$

with $\{\epsilon_i\}_{i=1}^{n+1}$ the standard orthonormal basis of $\mathbb{R}^{n+1}$. Let $R_0=\{\epsilon_i-\epsilon_j\}_{1\leq i\neq j\leq n+1}$ with ordered basis

$$\Delta_0=(\alpha_1,\ldots,\alpha_n)=(\epsilon_1-\epsilon_2,\ldots,\epsilon_n-\epsilon_{n+1}).$$

As lattices take $\Lambda=\mathbb{Z}^{n+1}=\Lambda^d$. Then indeed $D:=(R_0,\Delta_0,u,\mathbb{Z}^{n+1},\mathbb{Z}^{n+1})\in\mathcal{D}$.

The corresponding irreducible reduced affine root system is $R^u=\{mc+\alpha\}_{m\in\mathbb{Z},\alpha\in R_0}$, and the corresponding additional simple affine root is $a_0=c-\epsilon_1+\epsilon_{n+1}$. There is no nonreduced extension of R^u involved since $\big(\Lambda,a_i^\vee\big)=\mathbb{Z}$ for $i\in\{0,\ldots,n\}$. Hence $R=R^u$.

The fundamental weights $\varpi_j=\varpi_j^u$ $(1\leq j\leq n)$ are given by

$$\varpi_j=-\frac{j}{n+1}(\epsilon_1+\cdots+\epsilon_{n+1})+\epsilon_1+\epsilon_2+\cdots+\epsilon_j.$$

Define for $1\leq j\leq n+1$ the elements

$$\widetilde{\varpi}_j:=\epsilon_1+\cdots+\epsilon_j.$$

The orthocomplement $V^\perp$ of V in Z is $\mathbb{R}\widetilde{\varpi}_{n+1}$ and $\Lambda_0^d:=\Lambda^d\cap V^\perp=\mathbb{Z}\widetilde{\varpi}_{n+1}$. Then

$$\widetilde{\varpi}_j\in\big(\varpi_j+V^\perp\big)\cap\Lambda^d\qquad\forall\, j\in\{1,\ldots,n\}.$$

Since $-(Da_0)^{d\vee}=\epsilon_1-\epsilon_{n+1}=\sum_{i=1}^n\alpha_i$ we conclude that

$$J_{\Lambda^d}^+=\{1,\ldots,n\}.$$

The miniscule dominant weights in Λ^d are thus given by

$$\Lambda_{min}^{d+}=\Lambda_0^d\cup\bigcup_{j=1}^n(\widetilde{\varpi}_j+\Lambda_0^d).$$

We have $u(\epsilon_1)=\tau(\epsilon_1)s_1\cdots s_{n-1}s_n$ in the extended affine Weyl group $W\simeq S_{n+1}\ltimes\mathbb{Z}^{n+1}$ (where S_{n+1} is the symmetric group in $n+1$ letters). Note also that $u(\epsilon_1)(a_j)=a_{j+1}$ for $0\leq j<n$ and $u(\epsilon_1)(a_n)=a_0$. In addition we have $u(\epsilon_1)^j=u(\widetilde{\varpi}_j)$ for $1\leq j\leq n$ and $u(\epsilon_1)^{n+1}=u(\widetilde{\varpi}_{n+1})=\tau(\widetilde{\varpi}_{n+1})$ in W. Hence $\mathbb{Z}\simeq\Omega$ by $m\mapsto u(\epsilon_1)^m$.

Observe that $R=R^u$ has one $W(R^u)$-orbit, hence also one W-orbit. The affine Hecke algebra $H(W(R);\kappa)$ and the extended affine Hecke algebra $H(\kappa)=H(D;\kappa)$ thus depends on a single nonzero complex number κ. $\mathbb{Z}$ acts on $H(W(R);\kappa)$ by algebra automorphisms, where $1\in\mathbb{Z}$ acts on T_i by mapping it to T_{i+1} (reading the subscript modulo $n+1$). We write $\mathbb{Z}\ltimes H(W(R);\kappa)$ for the associated semidirect product algebra.

Proposition 3.26. **(i)** *$\mathbb{Z}\ltimes H(W(R);\kappa)\simeq H(\kappa)$ by mapping the generator $1\in\mathbb{Z}$ to $u(\widetilde{\varpi}_1)=u(\epsilon_1)$.*
(ii) *For $1\leq i\leq n+1$ we have*

$$Y^{\epsilon_i}=T_{i-1}^{-1}\cdots T_2^{-1}T_1^{-1}u(\epsilon_1)T_nT_{n-1}\cdots T_i.$$

in $H(\kappa)$.

We now turn to the explicit description of the Macdonald-Ruijsenaars q-difference operators and the orthogonality measure for the GL_{n+1} Macdonald polynomials.

The longest Weyl group element $w_0 \in W_0$ maps ϵ_i to ϵ_{n+2-i} for $1 \leq i \leq n+1$, hence

$$R^+ \cap \tau(w_0 \widetilde{\varpi}_j) R^- = \{\epsilon_r - \epsilon_s \,|\, 1 \leq r \leq n+1-j \;\&\; n+2-j \leq s \leq n+1\}.$$

Write $t_i = t^{\epsilon_i}$ for $t \in T_\Lambda$ and $1 \leq i \leq n+1$. Then, for $1 \leq j \leq n$ we obtain

$$c_{\tau(-w_0\widetilde{\varpi}_j)}(t) = \prod_{\substack{1 \leq j \leq n+1-i \\ n+2-j \leq s \leq n+1}} \frac{(1-\kappa^2 t_r t_s^{-1})}{(1-t_r t_s^{-1})}$$

and consequently

$$\begin{aligned} D_{w_0\widetilde{\varpi}_j} &= \sum_{\substack{I \subset \{1,\ldots,n+1\}: \\ \#I = n+1-j}} \left(\prod_{r \in I, s \notin I} \frac{(1-\kappa^2 t_r t_s^{-1})}{\kappa(1-t_r t_s^{-1})} \right) \tau\big(\sum_{s \notin I} \epsilon_s\big)_q \\ &= \sum_{\substack{I \subset \{1,\ldots,n+1\}: \\ \#I = j}} \left(\prod_{r \in I, s \notin I} \frac{\kappa^{-1} t_r - \kappa t_s}{t_r - t_s} \right) \tau\big(\sum_{r \in I} \epsilon_r\big)_q \end{aligned}$$

for $1 \leq j \leq n$ by Proposition 3.18**(i)**. These commutative q-difference operators were introduced by Ruijsenaars [73] as the quantum Hamiltonians of a relativistic version of the trigonometric quantum Calogero-Moser system. These operators also go by the name (trigonometric) Ruijsenaars q-difference operators, or Ruijsenaars-Macdonald q-difference operators.

The corresponding monic symmetric Macdonald polynomials $\{P_\lambda\}_{\lambda \in \Lambda^+}$ are parametrized by

$$\Lambda^+ = \{\lambda \in \mathbb{Z}^{n+1} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}\}.$$

The orthogonality weight function then becomes

$$v_+(t) = \prod_{1 \leq i \neq j \leq n+1} \frac{\big(t_i/t_j;q\big)_\infty}{\big(\kappa^2 t_i/t_j;q\big)_\infty}.$$

The GL_2 Macdonald polynomials are essentially the continuous q-ultraspherical polynomials (see [15, Chpt. 2] or [61, §6.3]),

$$P_\lambda^+(t) = t_1^{\lambda_1} t_2^{\lambda_2} {}_2\phi_1 \left(\begin{matrix} \kappa^2, q^{\lambda_2-\lambda_1} \\ q^{1+\lambda_2-\lambda_1}/\kappa^2 \end{matrix}; q, \frac{qt_2}{\kappa^2 t_1} \right),$$

for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geq \lambda_2$, where we use standard notations for the basic hypergeometric ${}_r\phi_s$ series (see, e.g., [28]).

3.8. **Koornwinder polynomials.** Take $Z = V = \mathbb{R}^n$ with standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$. We realize the root system $R_0 \subset V$ of type B_n as

$$R_0 = \{\pm\epsilon_i\}_{i=1}^n \cup \{\pm\epsilon_i \pm \epsilon_j\}_{1\le i<j\le n}$$

(all sign combinations allowed). The W_0-orbits are $\mathcal{O}_l = \{\pm\epsilon_i \pm \epsilon_j\}_{1\le i<j\le n}$ and $\mathcal{O}_s = \{\pm\epsilon_i\}_{i=1}^n$. As an ordered basis of R_0 take

$$\Delta_0 = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n).$$

Note that $\mathbb{Z}R_0 = \mathbb{Z}^n$. For $n = 1$ this should be interpreted as $R_0 = \{\pm\epsilon_1\}$, the root system of type A_1, with basis element ϵ_1. We consider in this subsection the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}^n, \mathbb{Z}^n) \in \mathcal{D}$.

The associated reduced affine root system is

$$R^t = \{\frac{m}{2}c + \alpha\}_{n\in\mathbb{Z},\alpha\in\mathcal{O}_s} \cup \{mc + \beta\}_{m\in\mathbb{Z},\beta\in\mathcal{O}_l}$$

(for $n = 1$ it should be read as $R^t = \{\frac{m}{2}c \pm \epsilon_1\}_{m\in\mathbb{Z}}$, the affine root system R^t with R_0 of type A_1). The associated ordered basis of R^t is

$$\Delta = (a_0, a_1, \ldots, a_n) = (\frac{1}{2}c - \theta, \alpha_1, \ldots, \alpha_n)$$

with $\theta = \epsilon_1 = \sum_{j=1}^n \alpha_j$ the highest short root of R_0 with respect to Δ_0. The affine root system R^t has three W-orbits,

$$R^t = \widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3,$$

where $\widehat{\mathcal{O}}_1 = Wa_0 = \big(\frac{1}{2} + \mathbb{Z}\big)c + \mathcal{O}_s$, $\widehat{\mathcal{O}}_2 = Wa_i = \mathbb{Z}c + \mathcal{O}_l$ (where $1 \le i < n$) and $\widehat{\mathcal{O}}_3 = Wa_n = \mathbb{Z}c + \mathcal{O}_s$. If $n = 1$ then $\Delta = (a_0, a_1) = (\frac{1}{2}c - \epsilon_1, \epsilon_1)$ and R^t has two W-orbits $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_3$.

Note that $\big(\Lambda, a_i^\vee\big) = 2\mathbb{Z}$ for $i = 0$ and $i = n$, hence $S = \{0, n\} = S^d$ and

$$R = R^t \cup W(2a_0) \cup W(2a_n) = \widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3 \cup \widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$$

with additional W-orbits $\widehat{\mathcal{O}}_4 = 2\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_5 = 2\widehat{\mathcal{O}}_3$. If $n = 1$ then R has four W-orbits $\widehat{\mathcal{O}}_i$ $(i = 1, 2, 4, 5)$. Since $D^d = D$ we have $R^d = R$, $\Delta^d = \Delta$ and $W^d = W$.

Suppose that $\kappa \in \mathcal{M}(D)$ and $q \in \mathbb{C}^*$ satisfy (3.5) or (3.6). Fix $1 \le i < n$. Then $\kappa \in \mathcal{M}(D)$ is determined by five (four in case $n = 1$) independent numbers $\kappa_{a_0}, \kappa_{a_i}, \kappa_{a_n}, \kappa_{2a_0}$ and κ_{2a_n}. The corresponding Askey-Wilson [1] parameters are defined by

$$(a, b, c, d, k) = (\kappa_{a_n}\kappa_{2a_n}, -\kappa_{a_n}\kappa_{2a_n}^{-1}, q^{\frac{1}{2}}\kappa_{a_0}\kappa_{2a_0}, -q^{\frac{1}{2}}\kappa_{a_0}\kappa_{2a_0}^{-1}, \kappa_{a_i}^2) \tag{3.17}$$

(the parameter k is dropping out in case $n = 1$). The dual multiplicity function κ^d on $R^d = R$ is then determined by $\kappa_{a_0}^d := \kappa_{2a_n}$, $\kappa_{a_i}^d := \kappa_{a_i}$, $\kappa_{a_n}^d := \kappa_{a_n}$, $\kappa_{2a_0}^d := \kappa_{2a_0}$ and $\kappa_{2a_n}^d := \kappa_{a_0}$. The corresponding Askey-Wilson parameters are

$$(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{k}) = (\kappa_{a_n}\kappa_{a_0}, -\kappa_{a_n}\kappa_{a_0}^{-1}, q^{\frac{1}{2}}\kappa_{2a_n}\kappa_{2a_0}, -q^{\frac{1}{2}}\kappa_{2a_n}\kappa_{2a_0}^{-1}, \kappa_{a_i}^2).$$

In terms of the original Askey-Wilson parameters this can be expressed as $\tilde{k}=k$ and

$$(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=\Big(\sqrt{q^{-1}abcd},\frac{ab}{\sqrt{q^{-1}abcd}},\frac{ac}{\sqrt{q^{-1}abcd}},\frac{ad}{\sqrt{q^{-1}abcd}}\Big).$$

Note that

$$-(Da_0)^{d\vee}=\theta^\vee=2\alpha_1^\vee+2\alpha_2^\vee+\cdots+2\alpha_{n-1}^\vee+\alpha_n^\vee.$$

Furthermore, denoting the fundamental weights of R_0 with respect to the ordered basis Δ_0 by $\{\varpi_i\}_{i=1}^n$, we have

$$\varpi_n=\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)\not\in\Lambda^d=\mathbb{Z}^n.$$

Consequently $J_{\Lambda^d}^+=\emptyset$. Hence the only explicit Macdonald q-difference operator obtainable from Corollary 3.18 is $D_{-\theta}=D_{-\epsilon_1}$.

Write

$$\mathcal{D}=\kappa_{\tau(\epsilon_1)}^{-1}\sum_{w\in W_0/W_{0,\epsilon_1}}w\big(c_{\tau(\epsilon_1)}\big)\big(\tau(-w\epsilon_1)_q-1\big),$$

so that $D_{-\epsilon_1}=\mathcal{D}+m_{-\epsilon_1}^d(\gamma_{0,q}^{-1})$. Write $t_i=t^{\epsilon_i}$ for $t\in T_\Lambda$ and $1\le i\le n$. Since

$$R^{t+}\cap\tau(-\epsilon_1)R^{t-}=\{\epsilon_1,\frac{1}{2}c+\epsilon_1\}\cup\{\epsilon_1\pm\epsilon_j\}_{j=2}^n$$

($=\{\epsilon_1,\frac{1}{2}c+\epsilon_1\}$ if $n=1$) it follows that

$$\begin{aligned}\kappa_{\tau(\epsilon_1)}^{-1}c_{\tau(\epsilon_1)}(t)&=\kappa_{\tau(\epsilon_1)}^{-1}c_{\epsilon_1}(t)c_{\frac{1}{2}c+\epsilon_1}(t)\prod_{j=2}^n c_{\epsilon_1-\epsilon_j}(t)c_{\epsilon_1+\epsilon_j}(t)\\&=\frac{1}{\sqrt{q^{-1}abcd}k^{n-1}}\frac{(1-at_1)(1-bt_1)(1-ct_1)(1-dt_1)}{(1-t_1^2)(1-qt_1^2)}\\&\quad\times\prod_{j=2}^n\frac{(1-kt_1t_j^{-1})(1-kt_1t_j)}{(1-t_1t_j^{-1})(1-t_1t_j)}.\end{aligned}\tag{3.18}$$

For $n=1$ the product over j is not present in (3.18). In particular, the operator $\mathcal{D}$ then only depends on q and on the four parameters a,b,c,d.

Hence $\mathcal{D}$ is Koornwinder's [51] second order q-difference operator

$$\begin{aligned}\mathcal{D}&=\frac{1}{\sqrt{q^{-1}abcd}k^{n-1}}\sum_{i=1}^n\big(A_i(t)(\tau(-\epsilon_i)_q-1)+A_i(t^{-1})(\tau(\epsilon_i)_q-1\big),\\A_i(t)&=\frac{(1-at_i)(1-bt_i)(1-ct_i)(1-dt_i)}{(1-t_i^2)(1-qt_i^2)}\prod_{j\neq i}\frac{(1-kt_it_j)(1-kt_it_j^{-1})}{(1-t_it_j)(1-t_it_j^{-1})},\end{aligned}$$

with the obvious adjustment for $n=1$, in which case $\mathcal{D}$ is the Askey-Wilson [1] second-order q-difference operator (this derivation is due to Noumi [63]). The weight function

$v_+(t)$ becomes

$$v_+(t) = \prod_{i=1}^{n} \frac{\big(t_i^2, t_i^{-2}; q\big)_\infty}{\big(at_i, at_i^{-1}, bt_i, bt_i^{-1}, ct_i, ct_i^{-1}, dt_i, dt_i^{-1}; q\big)_\infty} \prod_{1\le r\neq s\le n} \frac{\big(t_r t_s, t_r t_s^{-1}; q\big)_\infty}{\big(kt_r t_s, kt_r t_s^{-1}; q\big)_\infty}$$

The dominant elements Λ^+ in Λ are the partitions of length $\le n$,

$$\Lambda^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\}.$$

The corresponding monic symmetric Macdonald-Koornwinder polynomials $\{P_\lambda^+\}_{\lambda\in\Lambda^+}$ are the symmetric Koornwinder [51] polynomials.

For $n = 1$ the Koornwinder polynomials are the Askey-Wilson [1] polynomials. Concretely, using standard notations for basic hypergeometric series (cf. [28]), for $m \in \mathbb{Z}_{\ge 0} = \Lambda^+$,

$$P_m^+(t) = \mathrm{cst}_m {}_4\phi_3 \left(\begin{matrix} q^{-m}, q^{m-1}abcd, at, at^{-1} \\ ab, ac, ad \end{matrix} ; q, q \right),$$

where

$$\mathrm{cst}_m = \frac{\big(ab, ac, ad; q\big)_m}{a^m \big(q^{m-1}abcd; q\big)_m}.$$

The Cherednik-Macdonald theory associated to the Askey-Wilson polynomials was worked out in detail in [65].

Example 3.27. *Consider the initial data $D = (R_0, \Delta_0, t, \mathbb{Z}^n, P(R_0)) \in \mathcal{D}$ with the root system $R_0 \subset \mathbb{R}^n$ of type B_n ($n \ge 2$) and the ordered basis Δ_0 both defined in terms of the standard orthonormal basis $\{\epsilon_i\}_{i=1}^n$ of $\mathbb{R}^n$ as above. Compared to the Koornwinder setup just discussed we have thus chosen a different lattice $\Lambda^d = P(R_0)$, which contains $\mathbb{Z}R_0 = \mathbb{Z}^n$ as index two sublattice. Still $S(D) = \{0, n\}$, hence R is the nonreduced affine root system with five $W_0 \ltimes \mathbb{Z}^n$-orbits $\widehat{\mathcal{O}}_i$ ($1 \le i \le 5$) as introduced above. But the number $\nu = \nu(D)$ of $W_0 \ltimes P(R_0)$-orbits is three: they are given by $\widehat{\mathcal{O}}_2$, $\widehat{\mathcal{O}}_1 \cup \widehat{\mathcal{O}}_3$ and $\widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$.*

On the other hand, $R^d = R^t$ since $S(D^d) = \emptyset$, in particular R^d is reduced. It has the three $W^d = W_0 \ltimes \mathbb{Z}^n$-orbits $\widehat{\mathcal{O}}_i$ ($i = 1, 2, 3$).

Remark 3.28. Consider the initial data $D = (R_0, \Delta_0, u, \mathbb{Z}^n, \mathbb{Z}^n)$ with (R_0, Δ_0) of type B_n ($n \ge 3$) as above. Compared to initial data $D^{C^\vee C_n} := (R_0, \Delta_0, t, \mathbb{Z}^n, \mathbb{Z}^n)$ related to Koornwinder polynomials, we thus have only changed the type from twisted to untwisted. Then R^u is the affine root subsystem $\widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3$ of the affine root system $R^{C^\vee C_n}$ of type $C^\vee C_n$ as defined above, and $R = R(D)$ is the nonreduced irreducible affine root subsystem $\widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_3 \cup \widehat{\mathcal{O}}_5$ of $R(D^{C^\vee C_n})$. The dual affine root system $R^d = R(D^d)$ is the reduced irreducible affine root subsystem $R^d = \widehat{\mathcal{O}}_2 \cup \widehat{\mathcal{O}}_4 \cup \widehat{\mathcal{O}}_5$ (it is of untwisted type, with underlying finite root system $R_0^\vee$ of type C_n). The Macdonald-Koornwinder theory associated to the initial data D thus is the special case of the $C^\vee C_n$ theory when the multiplicity function $\kappa \in \mathcal{M}(D^{C^\vee C_n})$ takes the value one at the two orbits $\widehat{\mathcal{O}}_1 = W(a_0)$ and $\widehat{\mathcal{O}}_4 = W(2a_0)$ of $R^{C^\vee C_n} \setminus R$.

3.9. **Exceptional nonreduced rank two Macdonald-Koornwinder polynomials.** For this special case of the theory we take $D=(R_0,\Delta_0,u,\mathbb{Z}R_0,\mathbb{Z}R_0^\vee)$ with $R_0\subset Z=V=\mathbb{R}^2$ the root system of type C_2 (it is more natural to view it as type C_2 then of type B_2) given by $R_0=R_{0,s}\cup R_{0,l}$ with

$$\begin{aligned}R_{0,s}&=\{\pm(\epsilon_1+\epsilon_2),\pm(\epsilon_1-\epsilon_2)\},\\ R_{0,l}&=\{\pm 2\epsilon_1,\pm 2\epsilon_2\},\end{aligned}$$

where $\{\epsilon_1,\epsilon_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$. As ordered basis take $\Delta_0=(\alpha_1,\alpha_2)=(\epsilon_1-\epsilon_2,2\epsilon_2)$. Then $\varphi=2\epsilon_1$ and $\theta=\epsilon_1+\epsilon_2$. Hence the simple affine root a_0 of the associated reduced affine root system $R^u=\mathbb{Z}c+R_0$ is $c-2\epsilon_1$. Then $S=\{1\}$ hence

$$R=R^u\cup W(2\alpha_1)$$

with $W=W^u=W_0\ltimes\mathbb{Z}R_0^\vee$. Then R has four W-orbits,

$$\begin{aligned}W(a_0)&=(2\mathbb{Z}+1)c+R_{0,l},\\ W(\alpha_1)&=\mathbb{Z}c+R_{0,s},\\ W(\alpha_2)&=2\mathbb{Z}c+R_{0,l},\\ W(2\alpha_1)&=2\mathbb{Z}c+2R_{0,s}.\end{aligned}$$

For the dual initial data $D^d=(R_0^\vee,\Delta_0^\vee,u,\mathbb{Z}R_0^\vee,\mathbb{Z}R_0)$ we have $R^{ud}=\mathbb{Z}c+R_0^\vee$ and $R^d=R^{ud}\cup W^d(2\alpha_2^\vee)$ with $W^d=W^{ud}=W_0\ltimes\mathbb{Z}R_0$. The simple affine root a_0^d of R^{ud} is $a_0^d=c-\epsilon_1-\epsilon_2$. The four W^d-orbits of R^d are

$$\begin{aligned}W^d(a_0^d)&=(2\mathbb{Z}+1)c+R_{0,s}^\vee,\\ W^d(\alpha_1^\vee)&=2\mathbb{Z}c+R_{0,s}^\vee,\\ W^d(\alpha_2^\vee)&=\mathbb{Z}c+R_{0,l}^\vee,\\ W^d(2\alpha_2^\vee)&=2\mathbb{Z}c+2R_{0,l}^\vee.\end{aligned}$$

Note that $R\simeq R^d$ but not $(R,\Delta)\simeq(R^d,\Delta^d)$.

Let $\kappa\in\mathcal{M}(D)$. We write

$$\begin{aligned}\{a,b,c,d\}&:=\{\kappa_\theta\kappa_{2\theta},-\kappa_\theta\kappa_{2\theta}^{-1},\kappa_\varphi^2,q\kappa_0^2\},\\ \{\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\}&=\{\kappa_\varphi\kappa_0,-\kappa_\varphi\kappa_0^{-1},\kappa_\theta^2,q\kappa_{2\theta}^2\}\end{aligned}\tag{3.19}$$

(the dual parameters $(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d})$ are the parameters (a,b,c,d) with respect to the dual initial data D^d). We identify $T_{\mathbb{Z}R_0}\simeq(\mathbb{C}^*)^2$ by $t\mapsto(z_1,z_2):=(t^{\alpha_1},t^{\alpha_2})$. The W_0-action on $T_{\mathbb{Z}R_0}$ then reads $s_1(z_1,z_2)=(z_1^{-1},z_1^2z_2)$ and $s_2(z_1,z_2)=(z_1z_2,z_2^{-1})$, giving rise to a W_0-action on $\mathbb{C}[T_{\mathbb{Z}R_0}]\simeq\mathbb{C}[z_1^{\pm1},z_2^{\pm1}]$ by algebra automorphisms. The weight function $v_+(z_1,z_2)$ of the associated symmetric Macdonald-Koornwinder polynomials is given by $v_+(z_1,z_2)=\delta(z_1,z_2)\delta(z_1^{-1},z_2^{-1})$ with

$$\delta(z_1,z_2):=\frac{\big(z_1^2,z_1^2z_2^2;q^2\big)_\infty}{\big(az_1,az_1z_2,bz_1,bz_1z_2;q\big)_\infty}\frac{\big(z_1^2z_2,z_2;q\big)_\infty}{\big(cz_1^2z_2,cz_2,dz_1^2z_2,dz_2;q^2\big)_\infty}.$$

The symmetric Macdonald-Koornwinder polynomials associated to the initial data D form an orthogonal basis of $\mathbb{C}[z_1^{\pm 1}, z_1^{\pm 1}]^{W_0}$ with respect to the sesquilinear, positive pairing

$$\langle p, r\rangle_+ := \frac{1}{(2\pi i)^2}\int_{|z_1|=1}\int_{|z_2|=1} p(z_1,z_2)\overline{r(z_1,z_2)}v_+(z_1,z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

on $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]^{W_0}$ (note that the monic symmetric Macdonald-Koornwinder polynomials are parametrized by the $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 \in \Lambda^+$ with the $\lambda_i \in \mathbb{Z}$ satisfying $\lambda_1 + \lambda_2 \in 2\mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq 0$).

The associated explicit Macdonald operator is $D_{-\epsilon_1}$. Indeed,

$$-(Da_0)^{d\vee} = -(Da_0)^\vee = \epsilon_1 = \alpha_1^\vee + \alpha_2^\vee$$

hence

$$J_{\mathbb{Z}R_0^\vee} = \{i \in \{1,2\} \mid \varpi_i^d \in \mathbb{Z}R_0^\vee\} = \{1\}$$

since $\varpi_1^d = \epsilon_1 \in \mathbb{Z}R_0^\vee$ and $\varpi_2^d = \frac{1}{2}(\epsilon_1+\epsilon_2) \notin \mathbb{Z}R_0^\vee$. By a direct computation we obtain the explicit expression

$$\begin{aligned}\kappa_0\kappa_\theta^2\kappa_\varphi(D_{-\epsilon_1}f)(z_1,z_2) =& A(z_1,z_2)f(qz_1,z_2) + A(z_1^{-1},z_2)f(q^{-1}z_1,q^2z_2)\\ &+A(z_1^{-1}z_2^{-1},z_1^2z_2)f(qz_1,q^{-2}z_2) + A(z_1^{-1}z_2^{-1},z_2)f(q^{-1}z_1,z_2),\\ A(z_1,z_2) :=& \frac{(1-az_1)(1-az_1z_2)(1-bz_1)(1-bz_1z_2)(1-cz_1^2z_2)(1-dz_1^2z_2)}{(1-z_1^2)(1-z_1^2z_2^2)(1-z_1^2z_2)(1-qz_1^2z_2)}.\end{aligned}$$

4. Double affine Hecke algebras and normalized Macdonald-Koornwinder polynomials

Cherednik's [8, 9] double affine braid group and double affine Hecke algebra are fundamental for proving properties of the (non)symmetric Macdonald-Koornwinder polynomials such as evaluation formula, duality and quadratic norms (see [11, 12]). We discuss these results in this section.

The first part follows closely [32]. We define a double affine braid group and a double affine Hecke algebra depending on q, on the initial data D and, in case of the double affine Hecke algebra, on a choice of a multiplicity function $\kappa \in \mathcal{M}(D)$. We extend the duality $D \mapsto D^d$ on initial data to an antiisomorphism of the associated double affine braid groups, and to an algebra antiisomorphism of the associated double affine Hecke algebras (on the dual side, the double affine Hecke algebra is taken with respect to the dual multiplicity function $\kappa^d \in \mathcal{M}(D^d)$ as defined in Lemma 2.13). These antiisomorphisms are called duality antiisomorphisms and find their origins in the work of Cherednik [11, 12] (see also [37, 75, 61, 32] for further results and generalizations).

As a consequence of the duality antiisomorphism and of the theory of intertwiners we first derive the evaluation formula and duality of the Macdonald-Koornwinder polynomials following closely [12, 13, 75]. These results, together with quadratic norm formulas, were first conjectured by Macdonald (see, e.g., [60, 57]). For R_0 of type A the evaluation formula and duality were proven by Koornwinder [50] by different methods (see [58] for a detailed account). Subsequently Cherednik [11] established the evaluation formula and duality when $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$. The $C^\vee C$ case was established using Koornwinder's [50] methods by van Diejen [21] for a suitable subset of multiplicity functions. Cherednik's double affine Hecke algebra methods were extended to the $C^\vee C$ case in the work of Noumi [63] and Sahi [75], leading to the evaluation formula and duality for all multiplicity functions. Our uniform approach is close to Haiman's [32].

Following closely [12, 13] we use the duality antiisomorphism and intertwiners to establish quadratic norm formulas for the (non)symmetric Macdonald-Koornwinder polynomials.

We fix throughout this section initial data $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d) \in \mathcal{D}$, a deformation parameter $q \in \mathbb{R}_{>0} \setminus \{1\}$, and a multiplicity function $\kappa \in \mathcal{M}(D)$.

4.1. **Double affine braid groups, Weyl groups and Hecke algebras.** Consider the W-stable additive subgroup $\widehat{\Lambda} := \Lambda + \mathbb{R}c$ of $\widehat{Z}$. In the following definition it is convenient to write $X^{\widehat{\lambda}}$ ($\widehat{\lambda} \in \widehat{\Lambda}$) for the elements of $\widehat{\Lambda}$. In particular, $X^{\widehat{\lambda}} X^{\widehat{\mu}} = X^{\widehat{\lambda}+\widehat{\mu}}$ for $\widehat{\lambda}, \widehat{\mu} \in \widehat{\Lambda}$ and $X^0 = 1$.

We write $A = A(R, \Delta)$ and $A_0 = A(R_0, \Delta_0)$. The generators of the affine braid group $\mathcal{B}^\bullet = \mathcal{B}(A)$ are denoted by $T_0, T_1, \ldots, T_n$, the generators of $\mathcal{B}_0 = \mathcal{B}(A_0)$ by $T_1, \ldots, T_n$. Recall that elements $T_w \in \mathcal{B}$ ($w \in W^\bullet$) and $T_w \in \mathcal{B}_0$ ($w \in W_0$) can be defined using reduced expressions of w in the Coxeter groups $W^\bullet = W(A^\bullet)$ and $W_0 = W(A_0)$, respectively. Recall furthermore that $\mathcal{B} = \mathcal{B}(D) \simeq \Omega \ltimes \mathcal{B}^\bullet$ denotes the extended affine braid group (cf. Definition 2.17).

Definition 4.1. *The double affine braid group $\mathbb{B} = \mathbb{B}(D)$ is the group generated by the groups $\mathcal{B}$ and $\widehat{\Lambda}$ together with the relations:*

$$T_i X^{\widehat{\lambda}} = X^{\widehat{\lambda}} T_i \tag{4.1}$$

if $\widehat{\lambda} \in \widehat{\Lambda}$ and $0 \leq i \leq n$ such that $(\widehat{\lambda}, a_i^\vee) = 0$;

$$T_i X^{\widehat{\lambda}} T_i = X^{s_i \widehat{\lambda}} \tag{4.2}$$

if $\widehat{\lambda} \in \widehat{\Lambda}$ and $0 \leq i \leq n$ such that $(\widehat{\lambda}, a_i^\vee) = 1$;

$$\omega X^{\widehat{\lambda}} = X^{\omega \widehat{\lambda}} \omega \tag{4.3}$$

if $\widehat{\lambda} \in \widehat{\Lambda}$ and $\omega \in \Omega$.

Note that $X^{\mathbb{R}c}$ is contained in the center of $\mathbb{B}$. It is not necessarily true that any element $g \in \mathbb{B}$ can be written as $g = bX^{\widehat{\lambda}}$ (or $g = X^{\widehat{\lambda}} b$) with $b \in \mathcal{B}$ and $\widehat{\lambda} \in \widehat{\Lambda}$. The cases that this possibly fails to be true is if $S(D) \neq \emptyset$ (these are the cases for which $R \neq R^\bullet$, i.e. for which nonreduced extensions of $R^\bullet$ play a role in the theory, cf. Subsection 2.4). Straightening the elements of $\mathbb{B}$ is always possible as soon as quotients to double affine Weyl groups and double affine Hecke algebras are taken, as we shall see in a moment.

Recall from Subsection 3.1 the construction of commuting group elements $Y^\xi \in \mathcal{B}$ ($\xi \in \Lambda^d$). It leads to a presentation of $\mathcal{B}$ in terms of the group Λ^d (with its elements denoted by Y^ξ ($\xi \in \Lambda^d$)) and the braid group $\mathcal{B}_0 = \mathcal{B}(A_0)$, see e.g. [61, §3.3]. On the level of double affine braid groups it implies the following alternative presentation of $\mathbb{B}$ (recall that Da_0 equals $-\varphi$ if $\bullet = u$ and $-\theta$ if $\bullet = t$).

Proposition 4.2. *The double affine braid group $\mathbb{B}$ is isomorphic to the group generated by the groups $\mathcal{B}_0$, Λ^d and $\widehat{\Lambda}$, satisfying for $1 \leq i \leq n$, $\lambda \in \Lambda$ and $\xi \in \Lambda^d$,*

- **(a)** *$X^{\mathbb{R}c}$ is contained in the center,*
- **(b)** **1.** *$Y^{-\xi}T_i = T_iY^{-\xi}$ if $(\xi, \alpha_i^{d\vee}) = 0$,*
  2. *$T_iX^\lambda = X^\lambda T_i$ if $(\lambda, \alpha_i^\vee) = 0$,*
- **(c)** **1.** *$T_iY^{-\xi}T_i = Y^{-s_i\xi}$ if $(\xi, \alpha_i^{d\vee}) = 1$,*
  2. *$T_iX^\lambda T_i = X^{s_i\lambda}$ if $(\lambda, \alpha_i^\vee) = 1$,*
- **(d)** *if $(\lambda, a_0^\vee) = 0$,*
$$\big(Y^{-(Da_0)^d}T_{s_{Da_0}}^{-1}\big)X^\lambda = X^\lambda\big(Y^{-(Da_0)^d}T_{s_{Da_0}}^{-1}\big),$$
- **(e)** *if $(\lambda, a_0^\vee) = 1$,*
$$\big(Y^{-(Da_0)^d}T_{s_{Da_0}}^{-1}\big)X^\lambda\big(Y^{-(Da_0)^d}T_{s_{Da_0}}^{-1}\big) = p_{Da_0}^{-1}X^{s_{Da_0}\lambda}$$
  where $p_\alpha := X^{\mu_\alpha c}$ for $\alpha \in R_0$.

Recall that $u(\eta) \in W \simeq W_0 \ltimes \Lambda^d$ denotes the element of minimal length in $\tau(\eta)W_0$ for all $\eta \in \Lambda^d$. Then $u(\eta) = \tau(\eta)v(\eta)^{-1}$ with $v(\eta) \in W_0$ the element of minimal length such that $v(\eta)\eta = \eta_-$. Recall furthermore that $\Omega = \{u(\xi) \,|\, \xi \in \Lambda^{d+}_{min}\}$.

The identification of the two different sets of generators of $\mathbb{B}$ then is as follows: $T_0 = Y^{-(Da_0)^d}T_{s_{Da_0}}^{-1}$ and $u(\xi) = Y^\xi T_{v(\xi)}^{-1}$ for $\xi \in \Lambda^{d+}_{min}$. Conversely, for $\xi = \eta_1 - \eta_2 \in \Lambda^d$ with $\eta_1, \eta_2 \in \Lambda^{d+}$, $Y^\xi = Y^{\eta_1}\big(Y^{\eta_2}\big)^{-1}$ and
$$Y^{\eta_s} = T_{\tau(\eta_s)} = \omega T_{i_1}T_{i_2}\cdots T_{i_r}$$
if $\tau(\eta_s) = \omega s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ is a reduced expression ($\omega \in \Omega$ and $0 \leq i_j \leq n$).

Recall the set $S = S(D)$ given by (2.5). Write
$$V_i := X^{-a_i}T_i^{-1} \in \mathbb{B} \qquad \forall\, i \in S. \tag{4.4}$$
Recall furthermore that $R = R^\bullet \cup \bigcup_{i\in S} W^\bullet(2a_i)$.

Definition 4.3. **(i)** *The double affine Weyl group $\mathbb{W} = \mathbb{W}(D)$ is the quotient of $\mathbb{B}$ by the normal subgroup generated by T_j^2 ($j \in \{0, \ldots, n\}$) and V_i^2 ($i \in S$).*
(ii) *The double affine Hecke algebra $\mathbb{H}(\kappa, q) = \mathbb{H}(D; \kappa, q)$ is $\mathbb{C}[\mathbb{B}]/\widehat{I}_{\kappa,q}$, where $\widehat{I}_{\kappa,q}$ is the two-sided ideal of $\mathbb{C}[\mathbb{B}]$ generated by $X^{rc} - q^r$ ($r \in \mathbb{R}$), $(T_j - \kappa_j)(T_j + \kappa_j^{-1})$ ($j \in \{0, \ldots, n\}$) and $(V_i - \kappa_{2a_i})(V_i + \kappa_{2a_i}^{-1})$ ($i \in S$).*

Recall that W acts on $\widehat{\Lambda}$ by group automorphisms.

Proposition 4.4. $\mathbb{W} \simeq W \ltimes \widehat{\Lambda}$.

For the double affine Hecke algebra, note that we have canonical algebra homomorphisms $H(\kappa^\bullet) \to \mathbb{H}(\kappa, q)$ and $\mathbb{C}[\widehat{\Lambda}] \to \mathbb{H}(\kappa, q)$. We write $\widetilde{h}$, $\widetilde{X}_q^{\widehat{\lambda}}$ and $\widetilde{X}^\lambda$ for the images of $h \in \widehat{H}$, $X^{\widehat{\lambda}} \in \mathbb{C}[\widehat{\Lambda}]$ ($\widehat{\lambda} \in \widehat{\Lambda}$) and $X^\lambda \in \mathbb{C}[\Lambda]$ ($\lambda \in \Lambda$) in $\mathbb{H}(\kappa, q)$ respectively. Define a linear map

$$m : H(\kappa^\bullet) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \to \mathbb{H}(\kappa, q)$$

by $m(h \otimes X^\lambda) := \widetilde{h}\widetilde{X}^\lambda$ for $h \in H(\kappa^\bullet)$ and $\lambda \in \Lambda$.

If $\widehat{\lambda} = \lambda + rc \in \widehat{\Lambda}$ then we interpret $e_q^{\widehat{\lambda}}$ as the endomorphism of $\mathbb{C}[T_\Lambda]$ by

$$\big(e_q^{\widehat{\lambda}} p\big)(t) := t_q^{\widehat{\lambda}} p(t) = q^r t^\lambda p(t), \qquad r \in \mathbb{R},\ \lambda \in \Lambda.$$

Theorem 4.5. (i) *We have a unique algebra monomorphism* $\mathbb{H}(\kappa, q) \hookrightarrow \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_\Lambda]\big)$ *defined by*

$$\begin{aligned} \widetilde{h} &\mapsto \pi_{\kappa,q}(h) \qquad && \forall\, h \in H(\kappa^\bullet), \\ \widetilde{X}_q^{\widehat{\lambda}} &\mapsto e_q^{\widehat{\lambda}} && \forall\, \widehat{\lambda} \in \widehat{\Lambda}. \end{aligned}$$

(ii) *The linear map m defines a complex linear isomorphism* $H(\kappa^\bullet) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \xrightarrow{\sim} \mathbb{H}(\kappa, q)$.

To simplify notations, we omit the tilde when writing the elements in $\mathbb{H}$. With this convention $X_q^{\widehat{\lambda}} = q^r X^\lambda$ in $\mathbb{H}(\kappa, q)$ if $\widehat{\lambda} = \lambda + rc$.

Together with the Bernstein-Zelevinsky presentation of the extended affine Hecke algebra $H(\kappa^\bullet)$ (see Subsection 3.1) we conclude that

$$\mathbb{C}[T_{\Lambda^d}] \otimes_{\mathbb{C}} H_0(\kappa|_{R_0}) \otimes_{\mathbb{C}} \mathbb{C}[T_\Lambda] \simeq \mathbb{H}(\kappa, q)$$

as complex vector spaces by mapping $e^\xi \otimes h \otimes e^\lambda$ to $Y^\xi h X^\lambda$ ($\xi \in \Lambda^d$, $h \in H_0(\kappa|_{R_0})$ and $\lambda \in \Lambda$). This is the Poincaré-Birkhoff-Witt property of the double affine Hecke algebra $\mathbb{H}(\kappa, q)$.

The dual version of the Lusztig relations (3.2) and (3.3), now also including a commutation relation for T_0, is given as follows. Write $p(X) \in \mathbb{H}(\kappa, q)$ for the element corresponding to $p \in \mathbb{C}[T_\Lambda]$. In other words, $p(X) = \sum_{\lambda \in \Lambda} c_\lambda X^\lambda$ if $p(t) = \sum_{\lambda \in \Lambda} c_\lambda t^\lambda$.

Corollary 4.6. *Let $0 \le i \le n$ and $p \in \mathbb{C}[T_\Lambda] \subset \mathbb{H}(\kappa, q)$. Then*

$$T_i p(X) - (s_{i,q}p)(X) T_i = \left(\frac{\kappa_i - \kappa_i^{-1} + (\kappa_{2a_i} - \kappa_{2a_i}^{-1}) X_q^{a_i}}{1 - X_q^{2a_i}} \right) \big(p(X) - (s_{i,q}p)(X)\big) \tag{4.5}$$

in $\mathbb{H}(\kappa, q)$.

4.2. **Duality antiisomorphism.** Recall that we have associated to $D = (R_0, \Delta_0, \bullet, \Lambda, \Lambda^d)$ the dual initial data $D^d = (R_0^d, \Delta_0^d, \bullet, \Lambda^d, \Lambda)$, see (2.8).

We define the dual double affine braid group by $\mathbb{B}^d := \mathbb{B}(D^d)$. We add a superscript d to elements of $\mathbb{B}^d$ if confusion may arise. So we write ${}^dY^\lambda$ ($\lambda \in \Lambda$), ${}^dX^\xi$ ($\xi \in \Lambda^d$) and T_i^d ($0 \le i \le n$) in $\mathbb{B}^d$. The group generators T_i^d ($1 \le i \le n$) of the homomorphic image of $\mathcal{B}_0$ in $\mathbb{B}^d$ will usually be written without superscripts.

Recall that $Da_0 = -\varphi$ (respectively $-\theta$) if $\bullet = u$ (respectively $\bullet = t$). We have $s_0 = \tau(-(Da_0)^d) s_{Da_0} \in W^\bullet \subseteq W$ and $T_0 = Y^{-(Da_0)^d} T_{s_{Da_0}}^{-1} \in \mathbb{B}$. Dually, $D(a_0^d) = -\theta^d$,

$s_0^d = \tau(\theta)s_\theta \in W^d$ and $T_0^d = Y^\theta T_{s_\theta}^{-1} \in \mathbb{B}^d$. The following result, in the present generality, is from [32, §4] (in [32] the duality isomorphism is constructed, which is related to the duality antiisomorphism below via an elementary antiisomorphism). See also [11, 12, 37, 75, 61] for special cases.

Theorem 4.7. *There exists a unique antiisomorphism $\delta : \mathbb{B} \to \mathbb{B}^d$ satisfying*

$$\begin{aligned} \delta(X^{rc}) &= X^{rc} && \forall\, r \in \mathbb{R}, \\ \delta\big(Y^\xi\big) &= {}^dX^{-\xi} && \forall\, \xi \in \Lambda^d, \\ \delta\big(T_i\big) &= T_i && \forall\, i \in \{1,\ldots,n\}, \\ \delta\big(X^\lambda\big) &= {}^dY^{-\lambda} && \forall\, \lambda \in \Lambda. \end{aligned}$$

Note that $\delta^d = \delta^{-1}$, where δ^d is the duality antiisomorphism with respect to the dual initial data D^d.

Recall that for $\lambda \in \Lambda$, $v(\lambda)$ is the element in W_0 of smallest length such that $\lambda_- = v(\lambda)\lambda$. Then $\tau(\lambda) = u^d(\lambda)v(\lambda)$ in W^d, where $u^d(\lambda)$ is the shortest element (with respect to the length function l^d on W^d) of the coset $\tau(\lambda)W_0$ in W^d. Then $\Omega = \{u^d(\lambda) \,|\, \lambda \in \Lambda^+_{min}\}$ and in $\mathbb{B}^d$,

$${}^dY^\lambda = u^d(\lambda)T_{v(\lambda)}$$

hence

$$\delta^d(u^d(\lambda)) = T^{-1}_{v(\lambda)^{-1}}X^{-\lambda}$$

in $\mathbb{B}$. On the other hand, $\tau(\theta) = s_0^d s_\theta$ and $l^d(\tau(\theta)) = l(s_\theta) + 1$, hence

$$T_0^d = {}^dY^\theta T_{s_\theta}^{-1}$$

in $\mathbb{B}^d$ and

$$\delta^d(T_0^d) = T_{s_\theta}^{-1}X^{-\theta}$$

in $\mathbb{B}$.

The next result shows that the duality antiisomorphism from Theorem 4.7 descends to an antiisomorphism between double affine Hecke algebras. In order to do so, we use the isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^d$ from the complex torus $\mathcal{M} = \mathcal{M}(D)$ onto $\mathcal{M}^d = \mathcal{M}(D^d)$ from Lemma 2.13.

Theorem 4.8. *The antiisomorphism $\delta : \mathbb{B} \to \mathbb{B}^d$ descends to an antiisomorphism $\delta : \mathbb{H}(\kappa, q) \to \mathbb{H}^d(\kappa^d, q) := \mathbb{H}(D^d; \kappa^d, q)$.*

For instance, the cross relations (4.5) in $\mathbb{H}(\kappa, q)$ for $i \in \{1,\ldots,n\}$ match with the the cross relations (3.4) in $\mathbb{H}^d(\kappa^d, q)$ through the antiisomorphism δ. Note that $\delta^d = \delta^{-1}$ also on the level of the double affine Hecke algebra, where δ^d is the duality antiisomorphism with respect to the dual data (D^d, κ^d).

4.3. **Evaluation formulas.** We follow closely the arguments from [12] (which relates to the special case that $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d)))$. See also [79, 61, 32]. We assume in this subsection that q and $\kappa\in\mathcal{M}(D)$ satisfy either (3.5) or (3.6). Then we have the monic nonsymmetric Macdonald-Koornwinder polynomials $P_\lambda\in\mathbb{C}[T_\Lambda]$ $(\lambda\in\Lambda)$ associated to (D,κ,q), as well as the dual monic nonsymmetric Macdonald-Koornwinder polynomials $P_\xi^d\in\mathbb{C}[T_{\Lambda^d}]$ $(\xi\in\Lambda^d)$ associated to (D^d,κ^d,q).

Write $\gamma_{\xi,q}^d=\gamma_{\xi,q}(D^d;\kappa^{d\bullet})\in T_\Lambda$ $(\xi\in\Lambda^d)$ for the spectral points with respect to the dual initial data (D^d,κ^d,q). Concretely, they are given by

$$\gamma_{\xi,q}^d=q^\xi\prod_{\alpha\in R_0^+}{}^d\upsilon_\alpha^{\eta((\xi,\alpha^{d\vee}))\alpha^\vee}$$

with ${}^d\upsilon_\alpha:=(\kappa_{\alpha^d}^d)^{\frac{1}{2}}(\kappa^d_{\mu_{\alpha^d}c+\alpha^d})^{\frac{1}{2}}=\kappa_\alpha^{\frac{1}{2}}\kappa_{2\alpha}^{\frac{1}{2}}$ for $\alpha\in R_0^+$.

Cherednik's basic representations

$$\pi_{D;\kappa,q}:H(\kappa^\bullet)\hookrightarrow\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_\Lambda]\big),$$
$$\pi_{D^d;\kappa^d,q}:H(\kappa^{d\bullet})\hookrightarrow\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_{\Lambda^d}]\big)$$

extend to algebra maps

$$\widehat{\pi}:\mathbb{H}(\kappa,q)\hookrightarrow\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_\Lambda]\big),$$
$$\widehat{\pi}^d:\mathbb{H}(\kappa^d,q)\hookrightarrow\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}[T_{\Lambda^d}]\big)$$

by $\widehat{\pi}\big(X_q^{\widehat{\lambda}}\big)=e_q^{\widehat{\lambda}}$ for $\widehat{\lambda}\in\widehat{\Lambda}$ and $\widehat{\pi}^d\big({}^dX_q^{\widehat{\xi}}\big)=e_q^{\widehat{\xi}}$ for $\widehat{\xi}\in\widehat{\Lambda^d}$.

Note that

$$\gamma_{0,q}^d=\prod_{\alpha\in R_0^+}{}^d\upsilon_\alpha^{-\alpha^\vee}\in T_\Lambda,\qquad \gamma_{0,q}=\prod_{\alpha\in R_0^+}\upsilon_\alpha^{-\alpha^{d\vee}}\in T_{\Lambda^d}.$$

Definition 4.9. *Define evaluation maps* $\mathrm{Ev}:\mathbb{H}(\kappa,q)\to\mathbb{C}$ *and* $\mathrm{Ev}^d:{}^d\mathbb{H}(\kappa^d,q)\to\mathbb{C}$ *by*

$$\mathrm{Ev}(Z):=\big(\widehat{\pi}(Z)1\big)(\gamma_{0,q}^d),\qquad Z\in\mathbb{H}(\kappa,q),$$
$$\mathrm{Ev}^d(Z):=\big(\widehat{\pi}^d(Z)1\big)(\gamma_{0,q}),\qquad Z\in\mathbb{H}^d(\kappa^d,q).$$

The following lemma is crucial for the duality of the (nonsymmetric) Macdonald-Koornwinder polynomials.

Lemma 4.10. *For all* $Z\in\mathbb{H}(\kappa,q)$,

$$\mathrm{Ev}^d(\delta(Z))=\mathrm{Ev}(Z).$$

Write $\mathbb{H}=\mathbb{H}(\kappa,q)$ and $\mathbb{H}^d=\mathbb{H}^d(\kappa^d,q)$. Define bilinear forms

$$B:\mathbb{H}\times\mathbb{H}^d\to\mathbb{C},\qquad B(Z,\widetilde{Z})=\mathrm{Ev}(\delta^d(\widetilde{Z})Z),$$
$$B^d:\mathbb{H}^d\times\mathbb{H}\to\mathbb{C},\qquad B^d(\widetilde{Z},Z)=\mathrm{Ev}^d(\delta(Z)\widetilde{Z})$$

for $Z\in\mathbb{H}$ and $\widetilde{Z}\in\mathbb{H}^d$.

Corollary 4.11. $B(Z,\widetilde{Z})=B^d(\widetilde{Z},Z)$ *for all* $Z\in\mathbb{H}$ *and all* $\widetilde{Z}\in\mathbb{H}^d$.

The following elementary lemma provides convenient tools to derive the evaluation formula for nonsymmetric Macdonald-Koornwinder polynomials.

Lemma 4.12. *Let $p \in \mathbb{C}[T_\Lambda]$, $\widetilde{p} \in \mathbb{C}[T_{\Lambda^d}]$, $Z, Z_1, Z_2 \in \mathbb{H}$ and $\widetilde{Z}, \widetilde{Z}_1, \widetilde{Z}_2 \in \mathbb{H}^d$. Then*
(i) $B(Z_1 Z_2, \widetilde{Z}) = B(Z_2, \delta(Z_1)\widetilde{Z})$.
(ii) $B(ZT_i, \widetilde{Z}) = \kappa_i B(Z, \widetilde{Z})$ *for* $0 \leq i \leq n$.
(iii) $B((\widehat{\pi}(Z)(p))(X), \widetilde{Z}) = B(Zp(X), \widetilde{Z})$.

Lemma 4.12 and Corollary 4.11 imply

Proposition 4.13. **(i)** *For $\xi \in \Lambda^d$ and $p \in \mathbb{C}[T_\Lambda]$,*

$$P_\xi^d(\gamma_{0,q}) p(\gamma_{\xi,q}^d) = B(p, P_\xi^d).$$

(ii) *For $\lambda \in \Lambda$ and $\widetilde{p} \in \mathbb{C}[T_{\Lambda^d}]$,*

$$P_\lambda(\gamma_{0,q}^d)\widetilde{p}(\gamma_{\lambda,q}) = B^d(\widetilde{p}, P_\lambda).$$

Corollary 4.14 (duality)**.** *For $\lambda \in \Lambda$ and $\xi \in \Lambda^d$ we have*

$$P_\xi^d(\gamma_{0,q}) P_\lambda(\gamma_{\xi,q}^d) = P_\lambda(\gamma_{0,q}^d) P_\xi^d(\gamma_{\lambda,q}).$$

The next aim is to explicitly evaluate $\mathrm{Ev}(P_\lambda) = P_\lambda(\gamma_{0,q}^d)$.

Using the results on the pairing B above and using Proposition 3.5, one first derives an important intermediate result which shows how the action of generators of the double affine Hecke algebra on monic nonsymmetric Macdonald-Koornwinder polynomials P_λ can be explicitly expressed in terms of an action on the degree $\lambda \in \Lambda$ of P_λ.

Recall that W^d acts on Λ by $(w\tau(\lambda), \lambda') \mapsto w(\lambda + \lambda')$ ($w \in W_0$ and $\lambda, \lambda' \in \Lambda$).

Proposition 4.15. **(i)** *If $s_i^d \lambda = \lambda$ for $1 \leq i \leq n$ then*

$$\widehat{\pi}(T_i) P_\lambda = \kappa_i^d P_\lambda.$$

(ii) *If $s_0^d \lambda = \lambda$ then*

$$\widehat{\pi}(\delta^d(T_0^d)) P_\lambda = \kappa_0^d P_\lambda.$$

(iii) *Let $\lambda \in \Lambda_{min}^+$, then*

$$\widehat{\pi}\big(\delta^d(u^d(\lambda))\big) 1 = \kappa_{v(\lambda)}^d P_{-\lambda_-}.$$

(iv) *Suppose $1 \leq i \leq n$, $\lambda \in \Lambda$ such that $(\lambda, \alpha_i^\vee) > 0$. Then*

$$\widehat{\pi}(T_i) P_\lambda = \Big(\frac{\kappa_i^d - (\kappa_i^d)^{-1} + (\kappa_{2\alpha_i^d}^d - (\kappa_{2\alpha_i^d}^d)^{-1})\gamma_{\lambda,q}^{\alpha_i^d}}{1 - \gamma_{\lambda,q}^{2\alpha_i^d}}\Big) P_\lambda + (\kappa_i^d)^{-1} P_{s_i^d \lambda}.$$

(v) *Suppose $\lambda \in \Lambda$ such that $a_0^d(\lambda) > 0$. Then*

$$\widehat{\pi}\big(\delta^{-1}(T_0^d)\big) P_\lambda = \Big(\frac{\kappa_0^d - (\kappa_0^d)^{-1} + (\kappa_{2a_0^d}^d - (\kappa_{2a_0^d}^d)^{-1}) q_\theta \gamma_{\lambda,q}^{-\theta^d}}{1 - q_\theta^2 \gamma_{\lambda,q}^{-2\theta^d}}\Big) P_\lambda + \kappa_{v(s_0^d\lambda)}^d (\kappa_{v(\lambda)}^d)^{-1} P_{s_0^d \lambda}$$

(note that $a_0^d = \mu_\theta c - \theta^d = \mu_\theta(c - \theta^\vee)$ and $s_0^d \lambda = \lambda + (1 - (\lambda, \theta^\vee))\theta$).

The proposition gives the following recursion relations for $\mathrm{Ev}(P_\lambda) = P_\lambda(\gamma_{0,q}^d)$.

Corollary 4.16. **(i)** *If* $\lambda \in \Lambda^+_{min}$ *then*

$$\mathrm{Ev}(P_\lambda) = (\kappa^d_{v(\lambda)})^{-1}.$$

(ii) *If* $\lambda \in \Lambda$, $1 \leq i \leq n$ *and* $a^d_i(\lambda) > 0$ *then*

$$\mathrm{Ev}(P_{s^d_i\lambda}) = \frac{(1-\kappa^d_i\kappa^d_{2\alpha^d_i}\gamma^{\alpha^d_i}_{\lambda,q})(1+\kappa^d_i(\kappa^d_{2\alpha^d_i})^{-1}\gamma^{\alpha^d_i}_{\lambda,q})}{(1-\gamma^{2\alpha^d_i}_{\lambda,q})}\mathrm{Ev}(P_\lambda).$$

(iii) *If* $\lambda \in \Lambda$ *and* $a^d_0(\lambda) > 0$ *then*

$$\mathrm{Ev}(P_{s^d_0\lambda}) = (\kappa^d_0)^{-1}\kappa^d_{v(\lambda)}(\kappa^d_{v(s^d_0\lambda)})^{-1}\frac{(1-q_\theta\kappa^d_0\kappa^d_{2a^d_0}\gamma^{-\theta^d}_{\lambda,q})(1+q_\theta\kappa^d_0(\kappa^d_{2a^d_0})^{-1}\gamma^{-\theta^d}_{\lambda,q})}{(1-q^2_\theta\gamma^{-2\theta^d}_{\lambda,q})}\mathrm{Ev}(P_\lambda).$$

Recall the definition $c_a = c^{\kappa,q}_a(\cdot;D) \in \mathbb{C}(T_\Lambda)$ for $a \in R^\bullet$ from (3.10). The dual version is denoted by $c^d_a = c^{\kappa^d,q}_a(\cdot;D^d)$ ($a \in R^{d\bullet}$). Concretely, for $a \in R^{d\bullet}$, $c^d_a \in \mathbb{C}(T_{\Lambda^d})$ is given by

$$c^d_a(t) := \frac{(1-\kappa^d_a\kappa^d_{2a}t^a_q)(1+\kappa^d_a(\kappa^d_{2a})^{-1}t^a_q)}{1-t^{2a}_q}.$$

We also set for $w \in W^d$,

$$c^d_w := \prod_{a\in R^{d\bullet,+}\cap w^{-1}(R^{d\bullet,-})} c^d_a \in \mathbb{C}(T_{\Lambda^d}).$$

An induction argument gives now the explicit evaluation formula for the nonsymmetric Macdonald-Koornwinder polynomials (see [12, 79, 61]).

Theorem 4.17. *For* $\lambda \in \Lambda$ *we have*

$$\mathrm{Ev}\big(P_\lambda\big) = (\kappa^d_{\tau(\lambda)})^{-1}c^d_{u^d(\lambda)}(\gamma_{0,q}).$$

4.4. **Normalized nonsymmetric Macdonald-Koornwinder polynomials and duality.** The treatment in this subsection is close to [12] and [79], which deal with the case that $(\Lambda,\Lambda^d) = (P(R_0), P(R^d_0))$ and $C^\vee C$ case, respectively. We assume that q and $\kappa \in \mathcal{M}(D)$ satisfy (3.5) or (3.6). Then

$$P_\lambda(\gamma^d_{0,q}) \neq 0 \quad \& \quad P^d_\xi(\gamma_{0,q}) \neq 0$$

for all $\lambda \in \Lambda$ and $\xi \in \Lambda^d$ in view of the evaluation formula (Theorem 4.17).

Recall the Macdonald-Koornwinder polynomials $P_\lambda \in \mathbb{C}[T_\Lambda]$ ($\lambda \in \Lambda$) and $P^d_\xi \in \mathbb{C}[T_{\Lambda^d}]$ ($\xi \in \Lambda^d$) satisfy

$$\widehat{\pi}(p(Y))P_\lambda = p(\gamma^{-1}_{\lambda,q})P_\lambda,$$
$$\widehat{\pi}^d(r({}^dY))P^d_\xi = r(\gamma^{d\,-1}_{\xi,q})P^d_\xi$$

for all $p \in \mathbb{C}[T_{\Lambda^d}]$ and $r \in \mathbb{C}[T_\Lambda]$. This motivates the following notation for the normalized nonsymmetric Macdonald-Koornwinder polynomials.

Definition 4.18. *The normalized nonsymmetric Macdonald-Koornwinder polynomials are defined by*

$$E(\gamma_{\lambda,q}^{-1};\cdot) := \frac{P_\lambda}{P_\lambda(\gamma_{0,q}^d)} \in \mathbb{C}[T_\Lambda], \qquad \lambda \in \Lambda$$

and

$$E^d((\gamma_{\xi,q}^d)^{-1};\cdot) := \frac{P_\xi^d}{P_\xi^d(\gamma_{0,q})} \in \mathbb{C}[T_{\Lambda^d}], \qquad \xi \in \Lambda^d.$$

We denote by $E^\circ(\gamma_{\lambda,q};\cdot) \in \mathbb{C}[T_\Lambda]$ the normalized Macdonald-Koornwinder polynomial with respect to the inverted parameters (κ^{-1}, q^{-1}) (and similarly for $E^{d\circ}(\gamma_{\xi,q}^d;\cdot)$).

For $\lambda \in \Lambda$ we have $E(\gamma_{\lambda,q}^{-1};\gamma_{0,q}^d) = 1$. From the previous subsection we immediately get the self-duality of the normalized nonsymmetric Macdonald-Koornwinder polynomials (cf. [12, 75, 61, 32]).

Corollary 4.19. *For all $p \in \mathbb{C}[T_\Lambda]$, $r \in \mathbb{C}[T_{\Lambda^d}]$, $\lambda \in \Lambda$ and $\xi \in \Lambda^d$,*

$$B(p, E^d(\gamma_{\xi,q}^{d\,-1};\cdot)) = p(\gamma_{\xi,q}^d),$$
$$B(E(\gamma_{\lambda,q}^{-1};\cdot), r) = r(\gamma_{\lambda,q}).$$

In particular, $E(\gamma_{\lambda,q}^{-1};\gamma_{\xi,q}^d) = E^d(\gamma_{\xi,q}^{d\,-1};\gamma_{\lambda,q})$ for all $\lambda \in \Lambda$ and $\gamma \in \Lambda^d$.

4.5. **Polynomial Fourier transform.** For the remainder of the text we assume that the parameters q and $\kappa \in \mathcal{M}(D)$ satisfy the more restrictive parameter conditions (3.9).

In order to compute the norms of the normalized nonsymmetric Macdonald-Koornwinder polynomials it is convenient to formulate the explicit formulas from Proposition 4.15 in terms of properties of a Fourier transform whose kernel is given by the normalized nonsymmetric Macdonald-Koornwinder polynomial. For this we first need to consider the adjoint of the double affine Hecke algebra action with respect to the sesquilinear form

$$\langle p_1, p_2\rangle := \int_{T_\Lambda^u} p_1(t)\overline{p_2(t)}v(t)dt, \qquad p_1, p_2 \in \mathbb{C}[T_\Lambda]$$

and express it in terms of an explicit antilinear antiisomorphism of the double affine Hecke algebra.

Lemma 4.20. *There exists a unique antilinear antialgebra isomorphism $^\ddagger : \mathbb{H}(\kappa, q) \to \mathbb{H}(\kappa^{-1}, q^{-1})$ satisfying $T_w^\ddagger = T_w^{-1}$ $(w \in W)$ and $(X^\lambda)^\ddagger = X^{-\lambda}$ $(\lambda \in \Lambda)$. In addition,*

$$\langle \widehat{\pi}_{\kappa,q}(h)p_1, p_2\rangle = \langle p_1, \widehat{\pi}_{\kappa^{-1},q^{-1}}(h^\ddagger)p_2\rangle$$

for all $h \in \mathbb{H}$.

Define

$$\mathcal{S} := \{\gamma_{\lambda,q} \mid \lambda \in \Lambda\} \subset T_{\Lambda^d},$$
$$\mathcal{S}^d := \{\gamma_{\xi,q}^d \mid \xi \in \Lambda^d\} \subset T_\Lambda,$$

and write $F(\mathcal{S})$ (respectively $F(\mathcal{S}^d)$) for the space of finitely supported complex valued functions on $\mathcal{S}$ (respectively $\mathcal{S}^d$). The following lemma follows easily from the results in Subsection 2.2 and from Proposition 3.5.

Lemma 4.21. *There exists a unique algebra homomorphism* $\widehat{\rho}^d : \mathbb{H}^d \to \mathrm{End}_{\mathbb{C}}(F(\mathcal{S}))$ *satisfying for* $0 \leq i \leq n$,

$$
\begin{aligned}
(\widehat{\rho}^d(T_i^d)g)(\gamma_{\lambda,q}) &= \kappa_i^d g(\gamma_{\lambda,q}) + (\kappa_i^d)^{-1} c^d_{a_i^d}(\gamma_{\lambda,q})(g(\gamma_{s_i^d\lambda,q}) - g(\gamma_{\lambda,q})) \qquad && \text{if } s_i^d\lambda \neq \lambda,\\
(\widehat{\rho}^d(T_i^d)g)(\gamma_{\lambda,q}) &= \kappa_i^d g(\gamma_{\lambda,q}) && \text{if } s_i^d\lambda = \lambda,
\end{aligned}
$$

and satisfying for $\omega \in \Omega^d$,

$$(\widehat{\rho}^d(\omega)g)(\gamma_{\lambda,q}) = g(\gamma_{\omega^{-1}\lambda,q})$$

and for $\xi \in \Lambda^d$,

$$(\widehat{\rho}({}^dX^\xi)g)(\gamma_{\lambda,q}) = \gamma^\xi_{\lambda,q} g(\gamma_{\lambda,q}).$$

Definition 4.22. *We define the complex linear map*

$$\mathcal{F} : \mathbb{C}[T_\Lambda] \to F(\mathcal{S})$$

by

$$(\mathcal{F}p)(\gamma) := \langle p, E^\circ(\gamma; \cdot)\rangle, \qquad p \in \mathbb{C}[T_\Lambda],\ \gamma \in \mathcal{S}.$$

For generators $X \in \mathbb{H}$ and $p \in \mathbb{C}[T_\Lambda]$ we can re-express $\mathcal{F}(\widehat{\pi}(X)p)$ as an explicit linear operator acting on $\mathcal{F}p \in F(\mathcal{S})$ using Lemma 4.20, Proposition 4.15 and Theorem 4.17. It gives the following result (the first part of the theorem follows easily from the duality antiisomorphism).

Theorem 4.23. **(i)** *The following formulas define an algebra isomorphism* $\Phi : \mathbb{H} \to \mathbb{H}^d$,

$$
\begin{aligned}
\Phi(T_i) &= T_i^d, && 1 \leq i \leq n,\\
\Phi(T_{s_\theta}^{-1}X^{-\theta}) &= T_0^d, &&\\
\Phi(T^{-1}_{v(\lambda)^{-1}}X^{-\lambda}) &= u^d(\lambda)^{-1}, && \lambda \in \Lambda^+_{min},\\
\Phi(Y^\xi) &= {}^dX^{-\xi}, && \xi \in \Lambda^d.
\end{aligned}
$$

(ii) *We have for all* $h \in \mathbb{H}$,

$$\mathcal{F} \circ \widehat{\pi}(h) = \widehat{\rho}^d(\Phi(h)) \circ \mathcal{F}.$$

4.6. **Intertwiners and norm formulas.** Define for $0 \leq i \leq n$,

$$I_i^d := [T_i^d, {}^dX_q^{a_i^d}] \in \mathbb{H}^d,$$

and $I_\omega^d := \omega \in \mathbb{H}^d$ ($\omega \in \Omega^d$). The following theorem extends results from [13, 75, 79].

Theorem 4.24. *For a reduced expression* $w = \omega s_{i_1}^d s_{i_2}^d \cdots s_{i_r}^d \in W^d$ ($\omega \in \Omega^d$, $0 \leq i_j \leq n$), *the expression*

$$I_w^d := I_\omega^d I_{i_1}^d I_{i_2}^d \cdots I_{i_r}^d \in \mathbb{H}^d$$

is well defined (independent of the choice of reduced expression). In addition,

$$\widehat{\pi}^d(I_w^d) := r_w^d \cdot w_q \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T_{\Lambda^d}]),$$

where $r_w^d := \prod_{a \in {}^dR^{\bullet +} \cap w({}^dR^{\bullet -})} r_a^d \in \mathbb{C}[T_{\Lambda^d}]$ *and* $r_a^d \in \mathbb{C}[T_{\Lambda^d}]$ *is defined by*

$$r_a^d(t) := (\kappa_a^d)^{-1}(1 - \kappa_a^d\kappa_{2a}^d t_q^a)(1 + \kappa_a^d(\kappa_{2a}^d)^{-1}t_q^a)$$

for $t \in T_{\Lambda^d}$. In addition, in $\mathbb{H}^d$,

$$I_w^d I_{w^{-1}}^d = r_w^d({}^dX)(w_q r_{w^{-1}}^d)({}^dX)$$

and

$$I_w^d p({}^dX) = (w_q p)({}^dX) I_w^d$$

for all $p \in \mathbb{C}[T_{\Lambda^d}]$.

The $I_w^d \in \mathbb{H}^d$ ($w \in W^d$) are called the dual intertwiners. The intertwiners are defined by

$$\mathcal{I}_w := \delta^d(I_w^d) \in \mathbb{H}, \qquad w \in W^d$$

Then in $\mathbb{H}$,

$$\mathcal{I}_{w^{-1}}\mathcal{I}_w = r_w^d(Y^{-1})(w_q r_{w^{-1}}^d)(Y^{-1})$$

and

$$p(Y^{-1})\mathcal{I}_w = \mathcal{I}_w(w_q p)(Y^{-1})$$

for all $p \in \mathbb{C}[T_{\Lambda^d}]$. Proposition 4.15 and Theorem 4.17 give the following result.

Proposition 4.25. **(i)** *If $\lambda \in \Lambda$ and $0 \leq i \leq n$ satisfy $s_i^d\lambda \neq \lambda$ then*

$$\widehat{\pi}(\mathcal{I}_i)E(\gamma_{\lambda,q}^{-1};\cdot) = \gamma_{\lambda,q}^{-a_i^d} r_{a_i^d}^d(\gamma_{\lambda,q})E(\gamma_{s_i^d\lambda,q}^{-1};\cdot).$$

(ii) *If $\lambda \in \Lambda$ and $0 \leq i \leq n$ satisfy $s_i^d\lambda = \lambda$ then*

$$\widehat{\pi}(\mathcal{I}_i)E(\gamma_{\lambda,q}^{-1};\cdot) = 0.$$

(iii) *If $\omega \in \Omega^d$ and $\lambda \in \Lambda$ then*

$$\widehat{\pi}(\mathcal{I}_\omega)E(\gamma_{\lambda,q}^{-1};\cdot) = E(\gamma_{\omega^{-1}\lambda,q}^{-1};\cdot).$$

This proposition shows that intertwiners can be used to create the nonsymmetric Macdonald-Koornwinder polynomial from the the constant polynomial $E(\gamma_{0,q}^{-1};\cdot) \equiv 1$ (cf., e.g., [61, §5.10]). We explore now this observation to express the norms of the nonsymmetric Macdonald-Koornwinder in terms of the nonzero constant term

$$\langle 1, 1\rangle = \int_{T_\Lambda^u} v(t)dt = \frac{\mathcal{C}(\gamma_{0,q}^d)}{\#W_0}\int_{T_\Lambda^u} v_+(t)dt.$$

The constant term $\langle 1, 1\rangle$ is a q-analog and root system generalization of the Selberg integral, whose explicit evaluation was conjectured by Macdonald [60] in case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$. By various methods it was evaluated in special cases (for references we refer to the detailed discussions in [60, 9]). A uniform proof in case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$ using shift operators was given in [9, Thm. 0.1] (see [79] for the $C^\vee C$ case).

Write for $\lambda \in \Lambda$,

$$N(\lambda) := \frac{\langle E(\gamma_{\lambda,q}^{-1};\cdot), E^\circ(\gamma_{\lambda,q};\cdot)\rangle}{\langle 1, 1\rangle}. \tag{4.6}$$

We write

$$c_w^d := \prod_{a\in R^{d\bullet,+}\cap w^{-1}(R^{d\bullet,-})} c_a^d \in \mathbb{C}[T_{\Lambda^d}],$$

$$c_{-w}^d := \prod_{a\in R^{d\bullet,+}\cap w^{-1}(R^{d\bullet,-})} c_{-a}^d \in \mathbb{C}[T_{\Lambda^d}]$$

for $w \in W^d$ (warning: $c_{-w}^d(t) = c_w^d(t^{-1})$ is only valid if $w \in W_0$). Part **(i)** of the following theorem should be compared with [15, Prop. 3.4.1] and [61, (5.2.11)] (it originates from [12] in case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$ and [75, 79] in the $C^\vee C$ case). The first part of the theorem follows from Proposition 4.25 and the fact that the intertwiners behave nicely with respect to the antiinvolution ‡ (see Lemma 4.20): for all $0 \leq i \leq n$,

$$\mathcal{I}_i^\ddagger = \mathcal{I}_i$$

in $\mathbb{H}(\kappa^{-1}, q^{-1})$ and for all $\omega \in \Omega^d$,

$$\mathcal{I}_\omega^\ddagger = \mathcal{I}_{\omega^{-1}}$$

in $\mathbb{H}(\kappa^{-1}, q^{-1})$. The second part of the theorem is immediate from the first and from the biorthogonality of the nonsymmetric Macdonald-Koornwinder polynomials (see Theorem 3.13).

Theorem 4.26. **(i)** *For $\lambda \in \Lambda$,*

$$N(\lambda) = \frac{c_{-u^d(\lambda)}^d(\gamma_{0,q})}{c_{u^d(\lambda)}^d(\gamma_{0,q})},$$

which is nonzero for all $\lambda \in \Lambda$.
(ii) *The transform $\mathcal{F} : \mathbb{C}[T_\Lambda] \to F(\mathcal{S})$ is a linear bijection with inverse $\mathcal{G} : F(\mathcal{S}) \to \mathbb{C}[T_\Lambda]$ given by*

$$(\mathcal{G}f)(t) := \frac{1}{\langle 1, 1\rangle} \sum_{\lambda\in\Lambda} N(\lambda)^{-1} f(\gamma_{\lambda,q}) E(\gamma_{\lambda,q}^{-1}; t)$$

for $f \in F(\mathcal{S})$ and $t \in T_\Lambda$.

Combined with the evaluation formula (Theorem 4.17) we get

Corollary 4.27. *For all $\lambda \in \Lambda$,*

$$\frac{\langle P_\lambda, P_\lambda^\circ\rangle}{\langle 1, 1\rangle} = \big(\kappa_{u^d(\lambda)}^d\big)^2 c_{u^d(\lambda)}^d(\gamma_{0,q}) c_{-u^d(\lambda)}^d(\gamma_{0,q}).$$

4.7. **Normalized symmetric Macdonald-Koornwinder polynomials.** We keep the same assumptions on q and on $\kappa \in \mathcal{M}(D)$ as in the previous subsection. The results in this subsection are from [12] in case $(\Lambda, \Lambda^d) = (P(R_0), P(R_0^d))$ and from [75, 79] in the $C^\vee C$ case.

For $\lambda \in \Lambda^-$ define

$$E^+(\gamma_{\lambda,q}^{-1}; \cdot) \in \mathbb{C}[T_\Lambda]^{W_0}$$

by

$$E^+(\gamma_{\lambda,q}^{-1};\cdot) := \widehat{\pi}_{\kappa,q}(C_+)E(\gamma_{\lambda,q}^{-1};\cdot),$$

where (recall)

$$C_+ := \frac{1}{\sum_{w\in W_0}\kappa_w^2}\sum_{w\in W_0}\kappa_w T_w.$$

We call $E^+(\gamma_{\lambda,q}^{-1};\cdot)$ the normalized symmetric Macdonald-Koornwinder polynomial of degree $\lambda\in\Lambda^-$.

Lemma 4.28. **(i)** *For all $w\in W_0$ and $\lambda\in\Lambda^-$,*

$$E^+(\gamma_{\lambda,q}^{-1};\cdot) = \widehat{\pi}(C_+)E(\gamma_{w\lambda,q}^{-1};\cdot). \tag{4.7}$$

(ii) $E^+(\gamma_{\lambda,q}^{-1};\gamma_{0,q}^d)=1$ *for all $\lambda\in\Lambda^-$.*
(iii) $\{E^+(\gamma_{\lambda,q}^{-1};\cdot)\}_{\lambda\in\Lambda^-}$ *is a basis of $\mathbb{C}[T_\Lambda]^{W_0}$ satisfying, for all $p\in\mathbb{C}[T_{\Lambda^d}]^{W_0}$,*

$$D_p(E^+(\gamma_{\lambda,q}^{-1};\cdot)) = p(\gamma_{\lambda,q}^{-1})E^+(\gamma_{\lambda,q}^{-1};\cdot) = p(q^{-\lambda}\gamma_{0,q}^{-1})E^+(\gamma_{\lambda,q}^{-1};\cdot).$$

(iv) *For all $\lambda\in\Lambda^-$,*

$$E^+(\gamma_{\lambda,q}^{-1};\cdot) = \frac{P_{\lambda_+}^+(\cdot)}{P_{\lambda_+}^+(\gamma_{0,q}^d)}.$$

(iv) *For all $\lambda\in\Lambda^-$ and $\xi\in\Lambda^{d-}$,*

$$E^+(\gamma_{\lambda,q}^{-1};\gamma_{\xi,q}^d) = E^{+\,d}(\gamma_{\xi,q}^{d\,-1};\gamma_{\lambda,q}).$$

As before we write superindex $\circ$ to indicate that the parameters (κ,q) are inverted. The nonsymmetric and symmetric Macdonald-Koornwinder polynomials with inverted parameters (κ^{-1},q^{-1}) can be explicitly expressed in terms of the ones with parameters (κ,q). The result is as follows (see [15, §3.3.2] for $(\Lambda,\Lambda^d)=(P(R_0),P(R_0^d))$ and [82, (2.5)] in the twisted case).

Proposition 4.29. **(i)** *For all $\lambda\in\Lambda$,*

$$E^\circ(\gamma_{\lambda,q};t^{-1}) = \kappa_{w_0}^{-1}\big(\widehat{\pi}_{\kappa,q}(T_{w_0})E(\gamma_{-w_0\lambda,q}^{-1};\cdot)\big)(t)$$

in $\mathbb{C}[T_\Lambda]$, where $w_0\in W_0$ is the longest Weyl group element and $t\in T_\Lambda$.
(ii) *For all $\lambda\in\Lambda^-$ we have*

$$E^{+\,\circ}(\gamma_{\lambda,q};t^{-1}) = E^+(\gamma_{-w_0\lambda,q}^{-1};t),$$
$$E^{+\,\circ}(\gamma_{\lambda,q};t) = E^+(\gamma_{\lambda,q}^{-1};t)$$

in $\mathbb{C}[T_\Lambda]^{W_0}$.

Using intertwiners or Proposition 4.15 it is now possible to expand the normalized symmetric Macdonald-Koornwinder polynomials in nonsymmetric ones. This in turn leads to an explicit expression of the quadratic norms of the symmetric Macdonald-Koornwinder polynomials in terms of the nonsymmetric ones. Recall the rational function $\mathcal{C}(\cdot)=\mathcal{C}(\cdot;D;\kappa,q)\in\mathbb{C}(T_\Lambda)$ from (3.14). We write $\mathcal{C}^d(\cdot)=\mathcal{C}(\cdot;D^d;\kappa^d,q)\in\mathbb{C}(T_{\Lambda^d})$ for its dual version.

Theorem 4.30. **(i)** *For $\lambda \in \Lambda^-$ we have*

$$P_{\lambda_+}^+(t) = \sum_{\mu \in W_0\lambda} \left(\prod_{\alpha \in R_0^+ \cap v(\mu) R_0^-} c_\alpha^d(\gamma_{\lambda,q}) \right) P_\mu(t). \tag{4.8}$$

(ii) *For $\lambda \in \Lambda^-$ we have*

$$E^+(\gamma_{\lambda,q}^{-1}; t) = \sum_{\mu \in W_0\lambda} \frac{\mathcal{C}^d(\gamma_{\mu,q})}{\mathcal{C}^d(\gamma_{0,q})} E(\gamma_{\mu,q}^{-1}; t).$$

(iii) *For $\lambda, \mu \in \Lambda^-$ we have*

$$\frac{\langle E^+(\gamma_{\lambda,q}^{-1}; \cdot), E^+(\gamma_{\mu,q}^{-1}; \cdot) \rangle_+}{\langle 1, 1 \rangle_+} = \delta_{\lambda,\mu} \frac{\mathcal{C}^d(\gamma_{\lambda,q}) N(\lambda)}{\mathcal{C}^d(\gamma_{0,q})}$$

where $\langle p, r \rangle_+ := \int_{T_\Lambda^u} p(t)\overline{r(t)} v_+(t) dt$.

As a consequence we get the following explicit evaluation formulas and quadratic norm formulas for the monic symmetric Macdonald-Koornwinder polynomials. Set

$$N^+(\lambda) := \frac{\langle E^+(\gamma_{\lambda,q}^{-1}; \cdot), E^+(\gamma_{\lambda,q}^{-1}; \cdot) \rangle_+}{\langle 1, 1 \rangle_+}, \qquad \lambda \in \Lambda^-.$$

Corollary 4.31. **(i)** *For $\lambda \in \Lambda^-$ we have*

$$P_{\lambda_+}^+(\gamma_{0,q}^d) = \frac{\mathcal{C}^d(\gamma_{0,q}) c_{\tau(\lambda)}^d(\gamma_{0,q})}{\mathcal{C}^d(\gamma_{\lambda,q}) \kappa_{\tau(\lambda)}^d}.$$

(ii) *For $\lambda \in \Lambda^-$,*

$$N^+(\lambda) = \frac{\mathcal{C}^d(\gamma_{\lambda,q}) c_{-\tau(\lambda)}^d(\gamma_{0,q})}{\mathcal{C}^d(\gamma_{0,q}) c_{\tau(\lambda)}^d(\gamma_{0,q})}.$$

Remark 4.32. For $\lambda \in \Lambda^-$,

$$\sum_{\mu \in W_0\lambda} N(\mu)^{-1} = N^+(\lambda)^{-1}.$$

5. Explicit evaluation and norm formulas

We rewrite in this subsection the explicit evaluation formulas and the quadratic norm espressions for the symmetric Macdonald-Koornwinder polynomials in terms of q-shifted factorials (3.11). The explicit formulas for the GL_{n+1} symmetric Macdonald polynomials (see Subsection 3.7) can be immediately obtained as special cases of the explicit formulas below. We keep the conditions (3.9) on the parameters (κ, q).

5.1. **The twisted cases.** In case we have initial data $D=(R_0,\Delta_0,\bullet,\Lambda,\Lambda^d)$ with $\bullet=t$ the evaluation and norm formulas take the following explicit form.

Corollary 5.1. **(i)** *Suppose $\bullet=t$ and $S=\emptyset=S^d$. Then $R=R^\bullet=R^d$, $\kappa=\kappa^d$ and $\kappa_{m\mu_\alpha c+\alpha}=\kappa_\alpha$ for all $m\in\mathbb{Z}$ and $\alpha\in R_0$. Then for all $\lambda\in\Lambda^-$,*

$$P^+_{\lambda_+}(\gamma_{0,q})=\gamma_{0,q}^{-\lambda}\prod_{\alpha\in R_0^+}\frac{\big(\kappa_\alpha^2\gamma_{0,q}^{-\alpha};q_\alpha\big)_{-(\lambda,\alpha^\vee)}}{\big(\gamma_{0,q}^{-\alpha};q_\alpha\big)_{-(\lambda,\alpha^\vee)}},$$

$$N^+(\lambda)=\gamma_{0,q}^{2\lambda}\prod_{\alpha\in R_0^+}\left(\frac{1-\gamma_{0,q}^{-\alpha}}{1-q_\alpha^{-(\lambda,\alpha^\vee)}\gamma_{0,q}^{-\alpha}}\right)\frac{\big(q_\alpha\kappa_\alpha^{-2}\gamma_{0,q}^{-\alpha};q_\alpha\big)_{-(\lambda,\alpha^\vee)}}{\big(\kappa_\alpha^2\gamma_{0,q}^{-\alpha};q_\alpha\big)_{-(\lambda,\alpha^\vee)}}.$$

(ii) *Suppose $\bullet=t$ and $(\Lambda,\Lambda^d)=(\mathbb{Z}R_0,\mathbb{Z}R_0)$ (the $C^\vee C$ case), realized concretely as in Subsection 3.8. Recall the relabelling of the multiplicity functions κ and κ^d given by $k=\kappa_\varphi^2=\widetilde{k}$ and*

$$\{a,b,c,d\}=\{\kappa_\theta\kappa_{2\theta},-\kappa_\theta\kappa_{2\theta}^{-1},q_\theta\kappa_0\kappa_{2a_0},-q_\theta\kappa_0\kappa_{2a_0}^{-1}\},$$
$$\{\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\}=\{\kappa_\theta\kappa_0,-\kappa_\theta\kappa_0^{-1},q_\theta\kappa_{2\theta}\kappa_{2a_0},-q_\theta\kappa_{2\theta}\kappa_{2a_0}^{-1}\}.$$

Then for $\lambda\in\Lambda^-$ we have

$$P^+_{\lambda_+}(\gamma_{0,q}^d)=(\gamma_{0,q}^d)^{-\lambda}\prod_{\alpha\in R_{0,l}^+}\frac{\big(\kappa_\varphi^2\gamma_{0,q}^{-\alpha};q_\varphi\big)_{-(\lambda,\alpha^\vee)}}{\big(\gamma_{0,q}^{-\alpha};q_\varphi\big)_{-(\lambda,\alpha^\vee)}}$$
$$\times\prod_{\alpha\in R_{0,s}^+}\frac{\big(\widetilde{a}\gamma_{0,q}^{-\alpha},\widetilde{b}\gamma_{0,q}^{-\alpha},\widetilde{c}\gamma_{0,q}^{-\alpha},\widetilde{d}\gamma_{0,q}^{-\alpha};q_\theta^2\big)_{-(\lambda,\alpha^\vee)/2}}{\big(\gamma_{0,q}^{-2\alpha};q_\theta^2\big)_{-(\lambda,\alpha^\vee)}}$$

and

$$N^+(\lambda)=(\gamma_{0,q}^d)^{2\lambda}\prod_{\alpha\in R_{0,l}^+}\left(\frac{1-\gamma_{0,q}^{-\alpha}}{1-q_\varphi^{-(\lambda,\alpha^\vee)}\gamma_{0,q}^{-\alpha}}\right)\frac{\big(q_\varphi k_\varphi^{-2}\gamma_{0,q}^{-\alpha};q_\varphi\big)_{-(\lambda,\alpha^\vee)}}{\big(k_\varphi^2\gamma_{0,q}^{-\alpha};q_\varphi\big)_{-(\lambda,\alpha^\vee)}}$$
$$\times\prod_{\alpha\in R_{0,s}^+}\left(\frac{1-\gamma_{0,q}^{-2\alpha}}{1-q_\theta^{-2(\lambda,\alpha^\vee)}\gamma_{0,q}^{-2\alpha}}\right)\frac{\big(q_\theta^2\widetilde{a}^{-1}\gamma_{0,q}^{-\alpha},q_\theta^2\widetilde{b}^{-1}\gamma_{0,q}^{-\alpha},q_\theta^2\widetilde{c}^{-1}\gamma_{0,q}^{-\alpha},q_\theta^2\widetilde{d}^{-1}\gamma_{0,q}^{-\alpha};q_\theta^2\big)_{-(\lambda,\alpha^\vee)/2}}{\big(\widetilde{a}\gamma_{0,q}^{-\alpha},\widetilde{b}\gamma_{0,q}^{-\alpha},\widetilde{c}\gamma_{0,q}^{-\alpha},\widetilde{d}\gamma_{0,q}^{-\alpha};q_\theta^2\big)_{-(\lambda,\alpha^\vee)/2}},$$

where $R_{0,s}\subset R_0$ (respectively $R_{0,l}\subset R_0$) are the short (respectively long) roots in R_0. In addition, $q_\varphi=q_\theta^2$.

The formulas in the intermediate case $\bullet=t$ and $S=\emptyset\neq S^d$ (or $S\neq\emptyset=S^d$) are special cases of the $C^\vee C$ case (by choosing an appropriate specialization of the multiplicity function).

5.2. **The untwisted cases.** If the initial data is of the form $D=(R_0,\Delta_0,u,\Lambda,\Lambda^d)$ then there are essentially two cases to consider, namely the case that $S=\emptyset=S^d$ and the exceptional rank two case (see Subsection 3.9). Indeed, the cases corresponding to a nonreduced extension of the untwisted affine root system with underlying finite root system R_0 of type B_n $(n\geq 3)$ or BC_n $(n\geq 1)$ are special cases of the twisted $C^\vee C_n$ case.

Corollary 5.2. **(i)** *Suppose $\bullet=u$ and $S=\emptyset=S^d$. Then $R=\mathbb{Z}c+R_0$, $R^d=\mathbb{Z}c+R_0^\vee$ and $\kappa_{mc+\alpha}=\kappa_\alpha=\kappa^d_{mc+\alpha^\vee}$ for all $m\in\mathbb{Z}$ and $\alpha\in R_0$. Then for all $\lambda\in\Lambda^-$,*

$$P^+_{\lambda_+}(\gamma^d_{0,q})=\gamma_{0,q}^{-\lambda}\prod_{\alpha\in R_0^+}\frac{\big(\kappa_\alpha^2\gamma_{0,q}^{-\alpha^\vee};q\big)_{-(\lambda,\alpha^\vee)}}{\big(\gamma_{0,q}^{-\alpha^\vee};q\big)_{-(\lambda,\alpha^\vee)}},$$

$$N^+(\lambda)=\gamma_{0,q}^{2\lambda}\prod_{\alpha\in R_0^+}\left(\frac{1-\gamma_{0,q}^{-\alpha^\vee}}{1-q^{-(\lambda,\alpha^\vee)}\gamma_{0,q}^{-\alpha^\vee}}\right)\frac{\big(q\kappa_\alpha^{-2}\gamma_{0,q}^{-\alpha^\vee};q\big)_{-(\lambda,\alpha^\vee)}}{\big(\kappa_\alpha^2\gamma_{0,q}^{-\alpha^\vee};q\big)_{-(\lambda,\alpha^\vee)}}.$$

(ii) *Suppose $D=(R_0,\Delta_0,u,\mathbb{Z}R_0,\mathbb{Z}R_0^\vee)$ with R_0 of type C_2, realized concretely as in Subsection 3.9. Recall the relabelling of the multiplicity functions κ and κ^d given by (3.19). Then for $\lambda\in\Lambda^-$ we have*

$$P^+_{\lambda_+}(\gamma^d_{0,q})=(\gamma^d_{0,q})^{-\lambda}\prod_{\alpha\in R^+_{0,s}}\frac{\big(\widetilde{c}\gamma_{0,q}^{-\alpha^\vee},\widetilde{d}\gamma_{0,q}^{-\alpha^\vee};q^2\big)_{-(\lambda,\alpha^\vee)/2}}{\big(\gamma_{0,q}^{-\alpha^\vee};q\big)_{-(\lambda,\alpha^\vee)}}\prod_{\beta\in R^+_{0,l}}\frac{\big(\widetilde{a}\gamma_{0,q}^{-\beta^\vee},\widetilde{b}\gamma_{0,q}^{-\beta^\vee};q\big)_{-(\lambda,\beta^\vee)}}{\big(\gamma_{0,q}^{-2\beta^\vee};q^2\big)_{-(\lambda,\beta^\vee)}}$$

and

$$\begin{aligned}N^+(\lambda)=(\gamma^d_{0,q})^{2\lambda}&\prod_{\alpha\in R^+_{0,s}}\left(\frac{1-\gamma_{0,q}^{-\alpha^\vee}}{1-q^{-(\lambda,\alpha^\vee)}\gamma_{0,q}^{-\alpha^\vee}}\right)\frac{\big(q^2\widetilde{c}^{-1}\gamma_{0,q}^{-\alpha^\vee},q^2\widetilde{d}^{-1}\gamma_{0,q}^{-\alpha^\vee};q^2\big)_{-(\lambda,\alpha^\vee)/2}}{\big(\widetilde{c}\gamma_{0,q}^{-\alpha^\vee},\widetilde{d}\gamma_{0,q}^{-\alpha^\vee};q^2\big)_{-(\lambda,\alpha^\vee)/2}}\\&\times\prod_{\beta\in R^+_{0,l}}\left(\frac{1-\gamma_{0,q}^{-2\beta^\vee}}{1-q^{-2(\lambda,\beta^\vee)}\gamma_{0,q}^{-2\beta^\vee}}\right)\frac{\big(q\widetilde{a}^{-1}\gamma_{0,q}^{-\beta^\vee},q\widetilde{b}^{-1}\gamma_{0,q}^{-\beta^\vee};q\big)_{-(\lambda,\beta^\vee)}}{\big(\widetilde{a}\gamma_{0,q}^{-\beta^\vee},\widetilde{b}\gamma_{0,q}^{-\beta^\vee};q\big)_{-(\lambda,\beta^\vee)}},\end{aligned}$$

where $R_{0,s}\subset R_0$ (respectively $R_{0,l}\subset R_0$) are the short (respectively long) roots in R_0.

6. Appendix on affine root systems

6.1. **Affine root systems.** The main reference for this subsection is Macdonald [56].

Let E be a real affine space with the associated space of translations V of dimension $n\geq 1$. Fix a real scalar product $(\cdot,\cdot)$ on V and set $|v|^2=(v,v)$ for all $v\in V$. It turns E into a metric space, called an affine Euclidean space [4, §3].

A map $\Psi:E\rightarrow E$ is called an affine linear endomorphism of E if there exists a linear endomorphism $d\Psi$ of V such that $\Psi(e+v)=\Psi(e)+d\Psi(v)$ for all $e\in E$ and all $v\in V$. Set $\mathrm{O}(E)$ for the group of affine linear isometric automorphisms of E. Let $\tau_E:V\rightarrow\mathrm{O}(E)$ be the group monomorphism defined by $\tau_E(v)(e)=e+v$. Then we have a short exact sequence of groups

$$\tau_E(V)\hookrightarrow\mathrm{O}(E)\overset{d}{\twoheadrightarrow}\mathrm{O}(V).$$

For a subgroup $W \subseteq \mathrm{O}(E)$ let $L_W \subseteq V$ be the additive subgroup such that $\tau_E(L_W) = \mathrm{Ker}(d|_W)$.

Set $\widehat{E}$ for the real $n+1$-dimensional vector space of affine linear functions $a : E \to \mathbb{R}$. Let $c \in \widehat{E}$ be the constant function one. The gradient of an affine linear function $a \in \widehat{E}$ is the unique vector $Da \in V$ such that $a(e+v) = a(e) + (Da, v)$ for all $e \in E$ and $v \in V$. The gradient map $D : \widehat{E} \to V$ is linear, surjective, with kernel consisting of the constant functions on E.

For $a, b \in \widehat{E}$ set

$$\big(a, b\big) := \big(Da, Db\big),$$

which defines a semi-positive definite symmetric bilinear form on $\widehat{E}$. The radical consists of the constant functions on E. We write $|a|^2 := (a, a)$ for $a \in \widehat{E}$. Let $O(\widehat{E})$ be the form-preserving linear automorphisms of $\widehat{E}$ and $O_c(\widehat{E})$ its subgroup of automorphisms fixing the constant functions.

The contragredient action of $g \in O(E)$ on $\widehat{E}$, given by $(ga)(e) := a(g^{-1}e)$ $(a \in \widehat{E}, e \in E)$, realizes a group isomorphism $O(E) \simeq O_c(\widehat{E})$. Note that $\tau_E(v)a = a - (Da, v)c$ for $a \in \widehat{E}$ and $v \in V$.

A vector $a \in \widehat{E}$ is called nonisotropic if $Da \neq 0$. For a nonisotropic vector $a \in \widehat{E}$ let $s_a : E \to E$ be the orthogonal reflection in the affine hyperplane $a^{-1}(0)$ of E. It is explicitly given by

$$s_a(e) = e - a(e)Da^\vee, \qquad e \in E,$$

where $v^\vee := 2v/|v|^2 \in V$ is the covector of $v \in V \setminus \{0\}$. Viewed as element of $O_c(\widehat{E})$ it reads

$$s_a(b) = b - (a^\vee, b)a, \qquad b \in \widehat{E},$$

where $a^\vee := 2a/|a|^2 \in \widehat{E}$ is the covector of a. For a subset R of nonisotropic vectors in $\widehat{E}$ let $W(R)$ be the subgroup of $O(E) \simeq O_c(\widehat{E})$ generated by the orthogonal reflections s_a $(a \in R)$.

Definition 6.1. *A set R of nonisotropic vectors in $\widehat{E}$ is an affine root system on E if*

1. *R spans $\widehat{E}$,*
2. *$W(R)$ stabilizes R,*
3. *$(a^\vee, b) \in \mathbb{Z}$ for all $a, b \in R$,*
4. *$W(R)$ acts properly on E (i.e., if K_1 and K_2 are two compact subsets of E then $w(K_1) \cap K_2 \neq \emptyset$ for at most finitely many $w \in W(R)$),*
5. *$L_{W(R)}$ spans V.*

The elements $a \in R$ are called affine roots. The group $W(R)$ is called the affine Weyl group of R. The real dimension n of V is the rank of R.

Definition 6.2. *Let R be an affine root system on E. A nonempty subset $R' \subseteq R$ is called an affine root subsystem if $W(R')$ stabilizes R' and if $L_{W(R')}$ generates the real span V' of the set $\{Da\}_{a \in R'}$ of gradients of R' in V.*

Remark 6.3. With the notations from the previous definitions, let E' be the $V'^{\perp}$-orbits of E, where $V'^{\perp}$ is the orthocomplement of V' in V. It is an affine Euclidean space with V' the associated space of translations and with norm induced by the scalar product on V'. Let $F \subseteq \widehat{E}$ be the real span of R'. Then $F \overset{\sim}{\to} \widehat{E'}$ with form-preserving linear isomorphism $a \mapsto a'$ defined by $a'(e + V'^{\perp}) := a(e)$ for all $a \in F$ and $e \in E$. With this identification R' is an affine root system on E'. Furthermore, the corresponding affine Weyl group $W(R')$ is isomorphic to the subgroup of $W(R) \subset O(E)$ generated by the orthogonal reflections s_a ($a \in R'$).

We call an affine root system R irreducible if it cannot be written as a nontrivial orthogonal disjoint union $R' \cup R''$ (orthogonal meaning that $(a, b) = 0$ for all $a \in R'$ and all $b \in R''$). It is called reducible otherwise. In that case both R' and R'' are affine root subsystems of R. Each affine root system is an orthogonal disjoint union of irreducible affine root subsystems (cf. [56, §3]).

Remark 6.4. Macdonald's [56, §2] definition of an affine root system is (1)-(4) of Definition 6.1. Careful analysis reveals that Macdonald tacitly assumes condition (5), which only follows from the four axioms (1)-(4) if R is irreducible. Mark Reeder independently noted that an extra condition besides (1)-(4) is needed in order to avoid examples of affine root systems given as an orthogonal disjoint union of an affine root system R' and a finite crystallographic root system R'' (compare, on the level of affine Weyl groups, with [3, Chpt. V, §3] and [4, §1.3]). Mark Reeder proposes to add to (1)-(4) the axiom that for each $\alpha \in D(R)$ there exists at least two affine roots with gradient α. The resulting definition is equivalent to Definition 6.1, as well as to the notion of an échelonnage from [4, (1.4.1)] (take here [4, (1.3.2)] into account).

An affine root system R is called reduced if $\mathbb{R}a \cap R = \{\pm a\}$ for all $a \in R$, and nonreduced otherwise. If R is nonreduced then $R = R^{ind} \cup R^{unm}$ with R^{ind} (respectively R^{unm}) the reduced affine root subsystem of R consisting of indivisible (respectively unmultiplyable) affine roots.

If $R \subset \widehat{E}$ is an affine root system then $R_0 := D(R) \subset V$ is a finite crystallographic root system in V, called the gradient root system of R. The associated Weyl group $W_0 = W_0(R_0)$ is the subgroup of $\mathrm{O}(V)$ generated by the orthogonal reflections $s_\alpha \in O(V)$ in the hyperplanes $\alpha^\perp$ ($\alpha \in R_0$), which are explicitly given by $s_\alpha(v) = v - (\alpha^\vee, v)\alpha$ for $v \in V$. The Weyl group W_0 coincides with the image of $W(R)$ under the differential d.

We now define an appropriate equivalence relation between affine root systems, called similarity. It is a slightly weaker notion of similarity compared to the one used in [56, §3]. This is to render affine root systems that differ by a rescaling of the underlying gradient root system similar, see Remark 6.6.

Definition 6.5. *We call two affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ similar, $R \simeq R'$, if there exists a linear isomorphism $T : \widehat{E} \overset{\sim}{\longrightarrow} \widehat{E'}$ which restricts to a bijection of R onto R' preserving Cartan integers,*

$$\big((Ta)^\vee, Tb\big) = \big(a^\vee, b\big), \qquad \forall\, a, b \in R.$$

Similarity respects basic notions as affine root subsystems and irreducibility. If $R \simeq R'$, realized by the linear isomorphism T, then $Ts_aT^{-1} = s_{Ta}$ for all $a \in R$. In particular, $W(R) \simeq W(R')$. Note that T maps constant functions to constant functions. Replacing T by $-T$ if necessary, we may assume without loss of generality that $T(c) \in \mathbb{R}_{>0}c'$, where $c \in \widehat{E}$ and $c' \in \widehat{E'}$ denote the constant functions one on E and E' respectively. With this additional condition we call T a similarity transformation between R and R'.

If R is an affine root system and $\lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ then $\lambda R := \{\lambda a\}_{a \in R}$ is an affine root system similar to R (the similarity transformation realizing $R \xrightarrow{\sim} \lambda R$ is scalar multiplication by $|\lambda|$). We call λR a rescaling of the affine root system R. If two affine root systems R and R' are similar, then a similarity transformation T between R and a rescaling of R' exists such that $T(c) = c'$. In this case T arises as the contragredient of an affine linear isomorphism from E' onto E. For instance, each affine Weyl group element $w \in W(R)$ is a selfsimilarity transformation of R in this way.

If the affine root systems $R \subset \widehat{E}$ and $R' \subset \widehat{E'}$ are similar, then so are their gradients $R_0 \subset V$ and $R_0' \subset V'$ (i.e. there exists a linear isomorphism t of V onto V' restricting to a bijection of $R_0 \xrightarrow{\sim} R_0'$ and preserving Cartan integers). Indeed, if T is a similarity transformation between R and R', then the unique linear isomorphism $t : V \xrightarrow{\sim} V'$ such that $D \circ T = t \circ D$ realizes the similarity between R_0 and R_0'.

Remark 6.6. Let $R \subset \widehat{E}$ be an irreducible affine root system with associated gradient $R_0 \subset V$. Fix an origin $e \in E$. For $\lambda \in \mathbb{R}^* \setminus \{\pm 1\}$ and $a \in R$ define $a_\lambda \in \widehat{E}$ by $a_\lambda(e + v) := a(e) + \lambda(Da, v)$ for all $v \in V$. Then $R_\lambda = \{a_\lambda\}_{a \in R} \subset \widehat{E}$ is an affine root system similar to R, and $R_{\lambda,0} = \lambda R_0$. The affine root systems R and R_λ are not similar if one uses Macdonald's [56, §3] definition of similarity.

In the remainder of this section we assume that R is an irreducible affine root system of rank n. Since $W(R)$ acts properly on E, the set of regular elements

$$E_{reg} := E \setminus \bigcup_{a \in R} a^{-1}(0)$$

decomposes as the disjoint union of open n-simplices, called chambers of R. For a fixed chamber C there exists a unique $\mathbb{R}$-basis $\Delta = \Delta(C, R)$ of $\widehat{E}$ consisting of indivisible affine roots $a_i = a_{i,C}$ $(0 \le i \le n)$ such that $C = \{e \in E \,|\, a_i(e) > 0 \;\; \forall i \in \{0, \ldots, n\}\}$. The set of roots Δ is called the basis of R associated to the chamber C. The affine roots a_i are called simple affine roots. Any affine root $a \in R$ can be uniquely written as $a = \sum_{i=0}^n \lambda_i a_i$ with either all $\lambda_i \in \mathbb{Z}_{\ge 0}$ or all $\lambda_i \in \mathbb{Z}_{\le 0}$. The subset of affine roots of the first type is denoted by R^+ and is called the set of positive affine roots with respect to Δ. Then $R = R^+ \cup R^-$ (disjoint union) with $R^- := -R^+$ the subset of negative affine roots. The affine Weyl group $W(R)$ is a Coxeter group with Coxeter generators the simple reflections s_{a_i} $(0 \le i \le n)$.

A rank n *affine Cartan matrix* is a rational integral $(n+1) \times (n+1)$-matrix $A = (a_{ij})_{i,j=0}^n$ satisfying the four conditions

(1) $a_{ii} = 2$,
(2) $a_{ij} \in \mathbb{Z}_{\le 0}$ if $i \ne j$,

(3) $a_{ij}=0$ implies $a_{ji}=0$,
(4) $\det(A)=0$ and all the proper principal minors of A are strictly positive.
(5) A is indecomposable (i.e. the matrices obtained from A by a simultaneous permutation of its rows and columns are not the direct sum of two nontrivial blocks).

The larger class of rational integral matrices satisfying conditions (1)-(3) are called generalized Cartan matrices. They correspond to Kac-Moody Lie algebras, see [40]. The Kac-Moody Lie algebras related to the subclass of affine Cartan matrices are the affine Lie algebras. The affine Cartan matrices have been classified by Kac [40, Chpt. 4].

Fix an ordered basis $\Delta=(a_0,a_1,\ldots,a_n)$ of R. The matrix $A=A(R,\Delta)=(a_{ij})_{0\leq i,j\leq n}$ defined by

$$a_{ij}:=\big(a_i^\vee,a_j\big)$$

is an affine Cartan matrix. The coefficients a_{ij} $(0\leq i,j\leq n)$ are called the affine Cartan integers of R.

If $R\simeq R'$ with associated similarity transformation T, then the T-image of an ordered basis Δ of R is an ordered basis Δ' of R'. Let R and R' be irreducible affine root systems with ordered bases Δ and Δ' respectively. We say that (R,Δ) is similar to (R',Δ'), $(R,\Delta)\simeq(R',\Delta')$, if the irreducible affine root systems R and R' are similar and if there exists an associated similarity transformation T mapping the ordered basis Δ of R to the ordered basis Δ' of R'. For similar pairs the associated affine Cartan matrices coincide. Moreover, the affine Cartan matrix $A(R,\Delta)$ modulo simultaneous permutations of its row and columns does not depend on the choice of ordered basis Δ. This leads to a map from the set of similarity classes of reduced irreducible affine root systems to the set of affine Cartan matrices up to simultaneous permutations of rows and columns. Using Kac' [40, Chpt. 4] classification of the affine Cartan matrices and the explicit construction of reduced irreducible affine root systems from [56] (see also the next subsection), it follows that the map is surjective. It is injective by a straightforward adjustment of the proof of [35, Prop. 11.1] to the present affine setup. Hence we obtain the following classification result.

Theorem 6.7. *Reduced irreducible affine root systems up to similarity are in bijective correspondence to affine Cartan matrices up to simultaneous permutations of the rows and columns.*

Remark 6.8. Affine Cartan matrices also parametrize affine Lie algebras (see [40]). For a given affine Cartan matrix the set of real roots of the associated affine Lie algebra is the associated irreducible reduced affine root system.

Affine Cartan matrices up to simultaneous permutations of the rows and columns (and hence similarity classes of irreducible reduced affine root systems) can be naturally encoded by affine Dynkin diagrams [40, 56]. The affine Dynkin diagram associated to an affine Cartan matrix $A=(a_{ij})_{i,j=0}^n$ is the graph with $n+1$ vertices in which we join the ith and jth node $(i\neq j)$ by $a_{ij}a_{ji}$ edges. In addition we put an arrow towards the ith node if $|a_{ij}|>1$. At the end of the appendix we list all affine Dynkin diagrams and link it to Kac's [40] classification.

6.2. **Explicit constructions.** The main reference for the results in this subsection is [56]. For an irreducible finite crystallographic root system R_0 of type A, D, E or BC there is exactly one similarity class of reduced irreducible affine root systems whose gradient root system is similar to R_0. For the other types of root systems R_0, there are two such similarity classes of reduced irreducible affine root systems. We now proceed to realize them explicitly.

Let $R_0 \subset V$ be an irreducible finite crystallographic root system (possibly nonreduced). The affine space E is taken to be V with forgotten origin. We will write V for E in the sequel if no confusion is possible.

We identify the space $\widehat{E}$ of affine linear functions on E with $V \oplus \mathbb{R}c$ as real vector space, with c the constant function identically equal to one on V and with $V^* \simeq V$ the linear functionals on V (the identification with V is realized by the scalar product on V). With these identifications,

$$O_c(\widehat{E}) \simeq O(E) = O(V) \ltimes \tau(V),$$

with $\tau(v)(e) = e + v$. Regarding $\tau(v)$ as element of $O_c(\widehat{V})$ it is given by

$$\tau(v)a = -(Da, v)c + a, \qquad a \in \widehat{V}.$$

Note that the orthogonal reflection $s_a \in O(E)$ associated to $a = \lambda c + \alpha \in \widehat{E}$ ($\lambda \in \mathbb{R}$, $\alpha \in V \setminus \{0\}$) decomposes as $s_{\lambda c+\alpha} = \tau(-\lambda\alpha^\vee)s_\alpha$.

Consider the subset

$$\mathcal{S}(R_0) := \{mc + \alpha\}_{m\in\mathbb{Z},\alpha\in R_0^{ind}} \cup \{(2m+1)c + \beta\}_{m\in\mathbb{Z},\beta\in R_0\setminus R_0^{ind}}$$

of $\widehat{V}$, where $R_0^{ind} \subseteq R_0$ is the root subsystem of indivisible roots. Then $\mathcal{S}(R_0)$ and $\mathcal{S}(R_0^\vee)^\vee$ are reduced irreducible affine root systems with gradient root system R_0. We call $\mathcal{S}(R_0)$ (respectively $\mathcal{S}(R_0^\vee)^\vee$) the untwisted (respectively twisted) reduced irreducible affine root system associated to R_0. Note that $\mathcal{S}(R_0) \simeq \mathcal{S}(R_0^\vee)^\vee$ if R_0 is of type A, D, E or BC.

Proposition 6.9. *The following reduced irreducible affine root systems form a complete set of representatives of the similarity classes of reduced irreducible affine root systems:*

(1) $\mathcal{S}(R_0)$ *with* R_0 *running through the similarity classes of reduced irreducible finite crystallographic root systems (i.e.* R_0 *of type* $A, B, \ldots, G$*),*
(2) $\mathcal{S}(R_0^\vee)^\vee$ *with* R_0 *running through the similarity classes of reduced irreducible finite crystallographic root systems having two root lengths (i.e.* R_0 *of type* B_n $(n \geq 2)$*,* C_n $(n \geq 3)$*,* F_4 *and* G_2*),*
(3) $\mathcal{S}(R_0)$ *with* R_0 *a nonreduced irreducible finite crystallographic root system (i.e.* R_0 *of type* BC_n $(n \geq 1)$*).*

In view of the above proposition we use the following terminology: a reduced irreducible affine root system R is said to be of *untwisted type* if $R \simeq \mathcal{S}(R_0)$ with R_0 reduced, of *twisted type* if $R \simeq \mathcal{S}(R_0^\vee)^\vee$ with R_0 reduced, and of *mixed type* if $R \simeq \mathcal{S}(R_0)$ with R_0 nonreduced. Note that a reduced irreducible affine root system R with gradient root system of type A, D or E is of untwisted and of twisted type.

Suppose that $\Delta_0 = (\alpha_1, \ldots, \alpha_n)$ is an ordered basis of R_0. Let $\varphi \in R_0$ (respectively $\theta \in R_0$) be the associated highest root (respectively the highest short root). Then

$$\Delta := (a_0, a_1, \ldots, a_n) = (c - \varphi, \alpha_1, \ldots, \alpha_n)$$

is an ordered basis of $\mathcal{S}(R_0)$, while

$$\Delta := (a_0, a_1, \ldots, a_n) = (\frac{|\theta|^2}{2}c - \theta, \alpha_1, \ldots, \alpha_n)$$

is an ordered basis of $\mathcal{S}(R_0^\vee)^\vee$.

6.3. **Nonreduced irreducible affine root systems.** If R is an irreducible affine root system with ordered basis Δ, then Δ is also an ordered basis of the affine root subsystem R^{ind} of indivisible roots. Furthermore, $R \simeq R'$ implies $R^{ind} \simeq R'^{,ind}$. To classify nonreduced irreducible affine root systems up to similarity, one thus only needs to understand the possible ways to extend reduced irreducible affine root systems to nonreduced ones.

Let R' be a reduced irreducible affine root system with affine Weyl group $W = W(R')$. Choose an ordered basis $\Delta = (a_0, a_1, \ldots, a_n)$ of R'. Set

$$S := \{a \in \Delta \mid \big(\mathbb{Z}R', a^\vee\big) = 2\mathbb{Z}\}. \tag{6.1}$$

Let $1 \leq m \leq \#S + 1$ and choose a subset S_m of S of cardinality m. Then

$$R^{(m)} := R' \cup \bigcup_{a \in S_m} W(2a)$$

is an irreducible affine root system with $R^{(m),ind} \simeq R'$.

Considering the possible affine Dynkin diagrams associated to (R', Δ) (see Subsection 6.4) it follows that the set S (6.1) is of cardinality ≤ 2. It is of cardinality two iff $R' \simeq \mathcal{S}(R_0^\vee)^\vee$ with R_0 of type A_1 or with R_0 of type B_n ($n \geq 2$). It is of cardinality one iff $R' \simeq \mathcal{S}(R_0)$ with R_0 of type B_n ($n \geq 2$) or of type BC_n ($n \geq 1$). Hence the similarity class of $R^{(m)}$ does not depend on the choice of subset $S_m \subseteq S$ of cardinality m, and it does not depend on the choice of ordered basis Δ of R'. The number of W-orbits of $R^{(m)}$ equals the number of W-orbits of R' plus m. The number of similarity classes of irreducible affine root systems R satisfying $R^{ind} \simeq R'$ is $\#S + 1$.

If R_0 is of type A_1 or of type B_n ($n \geq 2$) we thus have a nonreduced irreducible affine root system in which two W-orbits are added to $\mathcal{S}(R_0^\vee)^\vee$. It is labelled as $C^\vee C_n$ by Macdonald [56]. In the rank one case it has four W-orbits, otherwise five. A detailed description of this affine root system is given in Subsection 3.8.

Irreducible affine root subsystems with underlying reduced affine root system $\mathcal{S}(R_0)$ having finite root system R_0 of type BC_n ($n \geq 1$) or of type B_n ($n \geq 3$) can be naturally viewed as affine root subsystems of the affine root system of type $C^\vee C_n$. This is not the case for the nonreduced extension of the affine root system $\mathcal{S}(R_0)$ with R_0 of type B_2. It can actually be better viewed as the rank two case of the family $\mathcal{S}(R_0)$ with R_0 of type C_n since, in the corresponding affine Dynkin diagram, the vertex labelled by the affine simple root a_0 is double bonded with the finite Dynkin diagram of R_0. The nonreduced extension of $\mathcal{S}(R_0)$ with R_0 of type C_2 was missing in Macdonald's [56] classification list. It was added in [61, (1.3.17)].

6.4. **Affine Dynkin diagrams.** In this subsection we list the connected affine Dynkin diagrams (cf. [56, Appendix 1]) which, as we have seen, are in one to one correspondence to similarity classes of irreducible reduced affine root systems. Each similarity class of irreducible reduced affine root systems has a representative of the form $\mathcal{S}(R_0)$ or $\mathcal{S}(R_0^\vee)^\vee$ for a unique irreducible finite crystallographic root system R_0 up to similarity, see Subsection 6.2. Recall that $\mathcal{S}(R_0^\vee)^\vee \simeq \mathcal{S}(R_0)$ if R_0 is of type A, D, E, BC. We label the connected affine Dynkin diagram by $\widehat{X}$ with X the type of the associated finite root system R_0 if $X \in \{A, D, E, BC\}$. If the associated finite root system R_0 is of type $X \in \{B, C, F, G\}$ then we label the connected affine Dynkin diagram by $\widehat{X}^u$ (respectively $\widehat{X}^t$) if the associated irreducible reduced affine root system is $\mathcal{S}(R_0)$ (respectively $\mathcal{S}(R_0^\vee)^\vee$). Since $A_1 \simeq B_1 \simeq C_1$ and $B_2 \simeq C_2$ there is some redundancy in the notations, we pick the one which is most convenient to fit it into an infinite family of affine Dynkin diagrams. In the terminology of Subsection 6.2, the irreducible reduced affine root systems corresponding to affine Dynkin diagrams labelled by $\widehat{X}$ with $X \in \{A, D, E\}$ are of untwisted and of twisted type, labelled by $\widehat{BC}$ of mixed type, labelled by $\widehat{X}^u$ of untwisted type and labelled by $\widehat{X}^t$ of twisted type. In [56, Appendix 1] the affine Dynkin diagrams labelled $\widehat{B}_n^t$ and $\widehat{C}_n^t$ are called of type $C_n^\vee$ and $B_n^\vee$ respectively. The remaining relations with the notations and terminologies in [56, Appendix 1] are self-explanatory.

We specify in each affine Dynkin diagram a particular vertex (the grey vertex) which is labelled by the unique affine simple root a_0 in the particular choice of ordered basis Δ of $\mathcal{S}(R_0)$ or $\mathcal{S}(R_0^\vee)^\vee$ as specified in Subsection 6.2.

In Kac's notations (see Tables Aff 1–3 in [40, §4.8]) the affine Dynkin diagrams are labelled differently: our label $\widehat{X}$ corresponds to $X^{(1)}$ if $X \in \{A, D, E\}$ and $\widehat{BC}_n$ corresponds to $A_{2n}^{(2)}$ $(n \geq 1)$. Our label $\widehat{X}^u$ corresponds to $X^{(1)}$ if $X \in \{B, C, F, G\}$. Finally, our label $\widehat{B}_n^t$ corresponds to $D_{n+1}^{(2)}$ $(n \geq 2)$, $\widehat{C}_n^t$ corresponds to $A_{2n-1}^{(2)}$ $(n \geq 3)$, $\widehat{F}_4^t$ to $E_6^{(2)}$ and $\widehat{G}_2^t$ to $D_4^{(3)}$.

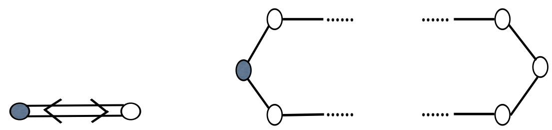

FIGURE 1. $\widehat{A}_1$ and $\widehat{A}_n$ $(n \geq 2)$

FIGURE 2. $\widehat{B}^u_n$ $(n \geq 3)$

FIGURE 3. $\widehat{B}^t_n$ $(n \geq 2)$

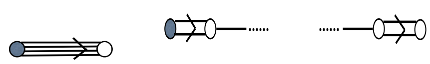

FIGURE 4. $\widehat{BC}_1$ and $\widehat{BC}_n$ $(n \geq 2)$

FIGURE 5. $\widehat{C}_n^u$ $(n \geq 2)$

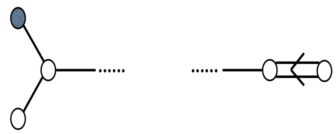

FIGURE 6. $\widehat{C}_n^t$ $(n \geq 3)$

FIGURE 7. $\widehat{D}_n$ $(n \geq 4)$

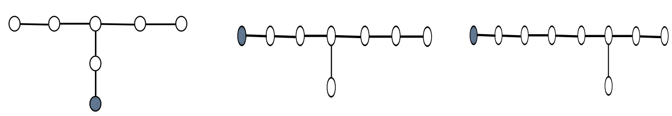

FIGURE 8. $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$

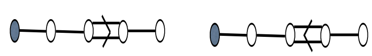

FIGURE 9. $\widehat{F}_4^u$ and $\widehat{F}_4^t$

FIGURE 10. $\widehat{G}_2^u$ and $\widehat{G}_2^t$

References

[1] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319.

[2] T.H. Baker, P.J. Forrester, *Multivariable Al-Salam & Carlitz polynomials associated with the type A q-Dunkl kernel*, Math. Nachr. **212** (2000), 5–35.

[3] N. Bourbaki, *Lie Groups and Lie Algebras*, Chapter 4-6.

[4] F. Bruhat, J. Tits, *Groupes réductives sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[5] O. Chalykh, *Macdonald polynomials and algebraic integrability*, Adv. Math. **166** (2002), no. 2, 193–259.

[6] O. Chalykh, P. Etingof, *Orthogonality relations and Cherednik identities for multivariable Baker-Akhiezer functions*, arXiv:1111.0515.

[7] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Comm. Math. Phys. **150** (1992), 109–136.

[8] I. Cherednik, *Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, Int. Math. Res. Not. IMRN **1992**, no. 9, 171–180.

[9] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. of Math. (2) **141** (1995), no. 1, 191–216.

[10] I. Cherednik, *Difference-elliptic operators and root systems*, Int. Math. Res. Not. IMRN **1995**, no. 1, 43–58.

[11] I. Cherednik, *Macdonald's evaluation conjectures and difference Fourier transform*, Invent. Math. **122** (1995), no. 1, 119–145.

[12] I. Cherednik, *Non-symmetric Macdonald polynomials*, Int. Math. Res. Not. IMRN **1995**, no. 10, 483–515.

[13] I. Cherednik, *Intertwining operators of double affine Hecke algebras*, Selecta Math. (N.S.) **3** (1997), no. 4, 459–495.

[14] I. Cherednik, *Difference Macdonald-Mehta conjecture*, Int. Math. Res. Not. IMRN **1997**, no. 10, 449–467.

[15] I. Cherednik, *Double affine Hecke algebras*, London Math. Soc. Lecture Note Series, **319**. Cambridge Univ. Press, Cambridge, 2005.

[16] I. Cherednik, *Double affine Hecke algebras and difference Fourier transforms*, Invent. Math. **152** (2003), no. 2, 213–303.

[17] I. Cherednik, *Nonsemisimple Macdonald polynomials*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 427–569.

[18] I. Cherednik, *Whittaker limits of difference spherical functions*, Int. Math. Res. Not. IMRN **2009**, no. 20, 3793–3842.

[19] J.-F. van Diejen, J.V. Stokman, *Multivariable q-Racah polynomials*, Duke Math. J. **91** (1998), 89–136.

[20] H. Coskun, R.A. Gustafson, *Well-poised Macdonald functions W_λ and Jackson coefficients ω_λ on BC_n*, in: "Jack, Hall-Littlewood and Macdonald polynomials", 127–155, Contem. Math., **417**, Amer. Math. Soc., Providence, RI, 2006.

[21] J.-F. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math. **126** (1996), 319–339.

[22] P.I. Etingof, A.A. Kirillov Jr., *Macdonald's polynomials and representations of quantum groups*, Math. Res. Lett. **1** (1994), no. 3, 279–296.

[23] P.I. Etingof, A.A. Kirillov Jr., *On the affine analogue of the Jack and Macdonald polynomials*, Duke Math. J. **78** (1995), no. 2, 229–256.

[24] P.I. Etingof, A.A. Kirillov Jr., *Representation-theoretic proof of the inner product and symmetry identities for Macdonald's polynomials*, Compositio Math. **102** (1996), no. 2, 179–202.

[25] P.I. Eingof, A.A. Kirillov Jr., *On Cherednik-Macdonald-Mehta identities*, Electron. Res. Announc. Amer. Math. Soc. **4** (1998), 43–47.

[26] , B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, *Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials*, Int. Math. Res. Not. **2003**, no. 18, 1015–1034.

[27] P. Di Francesco, P. Zinn-Justin, *Around the Razumov-Stroganov conjecture: proof of a multiparameter sum rule*, Electr. J. Combin. **12** (2005), Research Paper 6.

[28] G. Gasper, M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications **35**, Cambridge University Press (1990).

[29] J. Haglund, M. Haiman, N. Loehr, *A combinatorial formula for nonsymmetric Macdonald polynomials*, Amer. J. Math. **130** (2008), no. 2, 359–383.

[30] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006.

[31] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407.

[32] M. Haiman, *Cherednik algebras, Macdonald polynomials and combinatorics*, International Congress of Mathematicians. Vol. III, 843–872, Eur. Math. Soc., Zürich, 2006.

[33] G.J. Heckman, E.M. Opdam, *Yang's system of particles and Hecke algebras*, Ann. of Math. (2) **145**, no. 1 (1997), 139–173.

[34] G.J. Heckman, H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Mathematics, **16**. Academic Press, Inc., San Diego, CA, 1994.

[35] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer.

[36] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge studies in adv. math. **29**, Cambridge University Press 1990.

[37] B. Ion, *Involutions of double affine Hecke algebras*, Compositio Math. **139** (2003), no. 1, 67–84.

[38] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Math. J. **116** (2003), no. 2, 299–318.

[39] B. Ion, *Nonsymmetric Macdonald polynomials and matrix coefficients for unramified principal series*, Adv. Math. **201** (2006), no. 1, 36–62.

[40] V.G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press 1990.

[41] M. Kasatani, *Subrepresentations in the polynomial representation of the double affine Hecke algebra of type* GL_n *at* $t^{k+1}q^{r-1}=1$, Int. Math. Res. Not. **2005**, no. 28, 1717–1742.

[42] M. Kasatani, *Zeros of symmetric Laurent polynomials of type* $(BC)_n$ *and Koornwinder-Macdonald polynomials specialized at* $t^{k+1}q^{r-1}=1$, Compositio Math. **141** (2005), no. 6, 1589–1601.

[43] M. Kasatani, T. Miwa, A.N. Sergeev, A.P. Veselov, *Coincident root loci and Jack and Macdonald polynomials for special values of the parameters*, in: "Jack, Hall-Littlewood and Macdonald polynomials", 207–225, Contemp. Math., **417** (2006), Providence, RI, Amer. Math. Soc.

[44] M. Kasatani, V. Pasquier, *On polynomials interpolating between the stationary state of a* $O(n)$ *model and a Q.H.E. group state*, Comm. Math. Phys. **276** (2007), 397–435.

[45] M. Kasatani, Y. Takeyama, *The quantum Knizhnik-Zamolodchikov equation and non-symmetric Macdonald polynomials*, Funkcialaj Ekvacioj **50** (2007), 491–509.

[46] F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials*, Comment. Math. Helv. **72** (1997), no. 1, 84–100.

[47] F. Knop, S. Sahi, *Difference equations and symmetric polynomials defined by their zeros*, Int. Math. Res. Not. IMRN **1996**, no. 10, 473–486.

[48] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology, Faculty of Information Technology and Systems, Dept. of Technical Math. and Informatics, Report no. 98-17 (1998), see http://homepage.tudelft.nl/11r49/askey.html

[49] Y. Komori, K. Hikami, *Quantum integrability of the generalized Ruijsenaars models*, J. Phys. A **30** (1997), no. 12, 4341–4364.

[50] T.H. Koornwinder. *Self-duality for q-ultraspherical polynomials associated with the root system* A_n, unpublished manuscript (1988).

[51] T.H. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, in: "Hypergeometric functions on domains of positivity, Jack polynomials, and applications" (Tampa, FL, 1991), 189–204, Contemp. Math., **138**, Amer. Math. Soc., Providence, RI, 1992.

[52] A. Lascoux, E.M. Rains, S. Ole Warnaar, *Nonsymmetric interpolation Macdonald polynomials and* $\mathfrak{gl}_n$ *basic hypergeometric series*, Transform. Groups **14** (2009), no. 3, 613–647.

[53] G. Letzter, *Quantum zonal spherical functions and Macdonald polynomials*, Adv. Math. **189** (2004), no. 1, 88–147.

[54] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2**, 599–635 (1989).

[55] I.G. Macdonald, *The Poincaré Series of a Coxeter Group*, Math Ann. **199**, 161–174 (1972).

[56] I.G. Macdonald, *Affine Root Systems and Dedekind's* η*-Function*, Invent. Math. **15**, 91–143 (1972).

[57] I.G. Macdonald, *Some conjectures for Koornwinder's orthogonal polynomials*, unpublished manuscript (1991).

[58] I.G. Macdonald, *Symmetric functions and Hall polynomials.* Second edition, Oxford Mathematical Monographs. Oxford Science Publications. The Claredon press, Oxford Univ. Press, New York, 1995.

[59] I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Astérisque **237** (1996), 189–207.

[60] I.G. Macdonald, *Orthogonal polynomials associated with root systems*, Sém. Lothar. Combin. **45** (2000/01), Article B45a, 40pp.

[61] I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Math. **157**, Cambridge University Press 2003.

[62] M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123** (1996), no. 1, 16–77.

[63] M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings*, in: "Various aspects of hypergeometric functions" (Kyoto, 1994), Surikaisekikenkyusho Kokyuroku No. **895** (1995), 114–124.

[64] M. Noumi, M.S. Dijkhuizen, T. Sugitani, *Multivariable Askey-Wilson polynomials and quantum complex Grassmannians*, in: "Special functions, q-series and related topics" (Toronto, ON, 1995), 167–177, Fields Inst. Commun., **14**, Amer. Math. Soc., Providence, RI, 1997.

[65] M. Noumi, J.V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebra approach*, in: "Laredo Lectures on Orthogonal Polynomials and Special Functions", 111–144, Adv. Theory Spec. Funct. Orthogonal Polynomials, Nova Sci. Publ., Hauppauge, NY, 2004.

[66] M. Noumi, T. Sugitani, *Quantum symmetric spaces and related q-orthogonal polynomials*, in: "Group theoretical methods in physics" (Toyonaka, 1994), 28–40, World Sci. Publ., River Edge, NJ, 1995.

[67] A.A. Oblomkov, J.V. Stokman, *Vector calued spherical functions and Macdonald-Koornwinder polynomials*, Compositio Math. **141** (2005), no. 5, 1310–1350.

[68] A. Okounkov, *(Shifted) Macdonald polynomials: q-Integral representation and combinatorial formula*, Compositio Math. **112** (1998), 147–182.

[69] A. Okounkov, *BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials*, Transform. Groups **3** (1998), no. 2, 181–207.

[70] V. Pasquier, *Quantum incompressibility and Razumov-Stroganov type conjectures*, Ann. Henri Poincaré **7** (2006), 397–421.

[71] E.M. Rains, BC_n*-symmetruc Abelian functions*, Duke Math. J. **135** (2006), no. 1, 99–180.

[72] E.M. Rains, *Transformations of elliptic hypergeometric integrals*, Ann. of Math. (2) **171** (2010), no. 1, 169–243.

[73] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Comm. Math. Phys. **110** (1987), 191–213.

[74] S. Sahi, *Interpolation, integrality, and a generalization of Macdonald's polynomials*, Int. Math. Res. Not. IMRN **1996**, no. 10, 457–471.

[75] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) **150** (1999), no. 1, 267–282.

[76] O. Schiffmann, E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compositio Math. **147** (2011), 188–234.

[77] A.N. Sergeev, A.P. Veselov, *Deformed Macdonald-Ruijsenaars operators and super Macdonald polynomials*, Comm. Math. Phys. **288** (2009), no. 2, 653–675.

[78] J.V. Stokman, *On BC type basic hypergeometric orthogonal polynomials*, Trans. Amer. Math. Soc. **352** (2000), no. 4, 1527–1579.

[79] J.V. Stokman, *Koornwinder polynomials and affine Hecke algebras*, Internat. Math. Res. Notices **2000**, no. 19, 1005–1042.

[80] J.V. Stokman, *Difference Fourier transforms for nonreduced root systems*, Selecta Math. (N.S.) **9** (2003), no. 3, 409–494.

[81] J.V. Stokman, *Quantum affine Knizhnik-Zamolodchikov equations and quantum spherical functions, I*, Int. Math. Res. Not. IMRN **2011**, no. 5, 1023–1090.

[82] J.V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, arXiv:1109.0613.

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