Supplement to “On the Robustness of the Pooled CCE Estimator”: Proofs

Artūras Judodis[a,*], Hande Karabiyik[b], Joakim Westerlund[c,d]

[a] *Faculty of Economics and Business, University of Groningen*
[b] *School of Business and Economics, Vrije Universiteit Amsterdam*
[c] *Department of Economics, Lund University*
[d] *Center for Financial Econometrics, Deakin University*

Abstract

This supplementary material provides the proofs of the results stated in the main paper.

A.1. Auxiliary lemmas

Before we take the proofs of the main results of the paper, we report some auxiliary results that will be key in the sequel.

Lemma A.1. *Under Assumptions 2.1–2.5,*

(i) $\|T^{-1/2}\mathbf{v}_i'\mathbf{F}_m\| = O_p(1)$,

(ii) $\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}_m\| = O_p(1)$,

(iii) $\|T^{-1}\mathbf{F}_m'\mathbf{F}_m\| = O_p(1)$,

(iv) $\|T^{-1}\mathbf{v}_i'\overline{\mathbf{u}}\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$,

(v) $\|T^{-1}\overline{\mathbf{u}}'\overline{\mathbf{u}}\| = O_p(N^{-1})$,

(vi) $\|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_m\| = O_p((NT)^{-1/2})$,

(vii) $\|T^{-1}\overline{\mathbf{u}}'\boldsymbol{\varepsilon}_i\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$.

Proof of Lemma A.1.
The proof of this lemma is a direct consequence of the results provided in Pesaran (2006). It is therefore omitted. □

Lemma A.2. *Under Assumptions 2.1–2.5,*

(i) $\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$,

(ii) $\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$,

---

[*] Corresponding author. E–mail: a.juodis@rug.nl

(iii) $\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| = O_p(N^{-1})$,

(iv) $\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\mathbf{F}_m\| = O_p((NT)^{-1/2})$,

(v) $\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| = O_p(1)$,

(vi) $\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| = O_p(N^{-1/2})$,

(vii) $\|T^{-1}\boldsymbol{\varepsilon}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$,

(viii) $\|T^{-1}\boldsymbol{\varepsilon}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$,

(ix) $\|T^{-1}\mathbf{v}_i'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}\| = O_p(T^{-1/2})$,

(x) $\|T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\boldsymbol{\varepsilon}_i\| = O_p(T^{-1/2})$,

(xi) $\|T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| = O_p(N^{-1/2})$.

Proof of Lemma A.2.
Clearly, $\|\mathbf{Q}\mathbf{G}_m\| = O_p(1)$ and $\|\mathbf{G}_{-m}\| = O_p(\sqrt{N})$. Also, $\|\widetilde{\mathbf{C}}_{-y}\| = O_p(N^{-1/2})$ and $\|T^{-1}\mathbf{v}_i'\mathbf{F}_{-m,-y}\| = O_p(T^{-1/2})$, implying

$$\begin{aligned}\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| &\leq \|T^{-1}\mathbf{v}_i'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_m\| + \|T^{-1}\mathbf{v}_i'\overline{\mathbf{u}}\|\|\mathbf{Q}\mathbf{G}_m\| \\ &= O_p(N^{-1}) + O_p((NT)^{-1/2}), \end{aligned} \tag{A.1}$$

$$\begin{aligned}\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| &\leq \|T^{-1}\mathbf{v}_i'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_{-m}\| + \|T^{-1}\mathbf{v}_i'\overline{\mathbf{u}}\|\|\mathbf{Q}\mathbf{G}_{-m}\| \\ &= O_p(N^{-1/2}) + O_p(T^{-1/2}), \end{aligned} \tag{A.2}$$

as required for (i) and (ii), respectively. For (iii), we use $\|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_{-m,-y}\| = O_p(1)$ and $\|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_{-m,-y}\| = O_p((NT)^{-1/2})$, giving

$$\begin{aligned}\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| &\leq \|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|^2\|\mathbf{Q}\mathbf{G}_m\|^2 + \|T^{-1}\overline{\mathbf{u}}'\overline{\mathbf{u}}\|\|\mathbf{Q}\mathbf{G}_m\|^2 \\ &+ 2\|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_m\|^2 = O_p(N^{-1}), \end{aligned} \tag{A.3}$$

and since $\|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_m\| = 0$, we also have

$$\begin{aligned}\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\mathbf{F}_m\| &\leq \|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_m\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_m\| + \|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_m\|\|\mathbf{Q}\mathbf{G}_m\| \\ &= O_p((NT)^{-1/2}). \end{aligned} \tag{A.4}$$

We can similarly show that for (v) and (vi),

$$\begin{aligned}\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| &\leq \|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|^2\|\mathbf{Q}\mathbf{G}_{-m}\|^2 + \|T^{-1}\overline{\mathbf{u}}'\overline{\mathbf{u}}\|\|\mathbf{Q}\mathbf{G}_{-m}\|^2 \\ &+ 2\|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_{-m}\|^2 = O_p(1), \end{aligned} \tag{A.5}$$

$$\begin{aligned}\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| &\leq \|T^{-1}\mathbf{F}_{-m,-y}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|^2\|\mathbf{Q}\mathbf{G}_m\|\|\mathbf{Q}\mathbf{G}_{-m}\| \\ &+ \|T^{-1}\overline{\mathbf{u}}'\overline{\mathbf{u}}\|\|\mathbf{Q}\mathbf{G}_m\|\|\mathbf{Q}\mathbf{G}_{-m}\| \\ &+ 2\|T^{-1}\overline{\mathbf{u}}'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{Q}\mathbf{G}_m\|\|\mathbf{Q}\mathbf{G}_{-m}\| = O_p(N^{-1/2}). \end{aligned} \tag{A.6}$$

The results in (vii) and (viii) can be obtain in a similar way as in (i) and (ii), respectively. For (ix), we use $\widehat{\mathbf{F}}\mathbf{QG} = (\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)}) + \overline{\mathbf{E}}\mathbf{QG}$, which in view of $\|\mathbf{QG}\| = O_p(\sqrt{N})$ implies

$$\begin{aligned}\|T^{-1}\mathbf{v}_i'\widehat{\mathbf{F}}\mathbf{QG}\| &\leq \|T^{-1}\mathbf{v}_i'(\mathbf{F}_m, \mathbf{0}_{r_m\times(k+1-r_m)})\| + \|T^{-1}\mathbf{v}_i'\mathbf{F}_{-m,-y}\|\|\widetilde{\mathbf{C}}_{-y}\|\|\mathbf{QG}\| + \|T^{-1}\mathbf{v}_i'\overline{\mathbf{u}}\|\|\mathbf{QG}\| \\ &= O_p(T^{-1/2}). \end{aligned} \tag{A.7}$$

The results in (x) and (xi) can be obtained analogously. □

Before continuing to the next lemma, we note that by analogy to (S.29) in Karabiyik et al. (2017), we have

$$\begin{aligned} &T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}}) \\ &= \overline{\mathbf{E}}\mathbf{QG}[T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}(\overline{\mathbf{E}}\mathbf{QG})' + \overline{\mathbf{E}}\mathbf{QG}[T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}(\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})' \\ &+ (\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})[T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}(\overline{\mathbf{E}}\mathbf{QG})' \\ &+ (\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1} - [T^{-1}(\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})'(\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})]^{-1}) \\ &\times (\mathbf{F}_m, \mathbf{0}_{T\times(k+1-r_m)})' \\ &= \overline{\mathbf{E}}\mathbf{QG}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}(\overline{\mathbf{E}}\mathbf{QG}_m)' + \overline{\mathbf{E}}\mathbf{QG}_{-m}[T^{-1}(\mathbf{QG}_{-m})'\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{QG}_{-m}]^{-1}(\overline{\mathbf{E}}\mathbf{QG}_{-m})' \\ &+ \overline{\mathbf{E}}\mathbf{QG}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}\mathbf{F}_m' + \mathbf{F}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}(\overline{\mathbf{E}}\mathbf{QG}_m)' \\ &+ \widehat{\mathbf{F}}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1})(\widehat{\mathbf{F}}\mathbf{QG})', \end{aligned} \tag{A.8}$$

where

$$\begin{aligned}\mathbf{S}_{\mathbf{F}_m^+} &= \begin{pmatrix} T^{-1}\mathbf{F}_m'\mathbf{F}_m & \mathbf{0}_{r_m\times(k+1-r_m)} \\ \mathbf{0}_{(k+1-r_m)\times r_m} & T^{-1}(\mathbf{QG}_{-m})'\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{QG}_{-m} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{S}_{\mathbf{F}_m} & \mathbf{0}_{r_m\times(k+1-r_m)} \\ \mathbf{0}_{(k+1-r_m)\times r_m} & N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m} \end{pmatrix}, \end{aligned}$$

with $\mathbf{S}_{\mathbf{F}_m} = T^{-1}\mathbf{F}_m'\mathbf{F}_m$, $\mathbf{S}_{\overline{\mathbf{E}}} = NT^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}$ and $\mathbf{F}_m^+ = (\mathbf{F}_m, \overline{\mathbf{E}}\mathbf{QG}_{-m})$. It is important to note that the expansion in (A.8) implicitly assumes that $k+1 > r_m$. If $k+1 = r_m$, then $\mathbf{G} = \mathbf{G}_m = (\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}$, which is tantamount to dropping all terms involving $\mathbf{G}_{-m}$. Hence, if $k+1 = r_m$, (A.8) reduces to

$$\begin{aligned} &T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}}) \\ &= \overline{\mathbf{E}}\mathbf{QG}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}(\overline{\mathbf{E}}\mathbf{QG}_m)' + \overline{\mathbf{E}}\mathbf{QG}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}\mathbf{F}_m' + \mathbf{F}_m(T^{-1}\mathbf{F}_m'\mathbf{F}_m)^{-1}(\overline{\mathbf{E}}\mathbf{QG}_m)' \\ &+ \widehat{\mathbf{F}}\mathbf{QG}_m([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG}_m)'\widehat{\mathbf{F}}\mathbf{QG}_C]^{-1} - \mathbf{S}_{\mathbf{F}_m}^{-1})(\widehat{\mathbf{F}}\mathbf{QG}_m)'. \end{aligned} \tag{A.9}$$

Lemma A.3. *Under Assumptions 2.1–2.5,*

$$[T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1} = O_p(T^{-1/2}) + O_p(N^{-1/2}).$$

Proof of Lemma A.3.

This proof makes use of Theorem 1 of Karabiyik et al. (2017), which states that if

$$T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G} - \mathbf{S}_{\mathbf{F}_m^+} = O_p(T^{-1/2}) + O_p(N^{-1/2}) \tag{A.10}$$

and

$$\text{rk}[T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}] \overset{a.s.}{\to} \text{rk}\,\mathbf{S}_{\mathbf{F}_m^+}, \tag{A.11}$$

then

$$[T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m^+} = O_p(T^{-1/2}) + O_p(N^{-1/2}). \tag{A.12}$$

The condition in (A.10) is obviously true, since

$$\begin{aligned} T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G} &= \mathbf{S}_{\mathbf{F}_m} + T^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G} + T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G})'\mathbf{F}_m + T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G} \\ &= \mathbf{S}_{\mathbf{F}_m^+} + O_p(T^{-1/2}) + O_p(N^{-1/2}), \end{aligned} \tag{A.13}$$

where $\mathbf{S}_{\mathbf{F}_m^+} = O_p(1)$ by Assumption 2.1 (iii) and Assumption 2.5. Moreover, since $\text{rk}\,\widehat{\mathbf{F}} = \text{rk}\,(\mathbf{Q}\mathbf{G}) = k+1$, we can show that $\text{rk}[T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}] = k+1$. Next, consider $\text{rk}\,\mathbf{S}_{\mathbf{F}_m^+}$. Since $\mathbf{S}_{\mathbf{F}_m^+}$ is block-diagonal,

$$\text{rk}\,\mathbf{S}_{\mathbf{F}_m^+} = \text{rk}\,\mathbf{S}_{\mathbf{F}_m} + \text{rk}\,[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]. \tag{A.14}$$

Here, $\mathbf{F}_m = \mathbf{F}\mathbf{R}$, where $\text{rk}\,\mathbf{F} = r$, $\text{rk}\,\mathbf{R} = r_m$ and $r_m \leq r$. This gives $\text{rk}[T^{-1}(\mathbf{F}_m)'\mathbf{F}_m] = r_m$. We can show also that $N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}$ is positive definite for all finite N, including $N \to \infty$, from which it follows that $\text{rk}\,[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}] = k+1-r_m$. Hence, $\text{rk}\,\mathbf{S}_{\mathbf{F}_m^+} = k+1$, which means that (A.11) is satisfied. This proves the lemma. $\square$

The rate $O_p(T^{-1/2}) + O_p(N^{-1/2})$ in Lemma A.3 is sharp if $k+1 > r_m$, but not if $k+1 = r_m$. In fact, it is not difficult to show that

$$[T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}_m)'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}_m]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m} = O_p((NT)^{-1/2}) + O_p(N^{-1})$$

if $k+1 = r_m$. However, since the $O_p(T^{-1/2}) + O_p(N^{-1/2})$ rate is sufficient for our purposes, in this supplement we will uses this rate irrespective for both cases.

A.2. Proofs of main results

Proof of Theorem 3.1.

Letting

$$\mathbf{S}_x = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{x}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{x}_i,$$

it is clear that

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_P - \boldsymbol{\beta}) = \mathbf{S}_x^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}\boldsymbol{\lambda}_i)$$
$$= \mathbf{S}_x^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}\boldsymbol{\lambda}_i). \tag{A.1}$$

By using $\mathbf{F} = \mathbf{F}_m\mathbf{H}' + \mathbf{F}_{-m}$ and $\mathbf{F}_{-m} = (\mathbf{0}_{T\times r_y}, \mathbf{F}_{-m,-y})$, the summand in the above expression can be written as follows:

$$\begin{aligned}
&(\mathbf{v}_i + \mathbf{F}\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}\boldsymbol{\lambda}_i)\\
&= (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i' + \mathbf{F}_{-m}\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m}\boldsymbol{\lambda}_i)\\
&= (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i' + \mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&= (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i' + \mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i\\
&+ (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i' + \mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&= (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i + (\mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i\\
&+ (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i}) + (\mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&= (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i - (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i + (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i\\
&+ \boldsymbol{\Lambda}_{-y,i}\mathbf{F}_{-m,-y}'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i + (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&+ (\mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}')'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&= \mathbf{v}_i'\mathbf{M}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i - (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i + (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i\\
&+ \boldsymbol{\Lambda}_{-y,i}\mathbf{F}_{-m,-y}'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i}) + (\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i},
\end{aligned} \tag{A.2}$$

which in turn implies

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_P - \boldsymbol{\beta}) = \mathbf{S}_x^{-1}(\mathbf{B}_0 + \sqrt{N}T^{-1/2}\mathbf{B}_1 + \sqrt{T}N^{-1/2}\mathbf{B}_2 + \mathbf{B}_3), \tag{A.3}$$

where

$$\begin{aligned}
\mathbf{B}_0 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{v}_i'\boldsymbol{\varepsilon}_i,\\
\mathbf{B}_1 &= -\frac{1}{N}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{P}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i,\\
\mathbf{B}_2 &= \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i - \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i,\\
\mathbf{B}_3 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{-y,i}\mathbf{F}_{-m,-y}'\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})\\
&+ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i}.
\end{aligned}$$

In what follows we evaluate each of the terms on the right-hand side of $\sqrt{NT}(\widehat{\boldsymbol{\beta}}_P - \boldsymbol{\beta})$. We start with $\mathbf{B}_0$. Clearly, letting

$$\boldsymbol{\Sigma}_0 = \lim_{N,T\to\infty} E\left[\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{v}_i'\boldsymbol{\varepsilon}_i\right)\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{v}_i'\boldsymbol{\varepsilon}_i\right)'\right] = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\sigma_{\varepsilon,i}^2\boldsymbol{\Sigma}_{\mathbf{v},i},$$

by a central limit theorem (CLT) for independent but heterogenous variables,

$$\mathbf{B}_0 = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{v}_i'\boldsymbol{\varepsilon}_i \xrightarrow{d} \mathbf{b}_0 \overset{d}{=} N(\mathbf{0}_{k\times 1}, \boldsymbol{\Sigma}_0) \tag{A.4}$$

as N, $T \to \infty$.

Consider $\mathbf{B}_1$. Making use of the fact that $\mathbf{F}_m = \mathbf{F}\mathbf{R}$, this term can be written as

$$\begin{aligned}
\mathbf{B}_1 &= -\frac{1}{N}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{P}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i = -\frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{F}\mathbf{R}(\mathbf{R}'T^{-1}\mathbf{F}'\mathbf{F}\mathbf{R})^{-1}\mathbf{R}'\mathbf{F}'\boldsymbol{\varepsilon}_i \\
&= -\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\mathbf{v}_{i,t}\varepsilon_{i,s}\mathbf{F}_t'\mathbf{R}(\mathbf{R}'T^{-1}\mathbf{F}'\mathbf{F}\mathbf{R})^{-1}\mathbf{R}'\mathbf{F}_s. \qquad (A.5)
\end{aligned}$$

By Assumption 2.2,

$$\mathbf{S}_{\mathbf{F}_m} = \mathbf{R}'T^{-1}\mathbf{F}'\mathbf{F}\mathbf{R} = \mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R} + O_p(T^{-1/2}), \tag{A.6}$$

which in turn implies

$$\mathbf{S}_{\mathbf{F}_m}^{-1} = (\mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R})^{-1} + O_p(T^{-1/2}). \tag{A.7}$$

Hence, letting $\boldsymbol{\Gamma}_{\varepsilon\mathbf{v}}(h) = \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{\varepsilon\mathbf{v},i}(h)$, $\mathbf{J}_{\mathbf{F}} = \mathbf{R}(\mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R})^{-1}\mathbf{R}'$ and

$$\mathbf{b}_1 = -\sum_{h=1}^{\infty}\boldsymbol{\Gamma}_{\varepsilon\mathbf{v}}(-h)'\mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(h)\mathbf{J}_{\mathbf{F}}],$$

we have

$$\begin{aligned}
\mathbf{B}_1 &= -\frac{1}{T}\sum_{s=1}^{T-1}\sum_{t=s+1}^{T}\boldsymbol{\Gamma}_{\varepsilon\mathbf{v}}(s-t)'\mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(s-t)\mathbf{J}_{\mathbf{F}}] + o_p(1) \\
&= -\sum_{h=1}^{T-1}\left(1-\frac{h-1}{T}\right)\boldsymbol{\Gamma}_{\mathbf{v}\varepsilon}(-h)'\mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(-h)\mathbf{J}_{\mathbf{F}}] + o_p(1) \\
&= -\sum_{h=1}^{T-1}\left(1-\frac{h-1}{T}\right)\boldsymbol{\Gamma}_{\mathbf{v}\varepsilon}(-h)'\mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(h)\mathbf{J}_{\mathbf{F}}] + o_p(1) \\
&= \mathbf{b}_1 + o_p(1), \qquad (A.8)
\end{aligned}$$

where the third equality holds, because $\mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(-h)\mathbf{J}_{\mathbf{F}}] = \mathrm{tr}\,[\mathbf{J}_{\mathbf{F}}'\boldsymbol{\Gamma}_{\mathbf{F}}(-h)'] = \mathrm{tr}\,[\boldsymbol{\Gamma}_{\mathbf{F}}(h)\mathbf{J}_{\mathbf{F}}]$.

Next up is $\mathbf{B}_2$. We start with the first term. From $\mathbf{F}_m\overline{\mathbf{C}}_m = \widehat{\mathbf{F}} - \overline{\mathbf{E}}$, we obtain $\mathbf{F}_m = (\widehat{\mathbf{F}} - \overline{\mathbf{E}})\mathbf{Q}_m^0$ with $\mathbf{Q}_m^0 = \mathbf{Q}_m(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}$, which in turn implies that

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}[\mathbf{v}_i + (\widehat{\mathbf{F}} - \overline{\mathbf{E}})\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i']'\mathbf{M}_{\widehat{\mathbf{F}}}(\widehat{\mathbf{F}} - \overline{\mathbf{E}})\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= -\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= -\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{P}_{\mathbf{F}_m}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&+ \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \mathbf{B}_{21} + \mathbf{B}_{22} + \mathbf{B}_{23},
\end{aligned} \tag{A.9}$$

with obvious definitions of $\mathbf{B}_{21}$, $\mathbf{B}_{22}$ and $\mathbf{B}_{23}$. Here,

$$\begin{aligned}
\|\mathbf{B}_{21}\| &= \left\|\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i\right\| \\
&\leq \sum_{i=1}^{N}[\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\|\|\mathbf{Q}_m^0\|\|\mathbf{H}'\boldsymbol{\lambda}_i\| + \|\mathbf{H}'\boldsymbol{\Lambda}_i'\|\|\mathbf{Q}_m^0\|^2\|T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\|\|\mathbf{H}'\boldsymbol{\lambda}_i\|] \\
&= N([O_p(N^{-1}) + O_p((TN)^{-1/2})]O_p(1) + O_p(1)O_p(N^{-1})O_p(1)) = O_p(1),
\end{aligned} \tag{A.10}$$

and similarly for $\|\mathbf{B}_{22}\|$,

$$\begin{aligned}
\|\mathbf{B}_{22}\| &= \left\|\frac{1}{T^2}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{F}_m\mathbf{S}_{\mathbf{F}_m}^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i\right\| \\
&\leq \sum_{i=1}^{N}[\|T^{-1}\mathbf{v}_i'\mathbf{F}_m\|\|\mathbf{S}_{\mathbf{F}_m}^{-1}\|\|T^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\|\|\mathbf{Q}_m^0\|\|\mathbf{H}'\boldsymbol{\lambda}_i\| \\
&+ \|\mathbf{H}'\boldsymbol{\Lambda}_i'\|\|\mathbf{S}_{\mathbf{F}_m}^{-1}\|\|T^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\|^2\|\mathbf{Q}_m^0\|^2\|\mathbf{H}'\boldsymbol{\lambda}_i\|] \\
&= N[O_p(T^{-1/2})O_p(1)O_p((NT)^{-1/2})O_p(1) + O_p(1)O_p((NT)^{-1})O_p(1)] \\
&= O_p(\sqrt{N}T^{-1}) + O_p(T^{-1}) = O_p(\sqrt{N}T^{-1}).
\end{aligned} \tag{A.11}$$

By using the expansion of $\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}}$ in (A.8), $\mathbf{B}_{23}$ can be written in the following fashion:

$$\mathbf{B}_{23} = \frac{1}{T^2}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i = \sum_{k=1}^{10}\mathbf{B}_{23k},$$

where

$$\mathbf{B}_{231} = \frac{1}{T^2}\sum_{i=1}^{N} \mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}_{\mathbf{F}_m}^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{232} = \frac{1}{T^2}\sum_{i=1}^{N} \mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i$$

$$\mathbf{B}_{233} = \frac{1}{T^2}\sum_{i=1}^{N} \mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}_{\mathbf{F}_m}^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{234} = \frac{1}{T^2}\sum_{i=1}^{N} \mathbf{v}_i'\mathbf{F}_m\mathbf{S}_{\mathbf{F}_m}^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{235} = \frac{1}{T^2}\sum_{i=1}^{N} \mathbf{v}_i'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1})(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{236} = -\frac{1}{T^2}\sum_{i=1}^{N} \boldsymbol{\Lambda}_i\mathbf{H}(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}(\overline{\mathbf{E}}\mathbf{Q}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}_{\mathbf{F}_m}^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{237} = -\frac{1}{T^2}\sum_{i=1}^{N} \boldsymbol{\Lambda}_i\mathbf{H}(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}(\overline{\mathbf{E}}\mathbf{Q}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}$$
$$\times(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{238} = -\frac{1}{T^2}\sum_{i=1}^{N} \boldsymbol{\Lambda}_i\mathbf{H}(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}(\overline{\mathbf{E}}\mathbf{Q}_m)'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}_{\mathbf{F}_m}^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{239} = -\frac{1}{T^2}\sum_{i=1}^{N} \boldsymbol{\Lambda}_i\mathbf{H}(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}(\overline{\mathbf{E}}\mathbf{Q}_m)'\mathbf{F}_m\mathbf{S}_{\mathbf{F}_m}^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i,$$

$$\mathbf{B}_{2310} = -\frac{1}{T^2}\sum_{i=1}^{N} \boldsymbol{\Lambda}_i\mathbf{H}(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}(\overline{\mathbf{E}}\mathbf{Q}_m)'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1})(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i.$$

The orders of these terms can be worked out using the results of Lemmas A.1–A.3. The first five are

$$\begin{aligned}\|\mathbf{B}_{231}\| &\leq \sum_{i=1}^{N}\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\|\|\mathbf{S}_{\mathbf{F}_m}^{-1}\|\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\|\|\mathbf{Q}_m^0\|\|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= N[O_p(N^{-1}) + O_p((TN)^{-1/2})]O_p(1)O_p(N^{-1})O_p(1) \\ &= O_p(N^{-1}) + O_p((TN)^{-1/2}),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{232}\| &\leq \sum_{i=1}^{N}\|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\|\|[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}\|\|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\|\|\mathbf{Q}_m^0\|\|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= N[O_p(N^{-1/2}) + O_p(T^{-1/2})]O_p(1)O_p(N^{-1/2})O_p(1) \\ &= O_p(1) + O_p(\sqrt{N}T^{-1/2}),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{233}\| &\leq \sum_{i=1}^{N} \|T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\| \|\mathbf{Q}_m^0\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= N[O_p(N^{-1}) + O_p((TN)^{-1/2})]O_p(1)O_p((NT)^{-1/2})O_p(1) \\ &= O_p((TN)^{-1/2}) + O_p(T^{-1}),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{234}\| &\leq \sum_{i=1}^{N} \|T^{-1}\mathbf{v}_i'\mathbf{F}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\| \|\mathbf{Q}_m^0\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(T^{-1/2})O_p(1)O_p(N^{-1})O_p(1)O_p(1) = O_p(T^{-1/2}),\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{B}_{235}\| &\leq \sum_{i=1}^{N} \|T^{-1}\mathbf{v}_i'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}\| \|(T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1}\| \|T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\| \|\mathbf{Q}_m^0\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(T^{-1/2})[O_p(N^{-1}) + O_p((TN)^{-1/2})]O_p(N^{-1/2})O_p(1) \\ &= O_p((NT)^{-1/2}) + O_p(T^{-1}).\end{aligned}$$

Here, $\mathbf{B}_{232}$ dominates, with $\mathbf{B}_{234}$ being the leading remainder. We can similarly show that

$$\begin{aligned}\|\mathbf{B}_{236}\| &\leq \sum_{i=1}^{N} \|\boldsymbol{\Lambda}_i\mathbf{H}\| \|\mathbf{Q}_m^0\|^2 \|T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\mathbf{Q}_m\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(1)O_p(N^{-1})O_p(1)O_p(N^{-1})O_p(1) = O_p(N^{-1}),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{237}\| &\leq \sum_{i=1}^{N} \|\boldsymbol{\Lambda}_i\mathbf{H}\| \|\mathbf{Q}_m^0\|^2 \|T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}\| \|[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}\| \\ &\quad \times \|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(1)O_p(N^{-1/2})O_p(1)O_p(N^{-1/2})O_p(1) = O_p(1),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{238}\| &\leq \sum_{i=1}^{N} \|\boldsymbol{\Lambda}_i\mathbf{H}\| \|\mathbf{Q}_m^0\|^2 \|T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1}\mathbf{F}_m'\overline{\mathbf{E}}\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(1)O_p(N^{-1})O_p(1)O_p((NT)^{-1/2})O_p(1) = O_p(N^{-1/2}),\end{aligned}$$

$$\begin{aligned}\|\mathbf{B}_{239}\| &\leq \sum_{i=1}^{N} \|\boldsymbol{\Lambda}_i\mathbf{H}\| \|\mathbf{Q}_m^0\|^2 \|T^{-1}\overline{\mathbf{E}}'\mathbf{F}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{E}}\| \|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(1)O_p((NT)^{-1/2})O_p(1)O_p(N^{-1})O_p(1) = O_p((NT)^{-1/2}),\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{B}_{2310}\| &\leq \sum_{i=1}^{N} \|\boldsymbol{\Lambda}_i\mathbf{H}\|\|\mathbf{Q}_m^0\|^2\|T^{-1}\overline{\mathbf{E}}'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}\|\|(T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})^{-1} - \mathbf{S}_{\mathbf{F}_m^+}^{-1}\| \\ &\times \|T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}\|\|\mathbf{H}'\boldsymbol{\lambda}_i\| \\ &= NO_p(1)O_p(N^{-1/2})[O_p(N^{-1}) + O_p((TN)^{-1/2})]O_p(N^{-1/2})O_p(1) \\ &= O_p(N^{-1}) + O_p((TN)^{-1/2}),\end{aligned}$$

where $\mathbf{B}_{237}$ dominates and $\mathbf{B}_{238}$ is the leading remainder. Hence,

$$\begin{aligned}\mathbf{B}_{23} &= \mathbf{B}_{232} + \mathbf{B}_{237} + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\ &= \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{P}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}),\end{aligned} \tag{A.12}$$

which in turn implies

$$\begin{aligned}&\mathbf{B}_{21} + \mathbf{B}_{22} + \mathbf{B}_{23} \\ &= \mathbf{B}_{21} + \mathbf{B}_{232} + \mathbf{B}_{237} + O_p(N^{-1/2}) + O_p(T^{-1/2}) + O_p(\sqrt{N}T^{-1}) \\ &= -\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\ &+ O_p(\sqrt{N}T^{-1}).\end{aligned} \tag{A.13}$$

We now evaluate the limit of the first term on the right of the above equation. We start with $\mathbf{H}$. Since $\mathbf{R}$ has full column rank r_m, we have $\mathbf{R}^+ = (\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'$ such that $\mathbf{R}'(\mathbf{R}^+)' = \mathbf{I}_{r_m}$. This can be used to show that

$$\begin{aligned}\mathbf{H} &= \mathbf{F}'\mathbf{F}_m(\mathbf{F}_m'\mathbf{F}_m)^{-1} = T^{-1}\mathbf{F}'\mathbf{F}\mathbf{R}(\mathbf{R}'T^{-1}\mathbf{F}'\mathbf{F}\mathbf{R})^{-1} = \boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R}(\mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R})^{-1} + O_p(T^{-1/2}) \\ &= \mathbf{h} + O_p(T^{-1/2}).\end{aligned} \tag{A.14}$$

where $\mathbf{h} = \boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}(\mathbf{R}^+)'$. We also need $T^{-1}\sum_{i=1}^{N}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i$. Here, we make use of the fact that

$$E(\mathbf{v}_i'\mathbf{u}_i) = E(\mathbf{v}_i'\mathbf{e}_i)\mathbf{D} = (\mathbf{0}_{k\times 1}, \boldsymbol{\Sigma}_{\mathbf{v},i})\mathbf{D} = (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D},$$

giving

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\mathbf{v}_i'\sqrt{N}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}(\mathbf{v}_i'\mathbf{F}_{-m,-y}\sqrt{N}\widetilde{\mathbf{C}}_{-y} + \sqrt{N}\mathbf{v}_i'\overline{\mathbf{u}})\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{F}_{-m,-y}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{u}_i\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + \frac{1}{NT}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\mathbf{v}_i'\mathbf{u}_j\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{u}_i\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}),
\end{aligned}$$

and we can similarly show that

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&+ O_p(N^{-1/2}) + O_p(T^{-1/2}).
\end{aligned}$$

By inserting these results and $\mathbf{H} = \mathbf{h} + O_p(T^{-1/2})$ into the leading term of $\mathbf{B}_{21} + \mathbf{B}_{22} + \mathbf{B}_{23}$, we get

$$
\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{M}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i - \frac{1}{T}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'\mathbf{M}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= \sum_{i=1}^{N}T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&- \sum_{i=1}^{N}T^{-1}\mathbf{v}_i'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&- \sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&+ \sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}T^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i \\
&- (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}N^{-1/2}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}N^{-1/2}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i \\
&- \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}_m(\overline{\mathbf{C}}_m\mathbf{Q}_m)^{-1}\mathbf{h}'\boldsymbol{\lambda}_i \\
&+ \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{S}_{\overline{\mathbf{E}}}N^{-1/2}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}N^{-1/2}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i \\
&+ O_p(N^{-1/2}) + O_p(T^{-1/2}) + O_p(\sqrt{N}T^{-1}). \qquad \text{(A.15)}
\end{aligned}
$$

Consider $\mathbf{S}_{\overline{\mathbf{E}}}$. Because of independence, $\|\sqrt{N}T^{-1}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\| = \|\sqrt{N}T^{-1}\mathbf{F}'\mathbf{M}_{\mathbf{F}_m}\overline{\mathbf{u}}\| = O_p(T^{-1/2})$. Making use of this, $\|\sqrt{N}\widetilde{\mathbf{C}}_{-y}\| = O_p(1)$, $\overline{\mathbf{u}} = \overline{\mathbf{e}}\mathbf{D}$, the definition of $\overline{\mathbf{E}}$, and letting $\mathbf{S}_{\mathbf{F}_{-m,-y}} = T^{-1}\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}$ and $\mathbf{S}_{\overline{\mathbf{e}}} = NT^{-1}\overline{\mathbf{e}}'\overline{\mathbf{e}}$, we can show that

$$
\begin{aligned}
\mathbf{S}_{\overline{\mathbf{E}}} &= NT^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}} = NT^{-1}(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y} + \overline{\mathbf{u}})'(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y} + \overline{\mathbf{u}}) \\
&= T^{-1}\sqrt{N}\widetilde{\mathbf{C}}'_{-y}\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\sqrt{N}\widetilde{\mathbf{C}}_{-y} + \sqrt{N}\widetilde{\mathbf{C}}'_{-y}T^{-1}\sqrt{N}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}} \\
&+ T^{-1}\sqrt{N}\overline{\mathbf{u}}'\mathbf{F}_{-m,-y}\sqrt{N}\widetilde{\mathbf{C}}_{-y} + NT^{-1}\overline{\mathbf{u}}'\overline{\mathbf{u}} \\
&= \sqrt{N}\widetilde{\mathbf{C}}'_{-y}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y} + \mathbf{D}'\mathbf{S}_{\overline{\mathbf{e}}}\mathbf{D} + O_p(T^{-1/2}).
\end{aligned}
$$

Moreover, since

$$
\begin{aligned}
\mathbf{S}_{\overline{\mathbf{e}}} = NT^{-1}\overline{\mathbf{e}}'\overline{\mathbf{e}} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{e}_i'\mathbf{e}_i + \frac{1}{NT}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\mathbf{e}_i'\mathbf{e}_j = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{e}_i'\mathbf{e}_i + O_p(T^{-1/2}) \\
&= \boldsymbol{\Sigma}_{\mathbf{e}} + o_p(1), \qquad \text{(A.16)}
\end{aligned}
$$

we have

$$\mathbf{S}_{\overline{\mathbf{E}}} = \mathbf{\Sigma}_{\overline{\mathbf{E}}} + o_p(1), \tag{A.17}$$

where $\mathbf{\Sigma}_{\overline{\mathbf{E}}} = \sqrt{N}\widetilde{\mathbf{C}}'_{-y}\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y} + \mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}\mathbf{D}$ and $\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}} = \text{plim}_{T\to\infty} T^{-1}\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}$. Thus, defining $\mathbf{J}_{\overline{\mathbf{E}}} = \mathbf{Q}\mathbf{G}_{-m}[(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\mathbf{Q}\mathbf{G}_{-m})'$ for $k+1 > r_m$ and $\mathbf{J}_{\overline{\mathbf{E}}} = \mathbf{0}_{(k+1)\times(k+1)}$ for $k+1 = r_m$, where the latter reflects the fact that (A.8) simplifies under $k+1 = r_m$, we have

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i')'\mathbf{M}_{\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\lambda}_i \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Sigma}_{\mathbf{e},i}\mathbf{D}(\mathbf{\Sigma}_{\overline{\mathbf{E}}}^{-1} - \mathbf{J}_{\overline{\mathbf{E}}})\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i \\
&- \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{\Sigma}_{\overline{\mathbf{E}}}(\mathbf{\Sigma}_{\overline{\mathbf{E}}}^{-1} - \mathbf{J}_{\overline{\mathbf{E}}})\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i + o_p(1) \\
&= \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i}\mathbf{D} - \mathbf{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{\Sigma}_{\overline{\mathbf{E}}}](\mathbf{\Sigma}_{\overline{\mathbf{E}}}^{-1} - \mathbf{J}_{\overline{\mathbf{E}}})\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'\boldsymbol{\lambda}_i + o_p(1),
\end{aligned} \tag{A.18}$$

where the remainder absorbs the $O_p(\sqrt{N}T^{-1})$ term, which is $o_p(1)$ under our assumption that $N/T = O(1)$. Finally, since $\boldsymbol{\lambda}_{-y,i} \overset{a.s.}{=} \mathbf{0}_{(r-r_y)\times 1}$ and thus $\boldsymbol{\lambda}_i \overset{a.s.}{=} (\boldsymbol{\lambda}'_{y,i}, \mathbf{0}'_{(r-r_y)\times 1})'$, we arrive at

$$\begin{aligned}
\mathbf{B}_{21} + \mathbf{B}_{22} + \mathbf{B}_{23} = -\frac{1}{N}\sum_{i=1}^{N}&[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i}\mathbf{D} - \mathbf{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{\Sigma}_{\overline{\mathbf{E}}}](\mathbf{\Sigma}_{\overline{\mathbf{E}}}^{-1} - \mathbf{J}_{\overline{\mathbf{E}}}) \\
&\times \mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'(\boldsymbol{\lambda}'_{y,i}, \mathbf{0}'_{(r-r_y)\times 1})' + o_p(1).
\end{aligned} \tag{A.19}$$

Now onto the second term in $\mathbf{B}_2$. By using the fact that $\mathbf{F}_m = (\widehat{\mathbf{F}} - \overline{\mathbf{E}})\mathbf{Q}_m^0$, this term can be written as

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\mathbf{\Lambda}_i')'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i + \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{F}'_m(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i + \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}\mathbf{v}_i'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i - \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i + \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'\mathbf{M}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i \\
&= \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'\boldsymbol{\varepsilon}_i - \frac{1}{T}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{H}\mathbf{Q}_m^{0\prime}\overline{\mathbf{E}}'\mathbf{P}_{\mathbf{F}_m}\boldsymbol{\varepsilon}_i + \frac{1}{T}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i')'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i \\
&= \mathbf{B}_{24} + \mathbf{B}_{25} + \mathbf{B}_{26},
\end{aligned} \tag{A.20}$$

where $\|\mathbf{B}_{24}\| = O_p(1)$ can be shown by using the same steps as for $\|\mathbf{B}_{21}\|$ above. Also,

$$\begin{aligned}\|\mathbf{B}_{25}\| &\leq \sum_{i=1}^{N} \|\mathbf{\Lambda}_i \mathbf{H} \mathbf{Q}_m^0\| \|T^{-1} \overline{\mathbf{E}}' \mathbf{F}_m\| \|\mathbf{S}_{\mathbf{F}_m}^{-1}\| \|T^{-1} \mathbf{F}_m' \boldsymbol{\varepsilon}_i\| \\ &= N O_p(1) O_p((NT)^{-1/2}) O_p(1) O_p(T^{-1/2}) = O_p(\sqrt{N} T^{-1}). \end{aligned} \tag{A.21}$$

Analogous to $\mathbf{B}_{23}$, $\mathbf{B}_{26}$ can be written as a sum of 10 terms, where all but two are negligible. Specifically,

$$\mathbf{B}_{26} = \frac{1}{T} \sum_{i=1}^{N} (\mathbf{v}_i - \overline{\mathbf{E}} \mathbf{Q}_m^0 \mathbf{H}' \mathbf{\Lambda}_i')' \mathbf{P}_{\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m}} \boldsymbol{\varepsilon}_i + O_p(\sqrt{N} T^{-1}) + O_p(T^{-1/2}),$$

This implies

$$\begin{aligned} &\mathbf{B}_{24} + \mathbf{B}_{25} + \mathbf{B}_{26} \\ &= \frac{1}{T} \sum_{i=1}^{N} \mathbf{\Lambda}_i \mathbf{H} \mathbf{Q}_m^{0\prime} \overline{\mathbf{E}}' \boldsymbol{\varepsilon}_i + \frac{1}{T} \sum_{i=1}^{N} (\mathbf{v}_i - \overline{\mathbf{E}} \mathbf{Q}_m^0 \mathbf{H}' \mathbf{\Lambda}_i')' \mathbf{P}_{\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m}} \boldsymbol{\varepsilon}_i + O_p(\sqrt{N} T^{-1}) + O_p(T^{-1/2}) \\ &= \frac{1}{T} \sum_{i=1}^{N} \mathbf{v}_i' \mathbf{P}_{\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m}} \boldsymbol{\varepsilon}_i + \frac{1}{T} \sum_{i=1}^{N} \mathbf{\Lambda}_i \mathbf{H} \mathbf{Q}_m^{0\prime} \overline{\mathbf{E}}' \mathbf{M}_{\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m}} \boldsymbol{\varepsilon}_i + O_p(\sqrt{N} T^{-1}) + O_p(T^{-1/2}) \\ &= \sum_{i=1}^{N} T^{-1} \mathbf{v}_i' \overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m} [N^{-1} (\mathbf{Q} \mathbf{G}_{-m})' \mathbf{S}_{\overline{\mathbf{E}}} \mathbf{Q} \mathbf{G}_{-m}]^{-1} T^{-1} (\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m})' \boldsymbol{\varepsilon}_i \\ &+ \sum_{i=1}^{N} \mathbf{\Lambda}_i \mathbf{H} \mathbf{Q}_m^{0\prime} \overline{\mathbf{E}}' \boldsymbol{\varepsilon}_i - \sum_{i=1}^{N} \mathbf{\Lambda}_i \mathbf{H} \mathbf{Q}_m^{0\prime} \overline{\mathbf{E}}' \overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m} [N^{-1} (\mathbf{Q} \mathbf{G}_{-m})' \mathbf{S}_{\overline{\mathbf{E}}} \mathbf{Q} \mathbf{G}_{-m}]^{-1} T^{-1} (\overline{\mathbf{E}} \mathbf{Q} \mathbf{G}_{-m})' \boldsymbol{\varepsilon}_i \\ &+ O_p(\sqrt{N} T^{-1}) + O_p(T^{-1/2}). \end{aligned} \tag{A.22}$$

Further use of $E(\boldsymbol{\varepsilon}_i'\mathbf{u}_i) = (\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')$ and $E(\mathbf{v}_i'\mathbf{u}_i) = (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}$ gives

$$
\begin{aligned}
&\mathbf{B}_{24} + \mathbf{B}_{25} + \mathbf{B}_{26} \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}N^{-1/2}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1} \\
&\times N^{-1/2}(\mathbf{Q}\mathbf{G}_{-m})'(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' + \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \\
&- \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\mathbf{S}_{\overline{\mathbf{E}}}N^{-1/2}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}N^{-1/2}(\mathbf{Q}\mathbf{G}_{-m})'(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \\
&+ O_p(\sqrt{N}T^{-1}) + O_p(T^{-1/2}) + O_p(N^{-1/2}) \\
&= (\mathbf{0}_{k\times 1}, \mathbf{I}_k)\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D}\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' + \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \\
&- \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' + o_p(1) \\
&= \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D} - \boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}]\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \\
&+ \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' + o_p(1). \qquad \text{(A.23)}
\end{aligned}
$$

Hence, by adding the results,

$$
\mathbf{B}_2 = \mathbf{B}_{21} + \mathbf{B}_{22} + \mathbf{B}_{23} - \mathbf{B}_{24} - \mathbf{B}_{25} - \mathbf{B}_{26} = \mathbf{b}_2 + o_p(1), \qquad \text{(A.24)}
$$

where

$$
\begin{aligned}
\mathbf{b}_2 = &-\frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D} - \boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}](\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}^{-1} - \mathbf{J}_{\overline{\mathbf{E}}})\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}_m^0\mathbf{h}'(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' \\
&- \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\boldsymbol{\Sigma}_{\mathbf{e},i}\mathbf{D} - \boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}\boldsymbol{\Sigma}_{\overline{\mathbf{E}}}]\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' - \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_i\mathbf{h}\mathbf{Q}_m^{0\prime}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')'.
\end{aligned}
$$

Let us now consider $\mathbf{B}_3$. The last term is simplest. Indeed, since $\boldsymbol{\lambda}_{-y,i} \overset{a.s.}{=} \mathbf{0}_{(r-r_y)\times 1}$, we have

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i')'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i} \overset{a.s.}{=} \mathbf{0}_{k\times 1}. \qquad \text{(A.25)}
$$

Although the same result can be applied also to the first term, here it is more convenient to write $\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i} = \mathbf{F}\boldsymbol{\lambda}_i = \mathbf{F}_y\boldsymbol{\lambda}_{y,i}$, where the last equality is due to $\boldsymbol{\lambda}_{-y,i} \overset{a.s.}{=} \mathbf{0}_{(r-r_y)\times 1}$. It follows that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i+\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i+\mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})$$
$$=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_y\boldsymbol{\lambda}_{y,i}. \quad \text{(A.26)}$$

Consider the second term on the right. By using $\overline{\mathbf{C}}_{-y}\mathbf{Q}_y \overset{a.s.}{=} \mathbf{0}_{(r-r_y)\times r_y}$ and the positive definiteness of $\overline{\mathbf{C}}_y\mathbf{Q}_y$, letting $\mathbf{Q}_y^0=\mathbf{Q}_y(\overline{\mathbf{C}}_y\mathbf{Q}_y)^{-1}$, we obtain

$$\widehat{\mathbf{F}}\mathbf{Q}_y^0=(\mathbf{F}\overline{\mathbf{C}}+\overline{\mathbf{u}})\mathbf{Q}_y^0=(\mathbf{F}_y\overline{\mathbf{C}}_y+\mathbf{F}_{-y}\overline{\mathbf{C}}_{-y})\mathbf{Q}_y^0+\overline{\mathbf{u}}\mathbf{Q}_y^0\overset{a.s.}{=}\mathbf{F}_y+\overline{\mathbf{u}}\mathbf{Q}_y^0, \quad \text{(A.27)}$$

from which it follows that

$$\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_y\overset{a.s.}{=}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}(\widehat{\mathbf{F}}-\overline{\mathbf{u}})\mathbf{Q}_y^0=-\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\overline{\mathbf{u}}\mathbf{Q}_y^0$$
$$=-\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}_y^0+\mathbf{F}'_{-m,-y}\mathbf{P}_{\widehat{\mathbf{F}}}\overline{\mathbf{u}}\mathbf{Q}_y^0. \quad \text{(A.28)}$$

By using this and the fact that $\mathbf{F}'_{-m,-y}\mathbf{M}_{\mathbf{F}_m}=\mathbf{F}'_{-m,-y}$, we can show that

$$\mathbf{B}_3=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}(\boldsymbol{\varepsilon}_i+\mathbf{F}_m\mathbf{H}'\boldsymbol{\lambda}_i+\mathbf{F}_{-m,-y}\boldsymbol{\lambda}_{-y,i})+o_p(1)$$
$$=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_y\boldsymbol{\lambda}_{y,i}+o_p(1)$$
$$=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\boldsymbol{\varepsilon}_i-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}+o_p(1)$$
$$=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\boldsymbol{\varepsilon}_i-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i$$
$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}$$
$$+\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}+o_p(1) \quad \text{(A.29)}$$

Since $\mathbf{F}_{-m,-y}=\mathbf{M}_{\mathbf{F}_m}\mathbf{F}_{-y}$ with $\mathbf{F}_m=\mathbf{F}\mathbf{R}$, we have that $\mathbf{F}_{-m,-y}$ is orthogonal to all the columns of $\mathbf{C}$, including $\mathbf{\Lambda}$. Hence, $\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}=\widetilde{\mathbf{\Lambda}}_{-y,i}\mathbf{F}'_{-m,-y}$, which in turn implies

$$\mathbf{B}_3=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widetilde{\mathbf{\Lambda}}_{-y,i}\mathbf{F}'_{-m,-y}\boldsymbol{\varepsilon}_i-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i$$
$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widetilde{\mathbf{\Lambda}}_{-y,i}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}$$
$$+\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}+o_p(1)$$
$$=\mathbf{B}_{31}-\mathbf{B}_{32}-\mathbf{B}_{33}+\mathbf{B}_{34}+o_p(1). \quad \text{(A.30)}$$

Consider $\mathbf{B}_{33}$. Because it is a vector, we have

$$\begin{aligned}
\mathbf{B}_{33} &= -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\text{vec}\,(\widetilde{\boldsymbol{\Lambda}}_{-y,i}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}) \\
&= \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i})'\otimes\widetilde{\boldsymbol{\Lambda}}_{-y,i}]\text{vec}\,[(NT)^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}] \\
&= \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i})'\otimes\widetilde{\boldsymbol{\Lambda}}_{-y,i}]\text{vec}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t}\right), \qquad (A.31)
\end{aligned}$$

which is clearly mean zero. Conditional on factors the covariance matrix of the vectorized term is given by

$$\begin{aligned}
&E\left[\text{vec}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t}\right)\left(\text{vec}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t}\right)\right)'|\mathbf{F}\right] \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E[\text{vec}\,(\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t})(\text{vec}\,(\mathbf{F}_{-m,-y,s}\mathbf{u}'_{j,s}))'|\mathbf{F}] \qquad (A.32) \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E[\mathbf{u}_{i,s}\mathbf{u}'_{i,t}]\otimes\mathbf{F}_{-m,-y,t}\mathbf{F}'_{-m,-y,s} \\
&= \sum_{h=-\infty}^{\infty}\mathbf{D}'\boldsymbol{\Gamma}_{\mathbf{e}}(-h)\mathbf{D}\otimes\boldsymbol{\Gamma}_{\mathbf{F}_{-m,-y}}(h)+o_p(1), \qquad (A.33)
\end{aligned}$$

where in the second equality is due to the cross-sectional independence assumption, while the last equality makes use of $\boldsymbol{\Gamma}_{\mathbf{F}_{-m,-y}}(h) = \text{plim}_{T\to\infty}T^{-1}\sum_{t=h+1}^{T}\mathbf{F}_{-m,-y,t}\mathbf{F}'_{-m,-y,t-h}$. Hence, by a suitable CLT,

$$\mathbf{B}_{33} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}\xrightarrow{d}N(\mathbf{0}_{k\times 1},\boldsymbol{\Sigma}_{33}) \qquad (A.34)$$

as N, $T\to\infty$, where

$$\begin{aligned}
\boldsymbol{\Sigma}_{33} = {} & \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}E[\boldsymbol{\lambda}'_{y,i}(\mathbf{C}_y\mathbf{Q}_y)^{-1\prime}\mathbf{Q}'_y\otimes\widetilde{\boldsymbol{\Lambda}}_{-y,i}]\sum_{h=-\infty}^{\infty}\mathbf{D}'\boldsymbol{\Gamma}_{\mathbf{e}}(-h)\mathbf{D}\otimes\boldsymbol{\Gamma}_{\mathbf{F}_{-m,-y}}(h) \\
& \times\left(\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}E[\boldsymbol{\lambda}'_{y,i}(\mathbf{C}_y\mathbf{Q}_y)^{-1\prime}\mathbf{Q}'_y\otimes\widetilde{\boldsymbol{\Lambda}}_{-y,i}]\right)',
\end{aligned}$$

Similarly, $\mathbf{B}_{31}$ is mean zero and with covariance matrix

$$\begin{aligned}
E(\mathbf{B}_{311}\mathbf{B}'_{311}) &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}E(\widetilde{\boldsymbol{\Lambda}}_{-y,i}\mathbf{F}_{-m,-y,t}\varepsilon_{i,t}\varepsilon_{i,s}\mathbf{F}'_{-m,-y,s}\widetilde{\boldsymbol{\Lambda}}'_{-y,j}) \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}E(\widetilde{\boldsymbol{\Lambda}}_{-y,i}\mathbf{F}_{-m,-y,t}\varepsilon_{i,t}^2\mathbf{F}'_{-m,-y,t}\widetilde{\boldsymbol{\Lambda}}'_{-y,i}). \qquad (A.35)
\end{aligned}$$

Hence, by the same CLT arguments as for $\mathbf{B}_{33}$,

$$\mathbf{B}_{31} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \widetilde{\mathbf{\Lambda}}_{-y,i} \mathbf{F}'_{-m,-y} \boldsymbol{\varepsilon}_i \xrightarrow{d} N(\mathbf{0}_{k\times 1}, \mathbf{\Sigma}_{31}), \tag{A.36}$$

where

$$\mathbf{\Sigma}_{31} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sigma^2_{\varepsilon,i} E(\widetilde{\mathbf{\Lambda}}_{-y,i} \mathbf{\Sigma}_{\mathbf{F}_{-m,-y}} \widetilde{\mathbf{\Lambda}}'_{-y,i}).$$

As for the covariance between $\mathbf{B}_{31}$ and $\mathbf{B}_{33}$, we have conditional on $\mathbf{F}$

$$\begin{aligned} E[\mathbf{B}_{33}\mathbf{B}'_{31}|\mathbf{F}] &= \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\text{vec}\,(\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t})\varepsilon_{j,s}\mathbf{F}'_{-m,-y,s}\widetilde{\mathbf{\Lambda}}'_{-y,j}|\mathbf{F}] \\ &= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\text{vec}\,(\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t})\varepsilon_{i,s}\mathbf{F}'_{-m,-y,s}\widehat{\mathbf{\Lambda}}'_{-y,i}|\mathbf{F}] \\ &= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\text{vec}\,(\mathbf{F}_{-m,-y,t}\mathbf{u}'_{i,t})\varepsilon_{i,s}\mathbf{F}'_{-m,-y,s}|\mathbf{F}]E(\widetilde{\mathbf{\Lambda}}_{-y,i})' = \mathbf{0}_{k\times k}, \end{aligned} \tag{A.37}$$

where the last equality holds, because $\mathbf{\Lambda}_i$ is independent of $\mathbf{F}_{-m,-y,t}$, $\mathbf{u}_{i,t}$ and $\varepsilon_{i,t}$. Thus, $\mathbf{B}_{31}$ and $\mathbf{B}_{33}$ are asymptotically uncorrelated, and hence asymptotically independent by joint normality. It follows that

$$\mathbf{B}_{31} - \mathbf{B}_{33} \xrightarrow{d} \mathbf{b}_3 \overset{d}{=} N(\mathbf{0}_{k\times 1}, \mathbf{\Sigma}_3) \tag{A.38}$$

as $N, T \to \infty$, where $\mathbf{\Sigma}_3 = \mathbf{\Sigma}_{31} + \mathbf{\Sigma}_{33}$. We can similarly show that, as long as factors are mean zero, $\mathbf{b}_3$ is asymptotically uncorrelated with $\mathbf{b}_0$, and hence asymptotically independent by normality.

Next up is $\mathbf{B}_{34}$. From (A.8),

$$\begin{aligned}
&\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}})\overline{\mathbf{u}}\\
&=\mathbf{F}'_{-m,-y}\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}\mathbf{F}'_m\overline{\mathbf{u}}+\mathbf{F}'_{-m,-y}\mathbf{F}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{u}}\\
&=\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_m)'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_{-m}[N^{-1}(\mathbf{Q}\mathbf{G}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m}]^{-1}(\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_{-m})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}\mathbf{F}'_m\overline{\mathbf{u}}+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}_m\mathbf{S}^{-1}_{\mathbf{F}_m}\mathbf{F}'_m\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{u}}, \qquad \text{(A.39)}
\end{aligned}$$

which uses the fact that $\mathbf{F}'_{-m,-y}\mathbf{F}_m=\mathbf{0}_{(r-r_y)\times r_m}$. Note also how the last term can be written as

$$\begin{aligned}
&\mathbf{F}'_{-m,-y}\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\overline{\mathbf{u}}\\
&=\mathbf{F}'_{-m,-y}(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{E}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\overline{\mathbf{E}}'\overline{\mathbf{u}}\\
&=\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\widetilde{\mathbf{C}}'_{-y}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\widetilde{\mathbf{C}}'_{-y}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\overline{\mathbf{u}}'\overline{\mathbf{u}}\\
&+\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}\mathbf{G}([T^{-1}(\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G})'\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{Q}\mathbf{G})'\overline{\mathbf{u}}'\overline{\mathbf{u}}. \qquad \text{(A.40)}
\end{aligned}$$

Insertion and simplification yield

$$
\begin{aligned}
&\sqrt{N}T^{-2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}\mathbf{QG}})\overline{\mathbf{u}}\\
&= N^{-1}T^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{QG}_m)'\sqrt{N}\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+ N^{-1}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{QG}_m)'\mathbf{S}_{\overline{\mathbf{u}}}\\
&+ T^{-1}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{QG}_m)'\sqrt{N}\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+ N^{-1}T^{-1/2}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}(\mathbf{QG}_m)'\mathbf{S}_{\overline{\mathbf{u}}}\\
&+ T^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}(\mathbf{QG}_{-m})'\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}_{-m,-y}\overline{\mathbf{u}}\\
&+ T^{-1}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}N^{-1/2}\mathbf{G}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}\\
&\times (\mathbf{QG}_{-m})'\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+ \mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}N^{-1/2}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{u}}}\\
&+ T^{-1/2}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}N^{-1/2}\mathbf{G}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}N^{-1/2}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{u}}}\\
&+ (NT)^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}\sqrt{N}T^{-1/2}\mathbf{F}'_m\overline{\mathbf{u}}\\
&+ N^{-1/2}T^{-1}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{QG}_m\mathbf{S}^{-1}_{\mathbf{F}_m}\sqrt{N}T^{-1/2}\mathbf{F}'_m\overline{\mathbf{u}}\\
&+ \sqrt{N}T^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}N^{-1/2}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})\\
&\times N^{-1/2}(\mathbf{QG})'\sqrt{N}T^{-1/2}(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+ \sqrt{N}T^{-1}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}N^{-1/2}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})\\
&\times N^{-1/2}(\mathbf{QG})'\sqrt{N}T^{-1/2}(\mathbf{F}_m,\mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}}\\
&+ T^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})(\mathbf{QG})'\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+ T^{-1}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}N^{-1/2}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})\\
&\times (\mathbf{QG})'\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\\
&+ \mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})N^{-1/2}(\mathbf{QG})'\mathbf{S}_{\overline{\mathbf{u}}}\\
&+ T^{-1/2}\sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}N^{-1/2}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1}-\mathbf{S}^{-1}_{\mathbf{F}^+_m})N^{-1/2}(\mathbf{QG})'\mathbf{S}_{\overline{\mathbf{u}}}\\
&= \mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}N^{-1/2}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{u}}}+O_p(N^{-1/2})+O_p(T^{-1/2})\\
&= \mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}\mathbf{\Sigma}_{\overline{\mathbf{u}}}+o_p(1). \qquad \text{(A.41)}
\end{aligned}
$$

By using this and the fact that $\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}=\widetilde{\mathbf{\Lambda}}_{-y,i}\mathbf{F}'_{-m,-y}$, we obtain

$$
T^{-1/2}\mathbf{B}_{34}=\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\sqrt{N}T^{-2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m}-\mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{u}}\mathbf{Q}^0_y\boldsymbol{\lambda}_{y,i}=\mathbf{b}_4+o_p(1), \qquad \text{(A.42)}
$$

where

$$
\mathbf{b}_4=\frac{1}{N}\sum_{i=1}^{N}\widetilde{\mathbf{\Lambda}}_{-y,i}\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}\mathbf{D}\mathbf{Q}^0_y\boldsymbol{\lambda}_{y,i}.
$$

Hence, $\|\mathbf{B}_{34}\| = O_p(\sqrt{T})$, which means that $\sqrt{NT}(\widehat{\boldsymbol{\beta}}_P - \boldsymbol{\beta})$ is of the same order. This is true in S4 when $k+1 > r_m < r$. If, however, $r = r_m$ as in S1 and S2, then $\mathbf{F}_{-m,-y} = \mathbf{0}_{T\times(r-r_y)}$ in which case $\mathbf{B}_3$ is zero altogether. Another possibility is if $k+1 = r_m$ as in S3, which is tantamount to dropping all terms involving $\mathbf{G}_{-m}$, and to set $\mathbf{G}$, $(\mathbf{F}_m, \mathbf{0}_{T\times(r-r_m)})$ and $\mathbf{S}_{\mathbf{F}_m^+}$ equal to $\mathbf{G}_m$, $\mathbf{F}_m$ and $\mathbf{S}_{\mathbf{F}_m}$, respectively. Direct insertion into the above expression for $\sqrt{N}T^{-2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}\mathbf{QG}})\overline{\mathbf{u}}$ yields, after first dropping all terms that are $o_p(1)$ when scaled by $\sqrt{T}$,

$$
\begin{aligned}
&\sqrt{N}T^{-3/2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}\mathbf{QG}})\overline{\mathbf{u}} \\
&= \mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}(\mathbf{QG}_{-m})'\widetilde{\mathbf{C}}'_{-y}\sqrt{N}T^{-1/2}\mathbf{F}_{-m,-y}\overline{\mathbf{u}} \\
&+ \sqrt{T}\mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}N^{-1/2}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{u}}} \\
&+ \sqrt{N}T^{-1/2}\mathbf{F}'_{-m,-y}\overline{\mathbf{u}}\mathbf{Q}N^{-1/2}\mathbf{G}_{-m}[N^{-1}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{E}}}\mathbf{QG}_{-m}]^{-1}N^{-1/2}(\mathbf{QG}_{-m})'\mathbf{S}_{\overline{\mathbf{u}}} \\
&+ \sqrt{N}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}N^{-1/2}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m^+}) \\
&\times N^{-1/2}(\mathbf{QG})'\sqrt{N}T^{-1/2}(\mathbf{F}_m, \mathbf{0}_{T\times(r-r_m)})'\overline{\mathbf{u}} \\
&+ \sqrt{T}\mathbf{S}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG})'\widehat{\mathbf{F}}\mathbf{QG}]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m^+})N^{-1/2}(\mathbf{QG})'\mathbf{S}_{\overline{\mathbf{u}}} + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\
&= N^{-1/2}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_m([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG}_m)'\widehat{\mathbf{F}}\mathbf{QG}_m]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m})(\mathbf{QG}_m)'\sqrt{N}T^{-1/2}\mathbf{F}'_m\overline{\mathbf{u}} \\
&+ \sqrt{T}N^{-1}\mathbf{S}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{QG}_m([T^{-1}(\widehat{\mathbf{F}}\mathbf{QG}_m)'\widehat{\mathbf{F}}\mathbf{QG}_m]^{-1} - \mathbf{S}^{-1}_{\mathbf{F}_m})(\mathbf{QG}_m)'\mathbf{S}_{\overline{\mathbf{u}}} \\
&+ O_p(N^{-1/2}) + O_p(T^{-1/2}) \\
&= O_p(N^{-1/2}) + O_p(T^{-1/2}) + O_p(\sqrt{T}N^{-1}), \qquad \text{(A.43)}
\end{aligned}
$$

implying

$$
\begin{aligned}
\|\mathbf{B}_{34}\| &\leq \frac{1}{N}\sum_{i=1}^{N}\|\boldsymbol{\Lambda}_{-y,i}\|\|\sqrt{N}T^{-3/2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\overline{\mathbf{u}}\|\|\mathbf{Q}_y^0\boldsymbol{\lambda}_{y,i}\| \\
&= O_p(N^{-1/2}) + O_p(T^{-1/2}) + O_p(\sqrt{T}N^{-1}). \qquad \text{(A.44)}
\end{aligned}
$$

Hence, while in S4 $\|\mathbf{B}_{34}\| = O_p(\sqrt{T})$, in S1–S3 $\|\mathbf{B}_{34}\| = o_p(1)$.

Let us now consider $\mathbf{B}_{32}$. By using steps similar to those used above when analysing $\mathbf{B}_{32}$,

$$
\begin{aligned}
T^{-1/2}\mathbf{B}_{32} &= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{-y,i}\sqrt{N}T^{-2}\mathbf{F}'_{-m,-y}T(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})\boldsymbol{\varepsilon}_i \\
&= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}NT^{-1}\overline{\mathbf{u}}'\boldsymbol{\varepsilon}_i + o_p(1) \\
&= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}(\sigma^2_{\varepsilon,i}, \mathbf{0}'_{k\times 1})' + o_p(1). \qquad \text{(A.45)}
\end{aligned}
$$

Moreover, since $\mathbf{F}_{-m,-y} = \mathbf{M}_{\mathbf{F}_m}\mathbf{F}_{-y}$ with $\mathbf{F}_m = \mathbf{FR}$, we have that $\mathbf{F}_{-m,-y}$ is orthogonal to all the columns of $\mathbf{C}$, including $\boldsymbol{\Lambda}$. Hence, $\boldsymbol{\Lambda}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}} = \widetilde{\boldsymbol{\Lambda}}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}$, which in turn implies

$$\begin{aligned}
&\left\| \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\Lambda}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}} \sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \right\| \\
&= N^{-1/2} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \widetilde{\boldsymbol{\Lambda}}_{-y,i}\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}} \sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \right\| = O_p(N^{-1/2}). \qquad \text{(A.46)}
\end{aligned}$$

Hence,

$$\|T^{-1/2}\mathbf{B}_{32}\| = O_p(N^{-1/2}) + O_p(T^{-1/2}), \qquad \text{(A.47)}$$

or $\|\mathbf{B}_{32}\| = O_p(1)$. Again, this is true in the most general S4 specification, but then $\|\mathbf{B}_{34}\| = O_p(\sqrt{T})$ and so $\mathbf{B}_{34}$ is dominating anyways. Analogous to $\mathbf{B}_{34}$, $\mathbf{B}_{32} = \mathbf{0}_{k\times 1}$ if $r = r_m$. Moreover, by using the same steps as in the above, we can show that if $k + 1 = r_m$, then

$$\|\mathbf{B}_{32}\| = O_p(N^{-1/2}) + O_p(T^{-1/2}). \qquad \text{(A.48)}$$

Hence, $\|\mathbf{B}_{32}\| = o_p(1)$ regardless of the specification considered.

By adding the results,

$$\mathbf{B}_3 = \mathbf{B}_{31} - \mathbf{B}_{32} - \mathbf{B}_{33} + \mathbf{B}_{34} + o_p(1) = \mathbf{b}_3 + \sqrt{T}\mathbf{b}_4 + o_p(1). \qquad \text{(A.49)}$$

Strictly speaking, because $\mathbf{B}_{34} = \sqrt{T}\mathbf{b}_4 + O_p(1)$, the order of the remainder under S4 is not really $o_p(1)$ but rather $O_p(1)$. However, since in this case $\sqrt{T}\mathbf{b}_4$ dominates, for simplicity we drop the $O_p(1)$ term.

Let us now consider $\mathbf{S}_{\mathbf{x}}$. Making use of the fact that $\mathbf{F}_m = (\widehat{\mathbf{F}} - \overline{\mathbf{E}})\mathbf{Q}_m^0$, we have

$$\begin{aligned}
\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{x}_i &= \mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{v}_i + \mathbf{F}\boldsymbol{\Lambda}_i) = \mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i + \mathbf{F}_{-m}\boldsymbol{\Lambda}_i) = \mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{v}_i + \mathbf{F}_m\mathbf{H}'\boldsymbol{\Lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}) \\
&= \mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\boldsymbol{\Lambda}_i + \mathbf{F}_{-m,-y}\boldsymbol{\Lambda}_{-y,i}), \qquad \text{(A.50)}
\end{aligned}$$

which we can use to show that

$$
\begin{aligned}
\mathbf{S}_{\mathbf{x}} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{x}_i \\
&= \frac{1}{NT}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i + \mathbf{F}_{-m,-y}\mathbf{\Lambda}_{-y,i})'\mathbf{M}_{\widehat{\mathbf{F}}\mathbf{Q}\mathbf{G}}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i + \mathbf{F}_{-m,-y}\mathbf{\Lambda}_{-y,i}) \\
&= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{v}_i - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{P}_{\mathbf{F}_m}\mathbf{v}_i + \frac{1}{NT}\sum_{i=1}^{N}(\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i)'\mathbf{M}_{\mathbf{F}_m}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i \\
&- \frac{1}{NT}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i)'(\mathbf{M}_{\mathbf{F}_m} - \mathbf{M}_{\widehat{\mathbf{F}}})(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i) \\
&+ \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}_{-m,-y}'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i) \\
&+ \frac{1}{NT}\sum_{i=1}^{N}(\mathbf{v}_i - \overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_{-y,i})'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\mathbf{\Lambda}_{-y,i}' \\
&+ \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}_{-m,-y}'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\mathbf{\Lambda}_{-y,i}'. \qquad \text{(A.51)}
\end{aligned}
$$

We begin by noting now

$$
\frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{v}_i = \mathbf{\Sigma}_{\mathbf{v}} + o_p(1),
$$

$$
\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{v}_i'\mathbf{P}_{\mathbf{F}_m}\mathbf{v}_i\right\| \leq \frac{1}{N}\sum_{i=1}^{N}\|T^{-1}\mathbf{v}_i'\mathbf{F}_m\|\|\mathbf{S}_{\mathbf{F}_m}^{-1}\|\|T^{-1}\mathbf{F}_m'\mathbf{v}_i\| = O_p(T^{-1}),
$$

and

$$
\begin{aligned}
&\left\|\frac{1}{NT}\sum_{i=1}^{N}(\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i)'\mathbf{M}_{\mathbf{F}_m}\overline{\mathbf{E}}\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i\right\| \\
&\leq \frac{1}{N}\sum_{i=1}^{N}[\|\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i\|^2\|T^{-1}\overline{\mathbf{E}}'\overline{\mathbf{E}}\| + \|\mathbf{Q}_m^0\mathbf{H}'\mathbf{\Lambda}_i\|^2\|T^{-1}\overline{\mathbf{E}}'\mathbf{F}_m\|^2\|\mathbf{S}_{\mathbf{F}_m}^{-1}\|] \\
&= O_p(N^{-1}) + O_p((NT)^{-1}) = O_p(N^{-1}).
\end{aligned}
$$

We can similarly show that the remaining right-hand side terms are $o_p(1)$, except for the last, which is given by

$$
\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\mathbf{\Lambda}'_{-y,i} \\
&= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{F}'_{-m,-y}\mathbf{F}_{-m,-y}\mathbf{\Lambda}'_{-y,i} - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Lambda}_{i-y}\mathbf{F}'_{-m,-y}\mathbf{P}_{\widehat{\mathbf{F}}}\mathbf{F}_{-m,-y}\mathbf{\Lambda}'_{-y,i} \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\mathbf{\Lambda}'_{-y,i} \\
&- \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{i-y}\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}(\sqrt{N}\widetilde{\mathbf{C}}_{-y})\mathbf{J}_{\overline{\mathbf{E}}}(\sqrt{N}\widetilde{\mathbf{C}}_{-y})'\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\mathbf{\Lambda}'_{-y,i} + o_p(1). \qquad \text{(A.52)}
\end{aligned}
$$

It follows that if we let

$$
\mathbf{\Sigma}_{\mathbf{x}} = \mathbf{\Sigma}_{\mathbf{v}} + \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_{-y,i}(\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}} - \mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}\sqrt{N}\widetilde{\mathbf{C}}'_{-y}\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}})\mathbf{\Lambda}'_{-y,i}.
$$

then

$$
\mathbf{S}_{\mathbf{x}} = \mathbf{\Sigma}_{\mathbf{x}} + o_p(1). \qquad \text{(A.53)}
$$

Putting everything together,

$$
\begin{aligned}
\sqrt{NT}(\widehat{\boldsymbol{\beta}}_P - \boldsymbol{\beta}) &= \mathbf{S}_{\mathbf{x}}^{-1}[\mathbf{B}_0 + \sqrt{N}T^{-1/2}\mathbf{B}_1 + \sqrt{T}N^{-1/2}\mathbf{B}_2 + \mathbf{B}_3] \\
&= \mathbf{\Sigma}_{\mathbf{x}}^{-1}[\mathbf{b}_0 + \sqrt{\kappa}\mathbf{b}_1 + \kappa^{-1/2}\mathbf{b}_2 + \mathbf{b}_3 + \sqrt{T}\mathbf{b}_4] + o_p(1). \qquad \text{(A.54)}
\end{aligned}
$$

This completes the proof of the theorem. □

Proof of Proposition 3.1.
We start with the non-simplified version of $\mathbf{b}_2$, which in view of the definition of $\mathbf{\Sigma}_{\overline{\mathbf{E}}}$ can be written as follows:

$$
\begin{aligned}
\mathbf{b}_2 &= -\frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i}\mathbf{D} - \mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}'_m\mathbf{\Sigma}_{\overline{\mathbf{E}}}] \\
&\times \mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1}\mathbf{h}'(\boldsymbol{\lambda}'_{y,i}, \mathbf{0}'_{(r-r_y)\times 1})' - \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}'_m(\sigma^2_{\varepsilon,i}, \mathbf{0}'_{k\times 1})' \\
&= \sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}(\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m)'\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1}\mathbf{h}'(\boldsymbol{\lambda}'_{y,i}, \mathbf{0}'_{(r-r_y)\times 1})' \\
&- \frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i} - \mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}'_m\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}]\mathbf{D}\mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1}\mathbf{h}'(\boldsymbol{\lambda}'_{y,i}, \mathbf{0}'_{(r-r_y)\times 1})' \\
&- \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}'_m(\sigma^2_{\varepsilon,i}, \mathbf{0}'_{k\times 1})'. \qquad \text{(A.55)}
\end{aligned}
$$

Consider the first term on the right. Note that $(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' = (\mathbf{I}_{r_y}, \mathbf{0}_{(r-r_y)\times r_y}')' \boldsymbol{\lambda}_{y,i}$. Let now use decompose $\mathbf{R}$ as

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}_{r_y} & \mathbf{0}_{r_y\times(r_m-r_y)} \\ \mathbf{0}_{(r-r_y)\times r_y} & \mathbf{R}_{22} \end{pmatrix} = (\mathbf{R}_1, \, \mathbf{R}_2), \tag{A.56}$$

with obvious definitions of $\mathbf{R}_1$ and $\mathbf{R}_2$. It follows that $(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' = \mathbf{R}_1 \boldsymbol{\lambda}_{y,i}$. Moving on to $\mathbf{h}$, by using the definition of $\mathbf{J}_{\mathbf{F}}$, it is clear that

$$\mathbf{h} = \boldsymbol{\Sigma}_{\mathbf{F}} \mathbf{J}_{\mathbf{F}} \mathbf{R}(\mathbf{R}'\mathbf{R})^{-1} = \boldsymbol{\Sigma}_{\mathbf{F}} \mathbf{R}(\mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R})^{-1}\mathbf{R}'\mathbf{R}(\mathbf{R}'\mathbf{R})^{-1} = \boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R}(\mathbf{R}'\boldsymbol{\Sigma}_{\mathbf{F}}\mathbf{R})^{-1}. \tag{A.57}$$

Hence, $\mathbf{h}'\mathbf{R} = \mathbf{I}_{r_m}$, and therefore

$$\mathbf{h}'\mathbf{R}_1 = \begin{pmatrix} \mathbf{I}_{r_y} \\ \mathbf{0}_{(r_m-r_y)\times r_y} \end{pmatrix}. \tag{A.58}$$

Next up is $(\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m)'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m$. The first r_y columns of $\mathbf{Q}_m$ are given by $\mathbf{Q}_y$. Let us therefore write $\mathbf{Q}_m = (\mathbf{Q}_y, \mathbf{Q}_{m,-y})$, where $\mathbf{Q}_{m,-y}$ is $(k+1)\times(r_m - r_y)$. Making use of this and Assumption 2.4 (ii), which implies that $\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_y \overset{a.s.}{=} \mathbf{0}_{(r-r_y)\times r_y}$, we get

$$\begin{aligned}
&(\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m)'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m \\
&= \begin{pmatrix} (\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_y)'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_y & (\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_y)'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y} \\ (\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y})'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_y & (\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y})'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y} \end{pmatrix} \\
&\overset{a.s.}{=} \begin{pmatrix} \mathbf{0}_{r_y\times r_y} & \mathbf{0}_{r_y\times(r_m-r_y)} \\ \mathbf{0}_{(r_m-r_y)\times r_y} & (\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y})'\boldsymbol{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y} \end{pmatrix}.
\end{aligned} \tag{A.59}$$

Consider $\mathbf{R}^+ = (\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'$. Direct substitution using the definition of $\mathbf{R}$ and letting $\mathbf{R}_{22}^+ = (\mathbf{R}_{22}'\mathbf{R}_{22})^{-1}\mathbf{R}_{22}'$, we get

$$\mathbf{R}^+ = \begin{pmatrix} \mathbf{I}_{r_y} & \mathbf{0}_{r_y\times(r-r_y)} \\ \mathbf{0}_{(r_m-r_y)\times r_y} & \mathbf{R}_{22}^+ \end{pmatrix}. \tag{A.60}$$

Hence, since by Assumption 2.5 (ii) $\mathbf{C} = \mathbf{R}\mathbf{C}_m$, we have $\mathbf{R}^+\mathbf{C} = \mathbf{C}_m$. By using this, Assumption 2.4 (ii) and the above expression for $\mathbf{R}^+$,

$$\begin{aligned}
\mathbf{C}_m\mathbf{Q}_m = \mathbf{R}^+\mathbf{C}\mathbf{Q}_m = \mathbf{R}^+ \begin{pmatrix} \mathbf{C}_y\mathbf{Q}_y & \mathbf{C}_y\mathbf{Q}_{m,-y} \\ \mathbf{C}_{-y}\mathbf{Q}_y & \mathbf{C}_{-y}\mathbf{Q}_{m,-y} \end{pmatrix} \overset{a.s.}{=} \mathbf{R}^+ \begin{pmatrix} \mathbf{C}_y\mathbf{Q}_y & \mathbf{C}_y\mathbf{Q}_{m,-y} \\ \mathbf{0}_{(r-r_y)\times r_y} & \mathbf{C}_{-y}\mathbf{Q}_{m,-y} \end{pmatrix} \\
= \begin{pmatrix} \mathbf{C}_y\mathbf{Q}_y & \mathbf{C}_y\mathbf{Q}_{m,-y} \\ \mathbf{0}_{(r_m-r_y)\times r_y} & \mathbf{R}_{22}^+\mathbf{C}_{-y}\mathbf{Q}_{m,-y} \end{pmatrix}
\end{aligned} \tag{A.61}$$

Note that while $\mathbf{C}_y\mathbf{Q}_y$ is positive definite by Assumption 2.4 (i), the $(r_m - r_y)\times(r_m - r_y)$ matrix $\mathbf{R}_{22}^+\mathbf{C}_{-y}\mathbf{Q}_{m,-y}$ need not be. This means that we cannot rely on the conventional formula for the inverse of partitioned matrices to compute $(\mathbf{C}_m\mathbf{Q}_m)^{-1}$. However, by Theorem 1 of Meyer (1970), we have that the generalized inverse $(\mathbf{C}_m\mathbf{Q}_m)^{-}$ of $\mathbf{C}_m\mathbf{Q}_m$ is given by

$$(\mathbf{C}_m\mathbf{Q}_m)^{-} \overset{a.s.}{=} \begin{pmatrix} (\mathbf{C}_y\mathbf{Q}_y)^{-1} & -(\mathbf{C}_y\mathbf{Q}_y)^{-1}\mathbf{C}_y\mathbf{Q}_{m,-y}(\mathbf{R}_{22}^{+}\mathbf{C}_{-y}\mathbf{Q}_{m,-y})^{-} \\ \mathbf{0}_{(r_m-r_y)\times r_y} & (\mathbf{R}_{22}^{+}\mathbf{C}_{-y}\mathbf{Q}_{m,-y})^{-} \end{pmatrix}. \tag{A.62}$$

But $(\mathbf{C}_m\mathbf{Q}_m)^{-} = (\mathbf{C}_m\mathbf{Q}_m)^{-1}$, which we can use together with positive definiteness of $\mathbf{Q}$ to show that

$$(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}_m'\widetilde{\mathbf{C}}_{-y}'\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1} \overset{a.s.}{=} \begin{pmatrix} \mathbf{0}_{r_y\times r_y} & \mathbf{0}_{r_y\times(r-r_y)} \\ \mathbf{0}_{(k+1-r_y)\times r_y} & (\mathbf{R}_{22}^{+}\mathbf{C}_{-y}\mathbf{Q}_{m,-y})^{-\prime}(\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y})'\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_{m,-y}(\mathbf{R}_{22}^{+}\mathbf{C}_{-y}\mathbf{Q}_{m,-y})^{-} \end{pmatrix}. \tag{A.63}$$

By using this and the fact that $\mathbf{h}'\mathbf{R}_1 = (\mathbf{I}_{r_y}, \mathbf{0}_{(r_m-r_y)\times r_y}')'$, we can show that

$$(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}_m'\widetilde{\mathbf{C}}_{-y}'\mathbf{\Sigma}_{\mathbf{F}_{-m,-y}}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1}\mathbf{h}'(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' \overset{a.s.}{=} \mathbf{0}_{r\times 1}, \tag{A.64}$$

and so we obtain

$$\begin{aligned}
\mathbf{b}_2 &= -\frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i} - \mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}_m'\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}]\mathbf{D}\mathbf{Q}_m(\mathbf{C}_m\mathbf{Q}_m)^{-1}\mathbf{h}'(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' \\
&\quad - \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}(\mathbf{C}_m\mathbf{Q}_m)^{-1\prime}\mathbf{Q}_m'(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')' \\
&= -\frac{1}{N}\sum_{i=1}^{N}[(\mathbf{0}_{k\times 1}, \mathbf{I}_k)\mathbf{\Sigma}_{\mathbf{e},i} - \mathbf{\Lambda}_i\mathbf{h}\mathbf{C}_m^{-1\prime}\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}]\mathbf{D}\mathbf{C}_m^{-1}\mathbf{h}'(\boldsymbol{\lambda}_{y,i}', \mathbf{0}_{(r-r_y)\times 1}')' \\
&\quad - \frac{1}{N}\sum_{i=1}^{N}\mathbf{\Lambda}_i\mathbf{h}\mathbf{C}_m^{-1\prime}(\sigma_{\varepsilon,i}^2, \mathbf{0}_{k\times 1}')',
\end{aligned} \tag{A.65}$$

as required. □

The proofs of Corollaries 3.1–3.4 and Proposition 3.2 are obvious given Theorem 3.1 and the arguments given in the text. They are therefore omitted.

Remark A.1 (High-level conditions)**.** In the discussion following Example 2.2, we pointed to the fact that $\pi_1 = 0$ is enough to ensure $\sqrt{NT}$-consistency. This is true in the particular DGP considered in Example 2.2. With more than one regressor it is necessary to also restrict $\mathbf{J}_{\overline{\mathbf{E}}}$. In this remark we provide a set of high-level conditions that ensures that $\mathbf{b}_4 = \mathbf{0}_{k\times 1}$ in general. In so doing, it is useful to denote by $\mathbf{C}_{m,i}$ the summand of $\overline{\mathbf{C}}_m$, whose dimension is given by $r_m \times (k+1)$. Let us further define

$$\mathbf{D}_i = \begin{pmatrix} \mathbf{D}_{y,m,i} & \mathbf{D}_{y,-m,i} \\ \mathbf{D}_{-y,m,i} & \mathbf{D}_{-y,-m,i} \end{pmatrix} = \mathbf{C}_{m,i}\mathbf{Q}, \tag{A.66}$$

with $\mathbf{D}_{y,m,i}$, $\mathbf{D}_{y,-m,i}$, $\mathbf{D}_{-y,m,i}$ and $\mathbf{D}_{-y,-m,i}$ being $r_y \times r_m$, $r_y \times (k+1-r_m)$, $(r_m - r_y) \times r_m$ and $(r_m - r_y) \times (k+1-r_m)$, respectively. We similarly decompose $\mathbf{Q}'\mathbf{e}_{i,t} = (\mathbf{e}_{i,t}'\mathbf{Q}_m, \mathbf{e}_{i,t}'\mathbf{Q}_{-m})' = (\mathbf{e}_{m,i,t}', \mathbf{e}_{-m,i,t}')$, where $\mathbf{e}_{m,i,t}$ and $\mathbf{e}_{-m,i,t}$ are $r_m \times 1$ and $(k+1-r_m) \times 1$, respectively, and where the first r_y elements of $\mathbf{e}_{m,i,t}$ are given by $\mathbf{e}_{y,i,t} = \mathbf{Q}_y'\mathbf{e}_{i,t}$. The assumption that we need to ensure $\sqrt{NT}$-consistency in S4 is that one of the following conditions holds (for all i):

(HL1) $E(\mathbf{D}_{y,-m,i}) = \mathbf{0}_{r_y \times (k+1-r_m)}$ and $\mathbf{D}_{-y,-m,i} \overset{a.s.}{=} \mathbf{0}_{(r_m - r_y) \times (k+1-r_m)}$.

(HL2) $E(\mathbf{D}_{y,-m,i}) = \mathbf{0}_{r_y \times (k+1-r_m)}$, $E(\mathbf{D}_{-y,-m,i}) = \mathbf{0}_{(r_m - r_y) \times (k+1-r_m)}$ and $E(\mathbf{e}_{y,i,t}\mathbf{e}_{-m,i,t}') = \mathbf{0}_{r_y \times (k+1-r_m)}$.

The way that the above conditions work is to ensure that the upper $r_m \times (k+1-r_m)$ block of $\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}N^{-1/2}\mathbf{G}_{-m}$ is $O_p(N^{-1/2})$, that the off-diagonal blocks of $(N^{-1}\mathbf{G}_{-m}'\mathbf{Q}'\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m})^{-1}$ are $O_p(T^{-1/2}) + O_p(N^{-1/2})$, and that the lower $(k+1-r_m) \times r_y$ block of $N^{-1/2}\mathbf{G}_{-m}'\mathbf{Q}'\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}\mathbf{D}\mathbf{Q}_y$ is $O_p(T^{-1/2}) + O_p(N^{-1/2})$. This implies that

$$
\begin{aligned}
&\sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{J}_{\overline{\mathbf{E}}}\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}\mathbf{D}\mathbf{Q}_y \\
&= \sqrt{N}\widetilde{\mathbf{C}}_{-y}\mathbf{Q}N^{-1/2}\mathbf{G}_{-m}(N^{-1}\mathbf{G}_{-m}'\mathbf{Q}'\mathbf{\Sigma}_{\overline{\mathbf{E}}}\mathbf{Q}\mathbf{G}_{-m})^{-1}N^{-1/2}\mathbf{G}_{-m}'\mathbf{Q}'\mathbf{D}'\mathbf{\Sigma}_{\mathbf{e}}\mathbf{D}\mathbf{Q}_y \\
&= O_p(T^{-1/2}) + O_p(N^{-1/2}),
\end{aligned} \tag{A.67}
$$

which means that $\mathbf{b}_4$ is of the same order.

References

KARABIYIK, H., S. REESE, AND J. WESTERLUND (2017): "On The Role of The Rank Condition in CCE Estimation of Factor-augmented Panel Regressions," *Journal of Econometrics*, 197, 60–64.

MEYER, C. D. (1970): "Generalized Inverses of Block Triangular Matrices," *SIAM Journal on Applied Mathematics*, 19, 741–750.

PESARAN, M. H. (2006): "Estimation and Inference in Large Heterogeneous Panels with a Multifactor Error Structure," *Econometrica*, 74, 967–1012.