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CHARACTERIZING EXISTENCE OF A MEASURABLE CARDINAL VIA MODAL LOGIC

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Abstract. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic coincides with the modal logic of the Kripke frame isomorphic to the powerset of a two element set.

1. Introduction

Over the years there have been discovered several intriguing connections between set theory and modal logic. To name a few:

(1) There is an interesting connection between non-well-founded set theory and infinitary modal logic [1, 3, 2].
(2) The modal logic $\mathsf{S4.2}$ turns out to be the logic of forcing extensions of ZFC [16].
(3) The only existing proof that the modal logic $\mathsf{S4.1.2}$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space ω requires that each MAD family has cardinality 2^ω, a principle that is not provable in ZFC, and it remains an open problem whether this principle is necessary [8].

To these results we add the following. Let the diamond $\mathfrak{D} = (D, \leq)$ be the partially ordered Kripke frame shown in Figure 1. It is clear that $\mathfrak{D}$ is isomorphic to the powerset of a two element set. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic is the modal logic of $\mathfrak{D}$.

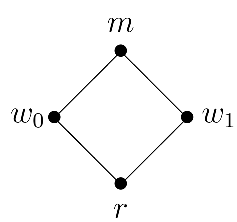

Figure 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.

We recall that topological semantics generalizes Kripke semantics for the well-known modal logic $\mathsf{S4}$. Thus, Kripke completeness implies topological completeness for logics above $\mathsf{S4}$. However, topological spaces arising from Kripke frames are usually not even T_1. Therefore, it is nontrivial to prove topological completeness results above $\mathsf{S4}$ with respect to spaces satisfying higher separation axioms. One such class is the class of Tychonoff spaces. By a celebrated theorem of Tychonoff, these are exactly subspaces of compact Hausdorff spaces. In [5] we initiated the study of modal logics arising from Tychonoff spaces. On the one hand, this yielded a new notion of dimension in topology, called modal Krull dimension. On the

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other hand, it provided a new concept of zemanian logics which generalize the well-known modal logic of Zeman.

It is known that extremally disconnected spaces are topological models of the modal logic $\mathsf{S4.2}$, and hereditarily extremally disconnected spaces are topological models of the modal logic $\mathsf{S4.3}$. In [6] we showed that a modal logic above $\mathsf{S4.3}$ is a zemanian logic iff it is the logic of an hereditarily extremally disconnected Tychonoff space. The simplest modal logic above $\mathsf{S4.2}$ that is not above $\mathsf{S4.3}$ is the logic of $\mathfrak{D}$. In this paper we show that topological completeness of the logic of $\mathfrak{D}$ with respect to a normal space is equivalent to the existence of a measurable cardinal. Whether normal can be weakened to Tychonoff remains an open problem.

We conclude the introduction by briefly describing the key ingredients of the proof. If there exists a measurable cardinal κ, using a countably complete ultrafilter on κ, we first build a normal P-space Y. Combining the results of [12] and [13] then allows us to embed Y into the remainder of the Čech-Stone compactification $\beta\mu$ of a cardinal μ viewed as a discrete space. Letting $Z = Y \cup \mu$ yields a normal space whose logic we prove is the logic of the diamond $\mathfrak{D}$. This we do by showing that a finite rooted Kripke frame $\mathfrak{F}$ is an interior image of Z iff $\mathfrak{F}$ is an interior image of $\mathfrak{D}$.

Conversely, suppose there exists a normal space Z whose logic is the logic of the diamond $\mathfrak{D}$. We first show that $\mathfrak{D}$ is an interior image of Z. We then prove that without loss of generality the inverse image of the root r of $\mathfrak{D}$ is a singleton $\{a\}$. We next prove that a is a P-point of an appropriately chosen subspace of Z. This allows us to define a family of subsets of Z whose cardinal is Ulam-measurable. Finally, it is well known that this implies the existence of a measurable cardinal.

2. Preliminaries

In this section we recall the necessary background from modal logic, its topological semantics, and measurable cardinals.

2.1. **Modal logic.** We use [10] as the main reference for modal logic. Modal formulas are built in the usual way using countably many propositional letters, the classical connectives $\neg$ (negation) and $\to$ (implication), the modal connective $\Box$ (necessity), and parentheses. We employ the standard abbreviations: $\wedge$ (conjunction), $\vee$ (disjunction), and $\Diamond$ (possibility).

The well-known modal system $\mathsf{S4}$ of Lewis is the least set of formulas containing the classical tautologies, the axioms

$$\Box(p \to q) \to (\Box p \to \Box q),$$
$$\Box p \to p,$$
$$\Box p \to \Box\Box p,$$

and closed under the inference rules of

Modus Ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$,
substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$,
necessitation $\frac{\varphi}{\Box\varphi}$.

A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set and R is a binary relation on W. As usual, for $w \in W$ we let

$$R(w) = \{v \in W \mid wRv\} \text{ and } R^{-1}(w) = \{v \in W \mid vRw\};$$

and for $A \subseteq W$ we let

$$R(A) = \bigcup\{R(w) \mid w \in A\} \text{ and } R^{-1}(A) = \bigcup\{R^{-1}(w) \mid w \in A\}.$$

Kripke semantics of modal logic recursively assigns to each formula a subset of a Kripke frame $\mathfrak{F}$ by interpreting each propositional letter as a subset of W, the classical connectives as Boolean operations in the powerset $\wp(W)$, and $\Box$ as the operation $\Box_R$ on $\wp(W)$ defined by

$$\Box_R(A) = \{w \in W \mid R(w) \subseteq A\}.$$

Consequently, $\Diamond$ is interpreted as the operation $\Diamond_R$ on $\wp(W)$ defined by

$$\Diamond_R(A) = R^{-1}(A).$$

Let φ be a modal formula and $\mathfrak{F} = (W, R)$ a Kripke frame. Call φ *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, provided φ evaluates to W for every assignment of the propositional letters. If φ is not valid in $\mathfrak{F}$, then we say that φ is *refuted* in $\mathfrak{F}$, and write $\mathfrak{F} \not\models \varphi$. The *logic* of $\mathfrak{F}$ is the set of modal formulas valid in $\mathfrak{F}$; in symbols $\mathsf{L}(\mathfrak{F}) = \{\varphi \mid \mathfrak{F} \models \varphi\}$.

A Kripke frame $\mathfrak{F}$ is called an $\mathsf{S4}$-*frame* if R is reflexive and transitive. The name is justified by the well-known fact that $\mathsf{S4}$ is sound and complete with respect to $\mathsf{S4}$-frames. In this paper we are mainly interested in the following logic.

Definition 2.1. Let $\mathsf{L} := \mathsf{L}(\mathfrak{D})$ be the logic of the diamond $\mathfrak{D}$ shown in Figure 1.

2.2. **Topological semantics.** Topological semantics interprets $\Box$ as topological interior (and consequently $\Diamond$ as topological closure). Specifically, for a topological space X, the propositional letters are assigned to subsets of X, the classical connectives are computed as the Boolean operations in $\wp(X)$, and $\Box$ is interpreted as the interior operator $\mathsf{i} : \wp(X) \to \wp(X)$, where $\mathsf{i}A$ is the greatest open subset of X contained in A. Consequently, $\Diamond$ is interpreted as the closure operator $\mathsf{c} : \wp(X) \to \wp(X)$, where $\mathsf{c}A$ is the least closed subset of X containing A.

Let φ be a modal formula and X a space. Call φ *valid* in X, denoted $X \models \varphi$, provided φ evaluates to X for every assignment of the propositional letters. If φ is not valid in X, then we say that φ is *refuted* in X, and write $X \not\models \varphi$. The *logic* of X is the set of formulas valid in X; symbolically, $\mathsf{L}(X) = \{\varphi \mid X \models \varphi\}$. It is well known that $\mathsf{S4}$ is sound and complete with respect to topological spaces.

There is a close connection between topological semantics and Kripke semantics for $\mathsf{S4}$. Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. Call $U \subseteq W$ an R-*upset* of $\mathfrak{F}$ if $w \in U$ and wRv imply $v \in U$. The set of R-upsets of $\mathfrak{F}$ is a topology τ_R on W in which every point w has a least neighborhood, namely $R(w)$. Such spaces are called *Alexandroff spaces*. We call (W, τ_R) the *Alexandroff space* of $\mathfrak{F}$. For a modal formula φ, we have

$$\mathfrak{F} \models \varphi \text{ iff } (W, \tau_R) \models \varphi.$$

Thus, topological semantics generalizes Kripke semantics for $\mathsf{S4}$, and hence Kripke completeness for logics above $\mathsf{S4}$ implies topological completeness. However, since Alexandroff spaces are usually not even T_1-spaces, such topological completeness is not guaranteed with respect to, for example, normal spaces.

We recall that a topological space X is

- *extremally disconnected* (ED) if the closure of each open set is open;
- *resolvable* if X is the union of two disjoint dense subsets of X;
- *irresolvable* if X is not resolvable;
- *hereditarily irresolvable* (HI) if every subspace of X is irresolvable.

Let

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p$$

be the *Grzegorczyk axiom* and

$$\mathsf{ga} = \Diamond\Box p \to \Box\Diamond p$$

the *Geach axiom* (see, e.g., [10]). It is well known that

$$\begin{array}{lll} X \text{ is ED} & \text{iff} & X \models \mathsf{grz}; \\ X \text{ is HI} & \text{iff} & X \models \mathsf{ga}. \end{array}$$

We next recall the definition of modal Krull dimension. For this we recall that a subset N of a space X is *nowhere dense* if $\mathsf{ic}N = \varnothing$.

Definition 2.2. ([5, Sec. 3]) Define the *modal Krull dimension* $\mathrm{mdim}(X)$ of a topological space X recursively as follows:

$$\begin{array}{lll} \mathrm{mdim}(X) = -1 & \text{if} & X = \varnothing, \\ \mathrm{mdim}(X) \leq n & \text{if} & \mathrm{mdim}(N) \leq n-1 \text{ for each } N \text{ nowhere dense in } X, \\ \mathrm{mdim}(X) = n & \text{if} & \mathrm{mdim}(X) \leq n \text{ but } \mathrm{mdim}(X) \not\leq n-1, \\ \mathrm{mdim}(X) = \infty & \text{if} & \mathrm{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \dots \end{array}$$

Let

$$\begin{array}{lll} \mathsf{bd}_1 & = & \Diamond\Box p_1 \to p_1 \\ \mathsf{bd}_{n+1} & = & \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1} \text{ for } n \geq 1. \end{array}$$

Theorem 2.3. ([5, Thm. 3.6]) *Let X be a nonempty space and $n \geq 1$. Then*

$$\mathrm{mdim}(X) \leq n-1 \textit{ iff } X \models \mathsf{bd}_n.$$

For nonempty scattered Hausdorff spaces, there is a close connection between finite modal Krull dimension and Cantor-Bendixson rank. For $Y \subseteq X$, let $\mathsf{d}Y$ be the set of limit points of Y and for an ordinal α, let $\mathsf{d}^\alpha Y$ be defined recursively as follows:

$$\begin{array}{lll} \mathsf{d}^0 Y & = & Y, \\ \mathsf{d}^{\alpha+1} Y & = & \mathsf{d}(\mathsf{d}^\alpha Y), \\ \mathsf{d}^\alpha Y & = & \bigcap\{\mathsf{d}^\beta Y \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \end{array}$$

The *Cantor-Bendixson rank* of X is the least ordinal γ satisfying $\mathsf{d}^\gamma X = \mathsf{d}^{\gamma+1} X$. It is well known that a space X is scattered iff there is an ordinal α such that $\mathsf{d}^\alpha X = \varnothing$. Thus, the Cantor-Bendixson rank of a scattered space X is the least ordinal γ such that $\mathsf{d}^\gamma X = \varnothing$.

Let X be a nonempty scattered Hausdorff space and $n \in \omega$. Then the Cantor-Bendixson rank of X is $n+1$ iff $\mathsf{d}^n X \neq \varnothing$ and $\mathsf{d}^{n+1} X = \varnothing$, which by [7, Thm. 4.9] happens iff $\mathrm{mdim}(X) = n$.

2.3. **Measurable cardinals.** We use [17, 18] as standard references for set theory, and also rely on [11] as the main reference for measurable cardinals. Let S be a set and $\mathfrak{p}$ a free ultrafilter on S. We denote infinite cardinals by κ, the first uncountable cardinal by ω_1, and recall that $\mathfrak{p}$ is

- κ*-complete* if $\bigcap \mathcal{K} \in \mathfrak{p}$ for any family $\mathcal{K} \subseteq \mathfrak{p}$ of cardinality $< \kappa$;
- *countably complete* if $\mathfrak{p}$ is ω_1-complete (that is, $\mathfrak{p}$ is closed under countable intersections).

Definition 2.4. ([11, Ch. 8]) An uncountable cardinal κ is

- *measurable* if there exists a κ-complete free ultrafilter on κ;
- *Ulam-measurable* if there exists a countably complete free ultrafilter on κ.

Remark 2.5. While in [11] it is not assumed that measurable cardinals are uncountable, it is common to make such an assumption.

It is clear that every measurable cardinal is Ulam-measurable, and it is well known (see, e.g., [11, Thm. 8.31]) that the existence of an Ulam-measurable cardinal implies the existence of a measurable cardinal.

3. Existence of a measurable cardinal is sufficient

In this section we prove that the existence of a measurable cardinal implies that there is a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$. We build Z in stages. Let κ be a measurable cardinal. Then κ is Ulam-measurable, and so there is a countably complete free ultrafilter $\mathfrak{p}$ on κ. Let $Y = (\kappa \times \{0,1\}) \cup \{\mathfrak{p}\}$. Consider the following family of subsets of Y:

$$\tau = \{U \subseteq Y \mid U \subseteq Y \setminus \{\mathfrak{p}\} \text{ or } \exists V, W \in \mathfrak{p} : U = (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})\}.$$

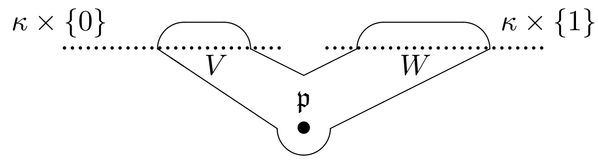

Figure 2. The space Y and an open neighborhood of $\mathfrak{p}$.

Lemma 3.1. *The family τ is a topology on Y that is closed under countable intersections.*

Proof. Clearly $\varnothing, Y \in \tau$. Let $\{U_i \mid i \in I\} \subseteq \tau$ and let $U = \bigcup\{U_i \mid i \in I\}$. If $\mathfrak{p} \notin U$, then $U \in \tau$. Suppose $\mathfrak{p} \in U$. Then $\mathfrak{p} \in U_i$ for some $i \in I$. Since $U_i \in \tau$ and $\mathfrak{p} \in U_i$, there are $V_0, V_1 \in \mathfrak{p}$ such that $U_i = (V_0 \times \{0\}) \cup \{\mathfrak{p}\} \cup (V_1 \times \{1\})$. For $n \in \{0,1\}$, set $W_n = \{\alpha \in \kappa \mid (\alpha, n) \in U\}$. Let $n \in \{0,1\}$ and $\alpha \in V_n$. Then $(\alpha, n) \in V_n \times \{n\} \subseteq U_i \subseteq U$, giving that $\alpha \in W_n$. Therefore, $V_n \subseteq W_n$. Since $V_n \in \mathfrak{p}$ and $\mathfrak{p}$ is an ultrafilter, $W_n \in \mathfrak{p}$. It follows from the definition of W_n that $W_n \times \{n\} = U \cap (\kappa \times \{n\})$. Thus,

$$\begin{aligned} U &= U \cap Y = U \cap ((\kappa \times \{0\}) \cup \{\mathfrak{p}\} \cup (\kappa \times \{1\})) \\ &= (U \cap (\kappa \times \{0\})) \cup (U \cap \{\mathfrak{p}\}) \cup (U \cap (\kappa \times \{1\})) \\ &= (W_0 \times \{0\}) \cup \{\mathfrak{p}\} \cup (W_1 \times \{1\}) \in \tau. \end{aligned}$$

Consequently, τ is closed under union.

Let $\{U_i \mid i \in \omega\} \subseteq \tau$ and let $U = \bigcap\{U_i \mid i \in \omega\}$. If $\mathfrak{p} \notin U$, then $U \in \tau$. Suppose $\mathfrak{p} \in U$. Let $i \in \omega$. Since $\mathfrak{p} \in U_i$ and $U_i \in \tau$, there are $V_i, W_i \in \mathfrak{p}$ such that $U_i = (V_i \times \{0\}) \cup \{\mathfrak{p}\} \cup (W_i \times \{1\})$. Put $V = \bigcap\{V_i \mid i \in \omega\}$ and $W = \bigcap\{W_i \mid i \in \omega\}$. As $\mathfrak{p}$ is countably complete, we have that $V, W \in \mathfrak{p}$.

Claim 3.2. $U = (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})$.

Proof. Let $\alpha \in \kappa$. We have

$$\begin{aligned} (\alpha, 0) \in U \quad &\text{iff} \quad (\alpha, 0) \in U_i \text{ for all } i \in \omega \\ &\text{iff} \quad \alpha \in V_i \text{ for all } i \in \omega \\ &\text{iff} \quad \alpha \in V \\ &\text{iff} \quad (\alpha, 0) \in V \times \{0\} \\ &\text{iff} \quad (\alpha, 0) \in (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\}). \end{aligned}$$

Similarly, $(\alpha, 1) \in U$ iff $(\alpha, 1) \in (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})$. The claim follows. □

We conclude that τ is a topology on Y that is closed under countable intersections. □

Remark 3.3. That κ is a measurable cardinal is used to see that τ is closed under countable intersections. In fact, this is the only place where we use that κ is a measurable cardinal.

Definition 3.4. (See, e.g., [20, p. 37]) A Tychonoff space is a *P-space* if every G_δ-set in X is open.

Lemma 3.5. *The space Y is a normal P-space.*

Proof. It is easy to see that each singleton in Y is closed, so Y is a T_1-space. Let A, B be disjoint closed subsets of Y. Either $\mathfrak{p} \notin A$ or $\mathfrak{p} \notin B$, and we may assume without loss of generality that $\mathfrak{p} \notin A$. Then $A \subseteq Y \setminus \{\mathfrak{p}\}$, hence A is open. Therefore, $U := A$ and $V := Y \setminus A$ are disjoint open subsets of Y separating A and B. Thus, Y is normal, and hence it follows from Lemma 3.1 that Y is a P-space. □

Since Y is a P-space, it follows from [12, Sec. 2] that the Čech-Stone compactification βY of Y can be embedded into a compact Hausdorff ED-space, say E. By Efimov's Theorem [13, Sec. 1], there is a cardinal μ, equipped with the discrete topology, such that the space E can be embedded into $\beta\mu$. It is well known (see, e.g., [14, Exercise 3.6.B.b]) that $\beta\mu$ can be embedded in the remainder $\beta\mu \setminus \mu$. Combining these results yields a sequence of embeddings

$$Y \hookrightarrow \beta Y \hookrightarrow E \hookrightarrow \beta\mu \hookrightarrow \beta\mu \setminus \mu \tag{1}$$

that gives an embedding of Y into $\beta\mu \setminus \mu$. We identify Y with its image in $\beta\mu$; see Figure 3.

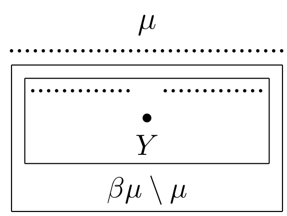

FIGURE 3. Y as a subspace of $\beta\mu$.

Definition 3.6. Let Z be the subspace $\mu \cup Y$ of $\beta\mu$.

Our goal is to show that Z is a normal space such that $\mathsf{L}(Z) = \mathsf{L}$.

Lemma 3.7. *The space Z is a scattered ED-space of Cantor-Bendixson rank 3.*

Proof. Since $Z \supseteq \mu$ and μ is dense in $\beta\mu$, we have that Z is dense in $\beta\mu$. As $\beta\mu$ is an ED-space (see, e.g., [14, Cor. 6.2.28]) and a dense subspace of an ED-space is an ED-space (see, e.g., [14, Exercise 6.2.G.c]), it follows that Z is an ED-space.

We have $\mathsf{d}^3 Z = \mathsf{d}^2 Y = \mathsf{d}\{\mathfrak{p}\} = \varnothing$ and $\mathsf{d}^2 Z = \mathsf{d} Y = \{\mathfrak{p}\} \neq \varnothing$. Therefore, Z is scattered and of Cantor-Bendixson rank 3. □

Lemma 3.8. *The space Z is normal.*

Proof. Clearly Z is T_1 since it is a subspace of a T_1-space. Let A and B be disjoint closed subsets of Z. Since μ is the set of isolated points of Z, we have that $A \cap \mu$ and $B \cap \mu$ are disjoint open subsets of Z. Let $A_0 = \mathsf{c}(A \cap \mu)$ and $B_0 = \mathsf{c}(B \cap \mu)$. Because Z is ED, A_0 and B_0 are disjoint clopen subsets of Z. Let $A_1 = A \setminus A_0$ and $B_1 = B \setminus B_0$. Then A_1 and B_1 are disjoint closed subsets of Y. Since Y is normal, it follows from [14, Cor. 3.6.4] that $\mathsf{c}_{\beta Y}(A_1)$ and $\mathsf{c}_{\beta Y}(B_1)$ are disjoint, where $\mathsf{c}_{\beta Y}$ is the closure in βY. Because βY is (up to homeomorphism) a closed subspace of $\beta\mu$, we have

$$\mathsf{c}_{\beta\mu}(A_1) \cap \mathsf{c}_{\beta\mu}(B_1) = \mathsf{c}_{\beta Y}(A_1) \cap \mathsf{c}_{\beta Y}(B_1) = \varnothing.$$

Since $\beta\mu$ is normal, there are disjoint open subsets U_1 and V_1 of $\beta\mu$ such that $\mathsf{c}_{\beta\mu}(A_1) \subseteq U_1$ and $\mathsf{c}_{\beta\mu}(B_1) \subseteq V_1$.

Clearly $U := U_1 \cap Z$ and $V := V_1 \cap Z$ are disjoint open subsets of Z. As both A_0 and B_0 are clopen in Z, it follows that both $U \setminus B_0$ and $V \setminus A_0$ are open in Z, and hence $U_0 := A_0 \cup (U \setminus B_0)$ and $V_0 := B_0 \cup (V \setminus A_0)$ are disjoint open subsets of Z. It is clear that $A_1 \subseteq U_1 \cap Z = U$.

Because A_1 and B_0 are disjoint, $A_1 \subseteq U \setminus B_0$, so $A = A_0 \cup A_1 \subseteq A_0 \cup (U \setminus B_0) = U_0$. Similarly, $B \subseteq V_0$. Thus, Z is normal. □

We recall that a map $f : X \to X'$ between spaces is *interior* if f is both continuous and open. If in addition f is onto, then we call X' an *interior image* of X. If X' is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{F}$, then we say that $\mathfrak{F}$ is an *interior image* of X. Finally, if X is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{G}$, then we say that $\mathfrak{F}$ is an *interior image* of $\mathfrak{G}$.

Remark 3.9. It is well known that $\mathfrak{F} = (W, R)$ is an interior image of $\mathfrak{G} = (V, S)$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$, where we recall that a *p-morphism* is a map $f : V \to W$ such that $f^{-1}R^{-1}(w) = S^{-1}f^{-1}(w)$ for each $w \in W$.

Convention 3.10. Since the diamond $\mathfrak{D} = (D, \leq)$ is a poset (partially ordered set), for $w \in D$ we write $\uparrow w$ and $\downarrow w$ instead of $R(w)$ and $R^{-1}(w)$, respectively.

Lemma 3.11. *The diamond $\mathfrak{D}$ is an interior image of Z.*

Proof. Define $f : Z \to D$ by

$$f(z) = \begin{cases} m & \text{if } z \in \mu \\ w_0 & \text{if } z \in \kappa \times \{0\} \\ w_1 & \text{if } z \in \kappa \times \{1\} \\ r & \text{if } z = \mathfrak{p} \end{cases}$$

It is clear that f is a well-defined onto mapping. To prove that f is interior, it is sufficient to show that $f^{-1}{\downarrow}w = \mathsf{c}f^{-1}(w)$ for each $w \in D$. Since μ is dense in Z, we have

$$f^{-1}{\downarrow}m = f^{-1}(D) = Z = \mathsf{c}\mu = \mathsf{c}f^{-1}(m).$$

Because Z is T_1, we have

$$f^{-1}{\downarrow}r = f^{-1}(r) = \{\mathfrak{p}\} = \mathsf{c}\{\mathfrak{p}\} = \mathsf{c}f^{-1}(r).$$

Since Y is closed in Z, we have that $\mathsf{c}_Y A = \mathsf{c}A$ for any $A \subseteq Y$, where $\mathsf{c}_Y A$ is closure in Y. Let $n \in \{0, 1\}$. Then $(\kappa \times \{n\}) \cup \{\mathfrak{p}\}$ is closed in Y. Therefore, $\mathfrak{p} \in \mathsf{c}_Y(\kappa \times \{n\})$. Thus, $\mathsf{c}(\kappa \times \{n\}) = \mathsf{c}_Y(\kappa \times \{n\}) = (\kappa \times \{n\}) \cup \{\mathfrak{p}\}$. This yields

$$f^{-1}{\downarrow}w_n = f^{-1}(\{w_n, r\}) = (\kappa \times \{n\}) \cup \{\mathfrak{p}\} = \mathsf{c}(\kappa \times \{n\}) = \mathsf{c}f^{-1}(w_n).$$

Consequently, f is interior. □

We are ready for the main lemma of this section. For this we recall that an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ is *rooted* if there is $w \in W$ (a *root* of $\mathfrak{F}$) such that $W = R(w)$.

Lemma 3.12. *Let $\mathfrak{F} = (W, R)$ be a finite rooted $\mathsf{S4}$-frame. If $\mathfrak{F}$ is an interior image of Z, then $\mathfrak{F}$ is an interior image of $\mathfrak{D}$.*

Proof. We start by observing some properties of $\mathfrak{F}$. Since Z is scattered, it is HI. Because Z is also of Cantor-Bendixson rank 3, it follows from Section 2.2 that the formulas grz and bd_3 are valid in Z. As $\mathfrak{F}$ is an interior image of Z, these formulas are also valid in $\mathfrak{F}$ (see, e.g., [4, Prop. 2.9(2)]). Therefore, R is a partial order and the R-depth of $\mathfrak{F}$ is ≤ 3 (see, e.g., [10, Props. 3.48 & 3.44]). In addition, since Z is ED, so is $\mathfrak{F}$. Thus, as $\mathfrak{F}$ is rooted, $\mathfrak{F}$ has a maximum (see, e.g., [10, Cor. 3.38]).

We consider three cases based on the depth of $\mathfrak{F}$. First, suppose that the depth of $\mathfrak{F}$ is 1. Then W is a singleton and it is clear that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Next suppose that the depth of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a rooted poset with a maximum, $\mathfrak{F}$ is isomorphic to the two element chain (see Figure 4). It is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$ and all the other points of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$ is an onto interior map.

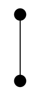

FIGURE 4. The two element chain.

Finally, suppose that the depth of $\mathfrak{F}$ is 3. Then $\mathfrak{F}$ is isomorphic to the frame depicted in Figure 5 where $W = \{0, v_0, \ldots, v_m, 1\}$ and $m \in \omega$.

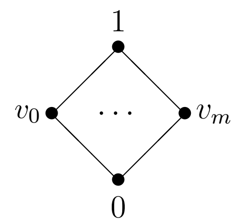

FIGURE 5. The poset $\mathfrak{F}$ of depth 3.

If $m = 0$, then it is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$, the maximum of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$, and w_0, w_1 to v_0 is an onto interior map. If $m = 1$, then $\mathfrak{D}$ is isomorphic to $\mathfrak{F}$, so it is obvious that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Thus, to complete the proof, it suffices to show that $m \not\geq 2$.

Suppose that $m \geq 2$ and let $f : Z \to W$ be an interior mapping onto $\mathfrak{F}$.

Claim 3.13.

(1) $\mu \subseteq f^{-1}(1)$.
(2) $\{\mathfrak{p}\} = f^{-1}(0)$.
(3) $f^{-1}(\{v_0, \ldots, v_m\}) \subseteq Y \setminus \{\mathfrak{p}\}$.
(4) $\mathfrak{p} \in \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{1\}))$ *for each* $i \in \{0, \ldots, m\}$.

Proof. (1) Since each $z \in \mu$ is isolated and f is interior, we have that $f(z)$ is the maximum of $\mathfrak{F}$. Thus, $f(z) = 1$.

(2) Because f is onto, there is $z \in f^{-1}(0)$. By (1), we have that $z \in Y$. If $z \neq \mathfrak{p}$, then z is an isolated point of Y, so there is an open subset U of Z such that $\{z\} = U \cap Y$. As f is interior and U is open, $f(U)$ is an R-upset of $\mathfrak{F}$. Therefore, $f(U) = W$ since $0 = f(z) \in f(U)$. On the other hand,

$$f(U) = f((U \cap Y) \cup (U \cap \mu)) \subseteq f(\{z\} \cup \mu) = f(\{z\}) \cup f(\mu) = \{0\} \cup \{1\} \neq W.$$

The obtained contradiction proves that $z = \mathfrak{p}$. Thus, $f^{-1}(0) = \{\mathfrak{p}\}$.

(3) Follows immediately from (1) and (2) since $\mu \cup \{\mathfrak{p}\} \subseteq f^{-1}(\{0, 1\})$.

(4) Let $i \in \{0, \ldots, m\}$. Because f is interior, it follows from (2) and (3) that

$$\begin{aligned}
\{\mathfrak{p}\} &\subseteq f^{-1}(\{0, v_i\}) = f^{-1}R^{-1}(v_i) \\
&= \mathsf{c}f^{-1}(v_i) = \mathsf{c}\left(f^{-1}(v_i) \cap (Y \setminus \{\mathfrak{p}\})\right) \\
&= \mathsf{c}\left(f^{-1}(v_i) \cap [(\kappa \times \{0\}) \cup (\kappa \times \{1\})]\right) \\
&= \mathsf{c}\left(\left[f^{-1}(v_i) \cap (\kappa \times \{0\})\right] \cup \left[f^{-1}(v_i) \cap (\kappa \times \{1\})\right]\right) \\
&= \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{0\})\right) \cup \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{1\})\right).
\end{aligned}$$

$\square$

Let

$$\mathscr{F}_0 = \left\{f^{-1}(1) \cap (\kappa \times \{0\}), f^{-1}(v_0) \cap (\kappa \times \{0\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{0\})\right\}$$

and

$$\mathscr{F}_1 = \left\{f^{-1}(1) \cap (\kappa \times \{1\}), f^{-1}(v_0) \cap (\kappa \times \{1\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{1\})\right\}.$$

Then both $\mathscr{F}_0$ and $\mathscr{F}_1$ are pairwise disjoint families of sets, $\bigcup \mathscr{F}_0 = \kappa \times \{0\}$, and $\bigcup \mathscr{F}_1 = \kappa \times \{1\}$. We prove that there is a unique $A_0 \in \mathscr{F}_0$ such that $\mathfrak{p} \in \mathsf{c}A_0$. A similar proof yields a unique $A_1 \in \mathscr{F}_1$ such that $\mathfrak{p} \in \mathsf{c}A_1$.

Because $\mathscr{F}_0$ is finite, we have

$$\mathfrak{p} \in \mathsf{c}(\kappa \times \{0\}) = \mathsf{c}\left(\bigcup \mathscr{F}_0\right) = \bigcup\nolimits_{A \in \mathscr{F}_0} \mathsf{c}A.$$

Therefore, there is $A_0 \in \mathscr{F}_0$ such that $\mathfrak{p} \in \mathsf{c}A_0$. Since $\mathfrak{p}$ is an ultrafilter,

$$\mathfrak{p} \notin \mathsf{c}((\kappa \times \{0\}) \setminus A_0) = \mathsf{c}\left(\bigcup (\mathscr{F}_0 \setminus \{A_0\})\right) = \bigcup\nolimits_{A \in \mathscr{F}_0 \setminus \{A_0\}} \mathsf{c}A.$$

Thus, A_0 is the unique member A of $\mathscr{F}_0$ satisfying the property that $\mathfrak{p} \in \mathsf{c}A$.

Since $m \geq 2$, by the Pigeonhole Principle, there is $i \in \{0, 1, 2, \ldots, m\}$ such that $A_0 \neq f^{-1}(v_i) \cap (\kappa \times \{0\})$ and $A_1 \neq f^{-1}(v_i) \cap (\kappa \times \{1\})$. Thus, $\mathfrak{p} \notin \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{0\})\right)$ and $\mathfrak{p} \notin \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{1\}))$, which contradicts Claim 3.13(4). Consequently, $m \ngeq 2$, completing the proof. $\square$

Lemma 3.14. *The logic of Z is L.*

Proof. By Lemma 3.11, $\mathfrak{D}$ is an interior image of Z. Therefore, $\mathsf{L}(Z) \subseteq \mathsf{L}(\mathfrak{D}) = \mathsf{L}$ (see, e.g., [4, Prop. 2.9(2)]). Conversely, suppose that $\mathsf{L}(Z) \nvdash \varphi$. Since Z is of Cantor-Bendixson rank 3, bd_3 is a theorem of $\mathsf{L}(Z)$. Therefore, by Segerberg's theorem (see, e.g., [10, Thm. 8.85]), $\mathsf{L}(Z)$ is complete with respect to finite rooted $\mathsf{L}(Z)$-frames. Thus, there is a finite rooted $\mathsf{L}(Z)$-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. As $\mathfrak{F}$ is an $\mathsf{L}(Z)$-frame, by [6, Lem 6.2], $\mathfrak{F}$ is an interior image of an open subspace U of Z. Let $f : U \to \mathfrak{F}$ be an interior map, and let $z \in U$ map to the root of $\mathfrak{F}$. Since Z is zero-dimensional, there is a clopen subset V of Z such that $z \in V$ and $V \subseteq U$. Then the restriction of f to V is an interior mapping of V onto $\mathfrak{F}$. Because $\mathfrak{F}$ has a maximum, we have that $\mathfrak{F}$ is an interior image of Z by [7, Lem. 5.4]. By Lemma 3.12, $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Therefore, $\mathfrak{D} \not\models \varphi$, and hence $\mathsf{L}(\mathfrak{D}) \nvdash \varphi$. Thus, $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D}) = \mathsf{L}$. $\square$

As a consequence of Lemmas 3.8 and 3.14 we arrive at the main result of this section.

Theorem 3.15. *If there exists a measurable cardinal, then there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$.*

4. Existence of a measurable cardinal is necessary

In this section we prove that the existence of a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$ implies the existence of a measurable cardinal. Let Z be a normal space such that $\mathsf{L}(Z) = \mathsf{L}$.

Lemma 4.1. *The space Z is an ED-space of modal Krull dimension 2 such that $\mathfrak{D}$ is an interior image of Z.*

Proof. As $\mathsf{L}(Z) = \mathsf{L}$, for each modal formula φ we have $Z \models \varphi$ iff $\mathfrak{D} \models \varphi$. Since $\mathfrak{D}$ has a maximum and is of depth 3, we have that

$$\begin{array}{rcl} \mathfrak{D} & \models & \mathsf{ga} \\ \mathfrak{D} & \models & \mathsf{bd}_3 \\ \mathfrak{D} & \not\models & \mathsf{bd}_2 \end{array}$$

Therefore, Z is an ED-space of modal Krull dimension 2 (see Section 2.2).

Because $\mathfrak{D} \models \mathsf{L}(Z)$, [6, Lem. 6.2] yields an open subspace U of Z and an onto interior map $g : U \to D$. Then there is $z \in U$ with $f(z) = r$. Since Z is normal and ED, it is zero-dimensional. Hence, there is clopen V in Z such that $z \in V \subseteq U$. Noting that the restriction of g to V is an interior mapping onto $\mathfrak{D}$, it follows from [7, Lem. 5.4] that $\mathfrak{D}$ is an interior image of Z. $\square$

Remark 4.2.

(1) Since $\mathfrak{D}$ is a finite poset, $\mathfrak{D}$ validates grz. Therefore, so does Z, and hence Z is HI.

(2) Observe that $\mathfrak{D}$ is not hereditarily ED since the subspace $\{r, w_0, w_1\}$ is not ED. Because $\mathfrak{D}$ is an interior image of Z, it follows that Z is not hereditarily ED.

(3) Since Z is a Hausdorff ED-space that is not hereditarily ED, Z must be uncountable (see, e.g., [9, Cor. 2.1]).

Definition 4.3. Let $f : Z \to \mathfrak{D}$ be an onto interior mapping. Denote the fibers of f by

$$\begin{array}{rcl} M & = & f^{-1}(m) \\ B_0 & = & f^{-1}(w_0) \\ B_1 & = & f^{-1}(w_1) \\ A & = & f^{-1}(r) \end{array}$$

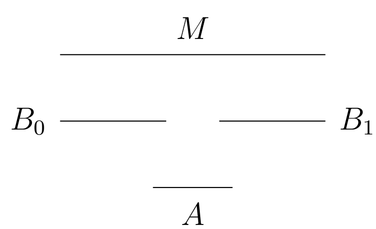

FIGURE 6. Depiction of Z partitioned by the fibers of f.

Remark 4.4.

(1) Clearly M is an open dense subset of Z (which is infinite as it is a dense subset of an infinite T_1-space).

(2) We also have that A is a closed nowhere dense subset of $Z \setminus M$. Therefore, A is discrete. More generally, any nonempty nowhere dense subset N of $Z \setminus M$ is discrete. To see this, since $\mathrm{mdim}(Z) = 2$, the definition of modal Krull dimension gives that $\mathrm{mdim}(Z \setminus M) \le 1$ and $\mathrm{mdim}(N) \le 0$. As $N \ne \varnothing$, we have that $\mathrm{mdim}(N) = 0$. Thus, N is discrete by [5, Rem. 4.8 & Thm. 4.9].

Lemma 4.5. *There is a normal subspace U of Z such that $U \cap A$ is a singleton and $\mathsf{L}(U) = \mathsf{L}$.*

Proof. Let $a \in A$. Since A is discrete and Z is zero-dimensional, there is a clopen subset U of Z such that $\{a\} = U \cap A$. As U is closed in Z, the subspace U is normal. Because U is open in Z, the restriction $f|_U$ of f to U is interior. Since $U \cap A \ne \varnothing$, we have that $r \in f(U)$. As $f(U)$ is an upset, $D = {\uparrow} r \subseteq f(U) \subseteq D$. Therefore, $f|_U$ is onto and $\mathfrak{D}$ is an interior image of U. By [4, Prop. 2.9], $\mathsf{L}(U) \subseteq \mathsf{L} = \mathsf{L}(Z) \subseteq \mathsf{L}(U)$, so $\mathsf{L}(U) = \mathsf{L}$, completing the proof. $\square$

By Lemma 4.5, we may assume without loss of generality that A is a singleton, say $\{a\}$, yielding that $Z = B_0 \cup \{a\} \cup B_1 \cup M$ (see Figure 7).

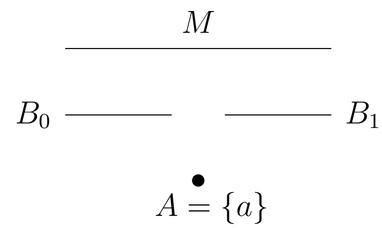

FIGURE 7. Reducing A to a singleton.

Lemma 4.6. *We have that $a \notin \mathsf{c}N$ for any nowhere dense subset N of the subspace $B_0 \cup B_1$.*

Proof. We first show that $N \cup A$ is nowhere dense in $Z \setminus M$. Let U be open in $Z \setminus M$ with $U \subseteq \mathsf{c}(N \cup A)$. Since A is closed, $U \subseteq \mathsf{c}(N) \cup A$. Therefore, $U \setminus A \subseteq \mathsf{c}(N) \setminus A = \mathsf{c}(N) \cap (B_0 \cup B_1)$, which is the closure of N relative to $B_0 \cup B_1$. Because $U \setminus A$ is open and N is nowhere dense in $B_0 \cup B_1$, we have that $U \setminus A = \varnothing$, so $U \subseteq A$. By Remark 4.4(2), A is a closed nowhere dense subset of $Z \setminus M$, hence $U = \varnothing$. Thus, $N \cup A$ is nowhere dense in $Z \setminus M$. Applying Remark 4.4(2) again yields that $N \cup A$ is discrete. Consequently, there is an open set V in Z such that $\{a\} = V \cap (N \cup A)$. As

$$V \cap N \subseteq V \cap (N \cup A) = \{a\} \subseteq Z \setminus (B_0 \cup B_1) \subseteq Z \setminus N,$$

it must be the case that $V \cap N = \varnothing$, so $a \notin \mathsf{c}N$. □

We recall that a normal space X is an *F-space* if any two disjoint open F_σ-sets in X have disjoint closures in X (see, e.g., [19, Lem. 1.2.2(b)]). Being a normal ED-space, it follows from [15, Exercise 14N.4] that Z is an F-space.

Definition 4.7. Let Y denote the subspace $B_0 \cup \{a\} \cup B_1$ of Z.

Because $Y = Z \setminus M$ is closed in Z, we have that Y is a normal F-space by [19, Lem. 1.2.2(d)]. We require the following definition.

Definition 4.8. (See, e.g., [20, p. 37]) A point x of a space X is called a *P-point* provided for any G_δ-set S in X we have that $x \in S$ implies $x \in \mathsf{i}S$.

Remark 4.9. By taking complements we obtain that $x \in X$ is a P-point iff for each F_σ-set S in X we have that $x \notin S$ implies $x \notin \mathsf{c}S$. This will be utilized in Lemma 4.16(5).

Lemma 4.10. *Either a is a P-point in the subspace $B_0 \cup \{a\}$ or a P-point in the subspace $B_1 \cup \{a\}$.*

Proof. Suppose not. Then we show that there are disjoint open F_σ-sets U_0 and U_1 of Y whose closures have nonempty intersection, which is a contradiction since Y is a normal F-space. We only show how to construct U_0 because U_1 is constructed similarly. Since a is not a P-point in $B_0 \cup \{a\}$, for each $n \in \omega$, there is W_n open in $B_0 \cup \{a\}$ such that $a \in \bigcap_{n\in\omega} W_n$ but $a \notin \mathsf{i}\left(\bigcap_{n\in\omega} W_n\right)$, where i is taken in $B_0 \cup \{a\}$. As Z is zero-dimensional, $B_0 \cup \{a\}$ is zero-dimensional. Thus, for each $n \in \omega$, there is V_n clopen in $B_n \cup \{a\}$ such that $a \in V_n \subseteq W_n$. Clearly, $a \in V := \bigcap_{n\in\omega} V_n$ and V is a closed G_δ-set in $B_0 \cup \{a\}$. Moreover, $a \notin \mathsf{i}V$ since $V \subseteq \bigcap_{n\in\omega} W_n$ and $a \notin \mathsf{i}\left(\bigcap_{n\in\omega} W_n\right)$. Put $U_0 = (B_0 \cup \{a\}) \setminus V$. Then U_0 is an open F_σ-set in $B_0 \cup \{a\}$ such that $a \notin U_0$ and $a \in \mathsf{c}U_0$. Clearly $U_0 \subseteq B_0$, and so U_0 is open in B_0. As $B_0 = Y \cap f^{-1}{\uparrow}w_0$ is open in Y, it follows that U_0 is open in Y. Because $B_0 \cup \{a\}$ is closed in Y and U_0 is an F_σ-set in $B_0 \cup \{a\}$, we have that U_0 is an F_σ-set in Y. Thus, U_0 is an open F_σ-set in Y such that $a \in \mathsf{c}U_0$. Analogously, there is an open F_σ-set U_1 in Y such that $a \in \mathsf{c}U_1$. By construction, $U_0 \subseteq B_0$ and $U_1 \subseteq B_1$, so U_0 and U_1 are disjoint. On the other hand, $a \in \mathsf{c}U_0 \cap \mathsf{c}U_1$, yielding the desired contradiction. □

Convention 4.11. Without loss of generality we assume that a is a P-point in $X := B_0 \cup \{a\}$.

Remark 4.12. Since X is closed in Z, the closure in X of any subset S of X coincides with the closure of S in Z. Therefore, there is no ambiguity in writing $\mathsf{c}S$ whenever $S \subseteq X$.

The following lemma is an easy consequence of Zorn's lemma, and we skip its proof.

Lemma 4.13. *There is a family $\mathscr{F}$ of subsets of X that is maximal with respect to the following two properties:*

(1) *Each $F \in \mathscr{F}$ is a nonempty clopen in X such that $a \notin F$;*
(2) *The family $\mathscr{F}$ is pairwise disjoint.*

Lemma 4.14. *Let $N = B_0 \setminus \bigcup \mathscr{F}$. Then we have:*

(1) *$\bigcup \mathscr{F}$ is open in both X and B_0.*
(2) *$\bigcup \mathscr{F}$ is dense in both B_0 and X.*
(3) *N is closed in Z.*
(4) *There is a clopen subspace U of Z such that $U \cap N = \varnothing$ and $\mathsf{L}(U) = \mathsf{L}$.*

Proof. (1) Since $\bigcup \mathscr{F}$ is a union of clopen subsets of X, it is open in X. Also, since $a \notin F$ for each $F \in \mathscr{F}$, we have that $\bigcup \mathscr{F} \subseteq B_0$, and hence it is also open in B_0.

(2) Let $z \in B_0$. If $z \notin \mathsf{c}\left(\bigcup \mathscr{F}\right)$, then as X is zero-dimensional, there is clopen V in X such that $z \in V$ and $V \cap \bigcup \mathscr{F} = \varnothing$. Since $z \neq a$, we may assume that $a \notin V$ (by shrinking V further if necessary). But this contradicts the maximality of $\mathscr{F}$ because the family $\{V\} \cup \mathscr{F}$ satisfies the conditions of Lemma 4.13. Thus, $z \in \mathsf{c}\left(\bigcup \mathscr{F}\right)$, and so $\bigcup \mathscr{F}$ is dense in B_0. Finally, since $a \in \mathsf{c}B_0$, we conclude that $\bigcup \mathscr{F}$ is dense in X.

(3) It suffices to show that N is closed in X. For any $z \in B_0 \setminus N$, we have that $\bigcup \mathscr{F}$ is open in X and $z \in \bigcup \mathscr{F}$. Since $N \cap \bigcup \mathscr{F} = \varnothing$, it follows that $z \notin \mathsf{c}N$. Because $\{B_0 \setminus N, N, \{a\}\}$ is a partition of X, it remains to show that $a \notin \mathsf{c}N$. But (1) and (2) imply that N is nowhere dense in B_0, hence nowhere dense in $B_0 \cup B_1$. This yields that $a \notin \mathsf{c}N$ by Lemma 4.6.

(4) Since $\{a\}$ and N are closed in the zero-dimensional normal space Z, there is U clopen in Z such that $a \in U$ and $U \cap N = \varnothing$. Because U is open, the restriction of f as defined in Definition 4.3 is an interior map from U to $\mathfrak{D}$. To see that it is onto, observe that $U \cap M \neq \varnothing$ since M is dense in Z, and both $U \cap B_0$ and $U \cap B_1$ are nonempty because $a \in \mathsf{c}B_0, \mathsf{c}B_1$ and $a \in U$. Therefore, $\mathfrak{D}$ is an interior image of Z, and so $\mathsf{L}(U) \subseteq \mathsf{L} = \mathsf{L}(Z) \subseteq L(U)$ by [4, Prop. 2.9]. Thus, $\mathsf{L}(U) = \mathsf{L}$. ☐

Let U be the clopen subspace of Z constructed in the proof of Lemma 4.14(4). Then U is normal since it is a closed subspace of a normal space. In addition, a remains a P-point of $X \cap U$ because $X \cap U$ is an open subspace of X and a is a P-point of X. Therefore, without loss of generality we may assume that $Z = U$. Thus, $B_0 = \bigcup \mathscr{F}$ and $N = \varnothing$.

Definition 4.15.

(1) Let κ be the cardinality of $\mathscr{F}$, and let $\varphi : \kappa \to \mathscr{F}$ be a bijection. Denoting $\varphi(\alpha)$ by F_α, we may write $\mathscr{F} = \{F_\alpha \mid \alpha \in \kappa\}$.
(2) Let
$$\mathscr{G} = \left\{ \Gamma \subseteq \kappa \mid a \in \mathsf{c}\left(\bigcup_{\alpha \in \Gamma} F_\alpha \right) \right\}.$$

We are ready to prove the main lemma of this section.

Lemma 4.16.

(1) *If $\Gamma \in \mathscr{G}$ and $\Gamma \subseteq \Lambda$, then $\Lambda \in \mathscr{G}$.*
(2) *For any $\Gamma \subseteq \kappa$, exactly one of $\Gamma, \kappa \setminus \Gamma$ belongs to $\mathscr{G}$.*

(3) *If* $\Gamma, \Lambda \in \mathscr{G}$, *then* $\Gamma \cap \Lambda \in \mathscr{G}$.
(4) $\mathscr{G}$ *is a free ultrafilter on* κ.
(5) $\mathscr{G}$ *is countably complete.*

Proof. (1) Let $\Gamma \in \mathscr{G}$ and $\Gamma \subseteq \Lambda$. Then $\bigcup_{\alpha\in\Gamma} F_\alpha \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, yielding

$$a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) \subseteq \mathsf{c}\left(\bigcup_{\alpha\in\Lambda} F_\alpha\right).$$

Thus, $\Lambda \in \mathscr{G}$.

(2) Let $\Gamma \subseteq \kappa$. We have that

$$a \in \mathsf{c}B_0 = \mathsf{c}\left(\bigcup_{\alpha\in\kappa} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha \cup \bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) \cup \mathsf{c}\left(\bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha\right).$$

Therefore, $\Gamma \in \mathscr{G}$ or $\kappa \setminus \Gamma \in \mathscr{G}$. Suppose that both Γ and $\kappa \setminus \Gamma$ belong to $\mathscr{G}$. Then the frame $\mathfrak{F}$ depicted in Figure 5 with $m = 2$ is an interior image of Z via the mapping $g : Z \to W$ given by

$$g(z) = \begin{cases} 1 & \text{if } z \in M \\ v_0 & \text{if } z \in \bigcup_{\alpha\in\Gamma} F_\alpha \\ v_1 & \text{if } z \in \bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha \\ v_2 & \text{if } z \in B_1 \\ 0 & \text{if } z = a \end{cases}$$

The function g is depicted in Figure 8 where each fiber of g is labeled to the right by its image in W.

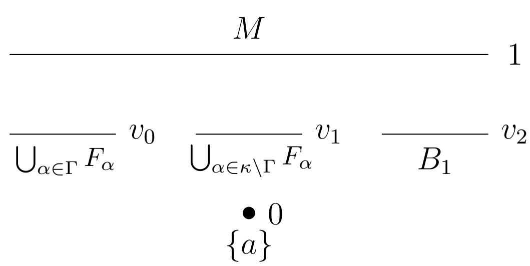

FIGURE 8. The function $g : Z \to W$.

This yields that $\mathfrak{F} \models \mathsf{L}(Z) = \mathsf{L}$, which is a contradiction since $\mathfrak{F} \not\models \mathsf{L}$. Thus, exactly one of Γ or $\kappa \setminus \Gamma$ is a member of $\mathscr{G}$.

(3) If $\Gamma \cap \Lambda \notin \mathscr{G}$, then $a \notin \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha\right)$. On the other hand,

$$a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha \cup \bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha\right) \cup \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right).$$

Therefore, $a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right)$. Thus, $\Gamma \setminus \Lambda \in \mathscr{G}$. Since $\Gamma \setminus \Lambda \subseteq \kappa \setminus \Lambda$, (1) implies that $\kappa \setminus \Lambda \in \mathscr{G}$. However, as $\Lambda \in \mathscr{G}$, (2) implies that $\kappa \setminus \Lambda \notin \mathscr{G}$. The obtained contradiction proves that $\Gamma \cap \Lambda \in \mathscr{G}$.

(4) That $\mathscr{G}$ is an ultrafilter follows from (1), (2), and (3). To see that $\mathfrak{G}$ is free, let $\alpha \in \kappa$. Then F_α is clopen in X and $a \notin F_\alpha$. Therefore, $a \notin \mathsf{c}F_\alpha$, yielding that $\{\alpha\} \notin \mathscr{G}$. Thus, $\mathscr{G}$ is a free ultrafilter.

(5) Let $\Lambda_n \in \mathscr{G}$ for each $n \in \omega$ and let $\Gamma := \bigcap_{n\in\omega} \Lambda_n \notin \mathscr{G}$. For $n \in \omega$ set $\Gamma_n = \bigcap_{i=0}^n \Lambda_i$. Then $\Gamma_n \in \mathscr{G}$ by (3), $\Gamma_{n+1} \subseteq \Gamma_n$, and $\Gamma = \bigcap_{n\in\omega} \Gamma_n$. For $n \in \omega$ set $\Delta_n = \Gamma_n \setminus \Gamma_{n+1}$. Since $\mathscr{G}$ is an ultrafilter, $\Delta_n \notin \mathscr{G}$ for each $n \in \omega$.

Claim 4.17. *The set $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is clopen in X.*

Proof. Clearly $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is open in X since each $F \in \mathscr{F}$ is clopen in X. To see that $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is closed in X we show that $\mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right) = \bigcup_{\alpha\in\Delta_n} F_\alpha$. As X is closed in Z, we have that $\mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right) \subseteq X$. Let $z \in X \setminus \bigcup_{\alpha\in\Delta_n} F_\alpha$. We show that $z \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. Either $z = a$ or $z \in B_0$. The former case is clear since $\Delta_n \notin \mathscr{G}$ implies that $z = a \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. Suppose $z \in B_0$. Then there is $\beta \in \kappa$ such that $z \in F_\beta$. Since $z \notin \bigcup_{\alpha\in\Delta_n} F_\alpha$, it follows that $\beta \notin \Delta_n$. Because F_β is clopen in X, there is U open in Z such that $F_\beta = U \cap X$. Clearly $z \in U$. As $\mathscr{F}$ is pairwise disjoint, we have that

$$U \cap \bigcup_{\alpha\in\Delta_n} F_\alpha = U \cap \bigcup_{\alpha\in\Delta_n} (X \cap F_\alpha) = \bigcup_{\alpha\in\Delta_n} (U \cap X \cap F_\alpha) = \bigcup_{\alpha\in\Delta_n} (F_\beta \cap F_\alpha) = \varnothing.$$

Therefore, $z \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. ☐

As $\Gamma_0 \setminus \Gamma = \bigcup_{n\in\omega} \Delta_n$, it follows from Claim 4.17 that

$$\bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha = \bigcup_{n\in\omega}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$$

is an open F_σ-set in X. Moreover, $a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha\right)$ because $\Gamma_0 \setminus \Gamma \in \mathscr{G}$. But $a \notin \bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha$ since $a \notin F_\alpha$ for each $\alpha \in \kappa$. This implies that a is not a P-point of X (see Remark 4.9). The obtained contradiction proves that $\mathscr{G}$ is countably complete. ☐

As a consequence of Lemma 4.16 and Section 2.3, we obtain:

Lemma 4.18. *The cardinal κ is Ulam-measurable, and hence there exists a measurable cardinal.*

Consequently, we have proved the following result.

Theorem 4.19. *If there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$, then there exists a measurable cardinal.*

Putting Theorems 3.15 and 4.19 together yields the main result of the paper:

Theorem 4.20. *There exists a measurable cardinal iff there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$.*

We conclude the paper by the following open problem:

Problem 4.21. In Theorem 4.20 can 'normal' be replaced by 'Tychonoff'?

Clearly the interesting implication is to prove that the existence of a Tychonoff space whose logic is L implies the existence of a measurable cardinal.

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