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Geometric Rank of Tensors and Subrank of Matrix Multiplication

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Abstract

Motivated by problems in algebraic complexity theory (e.g., matrix multiplication) and extremal combinatorics (e.g., the cap set problem and the sunflower problem), we introduce the geometric rank as a new tool in the study of tensors and hypergraphs. We prove that the geometric rank is an upper bound on the subrank of tensors and the independence number of hypergraphs. We prove that the geometric rank is smaller than the slice rank of Tao, and relate geometric rank to the analytic rank of Gowers and Wolf in an asymptotic fashion. As a first application, we use geometric rank to prove a tight upper bound on the (border) subrank of the matrix multiplication tensors, matching Strassen's well-known lower bound from 1987.

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1 Introduction

Tensors play a central role in computer science and mathematics. Motivated by problems in algebraic complexity theory (e.g., the arithmetic complexity of matrix multiplication), extremal combinatorics (e.g., the cap set problem and the Erdős–Szemerédi sunflower problem) and quantum information theory (the resource theory of quantum entanglement), we introduce and study a new tensor parameter called *geometric rank*. Like the many widely studied notions of rank for tensors (rank, subrank, border rank, border subrank, flattening rank, slice rank, analytic rank), geometric rank of tensors generalizes the classical rank of matrices. In this paper, we:

- prove a number of basic properties and invariances of geometric rank,
- develop several tools to reason about, and sometimes exactly compute, the geometric rank,
- show intimate connections between geometric rank and the other important notions of rank for tensors,
- and as a simple application of the above, we answer an old question of Strassen by showing that the (border) subrank of $m \times m$ matrix multiplication is at most $\lceil 3m^2/4 \rceil$ (this is tight for border subrank; previously the border subrank of the matrix multiplication tensor was known to lie between $\frac{3}{4}m^2$ and $(1 - o(1))m^2$).

More generally, we believe that geometric rank provides an interesting new route to prove upper bounds on subrank of tensors (and hence independence numbers of hypergraphs). Such upper bounds are important in complexity theory in the context of matrix multiplication and barriers to matrix multiplication, and combinatorics in the context of specific natural hypergraphs (as in the cap set problem and the Erdős–Szemeredi sunflower problem).

1.1 Geometric rank

We define the geometric rank of a tensor as the codimension of the (possibly reducible) algebraic variety defined by the bilinear forms given by the slices of the tensor. Here we use the standard notions of dimension and codimension of affine varieties from algebraic geometry. That is, for any tensor $T = (T_{i,j,k})_{i,j,k} \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ with coefficients $T_{i,j,k}$ in an algebraically closed field $\mathbb{F}$ (e.g., the complex numbers $\mathbb{C}$) and with 3-slices $M_k = (T_{i,j,k})_{i,j} \in \mathbb{F}^{n_1 \times n_2}$ we define the geometric rank $\mathrm{GR}(T)$ as

$$\mathrm{GR}(T) = \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid x^T M_1 y = \cdots = x^T M_{n_3} y = 0\}.$$

Viewing T as the trilinear map $T : \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \to \mathbb{F} : (x, y, z) \mapsto \sum_{i,j,k} T_{i,j,k}\, x_i y_j z_k$, we can equivalently write the geometric rank of T as

$$\mathrm{GR}(T) = \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid \forall z \in \mathbb{F}^{n_3} : T(x, y, z) = 0\}.$$

The definition of geometric rank is expressed asymmetrically in x, y and z. We will see, however, that the codimensions of $\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid \forall z : T(x, y, z) = 0\}$, $\{(x, z) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_3} \mid \forall y : T(x, y, z) = 0\}$ and $\{(y, z) \in \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \mid \forall x : T(x, y, z) = 0\}$ coincide (Theorem 4).

The motivation for this definition is a bit hard to explain right away. We arrived at it while searching for a characteristic 0 analogue of the analytic rank of Gowers and Wolf [19] (see Section 8).

▶ **Example 1.** We give an example of how to compute the geometric rank. Let $T \in \mathbb{F}^{2\times 2\times 2}$ be the tensor with 3-slices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(This is sometimes called the W-tensor). One verifies that the algebraic variety $V = \{(x, y) \in \mathbb{F}^2 \times \mathbb{F}^2 \mid x_1 y_1 = 0, x_2 y_1 + x_1 y_2 = 0\}$ has the three irreducible components $\{(x, y) \in \mathbb{F}^2 \times \mathbb{F}^2 \mid x_1 = 0, x_2 = 0\}$, $\{(x, y) \in \mathbb{F}^2 \times \mathbb{F}^2 \mid x_1 = 0, y_1 = 0\}$ and $\{(x, y) \in \mathbb{F}^2 \times \mathbb{F}^2 \mid y_1 = 0, y_2 = 0\}$. Each irreducible component has dimension 2 and thus V has dimension 2. Hence $\mathrm{GR}(T) = \mathrm{codim}\, V = 4 - 2 = 2$. We will see more examples of geometric rank later (Theorem 17).

1.2 Overview: notions of tensor rank

Before discussing our results we give an introduction to some of the existing notions of rank and their usefulness. Several interesting notions of rank of tensors have been studied in mathematics and computer science, each with their own applications. As a warm-up we first discuss the familiar situation for matrices.

Matrices

For any two matrices $M \in \mathbb{F}^{m_1 \times m_2}$ and $N \in \mathbb{F}^{n_1 \times n_2}$ we write $M \leq N$ if there exist matrices A, B such that $M = ANB$. Defining the matrix rank $\mathrm{R}(M)$ of M as the smallest number r such that M can be written as a sum of r matrices that are outer products $(u_i v_j)_{ij}$ (i.e., rank-1 matrices), we see that in terms of the relation $\leq$ we can write the matrix rank as the minimisation

$$\mathrm{R}(M) = \min\{r \in \mathbb{N} \mid M \leq I_r\},$$

where I_r is the $r \times r$ identity matrix. Matrix rank thus measures the "cost" of M in terms of identity matrices. Let us define the subrank $\mathrm{Q}(M)$ of M as the "value" of M in terms of identity matrices,

$$\mathrm{Q}(M) = \max\{s \in \mathbb{N} \mid I_s \leq M\}.$$

It turns out that subrank equals rank for matrices,

$$\mathrm{Q}(M) = \mathrm{R}(M).$$

Namely, if $\mathrm{R}(M) = r$, then by using Gaussian elimination we can bring M in diagonal form with exactly r nonzero elements on the diagonal, and so $I_r \leq M$. In fact, $M \leq N$ if and only if $\mathrm{R}(M) \leq \mathrm{R}(N)$.

Tensors

For any two tensors $S \in \mathbb{F}^{m_1 \times m_2 \times m_3}$ and $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ we write $S \leq T$ if there are matrices A, B, C such that $S = (A, B, C) \cdot T$ where we define

$$(A, B, C) \cdot T \coloneqq \Big(\sum_{a,b,c} A_{ia} B_{jb} C_{kc} T_{a,b,c}\Big)_{i,j,k}.$$

Thus $(A, B, C) \cdot T$ denotes taking linear combinations of the slices of T in three directions according to A, B and C. Let $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ be a tensor. The tensor rank $\mathrm{R}(T)$ of T is defined as the smallest number r such that T can be written as a sum of r tensors that are outer products $(u_i v_j w_k)_{i,j,k}$. Similarly as for matrices, we can write tensor rank in terms of the relation $\leq$ as the "cost" minimisation

$$\mathrm{R}(T) = \min\{r \in \mathbb{N} \mid M \leq I_r\}$$

where I_r is the $r \times r \times r$ identity tensor (i.e., the diagonal tensor with ones on the main diagonal). Strassen defined the subrank of T as the "value" of T in terms of identity tensors,

$$\mathrm{Q}(T) = \max\{s \in \mathbb{N} \mid I_s \leq M\}.$$

Naturally, since $\leq$ is transitive, we have that value is at most cost: $\mathrm{Q}(T) \leq \mathrm{R}(T)$. Unlike the situation for matrices, however, there exist tensors for which this inequality is strict. One way to see this is using the fact that a random tensor in $\mathbb{F}^{n \times n \times n}$ has tensor rank close to n^2 whereas its subrank is at most n. Another way to see this is using the ranks $\mathrm{R}^{(i)}(T) := \mathrm{R}(T^{(i)})$ of the matrices $T^{(1)} = (T_{i,j,k})_{i,(j,k)} \in \mathbb{F}^{n_1 \times n_2 n_3}$, $T^{(2)} = (T_{i,j,k})_{j,(i,k)} \in \mathbb{F}^{n_2 \times n_1 n_3}$, and $T^{(3)} = (T_{i,j,k})_{k,(i,j)} \in \mathbb{F}^{n_3 \times n_1 n_2}$ obtained from T by grouping two of the three indices together, since

$$\mathrm{Q}(T) \leq \mathrm{R}^{(i)}(T) \leq \mathrm{R}(T).$$

Namely, it is not hard to find tensors T for which $\mathrm{R}^{(1)}(T) < \mathrm{R}^{(2)}(T)$. We will now discuss two upper bounds on the subrank $\mathrm{Q}(T)$ that improve on the flattening ranks $\mathrm{R}^{(i)}(T)$. Then we will discuss connections between subrank and problems in complexity theory and combinatorics.

Slice rank

In the context of the cap set problem, Tao [34] defined the slice rank of any tensor T as the minimum number r such that T can be written as a sum of r tensors of the form $(u_i V_{jk})_{i,j,k}$, $(u_j V_{ik})_{i,j,k}$ or $(u_k V_{ij})_{i,j,k}$ (i.e., an outer product of a vector and a matrix). In other words, $\mathrm{SR}(T) := \min\{\mathrm{R}^{(1)}(S_1) + \mathrm{R}^{(2)}(S_2) + \mathrm{R}^{(3)}(S_3) : S_1 + S_2 + S_3 = T\}$. Clearly slice rank is at most any flattening rank, and Tao proved that slice rank upper bounds subrank,

$$\mathrm{Q}(T) \leq \mathrm{SR}(T) \leq \mathrm{R}^{(i)}(T).$$

The lower bound connects slice rank to problems in extremal combinatorics, which we will discuss further in Section 1.3. The slice rank of large Kronecker powers of tensors was studied in [7] and [13], which lead to strong connections with invariant theory and moment polytopes, and with the asymptotic spectrum of tensors introduced by Strassen [32].

Analytic rank

Gowers and Wolf [19] defined the analytic rank of any tensor $T \in \mathbb{F}_p^{n_1 \times n_2 \times n_3}$ over the finite field $\mathbb{F}_p$ for a prime p as $\mathrm{AR}(T) := -\log_p \mathrm{bias}(T)$, where the bias of T is defined as $\mathrm{bias}(T) := \mathbb{E} \exp(2\pi i\, T(x, y, z)/p)$ with the expectation taken over all vectors $x \in \mathbb{F}_p^{n_1}$, $y \in \mathbb{F}_p^{n_2}$ and $z \in \mathbb{F}_p^{n_3}$. The analytic rank relates to subrank and tensor rank as follows:

$$\mathrm{Q}(T) \leq \frac{\mathrm{AR}(T)}{\mathrm{AR}(I_1)} \leq \mathrm{R}(T)$$

where $\mathrm{AR}(I_1) = -\log_p(1 - (1 - 1/p)^2)$. The upper bound was proven in [6]. Interestingly, the value of $\mathrm{AR}(T)/\mathrm{AR}(I_1)$ can be larger than $\max_i \mathrm{R}^{(i)}(T)$ for small p. The lower bound is essentially by Lovett [27]. Namely, Lovett proves that $\mathrm{AR}(T)/\mathrm{AR}(I_1)$ upper bounds the size of the largest principal subtensor of T that is diagonal. (We will discuss this further in Section 1.3.) Lovett moreover proved that $\mathrm{AR}(T) \leq \mathrm{SR}(T)$ and he thus proposes analytic rank as an effective upper bound tool for any type of problem where slice rank works well asymptotically. Lovett's result motivated us to study other parameters to upper bound the subrank, which led to geometric rank.

Another line of work has shown upper bounds on $\mathrm{SR}(T)$ in terms of $\mathrm{AR}(T)$. This was first proven by Bhowmick and Lovett [5], with an Ackerman-type dependence. The dependence was later improved significantly by Janzer [22]. Recently, Janzer [23] and Milićević [28] proved polynomial upper bounds of SR in terms of AR. It is not known whether these parameters can be related by a multiplicative constant.

1.3 Connections of subrank to complexity theory and combinatorics

Arithmetic complexity of matrix multiplication and barriers

A well-known problem in computer science concerning tensors is about the arithmetic complexity of matrix multiplication. Asymptotically how many scalar additions and multiplications are required to multiply two $m \times m$ matrices? The answer is known to be between n^2 and $Cn^{2.37\cdots}$, or in other words, the exponent of matrix multiplication ω is known to be between 2 and 2.37... [26]. The complexity of matrix multiplication turns out to be determined by the tensor rank of the matrix multiplication tensors $\langle m, m, m\rangle$ corresponding to taking the trace of the product of three $m \times m$ matrices. Explicitly, $\langle m, m, m\rangle$ corresponds to the trilinear map $\sum_{i,j,k=1}^{m} x_{ij} y_{jk} z_{ki}$. In practice, upper bounds on the rank of the matrix multiplication tensors are obtained by proving a chain of inequalities

$$\langle m, m, m\rangle \leq T \leq I_r$$

for some intermediate tensor T, which is usually taken to be a Coppersmith–Winograd tensor, and an r that is small relatively to m. It was first shown by Ambainis, Filmus and Le Gall [3] that there is a barrier for this strategy to give fast algorithms. This barrier was recently extended and simplified in several works [7, 8, 2, 1, 14] and can be roughly phrased as follows: if the asymptotic subrank of the intermediate tensor $\lim_{n\to\infty} \mathrm{Q}(T^{\otimes n})^{1/n}$ is strictly smaller than the asymptotic rank $\lim_{n\to\infty} \mathrm{R}(T^{\otimes n})^{1/n}$, then one cannot obtain $\omega = 2$ via T. These barriers rely on the fact that the asymptotic subrank of the matrix multiplication tensors is maximal. Summarizing, the rank of the matrix multiplication tensors corresponds to the complexity of matrix multiplication whereas the subrank of any tensor corresponds to the a priori suitability of that tensor for use as an intermediate tensor. The upper bounds on the asymptotic subrank used in the aforementioned results were obtained via slice rank or the related theory of support functionals and quantum functionals [13].

Cap sets, sunflowers and independent sets in hypergraphs

Several well-known problems in extremal combinatorics can be phrased in terms of the independence number of families of hypergraphs. One effective collection of upper bound methods proceeds via the subrank of tensors. (For other upper bound methods, see e.g. the recent work of Filmus, Golubev and Lifshitz [17].) A hypergraph is a a symmetric subset $E \subseteq V \times V \times V$. An independent set of E is any subset $S \subseteq V$ such that S does not induce any edges in E, that is, $E \cap (S \times S \times S) = \emptyset$. The independence number $\alpha(E)$ of E is the largest size of any independence set in E. For any hypergraph $E \subseteq [n] \times [n] \times [n]$, if $T \subseteq \mathbb{F}^{n\times n\times n}$ is any tensor supported on $E \cup \{(i, i, i) : i \in [n]\}$, then

$$\alpha(E) \leq \mathrm{Q}(T).$$

Indeed, for any independent set S of E the subtensor $T|_{S\times S\times S}$ is a diagonal tensor with nonzero diagonal and $T \geq T|_{S\times S\times S}$. For example, the resolution of the cap set problem by Ellenberg and Gijswijt [16], as simplified by Tao [34], can be thought of as upper bounding the subrank of tensors corresponding to strong powers of the hypergraph consisting of the edge $(1, 2, 3)$ and permutations. The Erdős–Szemerédi sunflower problem for three petals was resolved by Naslund and Sawin [29] by similarly considering the strong powers of the hypergraph consisting of the edge $(1, 1, 2)$ and permutations. In both cases slice rank was used to obtain the upper bound. Another result in extremal combinatorics via analytic rank was recently obtained by Briët [10].

1.4 Our results

We establish a number of basic properties of geometric rank. These imply close connections between geometric rank and other notions of rank, and thus bring in a new set of algebraic geometric tools to help reason about the various notions of rank. In particular, our new upper bounds on the (border) subrank of matrix multiplication follow easily from our basic results.

Subrank and slice rank

We prove that the geometric rank $\mathrm{GR}(T)$ is at most the slice rank $\mathrm{SR}(T)$ of Tao [34] and at least the subrank $\mathrm{Q}(T)$ of Strassen [31] (see Theorem 6).

▶ **Theorem 1.** *For any tensor T,*

$$\mathrm{Q}(T) \leq \mathrm{GR}(T) \leq \mathrm{SR}(T).$$

We thus add GR to the collection of tools to upper bound the subrank of tensors Q and in turn the independence number of hypergraphs. We prove these inequalities by proving that GR is monotone under $\leq$, additive under the direct sum of tensors, and has value 1 on the trivial I_1 tensor. We also give a second more direct proof of this inequality (Theorem 23).

Border subrank

We extend our upper bound on subrank to *border subrank*, the (widely studied) approximative version of subrank.

The main ingredient in this extension is the following fact (which itself exploits the algebraic-geometric nature of definition of GR): the set $\{T \in \mathbb{F}^{n\times n\times n} \mid \mathrm{GR}(T) \leq m\}$ is closed in the Zariski topology.[1] In other words, geometric rank is lower-semicontinuous. This implies that the geometric rank also upper bounds the border subrank $\underline{\mathrm{Q}}(T)$ (see Theorem 12).

▶ **Theorem 2.** *For any tensor T,*

$$\underline{\mathrm{Q}}(T) \leq \mathrm{GR}(T).$$

As far as we know, GR is a new tensor parameter. We show that GR is not the same parameter as Q, $\underline{\mathrm{Q}}$ or SR (Remark 20 and Remark 22).

Matrix multiplication

In the study of the complexity of matrix multiplication, Strassen [31] proved that for the matrix multiplication tensors $\langle m, m, m\rangle \in \mathbb{F}^{m^2\times m^2\times m^2}$ the border subrank is lower bounded by $\lceil \frac{3}{4}m^2\rceil \leq \underline{\mathrm{Q}}(\langle m,m,m\rangle)$. We prove that this lower bound is optimal by proving the following (see Theorem 17).

▶ **Theorem 3.** *For any positive integers $e \leq h \leq \ell$,*

$$\underline{\mathrm{Q}}(\langle e,h,\ell\rangle) = \mathrm{GR}(\langle e,h,\ell\rangle) = \begin{cases} eh - \lfloor \frac{(e+h-\ell)^2}{4} \rfloor & \text{if } e+h \geq \ell, \\ eh & \text{otherwise.} \end{cases}$$

In particular, we have $\mathrm{Q}(\langle m,m,m\rangle) \leq \underline{\mathrm{Q}}(\langle m,m,m\rangle) = \mathrm{GR}(\langle m,m,m\rangle) = \lceil \frac{3}{4}m^2\rceil$ for any $m \in \mathbb{N}$.

[1] That is, the statement $\mathrm{GR}(T) \leq m$ is characterized by the vanishing of a finite number of polynomials.

Our computation of GR here is a calcluation of the dimension of a variety. We do this by studying the dimension of various sections of that variety, which then reduces to linear algebraic questions about matrices (we are talking about matrix multiplication after all).

Our result improves the previously best known upper bound on the subrank of matrix multiplication of Christandl, Lucia, Vrana and Werner [12], which was $\mathrm{Q}(\langle m, m, m\rangle) \leq m^2 - m + 1$. In fact, our upper bound on $\mathrm{GR}(\langle e, h, \ell\rangle)$ exactly matches the lower bound on $\underline{\mathrm{Q}}(\langle e, h, \ell\rangle)$ of Strassen [31], for any nonnegative integers e, h, and ℓ. We thus solve the problem of determining the exact value of $\underline{\mathrm{Q}}(\langle e, h, \ell\rangle)$.

Analytic rank

Finally, we establish a strong connection between geometric rank and analytic rank.

We prove that for any tensor $T \in \mathbb{Z}^{n_1\times n_2\times n_3} \subseteq \mathbb{C}^{n_1\times n_2\times n_3}$ with integer coefficients, the geometric rank of T equals the *liminf* of the analytic rank of the tensors $T_p \in \mathbb{F}_p^{n_1\times n_2\times n_3}$ obtained from T be reducing all coefficients modulo p and letting p go to infinity over all primes (see Theorem 24).

▶ **Theorem 4.** *For every tensor T over $\mathbb{Z}$ we have*

$$\liminf_{p\to\infty} \mathrm{AR}(T_p) = \mathrm{GR}(T).$$

This result is in fact the source of our definition of geometric rank. The analytic rank of a tensor is defined as the bias of a certain polynomial on random inputs. By simple transformations, computing the analytic rank over $\mathbb{F}_p$ reduces to computing the number of solutions of a system of polynomial equations over $\mathbb{F}_p$. Namely,

$$\mathrm{AR}(T_p) = n_1 + n_2 - \log_p \left|\{(x, y) \in \mathbb{F}_p^{n_1} \times \mathbb{F}_p^{n_2} : T_p(x, y, \cdot) = 0\}\right|.$$

This system of polynomial equations defines a variety, and it is natural to expect that the dimension of the variety roughly determines the number of $\mathbb{F}_p$-points of the variety. This expectation is not true in general, but under highly controlled circumstances something like it is true. This is how we arrived at the definition of geometric rank (which eventually turned out to have very natural properties on its own, without this connection to analytic rank).

Actually establishing the above liminf result is quite roundabout, and requires a number of tools from algebraic geometry and number theory. In particular, we do not know whether this liminf can be replaced by a limit!

We stress that analytic rank is only defined for tensors over prime fields of positive characteristic, whereas geometric rank is defined for tensors over any field. By the aforementioned result, geometric rank over the complex numbers can be thought of as an extension of analytic rank to characteristic 0. Finding an extension of analytic rank beyond finite fields is mentioned as an open problem by Lovett [27, Problem 1.10].

Organization of this paper

In the next section we formally define geometric rank. In Section 3, we give some alternative definitions of geometric rank that help us reason about it. In Section 4 and Section 5 we show the relationship between geometric rank, slice rank, subrank and border subrank. In Section 6 we use the established properties of geometric rank to give a proof of our upper bound on the (border) subrank of matrix multiplication. In Section 7 we give a more direct proof of the inequality between slice rank and geometric rank. Finally, in Section 8 we establish the relationship between geometric and analytic ranks.

2 Geometric rank

In this section we set up some general notation and define geometric rank. Let $\mathbb{F}$ be an algebraically closed field.

Dimension and codimension

The notion of dimension that we use is the standard notion in algebraic geometry, and is defined as follows. Let $V \subseteq \mathbb{F}^n$ be a (possibly reducible) algebraic variety. The codimension $\operatorname{codim} V$ is defined as $n - \dim V$. The dimension $\dim V$ is defined as the length of a maximal chain of irreducible subvarieties of V [21]. In our proofs we will use basic facts about dimension: the dimension of a linear space coincides with the notion from linear algebra, the dimension is additive under the cartesian product, the dimension of a locally open set equals the dimension of its closure and dimension behaves well under projections $(x, y) \mapsto y$.

Notation about tensors

Let $\mathbb{F}^{n_1 \times n_2 \times n_3}$ be the set of all three-dimensional arrays

$$T = (T_{i,j,k})_{i \in [n_1], j \in [n_2], k \in [n_3]}$$

with $T_{i,j,k} \in \mathbb{F}$. We refer to the elements of $\mathbb{F}^{n_1 \times n_2 \times n_3}$ as the $n_1 \times n_2 \times n_3$ *tensors over* $\mathbb{F}$. To any tensor $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ we associate the polynomial $T(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3})$ in $\mathbb{F}[x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3}]$ defined by

$$T(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3}) = \sum_{i \in [n_1]} \sum_{j \in [n_2]} \sum_{k \in [n_3]} T_{i,j,k}\, x_i y_j z_k$$

and the trilinear map $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \to \mathbb{F}$ defined by

$$T(x, y, z) = T(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3}).$$

Geometric rank

▶ **Definition 2.** *The* geometric rank *of a tensor* $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$*, written* $\operatorname{GR}(T)$*, is the codimension of the set of elements* $(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ *such that* $T(x, y, z) = 0$ *for all* $z \in \mathbb{F}^{n_3}$*. That is,*

$$\operatorname{GR}(T) \coloneqq \operatorname{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid \forall z \in \mathbb{F}^{n_3} : T(x, y, z) = 0\}.$$

For any $(x, y) \in \mathbb{F}^{n_2} \times \mathbb{F}^{n_3}$ *we define the vector* $T(x, y, \cdot) = (T(x, y, e_k))_{k=1}^{n_3}$*, where* $e_1, \ldots, e_{n_3}$ *is the standard basis of* $\mathbb{F}^{n_3}$*. In this notation the geometric rank is given by*

$$\operatorname{GR}(T) = \operatorname{codim}\{(x, y) \mid T(x, y, \cdot) = 0\}.$$

For later use we also define the vectors $T(x, \cdot, z) = T(x, e_j, z)_j$ *and* $T(\cdot, y, z) = T(e_i, y, z)_i$*, and we define the matrices* $T(x, \cdot, \cdot) = T(x, e_j, e_k)_{j,k}$*,* $T(\cdot, y, \cdot) = T(e_i, y, e_k)_{i,k}$ *and* $T(\cdot, \cdot, z) = T(e_i, e_j, z)_{i,j}$*.*

We defined the geometric rank of tensors with coefficients in an algebraically closed field. For tensors with coefficients in an arbitrary field we naturally define the geometric rank via the embedding of the field in its algebraic closure.

Computer software

One can compute the dimension of an algebraic variety $V \subseteq \mathbb{F}^n$ using computer software like Macaulay2 [20] or Sage [30]. This allows us to easily compute the geometric rank of small tensors. For example, for Example 1 in the introduction over the field $\mathbb{F} = \mathbb{C}$, one verifies in Macaulay2 with the commands

```
R = CC[x1,x2,y1,y2];
dim ideal(x1*y1, x2*y1 + x1*y2)
```

or in Sage with the commands

```
A.<x1,x2,y1,y2> = AffineSpace(4, CC);
Ideal([x1*y1, x2*y1 + x1*y2]).dimension()
```

that $\dim V = 2$.

Computational complexity

Koiran [24] studied the computational complexity of the problem of deciding whether the dimension of an algebraic variety $V \subseteq \mathbb{C}^n$ is at least a given number. When V is given by polynomial equations over the integers the problem is in PSPACE, and assuming the Generalized Riemann Hypothesis the problem is in the Arthur–Merlin class AM. Thus the same upper bounds apply to computing GR.

In the other direction, Koiran showed that computing dimension of algebraic varieties in general is NP-hard. We know of no hardness results for computing GR.

Higher-order tensors

Our definition of geometric rank extends naturally from the set of 3-tensors $\mathbb{F}^{n_1\times n_2\times n_3}$ to the set of k-tensors $\mathbb{F}^{n_1\times\cdots\times n_k}$ for any $k \geq 2$ by defining the geometric rank of any k-tensor $T \in \mathbb{F}^{n_1\times\cdots\times n_k}$ as

$$\mathrm{GR}(T) \coloneqq \mathrm{codim}\{(x_1,\ldots,x_{k-1}) \in \mathbb{F}^{n_1}\times\cdots\times\mathbb{F}^{n_{k-1}} \mid \forall x_k \in \mathbb{F}^{n_k} : T(x_1,\ldots,x_{k-1},x_k) = 0\}.$$

For $k = 2$ geometric rank coincides with matrix rank. Our results extend naturally to k-tensors with this definition, but for clarity our exposition will be in terms of 3-tensors.

3 Alternative descriptions of geometric rank

We give two alternative descriptions of geometric rank that we will use later. The first description relates geometric rank to the matrix rank of the matrices $T(x,\cdot,\cdot) = (T(x,e_j,e_k))_{j,k}$. The second description shows that the geometric rank of $T(x,y,z)$ is symmetric under permuting the variables x, y and z. Both theorems rely on an understanding of the dimension of fibers of a (nice) map.

▶ **Theorem 3.** *For any tensor $T \in \mathbb{F}^{n_1\times n_2\times n_3}$,*

$$\begin{aligned}\dim\{(x,y) \mid T(x,y,\cdot) = 0\} &= \max_i \dim\big\{x \mid \dim\{y \mid T(x,y,\cdot) = 0\} = i\big\} + i\\ &= \max_i \dim\big\{x \mid \mathrm{corank}\, T(x,\cdot,\cdot) = i\big\} + i\end{aligned}$$

and therefore

$$\mathrm{GR}(T) = \mathrm{codim}\{(x,y) \mid T(x,y,\cdot) = 0\} = \min_j \mathrm{codim}\big\{x \mid \mathrm{rank}\, T(x,\cdot,\cdot) = j\big\} + j.$$

Proof. Let $V = \{(x, y) \mid T(x, y, \cdot) = 0\}$. Let $W = \mathbb{F}^{n_1}$. Let $\pi : V \to W$ map (x, y) to x. Define the sets $W_i = \{x \mid \operatorname{corank}(T(x, \cdot, \cdot)) = i\}$. The rank-nullity theorem for matrices gives for any fixed x that $\operatorname{corank}(T(x, \cdot, \cdot)) = \dim\{y \mid T(x, y, \cdot) = 0\}$. The sets W_i are locally closed, that is, each W_i is the intersection of an open set and a closed set. Let $V_i = \pi^{-1}(W_i)$. The set V_i is also locally closed. We have that $W = \cup_i W_i$ and so $V = \cup_i V_i$. Therefore, $\dim V = \max_i \dim V_i$. We claim that $\dim V_i = \dim W_i + i$. From this claim follows $\dim V = \max_i \dim W_i + i$, which finishes the proof.

We prove the claim that $\dim V_i = \dim W_i + i$. For every $x \in W_i$ the fiber dimension $\dim \pi^{-1}(x)$ equals i. Write V_i as a union of irreducible components V_{ij}. Let W_{ij} be the closure of $\pi(V_{ij})$. We now apply Theorem 5 (see the end of this section) with $X = V_i$ and $X_0 = V_{ij}$. For any $p = (x, y) \in X_0$ we have that $\pi^{-1}(\pi(p)) = \{(x, y') \mid T(x, y', \cdot) = 0\}$. The set $\{y' \mid T(x, y', \cdot) = 0\}$ is a linear subspace and thus irreducible. Therefore, $\pi^{-1}(\pi(p))$ is irreducible. Then Theorem 5 gives that $\dim V_{ij} = \dim W_{ij} + i$. We have that $\max_j \dim W_{ij} = \dim W_i$, so taking the j maximising $\dim W_{ij}$ gives $\dim V_i \leq \dim W_i + i$. Also $\max_j \dim V_{ij} = \dim V_i$, so taking the j maximising $\dim V_{ij}$ gives $\dim V_i \geq \dim W_i + i$. ◀

▶ **Theorem 4.** *For any tensor T,*

$$\begin{aligned}\operatorname{GR}(T) = \operatorname{codim}\{(x, y) \mid T(x, y, \cdot) = 0\} &= \operatorname{codim}\{(x, z) \mid T(x, \cdot, z) = 0\} \\ &= \operatorname{codim}\{(y, z) \mid T(\cdot, y, z) = 0\}.\end{aligned}$$

Proof. We apply Theorem 3 to T and to T after swapping y and z to get that the codimensions of $\{(x, y) \mid T(x, y, \cdot) = 0\}$ and $\{(x, z) \mid T(x, \cdot, z) = 0\}$ are equal to $\min_j \operatorname{codim}\{x \mid \operatorname{rank} T(x, \cdot, \cdot) = j\} + j$. This proves the first equality. The second equality is proven similarly. ◀

▶ **Theorem 5** ([21, special case of Theorem 11.12])**.** *Let $X \subseteq \mathbb{F}^{n_1} \times \mathbb{F}^{n_2}$ be the affine cone over a quasi-projective variety, that is,*

$$X = \{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid f_1(x, y) = 0, \ldots, f_k(x, y) = 0, g_1(x, y) \neq 0, \ldots, g_m(x, y) \neq 0\}$$

where the f_i and g_i are homogeneous polynomials. Let $\pi : X \to \mathbb{F}^{n_1}$ map (x, y) to x. Let $X_0 \subseteq X$ be an irreducible component. Suppose that the fiber $\pi^{-1}(\pi(p))$ is irreducible for every $p \in X_0$. Then

$$\dim X_0 = \dim \overline{\pi(X_0)} + \min_{p \in X_0} \dim \pi^{-1}(\pi(p)).$$

4 Geometric rank is between subrank and slice rank

Recall that the subrank $\operatorname{Q}(T)$ of T is the largest number s such that $I_s \leq T$ and the slice rank $\operatorname{SR}(T)$ is the smallest number r such that $T(x, y, z)$ can be written as a sum of r trilinear maps of the form $f(x)g(y, z)$ or $f(y)g(x, z)$ or $f(z)g(x, y)$.

▶ **Theorem 6.** *For any tensor T,*

$$\operatorname{Q}(T) \leq \operatorname{GR}(T) \leq \operatorname{SR}(T).$$

Theorem 6 will follow from the following basic properties of GR. We will give a more direct proof of the inequality $\operatorname{GR}(T) \leq \operatorname{SR}(T)$ in Section 7. Recall from the introduction that for any two tensors $S \in \mathbb{F}^{m_1 \times m_2 \times m_3}$ and $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ we write $S \leq T$ if there are matrices A, B, C such that $S = (A, B, C) \cdot T$ where we define $(A, B, C) \cdot T := (\sum_{a,b,c} A_{ia} B_{jb} C_{kc} T_{a,b,c})_{i,j,k}$.

▶ **Lemma 7.** GR *is* $\leq$*-monotone: if* $S \leq T$*, then* $\mathrm{GR}(S) \leq \mathrm{GR}(T)$.

Proof. Let $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$. We claim that $\mathrm{GR}((\mathrm{Id}, \mathrm{Id}, C) \cdot T) \leq \mathrm{GR}(T)$ for any $C \in \mathbb{F}^{m_3 \times n_3}$, where Id denotes an identity matrix of the appropriate size. From this claim and the symmetry of GR (Theorem 4), follows the inequalities $\mathrm{GR}((A, \mathrm{Id}, \mathrm{Id}) \cdot T) \leq T$ and $\mathrm{GR}((\mathrm{Id}, B, \mathrm{Id}) \cdot T) \leq \mathrm{GR}(T)$ for any matrices $A \in \mathbb{F}^{m_1 \times n_1}$ and $B \in \mathbb{F}^{m_2 \times n_2}$. Chaining these three inequalities gives that for any two tensors S and T, if $S \leq T$, then $\mathrm{GR}(S) \leq \mathrm{GR}(T)$.

We prove the claim. Let $S = (\mathrm{Id}, \mathrm{Id}, C) \cdot T$. Let $M_k = (T_{i,j,k})_{ij}$ be the 3-slices of T and let $N_k = (S_{i,j,k})_{ij}$ be the 3-slices of S. Since $S = (\mathrm{Id}, \mathrm{Id}, C) \cdot T$, the matrices $N_1, \ldots, N_{m_3}$ are in the linear span of the matrices $M_1, \ldots, M_{n_3}$. Thus $V = \{(x, y) \mid x^T M_1 y = \cdots = x^T M_{n_3} y = 0\}$ is a subset of $W = \{(x, y) \mid x^T N_1 y = \cdots = x^T N_{m_3} y = 0\}$. Therefore, $\dim V \leq \dim W$ and it follows that $\mathrm{GR}(S) = \operatorname{codim} W \leq \operatorname{codim} V = \mathrm{GR}(T)$. ◀

Let $T_1 \in \mathbb{F}^{m_1 \times m_2 \times m_3}$ and $T_2 \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ be tensors with 3-slices A_k and B_k respectively. The direct sum $T_1 \oplus T_2 \in \mathbb{F}^{(m_1+n_1) \times (m_2+n_2) \times (m_3+n_3)}$ is defined as the tensor with 3-slices $A_k \oplus 0_{n_1 \times n_2}$ for $k = 1, \ldots, m_3$ and $0_{m_1 \times m_2} \oplus B_k$ for $k = m_3 + 1, \ldots, m_3 + n_3$ where $0_{a \times b}$ denotes the zero matrix of size $a \times b$. In other words, $T_1 \oplus T_2$ is the block-diagonal tensor with blocks T_1 and T_2.

▶ **Lemma 8.** GR *is additive under direct sums:* $\mathrm{GR}(T_1 \oplus T_2) = \mathrm{GR}(T_1) + \mathrm{GR}(T_2)$.

Proof. Let A_k be the 3-slices of T_1 and let B_k be the 3-slices of T_2. Let $T = T_1 \oplus T_2$ be the direct sum with 3-slices M_k. Then

$$V = \{(x, y) \mid T(x, y, \cdot) = 0\} = \{(x, y) \mid x^T M_1 y = \cdots = x^T M_{m_3+n_3} y = 0\}$$

is the cartesian product of

$$V_1 = \{(x, y) \mid x^T A_1 y = \cdots = x^T A_{m_3} y = 0\}$$

and

$$V_2 = \{(x, y) \mid x^T B_1 y = \cdots = x^T B_{n_3} y = 0\}.$$

Thus $\dim V = \dim V_1 + \dim V_2$ [21, page 138]. Therefore, $\mathrm{GR}(T) = \mathrm{GR}(T_1) + \mathrm{GR}(T_2)$. ◀

▶ **Lemma 9.** GR *is sub-additive under element-wise sums:* $\mathrm{GR}(S + T) \leq \mathrm{GR}(S) + \mathrm{GR}(T)$.

Proof. Note that $S + T \leq S \oplus T$. Thus, $\mathrm{GR}(S+T) \leq \mathrm{GR}(S \oplus T) = \mathrm{GR}(S) + \mathrm{GR}(T)$, where the inequality uses Lemma 7, and the equality uses Lemma 8. ◀

▶ **Lemma 10.** *If* $\mathrm{SR}(T) = 1$*, then* $\mathrm{GR}(T) = 1$.

Proof. It is sufficient to consider a tensor $T \in \mathbb{F}^{1 \times n \times n}$ with one nonzero slice. Then we have that $T(0, \mathbb{F}^n, \mathbb{F}^n) = 0$, and so $\mathrm{GR}(T) = 1 + n - n = 1$. ◀

▶ **Lemma 11.** *For every* $r \in \mathbb{N}$ *we have* $\mathrm{GR}(I_r) = r$.

Proof. We have $\mathrm{SR}(I_1) = 1$ and so $\mathrm{GR}(I_1) = 1$ (Lemma 10). Since I_r is a direct sum of r copies of I_1 and geometric rank is additive under taking the direct sum $\oplus$ (Lemma 9), we find that $\mathrm{GR}(I_r) = r\,\mathrm{GR}(I_1) = r$. ◀

Proof of Theorem 6. We prove that $\mathrm{GR}(T) \leq \mathrm{SR}(T)$. Let $r = \mathrm{SR}(T)$. Then there are tensors $T_1, \ldots, T_r$ so that $T = \sum_{i=1}^r T_i$ and $\mathrm{SR}(T_i) = 1$. Then also $\mathrm{GR}(T_i) = 1$ (Lemma 10). Subadditivity of GR under element-wise sums (Lemma 9) gives

$$\mathrm{GR}(T) \leq \sum_{i=1}^{r} \mathrm{GR}(T_i) = r = \mathrm{SR}(T).$$

We prove that $\mathrm{Q}(T) \leq \mathrm{GR}(T)$. Let $s = \mathrm{Q}(T)$. Then $I_s \leq T$. We know $\mathrm{GR}(I_s) = s$ (Lemma 11). By the $\leq$-monotonicity of GR (Lemma 7), we have

$$\mathrm{Q}(T) = s = \mathrm{GR}(I_s) \leq \mathrm{GR}(T).$$ ◀

5 Geometric rank is at least border subrank

In this section we extend the inequality $\mathrm{Q}(T) \leq \mathrm{GR}(T)$ (Theorem 6) to the approximative version of subrank, called border subrank. To define border subrank we first define degeneration $\trianglelefteq$, which is the approximative version of restriction $\leq$. We write $S \trianglelefteq T$, and we say S is a *degeneration* of T, if for some $e \in \mathbb{N}$ we have

$$S + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots + \varepsilon^e S_e = (A(\varepsilon), B(\varepsilon), C(\varepsilon)) \cdot T$$

for some tensors S_i over $\mathbb{F}$ and for some matrices $A(\varepsilon), B(\varepsilon), C(\varepsilon)$ whose coefficients are Laurent polynomials in the formal variable ε. Equivalently, $S \trianglelefteq T$ if and only if S is in the orbit closure $\overline{G \cdot T}$ where G denotes the group $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{n_3}$, $G \cdot T$ denotes the natural group action that we also used in the definition of $\leq$, and the closure is taken in the Zariski topology [11, Theorem 20.24]. (When $\mathbb{F} = \mathbb{C}$ one may equivalently take the closure in the Euclidean topology.) Recall that the subrank of T is defined as $\mathrm{Q}(T) = \max\{n \in \mathbb{N} \mid I_n \leq T\}$. The *border subrank* of T is defined as

$$\underline{\mathrm{Q}}(T) = \max\{n \in \mathbb{N} \mid I_n \trianglelefteq T\}.$$

Clearly, $\mathrm{Q}(T) \leq \underline{\mathrm{Q}}(T)$.

▶ **Theorem 12.** *For any tensor T,*

$$\underline{\mathrm{Q}}(T) \leq \mathrm{GR}(T).$$

To prove Theorem 12 we use the following theorem on upper-semicontinuity of fiber dimension.

▶ **Theorem 13** ([21, special case of Corollary 11.13])**.** *Let X be the zero set of bi-homogeneous polynomials, that is,*

$$X = \{(a, b) \in \mathbb{F}^{m_1} \times \mathbb{F}^{m_2} \mid f_1(a, b) = \cdots = f_k(a, b) = 0\}$$

where the $f_i(a, b)$ are polynomials that are homogeneous in both a and b. Let $\pi : X \to \mathbb{F}^{m_2}$ map (a, b) to b. Let $Y = \pi(X)$ be its image. For any $q \in Y$, let $\lambda(q) = \dim(\pi^{-1}(q))$. Then $\lambda(q)$ is an upper-semicontinuous function of q, that is, the set $\{q \in Y \mid \lambda(q) \geq m\}$ is Zariski closed in Y.

▶ **Lemma 14.** GR *is lower-semicontinuous: for any $n_i, m \in \mathbb{N}$ the set $\{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \mid \mathrm{GR}(T) \leq m\}$ is Zariski closed.*

Proof. We define the set

$$X = \{(T, x, y) \in \mathbb{F}^{n_1 \times n_2 \times n_3} \times \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid T(x, y, \mathbb{F}^{n_3}) = 0\}.$$

Let $\pi : X \to \mathbb{F}^{n_1 \times n_2 \times n_3}$ map (T, x, y) to T. Let $Y = \pi(X) = \mathbb{F}^{n_1 \times n_2 \times n_3}$ be the image of π. For any $T \in Y$ let $\lambda(T) \coloneqq \dim(\pi^{-1}(T))$. Then $\lambda(T)$ is an upper-semicontinuous function of T in the Zariski topology on Y by Theorem 13. This means that the set $\{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \mid \lambda(T) \geq m\}$ is closed for every $m \in \mathbb{N}$. It follows that $\{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \mid \mathrm{GR}(T) \leq m\}$ is closed for every $m \in \mathbb{N}$. ◀

▶ Remark 15. A well-known example of a lower-semicontinuous function is matrix rank. Indeed, the set of matrices of rank at most m is the zero set of the determinants of all $(m + 1) \times (m + 1)$ submatrices. For geometric rank we do not know an explicit set of generators for the vanishing ideal of $\{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \mid \mathrm{GR}(T) \leq m\}$. For slice rank the set $\{T \in \mathbb{F}^{n_1 \times n_2 \times n_3} \mid \mathrm{SR}(T) \leq m\}$ is also known to be Zariski closed and explicit vanishing polynomials for this variety were recently obtained by Bläser, Ikenmeyer, Lysikov, Pandey and Schreyer [9].

▶ **Lemma 16.** GR *is* $\trianglelefteq$*-monotone: if* $S \trianglelefteq T$*, then* $\mathrm{GR}(S) \leq \mathrm{GR}(T)$

Proof. For all $g \in G$ we have $\mathrm{GR}(g \cdot T) = \mathrm{GR}(T)$ by Lemma 7. The set $\{T' \mid \mathrm{GR}(T') \leq \mathrm{GR}(T)\}$ is Zariski closed by Lemma 14. It contains the orbit $G \cdot T$ and hence also its Zariski closure $\overline{G \cdot T}$, that is,

$$\{T' \mid T' \trianglelefteq T\} = \overline{G \cdot T} \subseteq \{T' \mid \mathrm{GR}(T') \leq \mathrm{GR}(T)\}.$$

Therefore, $\mathrm{GR}(S) \leq \mathrm{GR}(T)$. ◀

Proof of Theorem 12. Let $n = \underline{\mathrm{Q}}(T)$. Then $I_n \trianglelefteq T$ by the definition of $\underline{\mathrm{Q}}$, and so $n \leq \mathrm{GR}(T)$ by Lemma 16. This proves the claim. ◀

6 The border subrank of matrix multiplication

In the context of constructing fast matrix multiplication algorithms, Strassen [31, Theorem 6.6] proved that for any positive integers $e \leq h \leq \ell$ the border subrank of the matrix multiplication tensor $\langle e, h, \ell\rangle$ is lower bounded by

$$\underline{\mathrm{Q}}(\langle e, h, \ell\rangle) \geq \begin{cases} eh - \lfloor \frac{(e+h-\ell)^2}{4} \rfloor & \text{if } e + h \geq \ell, \\ eh & \text{otherwise.} \end{cases} \tag{1}$$

Here $\langle e, h, \ell\rangle$ is the tensor that corresponds to taking the trace of the product of an $e \times h$ matrix, an $h \times \ell$ matrix and an $\ell \times e$ matrix. We prove using the geometric rank that this lower bound is optimal.

▶ **Theorem 17.** *For any positive integers* $e \leq h \leq \ell$

$$\underline{\mathrm{Q}}(\langle e, h, \ell\rangle) = \mathrm{GR}(\langle e, h, \ell\rangle) = \begin{cases} eh - \lfloor \frac{(e+h-\ell)^2}{4} \rfloor & \text{if } e + h \geq \ell, \\ eh & \text{otherwise.} \end{cases}$$

In particular, we have $\underline{\mathrm{Q}}(\langle m, m, m\rangle) = \mathrm{GR}(\langle m, m, m\rangle) = \lceil \frac{3}{4} m^2 \rceil$ *for any* $m \in \mathbb{N}$.

Proof. Since $\underline{\mathrm{Q}}(\langle e,h,\ell\rangle) \leq \mathrm{GR}(\langle e,h,\ell\rangle)$ (Theorem 12) and since we have the lower bound in (1), it suffices to show that $\mathrm{GR}(\langle e,h,\ell\rangle)$ is at most $eh - \lfloor (e+h-\ell)^2/4 \rfloor$ if $e+h \geq \ell$ and at most eh otherwise.

Let $T = \langle e,h,\ell\rangle$. Let $V = \{(x,y) \in \mathbb{F}^{eh} \times \mathbb{F}^{h\ell} \mid T(x,y,\cdot) = 0\}$. Then $\mathrm{GR}(T) = eh + h\ell - \dim V$. From Theorem 3 it follows that

$$\dim V = \max_i \dim\{x \in \mathbb{F}^{eh} \mid \dim\{y \in \mathbb{F}^{h\ell} \mid T(x,y,\cdot) = 0\} = i\} + i. \tag{2}$$

We now think of $\mathbb{F}^{eh}$, $\mathbb{F}^{h\ell}$ and $\mathbb{F}^{\ell e}$ as the matrix spaces $\mathbb{F}^{e\times h}$, $\mathbb{F}^{h\times\ell}$ and $\mathbb{F}^{\ell\times e}$. Then T gives the trilinear map $T : \mathbb{F}^{e\times h} \times \mathbb{F}^{h\times\ell} \times \mathbb{F}^{\ell\times e} \to \mathbb{F} : (X,Y,Z) \mapsto \mathrm{Tr}(XYZ)$. Therefore, $T(X,Y,\cdot) = 0$ if and only if $XY = 0$. If the rank of X as an $e \times h$ matrix equals r, then

$$\dim\{Y \in \mathbb{F}^{h\times\ell} \mid T(X,Y,\cdot) = 0\} = (h-r)\ell,$$

since Y is any matrix with columns from $\ker(X)$. We have

$$\dim\{X \in \mathbb{F}^{e\times h} \mid \mathrm{rank}(X) = r\} = er + (h-r)r.$$

Thus the relevant values of i in (2) are of the form $i = (h-r)\ell$ and we have that

$$\begin{aligned}\dim V &= \max_r \dim\{X \in \mathbb{F}^{e\times h} \mid \mathrm{rank}\, X = r\} + (h-r)\ell \\ &= \max_r er + (h-r)r + (h-r)\ell \\ &= \max_r f(r) + h\ell\end{aligned}$$

where $f(r) = r(\Delta - r)$ with $\Delta := e + h - \ell$. Thus,

$$\mathrm{GR}(T) = eh - \max_r f(r).$$

Over the integers, the function f attains its maximum at $\lfloor \frac{\Delta}{2} \rfloor$ (and at $\lceil \frac{\Delta}{2} \rceil$), but this may be outside the interval $[0,e]$ that we want to maximise over (recall $e \leq h \leq l$). Observe that if $\Delta \geq 0$ then $e \geq \Delta/2 \geq 0$, meaning that f does attain its global maximum in the interval $[0,e]$. On the other hand, if $\Delta \leq 0$ then $r(\Delta - r) \leq 0 = f(0)$ for every $r \geq 0$, so the maximum of f in the interval $[0,e]$ is at the endpoint $r = 0$. Summarizing,

$$\max_{0\leq r\leq e} f(r) = \begin{cases} \lfloor \frac{\Delta^2}{4} \rfloor & \text{if } \Delta \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

This completes the proof. ◀

▶ Remark 18. Theorem 17 gives the upper bound $\mathrm{Q}(\langle m,m,m\rangle) \leq \underline{\mathrm{Q}}(\langle m,m,m\rangle) = \lceil \frac{3}{4}m^2 \rceil$ on the subrank of matrix multiplication $\mathrm{Q}(\langle m,m,m\rangle)$. This improves the previously best known upper bound $\mathrm{Q}(\langle m,m,m\rangle) \leq m^2 - m + 1$ from [12, Equation 25].

▶ Remark 19. Geometric rank GR is not sub-multiplicative under the tensor Kronecker product $\otimes$. We give an example. The matrix multiplication tensor $\langle m,m,m\rangle$ can be written as the product $\langle m,m,m\rangle = \langle m,1,1\rangle \otimes \langle 1,m,1\rangle \otimes \langle 1,1,m\rangle$ and $\mathrm{GR}(\langle m,1,1\rangle) = \mathrm{GR}(\langle 1,m,1\rangle) = \mathrm{GR}(\langle 1,1,m\rangle) = 1$ whereas $\mathrm{GR}(\langle m,m,m\rangle) = \lceil \frac{3}{4}m^2 \rceil$ by Theorem 17.

▶ Remark 20. Geometric rank GR is not the same as subrank Q or border subrank $\underline{\mathrm{Q}}$. For example, for the trilinear map $W(x_1,x_2,y_1,y_2,z_1,z_2) = x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1$ we find $\mathrm{GR}(W) = 2$ (see the example in the introduction), whereas $\mathrm{Q}(W) = \underline{\mathrm{Q}}(W) = 1$. The latter follows from the fact that $\underset{\sim}{\mathrm{Q}}(W) = 1.81...$ [33], where $\underset{\sim}{\mathrm{Q}}(T) := \lim_{n\to\infty} \mathrm{Q}(T^{\otimes n})^{1/n}$ is the asymptotic subrank of T, since $\underline{\mathrm{Q}}(T) \leq \underset{\sim}{\mathrm{Q}}(T)$ [31].

▶ Remark 21. Geometric rank GR is not super-multiplicative under the tensor Kronecker product $\otimes$. Here is an example. Let $\underset{\sim}{\mathrm{SR}}(T) \coloneqq \lim_{n\to\infty} \mathrm{SR}(T^{\otimes n})^{1/n}$ and let $\underset{\sim}{\mathrm{GR}}(T) \coloneqq \lim_{n\to\infty} \mathrm{GR}(T^{\otimes n})^{1/n}$, whenever these limits are defined. From the fact that $\mathrm{Q}(T) \leq \mathrm{GR}(T) \leq \mathrm{SR}(T)$ and the fact that $\underset{\sim}{\mathrm{Q}}(W) = \underset{\sim}{\mathrm{SR}}(W) = 1.81...$ [13] it follows that $\underset{\sim}{\mathrm{GR}}(W) = 1.81..$, whereas $\mathrm{GR}(W) = 2$. We conclude that GR is not super-multiplicative. We have seen already in Remark 19 that GR is not sub-multiplicative.

▶ Remark 22. Geometric rank GR is not the same as slice rank SR. For example, for the matrix multiplication tensor $\langle m,m,m\rangle$ we find that $\mathrm{GR}(\langle m,m,m\rangle) = \lceil \frac{3}{4}m^2\rceil$ (Theorem 17), whereas it was known that $\mathrm{SR}(\langle m,m,m\rangle) = m^2$ [7, Remark 4.9].

7 Geometric rank versus slice rank

In Section 4 we proved, by chaining the basic properties of geometric rank, that geometric rank is at most slice rank, that is, $\mathrm{GR}(T) \leq \mathrm{SR}(T)$. What is the largest gap between $\mathrm{GR}(T)$ and $\mathrm{SR}(T)$? Motivated by this question, and motivated by the analogous question for analytic rank instead of geometric rank that we discussed in the introduction we give a direct proof of the inequality $\mathrm{GR}(T) \leq \mathrm{SR}(T)$.

In fact, we prove a chain of inequalities $\mathrm{GR}(T) \leq \mathrm{ZR}(T) \leq \mathrm{SR}(T)$ where $\mathrm{ZR}(T)$ is defined as follows. We will use the following notation for a tensor $T \in \mathbb{F}^{n_1\times n_2\times n_3}$;

$$\mathbf{V}(T) = \{(x,y) \in \mathbb{F}^{n_1\times n_2} \mid \forall z \in \mathbb{F}^{n_3} : T(x,y,z) = 0\}. \tag{4}$$

Moreover, we use the following standard notation for the variety cut out by polynomials $f_1,\ldots,f_s$;

$$\mathbf{V}(f_1,\ldots,f_s) = \{x \mid f_1(x) = \cdots = f_s(x) = 0\}. \tag{5}$$

Let $\mathbb{F}[\mathbf{x},\mathbf{y}] = \mathbb{F}[x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2}]$ and let

$$\mathbb{F}[\mathbf{x},\mathbf{y},\mathbf{z}] = \mathbb{F}[x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2},z_1,\ldots,z_{n_3}].$$

Let $\mathbb{F}[\mathbf{x},\mathbf{y}]_{\{(0,1),(1,0),(1,1)\}} \subseteq \mathbb{F}[\mathbf{x},\mathbf{y}]$ be the subset of polynomials that are bi-homogeneous of bi-degree $(0,1)$, $(1,0)$ or $(1,1)$. That is, the set $\mathbb{F}[\mathbf{x},\mathbf{y}]_{\{(0,1),(1,0),(1,1)\}}$ contains the polynomials in $\mathbb{F}[x_1,\ldots,x_{n_1}]$ that are homogeneous of degree 1, and the polynomials in $\mathbb{F}[y_1,\ldots,y_{n_2}]$ that are homogeneous of degree 1, and the polynomials in $\mathbb{F}[\mathbf{x},\mathbf{y}]$ that are homogeneous of degree 1 in $x_1,\ldots,x_{n_1}$ and homogeneous of degree 1 in $y_1,\ldots,y_{n_2}$. For any tensor T we define

$$\mathrm{ZR}(T) = \min\bigl\{s \in \mathbb{N} \mid \exists f_1,\ldots,f_s \in \mathbb{F}[\mathbf{x},\mathbf{y}]_{\{(0,1),(1,0),(1,1)\}} : \mathbf{V}(f_1,\ldots,f_s) \subseteq \mathbf{V}(T)\bigr\}.$$

▶ **Theorem 23.** *Let T be a tensor. Then* $\mathrm{GR}(T) \leq \mathrm{ZR}(T) \leq \mathrm{SR}(T)$.

Proof. We prove that $\mathrm{ZR}(T) \leq \mathrm{SR}(T)$. Let $r = \mathrm{SR}(T)$. We view T as a polynomial $T \in \mathbb{F}[\mathbf{x},\mathbf{y},\mathbf{z}]$. Write $T = \sum_{i=1}^r T_i$ with $\mathrm{SR}(T_i) = 1$ for every i. Then $T_i = f_i g_i$ for some $f_i \in \mathbb{F}[\mathbf{x},\mathbf{y}]_{\{(0,1),(1,0),(1,1)\}}$ and $g_i \in \mathbb{F}[\mathbf{x},\mathbf{y},\mathbf{z}]$. We claim that $\mathbf{V}(f_1,\ldots,f_r) \subseteq \mathbf{V}(T)$. Indeed, if $(x,y) \in \mathbf{V}(f_1,\ldots,f_r)$, then $T_i(x,y,z) = 0$ for every i and every z, and therefore $T(x,y,z) = 0$ for every z. We conclude that $\mathrm{ZR}(T) \leq r = \mathrm{SR}(T)$.

We prove that $\mathrm{GR}(T) \leq \mathrm{ZR}(T)$. Let $s = \mathrm{ZR}(T)$. Then there are s polynomials $f_1,\ldots,f_s \in \mathbb{F}[\mathbf{x},\mathbf{y}]_{\{(0,1),(1,0),(1,1)\}}$ such that $\mathbf{V}(f_1,\ldots,f_s) \subseteq \mathbf{V}(T)$. We have

$$\mathrm{GR}(T) = \operatorname{codim} \mathbf{V}(T) \leq \operatorname{codim} \mathbf{V}(f_1,\ldots,f_s) \leq s = \mathrm{ZR}(T),$$

where the first inequality follows from the containment $\mathbf{V}(f_1,\ldots,f_s) \subseteq \mathbf{V}(T)$ which implies that $\dim \mathbf{V}(f_1,\ldots,f_s) \leq \dim \mathbf{V}(T)$. ◀

8 Geometric rank as liminf of analytic rank

For a tensor T over $\mathbb{Z}$ and a prime number p, we denote by T_p the 3-tensor over $\mathbb{F}_p$ obtained by reducing all coefficients of T modulo p. In this section we prove the following tight relationship between $\mathrm{AR}(T_p)$ and $\mathrm{GR}(T)$.

▶ **Theorem 24.** *For every tensor T over $\mathbb{Z}$ we have*

$$\liminf_{p\to\infty} \mathrm{AR}(T_p) = \mathrm{GR}(T).$$

The starting point for the proof of Theorem 24 is the important observation that analytic rank can be written in terms of the number of $\mathbb{F}_p$-points of the algebraic variety $\mathbf{V}(T_p)$, that is, for any tensor $T \in \mathbb{Z}^{n_1\times n_2\times n_3}$,

$$\mathrm{AR}(T_p) = n_1 + n_2 - \log_p |\mathbf{V}(T_p)(\mathbb{F}_p)|\,.$$

For the proof of Theorem 24 we will need to prove three auxiliary results: that the Bertini–Noether Theorem can be extended to reducible varieties (Theorem 26 below), that prime fields are rich enough infinitely often to contain any finite set of algebraic numbers (Lemma 28 below), and that for any variety satisfying a mild assumption, its number of rational points in a finite field is determined by its dimension (Lemma 31 below).

8.1 Bertini–Noether Theorem

In this subsection we extend the Bertini–Noether Theorem to reducible varieties. The Bertini–Noether Theorem says that, roughly, if an variety is irreducible then applying a homomorphism on the defining equations – for example the modulo-p homomorphism – typically does not change its invariants (see Proposition 10.4.2 in [18]).

▶ **Theorem 25** (Bertini–Noether Theorem [18])**.** *Let $f_1,\dots,f_m \in R[\mathbf{x}]$, where R is an integral domain, such that $V = \mathbf{V}(f_1,\dots,f_m)$ is (absolutely) irreducible. There exists a nonzero $c \in R$ such that for every homomorphism $\phi\colon R \to \mathbb{K}$ into a field $\mathbb{K}$, if $\phi(c) \neq 0$ then $\mathbf{V}(\phi(f_1),\dots,\phi(f_m)) \subseteq \overline{\mathbb{K}}$ is (absolutely) irreducible of dimension $\dim V$ and degree $\deg V$.*[2] [3]

The version of the Berini-Noether Theorem that we need is as follows. We observe that any variety defined over a field $\mathbb{F}$, where $\mathbb{F}$ is the field of fractions of an integral domain R, can also be defined over R, by clearing denominators. For example, any variety defined over the algebraic numbers $\overline{\mathbb{Q}}$ can also be defined over the algebraic integers $\overline{\mathbb{Z}}$.

▶ **Theorem 26** (Extended Bertini–Noether Theorem)**.** *Let $f_1,\dots,f_m \in R[\mathbf{x}]$, where R is an integrally closed domain.*[4] *There exists a nonzero $C \in R$ such that for every homomorphism $\psi\colon R \to \mathbb{K}$ into a field $\mathbb{K}$, if $\psi(C) \neq 0$ then $V^\psi := \mathbf{V}(\psi(f_1),\dots,\psi(f_m)) \subseteq \overline{\mathbb{K}}$ is of dimension $\dim V$ and degree $\deg V$. Moreover, if the irreducible components of $\mathbf{V}(f_1,\dots,f_m)$ are $V_1,\dots,V_k$, where $\mathrm{I}(V_i) = \langle f_{i,j}\rangle_j$ with $f_{i,j} \in R[\mathbf{x}]$, then the irreducible components of V^ψ are $V_1^\psi,\dots,V_k^\psi$, where $V_i^\psi = \mathbf{V}(\psi(f_{i,j})_j)$.*

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[2] $\phi(f_i) \in \mathbb{K}[\mathbf{x}]$ is obtained by applying ϕ on each of the coefficients of f_i.
[3] That $\deg V$ remains unchanged follows along similar lines to the proof for $\dim V$ (see Corollary 9.2.2 in [18]).
[4] The field of fractions of the integral domain R is algebraically closed.

For the proof of Theorem 26 we will need some notation and a standard auxiliary result, as follows. Let R be a (commutative) ring. For a ideal I in R, the radical of I (in R) is the ideal $\sqrt{I} = \{f \in R \mid \exists n \in \mathbb{N}\colon f^n \in I\}$. Moreover, for a ring homomorphism $\psi\colon R \to R'$ we denote $\psi(I) = \langle \psi(f) \mid f \in I\rangle$, which is an ideal in R'.

▶ **Lemma 27.** *Let I be an ideal in a ring R, and let $\psi\colon R \to R'$ be a ring homomorphism. Then $\sqrt{\psi(\sqrt{I})} = \sqrt{\psi(I)}$.*

Proof. If $p \in \sqrt{\psi(I)}$ then there is an integer n such that $p^n \in \psi(I) \subseteq \psi(\sqrt{I})$, hence $p \in \sqrt{\psi(\sqrt{I})}$.

Let $p \in \sqrt{\psi(\sqrt{I})}$, meaning there is an integer n such that $p^n \in \psi(\sqrt{I})$. Thus, we have $p^n = \sum_{i=1}^m g_i \psi(f_i)$ for some $m \in \mathbb{N}$, $g_i \in R'$ and $f_i \in \sqrt{I}$. Note that for every i there is an integer k_i such that $f_i^{k_i} \in I$. Let $k = \max_{1\leq i\leq m} k_i$. Then

$$(p^n)^{km} = \sum_{\substack{d_1,\ldots,d_m\\ d_1+\cdots+d_m=km}} \prod_{i=1}^m (g_i\psi(f_i))^{d_i} \,.$$

Observe that every summand has a multiplicand $(g_i\psi(f_i))^{d_i}$ with $d_i \geq k \geq k_i$, which lies in $\psi(I)$ since $\psi(f_i)^{d_i} = \psi(f_i^{d_i})$ and $f_i^{d_i} = f_i^{d_i-k_i} f_i^{k_i} \in I$. We deduce that $p^{nkm} \in \psi(I)$, being a sum of members of the ideal $\psi(I)$. Hence $p \in \sqrt{\psi(I)}$, completing the proof. ◀

Proof of Theorem 26. We begin with some notation. Let $\mathbb{F}$ be the (algebraically closed) field of fractions of R. For any ideal J in $\mathbb{F}[\mathbf{x}]$ we denote by $J^R := J \cap R[\mathbf{x}]$ the corresponding ideal in $R[\mathbf{x}]$. With a slight abuse of notation, we abbreviate $\psi(J) := \psi(J^R)$ (which is an ideal in $\mathbb{K}[\mathbf{x}]$). Furthermore, we take $\sqrt{J^R}$ to mean the radical ideal of J in $R[\mathbf{x}]$. Observe that $\sqrt{J^R} = (\sqrt{J})^R$; indeed, $f \in (\sqrt{J})^R$ iff $f^n \in J$ and $f \in R[\mathbf{x}]$ iff $f \in \sqrt{J^R}$.

Let $I = \langle f_1, \ldots, f_m\rangle$ and $I_i = \langle f_{i,j}\rangle_j$ be ideals in $\mathbb{F}[\mathbf{x}]$. We will show that

$$\sqrt{\psi(I)} = \sqrt{\prod \psi(I_i)} \,. \tag{6}$$

We have $\mathbf{V}(I) = \bigcup_i \mathbf{V}(I_i) = \mathbf{V}(\prod_i I_i)$. By Hilbert's Nullstellensatz, $\sqrt{I} = \sqrt{\prod_i I_i}$. Next, and for the rest of this paragraph, we switch from ideals in $\mathbb{F}[\mathbf{x}]$ to ideals in $R[\mathbf{x}]$. We have

$$\sqrt{I^R} = (\sqrt{I})^R = \Big(\sqrt{\prod I_i}\Big)^R = \sqrt{\prod I_i^R} \,. \tag{7}$$

We deduce (6) as follows;

$$\begin{aligned}\sqrt{\psi(I)} = \sqrt{\psi(I^R)} = \sqrt{\psi(\sqrt{I^R})} = \sqrt{\psi\Big(\sqrt{\prod I_i^R}\Big)} = \sqrt{\psi\big(\prod I_i^R\big)} = \sqrt{\prod \psi(I_i^R)}\\ = \sqrt{\prod \psi(I_i)} \,,\end{aligned}$$

where the second equality follows from Lemma 27, the third follows from (7), the fourth again from Lemma 27, and the fifth using the fact that ψ is a homomorphism. It follows that

$$V^\psi := \mathbf{V}(\psi(I)) = \mathbf{V}(\sqrt{\psi(I)}) = \mathbf{V}\left(\sqrt{\prod \psi(I_i)}\right) = \mathbf{V}\left(\prod \psi(I_i)\right) = \bigcup \mathbf{V}(\psi(I_i)) = \bigcup V_i^\psi \,,$$

where (6) is used in the third equality.

Recall that V_i is an irreducible variety defined over R. For each i, applying Theorem 25 on any generating set of $\mathrm{I}(V_i)$ in $R[\mathbf{x}]$ and on ψ implies that there is a nonzero $c_i \in R$ such that if $\psi(c_i) \neq 0$ then V_i^{ψ} is irreducible, of dimension $\dim \mathbf{V}_i^{\psi} = \dim \mathbf{V}_i$ and degree $\deg V_i^{\psi} = \deg V_i$. Let $C = \prod_i c_i$. Thus, if $\psi(C) \neq 0$ then $\psi(c_i) \neq 0$ for all i, which implies that $V^{\psi} = \bigcup_i V_i^{\psi}$ is a union of irreducible varieties, and moreover,

$$\dim V^{\psi} = \max_i \dim V_i^{\psi} = \max_i \dim V_i = \dim V \quad \text{and}$$

$$\deg V^{\psi} = \sum_i \deg V_i^{\psi} = \sum_i \deg V_i = \deg V.$$

This completes the proof. ◀

8.2 Modular roots

In this subsection we prove that, intuitively, every finite set of algebraic integers is contained in $\mathbb{F}_p$, for infinitely many primes p. We say that there is a positive density of primes satisfying a property $\mathcal{P} \subseteq \mathbb{P}$ (here $\mathbb{P}$ is the set of prime numbers) if $\lim_{n\to\infty} |\mathcal{P} \cap [n]|/|\mathbb{P} \cap [n]| > 0$.

▶ **Lemma 28.** *For every finite set of algebraic integers S there is a positive density of primes p for which there is a homomorphism from $\mathbb{Z}[S]$ to $\mathbb{F}_p$.*

We will use (a special case of) the Primitive Element Theorem (see, e.g., Section 6.10 in [35]).

▶ **Theorem 29** (Primitive Element Theorem in Characteristic 0 [35])**.** *Let $\mathbb{K}$ be a finite extension of a field $\mathbb{F}$ of characteristic 0. Then $\mathbb{K} = \mathbb{F}(\alpha)$ for some $\alpha \in \mathbb{K}$.*

For example, $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

We will also rely on the following result (see Berend and Bilu [4], Theorem 2).

▶ **Theorem 30** ([4])**.** *For every polynomial $P \in \mathbb{Z}[x]$ there is a positive density of prime numbers p such that P has a root modulo p.*

Proof of Lemma 28. Consider $\mathbb{Q}(S)$, the field extension of the rationals $\mathbb{Q}$ obtained by adjoining all the elements of S. By the Primitive Element Theorem (Theorem 29) there exists $\alpha \in \mathbb{Q}(S)$ such that $\mathbb{Q}(S) = \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$. Thus, for every $\alpha_i \in S$ there is a (univariate) polynomial $f_i \in \mathbb{Q}[x]$ such that $\alpha_i = f_i(\alpha)$. We denote by P be the minimal polynomial of α over $\mathbb{Q}$; by clearing denominators, we assume without loss of generality that $P \in \mathbb{Z}[x]$.

Let p be a prime number such that P has a root a_p modulo p and, moreover, p is larger than the absolute value of the coefficient denominators of every f_i. By Theorem 30, applied on P, there is a positive density of primes satisfying both conditions. Note that $f_i \pmod{p}$ is a well-defined polynomial in $\mathbb{F}_p[x]$ by our second condition on p. Consider the function ϕ_p that maps each $\alpha_i = f_i(\alpha) \in S$ to $f_i(a_p) \pmod{p}$. Since every member of $\mathbb{Z}[S]$ is a multivariate polynomial in the variables α_i with integer coefficients, we deduce from our first condition on p that the function ϕ_p extends to a homomorphism $\phi_p \colon \mathbb{Z}[S] \to \mathbb{F}_p$. This completes the proof. ◀

8.3 Putting everything together

We will also need the following asymptotically-tight estimate on the number of rational points in a finite field.

▶ **Lemma 31.** *For every variety V defined over a finite field $\mathbb{F}$, if V has an irreducible component of dimension* $\dim V$ *that is also defined over $\mathbb{F}$ then*

$$|V(\mathbb{F})| = \Theta_{\deg V,\, n}(|\mathbb{F}|^{\dim V}).$$

The proof of Lemma 31 will follow by combining the Lang-Weil Theorem [25] with a Schwartz-Zippel-type upper bound (see Claim 7.2 in [15]).

▶ **Theorem 32** (Lang–Weil Bound [25])**.** *For every (absolutely) irreducible variety V defined over a finite field $\mathbb{F}$,*

$$|V(\mathbb{F})| = |\mathbb{F}|^{\dim V}(1 + O_{\deg V,\, n}(|\mathbb{F}|^{-1/2})).$$

▶ **Lemma 33** (Generalized Schwartz–Zippel lemma [15])**.** *For every variety V defined over a finite field $\mathbb{F}$, $|V(\mathbb{F})| \le \deg(V) \cdot |\mathbb{F}|^{\dim V}$.*

Proof of Lemma 31. For the upper bound, apply Lemma 33 on V. For the lower bound, let U be an irreducible component of V of dimension $\dim V$ that is defined over $\mathbb{F}$, as guaranteed by the statement, and apply Theorem 32 on U to obtain $|V(\mathbb{F})| \ge |U(\mathbb{F})| = \Omega_{\deg U,\, n}(|\mathbb{F}|^{\dim U}) = \Omega_{\deg V,\, n}(|\mathbb{F}|^{\dim V})$. ◀

We are now ready to prove the main result of this section.

Proof of Theorem 24. Put $d = \dim \mathbf{V}(T)$ and $r = \deg \mathbf{V}(T)$. We will use the notation in (4) and (5). We will show that $\mathbf{V}(T) \subseteq \overline{\mathbb{Q}}^N$ and $\mathbf{V}(T_p) \subseteq \overline{\mathbb{F}_p}^N$ (here $N = n_1 + n_2$) are related, for infinitely many prime numbers p, in the following sense;

$$|\,\mathbf{V}(T_p)(\mathbb{F}_p)| = \Theta_{r,\, N}(p^d). \tag{8}$$

This would complete the proof since for any such prime p,

$$\mathrm{AR}(T_p) = -\log_p\Big(\frac{|\,\mathbf{V}(T_p)(\mathbb{F}_p)|}{|\mathbb{F}_p|^N}\Big) = N - \log_p |\,\mathbf{V}(T_p)(\mathbb{F}_p)| = \mathrm{GR}(T) - \Theta_{r,\, N}\Big(\frac{1}{\log p}\Big),$$

where the last inequality follows from (8) using the fact that $N - d = \operatorname{codim} \mathbf{V}(T) = \mathrm{GR}(T)$. Thus, proving (8) would imply that $\liminf_{p\to\infty} \mathrm{AR}(T_p) = \mathrm{GR}(T)$, as needed.

Let U be an irreducible component of $\mathbf{V}(T)$ of dimension d. Note that U is defined over some finite extension $\mathbb{Z}[S]$ of the integers, where S is a finite set of algebraic integers. Lemma 28, applied on S, implies that for a positive density of prime numbers p there is a homomorphism $\phi_p\colon \mathbb{Z}[S] \to \mathbb{F}_p$. Thus, if $\mathrm{I}(U) = \mathbf{V}(f_j)_j$ with $f_j \in \mathbb{Z}[S][\mathbf{x}]$ then $U^{\phi_p} := \mathbf{V}(\phi_p(f_j)_j)$ is defined over $\mathbb{F}_p$ (rather than $\overline{\mathbb{F}_p}$). Let p be any such prime. Theorem 26, applied on $R = \overline{\mathbb{Z}}$, $\mathbb{K} = \overline{\mathbb{F}_p}$ and on any extension ψ_p of ϕ_p to a homomorphism from $\overline{\mathbb{Z}}$ to $\overline{\mathbb{F}_p}$, implies that there is $0 \ne C \in \overline{\mathbb{Z}}$ such that for any prime p with $\psi_p(C) \ne 0$, we have that $\dim \mathbf{V}(T_p) = d$, $\deg \mathbf{V}(T_p) = r$, and that $U^{\psi_p} = U^{\phi_p}$ is an irreducible component of $\mathbf{V}(T_p)$ of dimension $d = \dim \mathbf{V}(T_p)$. We claim that the condition $\psi_p(C) \ne 0$ is satisfied for all but finitely many primes p; indeed, since $\psi_p(C)$ is a root modulo p of the minimal polynomial of C over $\mathbb{Z}$, it holds that $\psi_p(C) = 0$ if and only if the constant term c of that polynomial is 0 modulo p, which is never the case for $p > |c|$ (as $c \ne 0$). Lemma 31 therefore implies, together with all of the above, that for a positive density of primes p we have

$$|\,\mathbf{V}(T_p)(\mathbb{F}_p)| = \Theta_{\deg \mathbf{V}(T_p),\, N}(p^{\dim \mathbf{V}(T_p)}) = \Theta_{r,\, N}(p^d).$$

This proves (8), and thus we are done. ◀

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