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CHARACTERIZING EXISTENCE OF A MEASURABLE CARDINAL VIA MODAL LOGIC

G. BEZHANISHVILI, N. BEZHANISHVILI, J. LUCERO-BRYAN, J. VAN MILL

Abstract. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic coincides with the modal logic of the Kripke frame isomorphic to the powerset of a two element set.

1. Introduction

Over the years there have been discovered several intriguing connections between set theory and modal logic. To name a few:

(1) There is an interesting connection between non-well-founded set theory and infinitary modal logic [1, 3, 2].
(2) The modal logic $S_4$.2 turns out to be the logic of forcing extensions of $\text{ZFC}$ [16].
(3) The only existing proof that the modal logic $S_4.1.2$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space $\omega$ requires that each MAD family has cardinality $2^\omega$, a principle that is not provable in $\text{ZFC}$, and it remains an open problem whether this principle is necessary [8].

To these results we add the following. Let the diamond $\mathfrak{D} = (D, \leq)$ be the partially ordered Kripke frame shown in Figure 1. It is clear that $\mathfrak{D}$ is isomorphic to the powerset of a two element set. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic is the modal logic of $\mathfrak{D}$.

![Figure 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.](image)

We recall that topological semantics generalizes Kripke semantics for the well-known modal logic $S_4$. Thus, Kripke completeness implies topological completeness for logics above $S_4$. However, topological spaces arising from Kripke frames are usually not even $T_1$. Therefore, it is nontrivial to prove topological completeness results above $S_4$ with respect to spaces satisfying higher separation axioms. One such class is the class of Tychonoff spaces. By a celebrated theorem of Tychonoff, these are exactly subspaces of compact Hausdorff spaces. In [5] we initiated the study of modal logics arising from Tychonoff spaces. On the one hand, this yielded a new notion of dimension in topology, called modal Krull dimension. On the

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other hand, it provided a new concept of zemanian logics which generalize the well-known modal logic of Zeman.

It is known that extremally disconnected spaces are topological models of the modal logic $S4.2$, and hereditarily extremally disconnected spaces are topological models of the modal logic $S4.3$. In [6] we showed that a modal logic above $S4.3$ is a zemanian logic iff it is the logic of an hereditarily extremally disconnected Tychonoff space. The simplest modal logic above $S4.2$ that is not above $S4.3$ is the logic of $D$. In this paper we show that topological completeness of the logic of $D$ with respect to a normal space is equivalent to the existence of a measurable cardinal. Whether normal can be weakened to Tychonoff remains an open problem.

We conclude the introduction by briefly describing the key ingredients of the proof. If there exists a measurable cardinal $\kappa$, using a countably complete ultrafilter on $\kappa$, we first build a normal $P$-space $Y$. Combining the results of [12] and [13] then allows us to embed $Y$ into the remainder of the Čech-Stone compactification $\beta \mu$ of a cardinal $\mu$ viewed as a discrete space. Letting $Z = Y \cup \mu$ yields a normal space whose logic we prove is the logic of the diamond $D$. This we do by showing that a finite rooted Kripke frame $\mathfrak{F}$ is an interior image of $Z$ iff $\mathfrak{F}$ is an interior image of $D$.

Conversely, suppose there exists a normal space $Z$ whose logic is the logic of the diamond $D$. We first show that $D$ is an interior image of $Z$. We then prove that without loss of generality the inverse image of the root $r$ of $D$ is a singleton $\{a\}$. We next prove that $a$ is a $P$-point of an appropriately chosen subspace of $Z$. This allows us to define a family of subsets of $Z$ whose cardinal is Ulam-measurable. Finally, it is well known that this implies the existence of a measurable cardinal.

2. Preliminaries

In this section we recall the necessary background from modal logic, its topological semantics, and measurable cardinals.

2.1. Modal logic. We use [10] as the main reference for modal logic. Modal formulas are built in the usual way using countably many propositional letters, the classical connectives $\neg$ (negation) and $\to$ (implication), the modal connective $\Box$ (necessity), and parentheses. We employ the standard abbreviations: $\land$ (conjunction), $\lor$ (disjunction), and $\Diamond$ (possibility).

The well-known modal system $S4$ of Lewis is the least set of formulas containing the classical tautologies, the axioms

\begin{align*}
\Box(p \to q) & \to (\Box p \to \Box q), \\
\Box p & \to p, \\
\Box p & \to \Box \Box p,
\end{align*}

and closed under the inference rules of

\begin{align*}
\text{Modus Ponens} & \quad \varphi, \varphi \to \psi \vdash \psi, \\
\text{substitution} & \quad \varphi(p_1, \ldots, p_n), \\
\text{necessitation} & \quad \Box \varphi.
\end{align*}

A Kripke frame is a pair $\mathfrak{F} = (W, R)$ where $W$ is a nonempty set and $R$ is a binary relation on $W$. As usual, for $w \in W$ we let

\[ R(w) = \{ v \in W \mid wRv \} \quad \text{and} \quad R^{-1}(w) = \{ v \in W \mid vRw \}; \]

and for $A \subseteq W$ we let

\[ R(A) = \bigcup \{ R(w) \mid w \in A \} \quad \text{and} \quad R^{-1}(A) = \bigcup \{ R^{-1}(w) \mid w \in A \}. \]
Kripke semantics of modal logic recursively assigns to each formula a subset of a Kripke frame \( \mathfrak{F} \) by interpreting each propositional letter as a subset of \( W \), the classical connectives as Boolean operations in the powerset \( \wp(W) \), and \( \square \) as the operation \( \square_R \) on \( \wp(W) \) defined by
\[
\square_R(A) = \{ w \in W \mid R(w) \subseteq A \}.
\]
Consequently, \( \Diamond \) is interpreted as the operation \( \Diamond_R \) on \( \wp(W) \) defined by
\[
\Diamond_R(A) = R^{-1}(A).
\]
Let \( \varphi \) be a modal formula and \( \mathfrak{F} = (W, R) \) a Kripke frame. Call \( \varphi \) valid in \( \mathfrak{F} \), written \( \mathfrak{F} \models \varphi \), provided \( \varphi \) evaluates to \( W \) for every assignment of the propositional letters. If \( \varphi \) is not valid in \( \mathfrak{F} \), then we say that \( \varphi \) is refuted in \( \mathfrak{F} \), and write \( \mathfrak{F} \nvdash \varphi \). The logic of \( \mathfrak{F} \) is the set of modal formulas valid in \( \mathfrak{F} \); in symbols \( L(\mathfrak{F}) = \{ \varphi \mid \mathfrak{F} \models \varphi \} \).

A Kripke frame \( \mathfrak{F} \) is called an S4-frame if \( R \) is reflexive and transitive. The name is justified by the well-known fact that S4 is sound and complete with respect to S4-frames. In this paper we are mainly interested in the following logic.

**Definition 2.1.** Let \( L := L(\mathfrak{D}) \) be the logic of the diamond \( \mathfrak{D} \) shown in Figure 1.

2.2. **Topological semantics.** Topological semantics interprets \( \square \) as topological interior (and consequently \( \Diamond \) as topological closure). Specifically, for a topological space \( X \), the propositional letters are assigned to subsets of \( X \), the classical connectives are computed as the Boolean operations in \( \wp(X) \), and \( \square \) is interpreted as the interior operator \( i : \wp(X) \to \wp(X) \), where \( iA \) is the greatest open subset of \( X \) contained in \( A \). Consequently, \( \Diamond \) is interpreted as the closure operator \( c : \wp(X) \to \wp(X) \), where \( cA \) is the least closed subset of \( X \) containing \( A \).

Let \( \varphi \) be a modal formula and \( X \) a space. Call \( \varphi \) valid in \( X \), denoted \( X \models \varphi \), provided \( \varphi \) evaluates to \( X \) for every assignment of the propositional letters. If \( \varphi \) is not valid in \( X \), then we say that \( \varphi \) is refuted in \( X \), and write \( X \nvdash \varphi \). The logic of \( X \) is the set of formulas valid in \( X \); symbolically, \( L(X) = \{ \varphi \mid X \models \varphi \} \). It is well known that S4 is sound and complete with respect to topological spaces.

There is a close connection between topological semantics and Kripke semantics for S4. Let \( \mathfrak{F} = (W, R) \) be an S4-frame. Call \( U \subseteq W \) an R-upset of \( \mathfrak{F} \) if \( w \in U \) and \( wRv \) imply \( v \in U \). The set of R-upsets of \( \mathfrak{F} \) is a topology \( \tau_R \) on \( W \) in which every point \( w \) has a least neighborhood, namely \( R(w) \). Such spaces are called Alexandroff spaces. We call \( (W, \tau_R) \) the Alexandroff space of \( \mathfrak{F} \). For a modal formula \( \varphi \), we have
\[
\mathfrak{F} \models \varphi \text{ iff } (W, \tau_R) \models \varphi.
\]
Thus, topological semantics generalizes Kripke semantics for S4, and hence Kripke completeness for logics above S4 implies topological completeness. However, since Alexandroff spaces are usually not even T1-spaces, such topological completeness is not guaranteed with respect to, for example, normal spaces.

We recall that a topological space \( X \) is
- **extremally disconnected** (ED) if the closure of each open set is open;
- **resolvable** if \( X \) is the union of two disjoint dense subsets of \( X \);
- **irresolvable** if \( X \) is not resolvable;
- **hereditarily irresolvable** (HI) if every subspace of \( X \) is irresolvable.

Let
\[
\text{grz} = \square(\square(p \to \square p) \to p)
\]
be the Grzegorczyk axiom and
\[
\text{ga} = \Diamond \square p \to \square \Diamond p
\]
the Geach axiom (see, e.g., [10]). It is well known that
\[ X \text{ is } \text{ED} \iff X \models \text{grz}; \]
\[ X \text{ is } \text{HI} \iff X \models \text{ga}. \]

We next recall the definition of modal Krull dimension. For this we recall that a subset \( N \) of a space \( X \) is nowhere dense if \( \overline{iN} = \emptyset \).

**Definition 2.2.** ([5, Sec. 3]) Define the modal Krull dimension \( \text{mdim}(X) \) of a topological space \( X \) recursively as follows:

\[
\begin{align*}
\text{mdim}(X) &= -1 \text{ if } X = \emptyset, \\
\text{mdim}(X) &\leq n \text{ if } \text{mdim}(N) \leq n - 1 \text{ for each } N \text{ nowhere dense in } X, \\
\text{mdim}(X) &= n \text{ if } \text{mdim}(X) \leq n \text{ but } \text{mdim}(X) \not\leq n - 1, \\
\text{mdim}(X) &= \infty \text{ if } \text{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \ldots.
\end{align*}
\]

Let
\[
\begin{align*}
\text{bd}_1 &= \Diamond \Box p_1 \to p_1 \\
\text{bd}_{n+1} &= \Diamond (\Box p_{n+1} \land \neg \text{bd}_n) \to p_{n+1} \text{ for } n \geq 1.
\end{align*}
\]

**Theorem 2.3.** ([5, Thm. 3.6]) Let \( X \) be a nonempty space and \( n \geq 1 \). Then
\[
\text{mdim}(X) \leq n - 1 \iff X \models \text{bd}_n.
\]

For nonempty scattered Hausdorff spaces, there is a close connection between finite modal Krull dimension and Cantor-Bendixson rank. For \( Y \subseteq X \), let \( d^0Y \) be the set of limit points of \( Y \) and for an ordinal \( \alpha \), let \( d^{\alpha}Y \) be defined recursively as follows:

\[
\begin{align*}
\text{d}^0 Y &= Y, \\
\text{d}^{\alpha+1} Y &= \text{d}(\text{d}^{\alpha} Y), \\
\text{d}^\alpha Y &= \bigcap \{ \text{d}^\beta Y \mid \beta < \alpha \} \text{ if } \alpha \text{ is a limit ordinal.}
\end{align*}
\]

The Cantor-Bendixson rank of \( X \) is the least ordinal \( \gamma \) satisfying \( \text{d}^\gamma X = \text{d}^{\gamma+1} X \). It is well known that a space \( X \) is scattered iff there is an ordinal \( \alpha \) such that \( \text{d}^\alpha X = \emptyset \). Thus, the Cantor-Bendixson rank of a scattered space \( X \) is the least ordinal \( \gamma \) such that \( \text{d}^\gamma X = \emptyset \).

Let \( X \) be a nonempty scattered Hausdorff space and \( n \in \omega \). Then the Cantor-Bendixson rank of \( X \) is \( n + 1 \) iff \( \text{d}^n X \neq \emptyset \) and \( \text{d}^{n+1} X = \emptyset \), which by [7, Thm. 4.9] happens iff \( \text{mdim}(X) = n \).

### 2.3. Measurable cardinals.

We use [17, 18] as standard references for set theory, and also rely on [11] as the main reference for measurable cardinals. Let \( S \) be a set and \( p \) a free ultrafilter on \( S \). We denote infinite cardinals by \( \kappa \), the first uncountable cardinal by \( \omega_1 \), and recall that \( p \) is

- **\( \kappa \)-complete** if \( \bigcap K \in p \) for any family \( K \subseteq p \) of cardinality \( < \kappa \);
- **countably complete** if \( p \) is \( \omega_1 \)-complete (that is, \( p \) is closed under countable intersections).

**Definition 2.4.** ([11, Ch. 8]) An uncountable cardinal \( \kappa \) is

- **measurable** if there exists a \( \kappa \)-complete free ultrafilter on \( \kappa \);
- **Ulam-measurable** if there exists a countably complete free ultrafilter on \( \kappa \).

**Remark 2.5.** While in [11] it is not assumed that measurable cardinals are uncountable, it is common to make such an assumption.

It is clear that every measurable cardinal is Ulam-measurable, and it is well known (see, e.g., [11, Thm. 8.31]) that the existence of an Ulam-measurable cardinal implies the existence of a measurable cardinal.
3. Existence of a measurable cardinal is sufficient

In this section we prove that the existence of a measurable cardinal implies that there is a normal space $Z$ such that $L(Z) = L$. We build $Z$ in stages. Let $\kappa$ be a measurable cardinal. Then $\kappa$ is Ulam-measurable, and so there is a countably complete free ultrafilter $\mathcal{p}$ on $\kappa$. Let $Y = (\kappa \times \{0,1\}) \cup \{\mathcal{p}\}$. Consider the following family of subsets of $Y$:

$$\tau = \{U \subseteq Y \mid U \subseteq Y \setminus \{\mathcal{p}\} \text{ or } \exists V, W \in \mathcal{p}: U = (V \times \{0\}) \cup \{\mathcal{p}\} \cup (W \times \{1\})\}.$$ 

\[\begin{array}{c}
\kappa \times \{0\} \\
\mathcal{p} \\
\kappa \times \{1\}
\end{array}\]

Figure 2. The space $Y$ and an open neighborhood of $\mathcal{p}$.

Lemma 3.1. The family $\tau$ is a topology on $Y$ that is closed under countable intersections.

Proof. Clearly $\emptyset, Y \in \tau$. Let $\{U_i \mid i \in I\} \subseteq \tau$ and let $U = \bigcup \{U_i \mid i \in I\}$. If $\mathcal{p} \not\subseteq U$, then $U \in \tau$. Suppose $\mathcal{p} \in U$. Then $\mathcal{p} \in U_i$ for some $i \in I$. Since $U_i \in \tau$ and $\mathcal{p} \in U_i$, there are $V_0, V_i \in \mathcal{p}$ such that $U_i = (V_0 \times \{0\}) \cup \{\mathcal{p}\} \cup (V_i \times \{1\})$. For $n \in \{0,1\}$, set $W_n = \{\alpha \in \kappa \mid (\alpha, n) \in U\}$. Let $n \in \{0,1\}$ and $\alpha \in V_n$. Then $(\alpha, n) \in V_n \times \{n\} \subseteq U_i \subseteq U$, giving that $\alpha \in W_n$. Therefore, $V_n \subseteq W_n$. Since $V_n \in \mathcal{p}$ and $\mathcal{p}$ is an ultrafilter, $W_n \in \mathcal{p}$. It follows from the definition of $W_n$ that $W_n \times \{n\} = U \cap (\kappa \times \{n\})$. Thus,

$$U = U \cap Y = U \cap ((\kappa \times \{0\}) \cup \{\mathcal{p}\} \cup (\kappa \times \{1\}))$$

$$= (U \cap (\kappa \times \{0\})) \cup (U \cap \{\mathcal{p}\}) \cup (U \cap (\kappa \times \{1\}))$$

$$= (W_0 \times \{0\}) \cup \{\mathcal{p}\} \cup (W_1 \times \{1\}) \in \tau.$$ 

Consequently, $\tau$ is closed under union.

Let $\{U_i \mid i \in \omega\} \subseteq \tau$ and let $U = \bigcap \{U_i \mid i \in \omega\}$. If $\mathcal{p} \not\subseteq U$, then $U \in \tau$. Suppose $\mathcal{p} \in U$. Let $i \in \omega$. Since $\mathcal{p} \in U_i$ and $U_i \in \tau$, there are $V_i, W_i \in \mathcal{p}$ such that $U_i = (V_i \times \{0\}) \cup \{\mathcal{p}\} \cup (W_i \times \{1\})$. Put $V = \bigcap \{V_i \mid i \in \omega\}$ and $W = \bigcap \{W_i \mid i \in \omega\}$. As $\mathcal{p}$ is countably complete, we have that $V, W \in \mathcal{p}$.

Claim 3.2. $U = (V \times \{0\}) \cup \{\mathcal{p}\} \cup (W \times \{1\})$.

Proof. Let $\alpha \in \kappa$. We have

$$(\alpha, 0) \in U \quad \text{iff} \quad (\alpha, 0) \in U_i \text{ for all } i \in \omega$$

$$\text{iff} \quad \alpha \in V_i \text{ for all } i \in \omega$$

$$\text{iff} \quad \alpha \in V$$

$$\text{iff} \quad (\alpha, 0) \in V \times \{0\}$$

$$\text{iff} \quad (\alpha, 0) \in (V \times \{0\}) \cup \{\mathcal{p}\} \cup (W \times \{1\}) \text{.}$$

Similarly, $(\alpha, 1) \in U$ iff $(\alpha, 1) \in (V \times \{0\}) \cup \{\mathcal{p}\} \cup (W \times \{1\})$. The claim follows. \hfill \Box

We conclude that $\tau$ is a topology on $Y$ that is closed under countable intersections. \hfill \Box

Remark 3.3. That $\kappa$ is a measurable cardinal is used to see that $\tau$ is closed under countable intersections. In fact, this is the only place where we use that $\kappa$ is a measurable cardinal.

Definition 3.4. (See, e.g., [20, p. 37]) A Tychonoff space is a $P$-space if every $G_\delta$-set in $X$ is open.
Lemma 3.5. The space $Y$ is a normal $P$-space.

Proof. It is easy to see that each singleton in $Y$ is closed, so $Y$ is a $T_1$-space. Let $A, B$ be disjoint closed subsets of $Y$. Either $p \notin A$ or $p \notin B$, and we may assume without loss of generality that $p \notin A$. Then $A \subseteq Y \setminus \{p\}$, hence $A$ is open. Therefore, $U := A$ and $V := Y \setminus A$ are disjoint open subsets of $Y$ separating $A$ and $B$. Thus, $Y$ is normal, and hence it follows from Lemma 3.1 that $Y$ is a $P$-space. □

Since $Y$ is a $P$-space, it follows from [12, Sec. 2] that the Čech-Stone compactification $\beta Y$ of $Y$ can be embedded into a compact Hausdorff ED-space, say $E$. By Efimov’s Theorem [13, Sec. 1], there is a cardinal $\mu$, equipped with the discrete topology, such that the space $E$ can be embedded into $\beta \mu$. It is well known (see, e.g., [14, Exercise 6.2.B.b]) that $\beta \mu$ can be embedded in the remainder $\beta \mu \setminus \mu$. Combining these results yields a sequence of embeddings

$$(1) \quad Y \hookrightarrow \beta Y \hookrightarrow \varepsilon \hookrightarrow \beta \mu \hookrightarrow \beta \mu \setminus \mu$$

that gives an embedding of $Y$ into $\beta \mu \setminus \mu$. We identify $Y$ with its image in $\beta \mu$; see Figure 3.

\[\begin{array}{c}
\mu \\
\hline
\hline
\bullet \\
Y \\
\hline
\beta \mu \setminus \mu \\
\end{array}\]

Figure 3. $Y$ as a subspace of $\beta \mu$.

Definition 3.6. Let $Z$ be the subspace $\mu \cup Y$ of $\beta \mu$.

Our goal is to show that $Z$ is a normal space such that $L(Z) = L$.

Lemma 3.7. The space $Z$ is a scattered ED-space of Cantor-Bendixson rank 3.

Proof. Since $Z \supseteq \mu$ and $\mu$ is dense in $\beta \mu$, we have that $Z$ is dense in $\beta \mu$. As $\beta \mu$ is an ED-space (see, e.g., [14, Cor. 6.2.28]) and a dense subspace of an ED-space is an ED-space (see, e.g., [14, Exercise 6.2.G.c]), it follows that $Z$ is an ED-space.

We have $d^1Z = d^2Y = d\{p\} = \emptyset$ and $d^2Z = dY = \{p\} \neq \emptyset$. Therefore, $Z$ is scattered and of Cantor-Bendixson rank 3. □

Lemma 3.8. The space $Z$ is normal.

Proof. Clearly $Z$ is $T_1$ since it is a subspace of a $T_1$-space. Let $A$ and $B$ be disjoint closed subsets of $Z$. Since $\mu$ is the set of isolated points of $Z$, we have that $A \cap \mu$ and $B \cap \mu$ are disjoint open subsets of $Z$. Let $A_0 = c(A \cap \mu)$ and $B_0 = c(B \cap \mu)$. Because $Z$ is ED, $A_0$ and $B_0$ are disjoint closed subsets of $Z$. Let $A_1 = A \setminus A_0$ and $B_1 = B \setminus B_0$. Then $A_1$ and $B_1$ are disjoint closed subsets of $Y$. Since $Y$ is normal, it follows from [14, Cor. 3.6.4] that $c_{\beta Y}(A_1)$ and $c_{\beta Y}(B_1)$ are disjoint, where $c_{\beta Y}$ is the closure in $\beta Y$. Because $\beta Y$ is (up to homeomorphism) a closed subspace of $\beta \mu$, we have

\[c_{\beta \mu}(A_1) \cap c_{\beta \mu}(B_1) = c_{\beta Y}(A_1) \cap c_{\beta Y}(B_1) = \emptyset.\]

Since $\beta \mu$ is normal, there are disjoint open subsets $U_1$ and $V_1$ of $\beta \mu$ such that $c_{\beta \mu}(A_1) \subseteq U_1$ and $c_{\beta \mu}(B_1) \subseteq V_1$.

Clearly $U := U_1 \cap Z$ and $V := V_1 \cap Z$ are disjoint open subsets of $Z$. As both $A_0$ and $B_0$ are clopen in $Z$, it follows that both $U \setminus B_0$ and $V \setminus A_0$ are open in $Z$, and hence $U_0 := A_0 \cup (U \setminus B_0)$ and $V_0 := B_0 \cup (V \setminus A_0)$ are disjoint open subsets of $Z$. It is clear that $A_1 \subseteq U_1 \cap Z = U$.  

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Because $A_1$ and $B_0$ are disjoint, $A_1 \subseteq U \setminus B_0$, so $A = A_0 \cup A_1 \subseteq A_0 \cup (U \setminus B_0) = U_0$. Similarly, $B \subseteq V_0$. Thus, $Z$ is normal.

We recall that a map $f : X \to X'$ between spaces is interior if $f$ is both continuous and open. If in addition $f$ is onto, then we call $X'$ an interior image of $X$. If $X'$ is the Alexandroff space of an $S4$-frame $\mathcal{F}$, then we say that $\mathcal{F}$ is an interior image of $X$. Finally, if $X$ is the Alexandroff space of an $S4$-frame $\mathcal{G}$, then we say that $\mathcal{F}$ is an interior image of $\mathcal{G}$.

**Remark 3.9.** It is well known that $\mathcal{F} = (W,R)$ is an interior image of $\mathcal{G} = (V,S)$ iff $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$, where we recall that a $p$-morphism is a map $f : V \to W$ such that $f^{-1}R^{-1}(w) = S^{-1}f^{-1}(w)$ for each $w \in W$.

**Convention 3.10.** Since the diamond $\mathfrak{D} = (D,\leq)$ is a poset (partially ordered set), for $w \in D$ we write $\uparrow w$ and $\downarrow w$ instead of $R(w)$ and $R^{-1}(w)$, respectively.

**Lemma 3.11.** The diamond $\mathfrak{D}$ is an interior image of $Z$.

**Proof.** Define $f : Z \to D$ by

$$f(z) = \begin{cases} \ m & \text{if } z \in \mu \\ \ w_0 & \text{if } z \in \kappa \times \{0\} \\ \ w_1 & \text{if } z \in \kappa \times \{1\} \\ \ r & \text{if } z = p \end{cases}$$

It is clear that $f$ is a well-defined onto mapping. To prove that $f$ is interior, it is sufficient to show that $f^{-1}\downarrow w = c_{\mu} f^{-1}(w)$ for each $w \in D$. Since $\mu$ is dense in $Z$, we have

$$f^{-1}\downarrow m = f^{-1}(D) = Z = c_{\mu} = c_{\mu} f^{-1}(m).$$

Because $Z$ is $T_1$, we have

$$f^{-1}\downarrow r = f^{-1}(r) = \{p\} = c_{\{p\}} = c_{\{p\}} f^{-1}(r).$$

Since $Y$ is closed in $Z$, we have that $c_Y A = cA$ for any $A \subseteq Y$, where $c_Y A$ is closure in $Y$.

Let $n \in \{0,1\}$. Then $(\kappa \times \{n\}) \cup \{p\}$ is closed in $Y$. Therefore, $p \in c_Y (\kappa \times \{n\})$. Thus, $c (\kappa \times \{n\}) = c_Y (\kappa \times \{n\}) = (\kappa \times \{n\}) \cup \{p\}$. This yields

$$f^{-1}\downarrow w_n = f^{-1}(\{w_n, r\}) = (\kappa \times \{n\}) \cup \{p\} = c (\kappa \times \{n\}) = c f^{-1}(w_n).$$

Consequently, $f$ is interior. $\square$

We are ready for the main lemma of this section. For this we recall that an $S4$-frame $\mathcal{F} = (W,R)$ is rooted if there is $w \in W$ (a root of $\mathcal{F}$) such that $W = R(w)$.

**Lemma 3.12.** Let $\mathcal{F} = (W,R)$ be a finite rooted $S4$-frame. If $\mathcal{F}$ is an interior image of $Z$, then $\mathcal{F}$ is an interior image of $\mathfrak{D}$.

**Proof.** We start by observing some properties of $\mathcal{F}$. Since $Z$ is scattered, it is HI. Because $Z$ is also of Cantor-Bendixson rank 3, it follows from Section 2.2 that the formulas $grz$ and $bd_3$ are valid in $Z$. As $\mathcal{F}$ is an interior image of $Z$, these formulas are also valid in $\mathcal{F}$ (see, e.g., [4, Prop. 2.9(2)]). Therefore, $R$ is a partial order and the $R$-depth of $\mathcal{F}$ is $\leq 3$ (see, e.g., [10, Props. 3.48 & 3.44]). In addition, since $Z$ is ED, so is $\mathcal{F}$. Thus, as $\mathcal{F}$ is rooted, $\mathcal{F}$ has a maximum (see, e.g., [10, Cor. 3.38]).

We consider three cases based on the depth of $\mathcal{F}$. First, suppose that the depth of $\mathcal{F}$ is 1. Then $W$ is a singleton and it is clear that $\mathcal{F}$ is an interior image of $\mathfrak{D}$. Next suppose that the depth of $\mathcal{F}$ is 2. Since $\mathcal{F}$ is a rooted poset with a maximum, $\mathcal{F}$ is isomorphic to the two element chain (see Figure 4). It is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathcal{F}$ and all the other points of $\mathfrak{D}$ to the maximum of $\mathcal{F}$ is an onto interior map.
Finally, suppose that the depth of $\mathcal{F}$ is 3. Then $\mathcal{F}$ is isomorphic to the frame depicted in Figure 5 where $W = \{0, v_0, \ldots, v_m, 1\}$ and $m \in \omega$.

![Figure 5. The two element chain.](image)

![Figure 5. The poset $\mathcal{F}$ of depth 3.](image)

If $m = 0$, then it is easy to see that mapping the root of $\mathcal{O}$ to the root of $\mathcal{F}$, the maximum of $\mathcal{O}$ to the maximum of $\mathcal{F}$, and $w_0, w_1$ to $v_0$ is an onto interior map. If $m = 1$, then $\mathcal{O}$ is isomorphic to $\mathcal{F}$, so it is obvious that $\mathcal{F}$ is an interior image of $\mathcal{O}$. Thus, to complete the proof, it suffices to show that $m \neq 2$.

Suppose that $m \geq 2$ and let $f : Z \to W$ be an interior mapping onto $\mathcal{F}$.

Claim 3.13.

1. $\mu \subseteq f^{-1}(1)$.
2. $\{p\} = f^{-1}(0)$.
3. $f^{-1}(\{v_0, \ldots, v_m\}) \subseteq Y \setminus \{p\}$.
4. $p \in c(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup c(f^{-1}(v_i) \cap (\kappa \times \{1\}))$ for each $i \in \{0, \ldots, m\}$.

Proof. (1) Since each $z \in \mu$ is isolated and $f$ is interior, we have that $f(z)$ is the maximum of $\mathcal{F}$. Thus, $f(z) = 1$.

(2) Because $f$ is onto, there is $z \in f^{-1}(0)$. By (1), we have that $z \in Y$. If $z \neq p$, then $z$ is an isolated point of $Y$, so there is an open subset $U$ of $Z$ such that $\{z\} = U \cap Y$. As $f$ is interior and $U$ is open, $f(U)$ is an $R$-upset of $\mathcal{F}$. Therefore, $f(U) = W$ since $0 = f(z) \in f(U)$.

On the other hand,

$$f(U) = f((U \cap Y) \cup (U \cap \mu)) \subseteq f(Z) = f(\{z\} \cup f(\mu) = \{0\} \cup \{1\} \neq W.$$  

The obtained contradiction proves that $z = p$. Thus, $f^{-1}(0) = \{p\}$.

(3) Follows immediately from (1) and (2) since $\mu \cup \{p\} \subseteq f^{-1}(\{0, 1\})$.

(4) Let $i \in \{0, \ldots, m\}$. Because $f$ is interior, it follows from (2) and (3) that

$$\{p\} \subseteq f^{-1}(\{v_i\}) = f^{-1}R^{-1}(v_i)$$

$$= c(f^{-1}(v_i) \cap (Y \setminus \{p\}))$$

$$= c(f^{-1}(v_i) \cap ((\kappa \times \{0\}) \cup (\kappa \times \{1\})))$$

$$= c(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup c(f^{-1}(v_i) \cap (\kappa \times \{1\}))$$

$$= c(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup c(f^{-1}(v_i) \cap (\kappa \times \{1\})).$$

$\square$
Therefore, there is a unique $A_0 \in \mathcal{F}_0$ such that $p \in cA_0$. A similar proof yields a unique $A_1 \in \mathcal{F}_1$ such that $p \in cA_1$.

Because $\mathcal{F}_0$ is finite, we have

$$p \in c(\kappa \times \{0\}) = c\left(\bigcup_{A \in \mathcal{F}_0} A\right) = \bigcup_{A \in \mathcal{F}_0} cA.$$ 

Therefore, there is $A_0 \in \mathcal{F}_0$ such that $p \in cA_0$. Since $p$ is an ultrafilter,

$$p \notin c((\kappa \times \{0\}) \setminus A_0) = c\left(\bigcup_{A \in \mathcal{F}_0 \setminus \{A_0\}} \{A\}\right) = \bigcup_{A \in \mathcal{F}_0 \setminus \{A_0\}} cA.$$

Thus, $A_0$ is the unique member $A$ of $\mathcal{F}_0$ satisfying the property that $p \in cA$.

Since $m \geq 2$, by the Pigeonhole Principle, there is $i \in \{0, 1, 2, \ldots, m\}$ such that $A_0 \neq f^{-1}(v_i) \cap (\kappa \times \{0\})$ and $A_1 \neq f^{-1}(v_i) \cap (\kappa \times \{1\})$. Thus, $p \notin c(f^{-1}(v_i) \cap (\kappa \times \{0\}))$ and $p \notin c(f^{-1}(v_i) \cap (\kappa \times \{1\}))$, which contradicts Claim 3.13(4). Consequently, $m \neq 2$, completing the proof.

**Lemma 3.14.** The logic of $Z$ is $L$.

**Proof.** By Lemma 3.11, $\mathcal{D}$ is an interior image of $Z$. Therefore, $L(Z) \subseteq L(\mathcal{D}) = L$ (see, e.g., [4, Prop. 2.9(2)]). Conversely, suppose that $L(Z) \not\models \varphi$. Since $Z$ is of Cantor-Bendixson rank 3, $bd_3$ is a theorem of $L(Z)$. Therefore, by Segerberg’s theorem (see, e.g., [10, Thm. 8.85]), $L(Z)$ is complete with respect to finite rooted $L(Z)$-frames. Thus, there is a finite rooted $L(Z)$-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. As $\mathfrak{F}$ is an $L(Z)$-frame, by [6, Lem 6.2], $\mathfrak{F}$ is an interior image of an open subspace $U$ of $Z$. Let $f : U \to \mathfrak{F}$ be an interior map, and let $z \in U$ map to the root of $\mathfrak{F}$. Since $Z$ is zero-dimensional, there is a clopen subset $V$ of $Z$ such that $z \in V$ and $V \subseteq U$. Then the restriction of $f$ to $V$ is an interior mapping of $V$ onto $\mathfrak{F}$. Because $\mathfrak{F}$ has a maximum, we have that $\mathfrak{F}$ is an interior image of $Z$ by [7, Lem. 5.4]. By Lemma 3.12, $\mathfrak{F}$ is an interior image of $\mathcal{D}$. Therefore, $\mathcal{D} \not\models \varphi$, and hence $L(\mathcal{D}) \not\models \varphi$. Thus, $L(Z) = L(\mathcal{D}) = L$. □

As a consequence of Lemmas 3.8 and 3.14 we arrive at the main result of this section.

**Theorem 3.15.** If there exists a measurable cardinal, then there exists a normal space $Z$ such that $L(Z) = L$.

4. Existence of a Measurable Cardinal is Necessary

In this section we prove that the existence of a normal space $Z$ such that $L(Z) = L$ implies the existence of a measurable cardinal. Let $Z$ be a normal space such that $L(Z) = L$.

**Lemma 4.1.** The space $Z$ is an ED-space of modal Krull dimension 2 such that $\mathcal{D}$ is an interior image of $Z$.

**Proof.** As $L(Z) = L$, for each modal formula $\varphi$ we have $Z \models \varphi$ iff $\mathcal{D} \models \varphi$. Since $\mathcal{D}$ has a maximum and is of depth 3, we have that

$$\begin{align*}
\mathcal{D} &\models ga \\
\mathcal{D} &\models bd_3 \\
\mathcal{D} &\not\models bd_2
\end{align*}$$

Therefore, $Z$ is an ED-space of modal Krull dimension 2 (see Section 2.2).
Because $\mathfrak{D} \models L(Z)$, [6, Lem. 6.2] yields an open subspace $U$ of $Z$ and an onto interior map $g : U \to D$. Then there is $z \in U$ with $f(z) = r$. Since $Z$ is normal and ED, it is zero-dimensional. Hence, there is clopen $V$ in $Z$ such that $z \in V \subseteq U$. Noting that the restriction of $g$ to $V$ is an interior mapping onto $\mathfrak{D}$, it follows from [7, Lem. 5.4] that $\mathfrak{D}$ is an interior image of $Z$.

**Remark 4.2.**

(1) Since $D$ is a finite poset, $\mathfrak{D}$ validates grz. Therefore, so does $Z$, and hence $Z$ is HI.

(2) Observe that $\mathfrak{D}$ is not hereditarily ED since the subspace $\{r, w_0, w_1\}$ is not ED. Because $\mathfrak{D}$ is an interior image of $Z$, it follows that $Z$ is not hereditarily ED.

(3) Since $Z$ is a Hausdorff ED-space that is not hereditarily ED, $Z$ must be uncountable (see, e.g., [9, Cor. 2.1]).

**Definition 4.3.** Let $f : Z \to \mathfrak{D}$ be an onto interior mapping. Denote the fibers of $f$ by

\[
\begin{align*}
M &= f^{-1}(m) \\
B_0 &= f^{-1}(w_0) \\
B_1 &= f^{-1}(w_1) \\
A &= f^{-1}(r)
\end{align*}
\]

\[
\begin{array}{c}
\text{Figure 6. Depiction of } Z \text{ partitioned by the fibers of } f.
\end{array}
\]

**Remark 4.4.**

(1) Clearly $M$ is an open dense subset of $Z$ (which is infinite as it is a dense subset of an infinite $T_1$-space).

(2) We also have that $A$ is a closed nowhere dense subset of $Z \setminus M$. Therefore, $A$ is discrete. More generally, any nonempty nowhere dense subset $N$ of $Z \setminus M$ is discrete. To see this, since $\text{mdim}(Z) = 2$, the definition of modal Krull dimension gives that $\text{mdim}(Z \setminus M) \leq 1$ and $\text{mdim}(N) \leq 0$. As $N \neq \emptyset$, we have that $\text{mdim}(N) = 0$. Thus, $N$ is discrete by [5, Rem. 4.8 & Thm. 4.9].

**Lemma 4.5.** There is a normal subspace $U$ of $Z$ such that $U \cap A$ is a singleton and $L(U) = L$.

**Proof.** Let $a \in A$. Since $A$ is discrete and $Z$ is zero-dimensional, there is a clopen subset $U$ of $Z$ such that $\{a\} = U \cap A$. As $U$ is closed in $Z$, the subspace $U$ is normal. Because $U$ is open in $Z$, the restriction $f|_U$ of $f$ to $U$ is interior. Since $U \cap A \neq \emptyset$, we have that $r \in f(U)$. As $f(U)$ is an upset, $D = \uparrow r \subseteq f(U) \subseteq D$. Therefore, $f|_U$ is onto and $\mathfrak{D}$ is an interior image of $U$. By [4, Prop. 2.9], $L(U) \subseteq L = L(Z) \subseteq L(U)$, so $L(U) = L$, completing the proof. □

By Lemma 4.5, we may assume without loss of generality that $A$ is a singleton, say $\{a\}$, yielding that $Z = B_0 \cup \{a\} \cup B_1 \cup M$ (see Figure 7).
Lemma 4.6. We have that $a \not\in cN$ for any nowhere dense subset $N$ of the subspace $B_0 \cup B_1$.

Proof. We first show that $N \cup A$ is nowhere dense in $Z \setminus M$. Let $U$ be open in $Z \setminus M$ with $U \subseteq c(N \cup A)$. Since $A$ is closed, $U \subseteq c(N) \cup A$. Therefore, $U \setminus A \subseteq c(N) \setminus A = c(N) \cap (B_0 \cup B_1)$, which is the closure of $N$ relative to $B_0 \cup B_1$. Because $U \setminus A$ is open and $N$ is nowhere dense in $B_0 \cup B_1$, we have that $U \setminus A = \emptyset$, so $U \subseteq A$. By Remark 4.4(2), $A$ is a closed nowhere dense subset of $Z \setminus M$, hence $U = \emptyset$. Thus, $N \cup A$ is nowhere dense in $Z \setminus M$. Applying Remark 4.4(2) again yields that $N \cup A$ is discrete. Consequently, there is an open set $V$ in $Z$ such that $\{a\} = V \cap (N \cup A)$. As

$$V \cap N \subseteq V \cap (N \cup A) = \{a\} \subseteq Z \setminus (B_0 \cup B_1) \subseteq Z \setminus N,$$

it must be the case that $V \cap N = \emptyset$, so $a \not\in cN$. □

We recall that a normal space $X$ is an $F$-space if any two disjoint open $F_\sigma$-sets in $X$ have disjoint closures in $X$ (see, e.g., [19, Lem. 1.2.2(b)]). Being a normal ED-space, it follows from [15, Exercise 14N.4] that $Z$ is an $F$-space.

Definition 4.7. Let $Y$ denote the subspace $B_0 \cup \{a\} \cup B_1$ of $Z$.

Because $Y = Z \setminus M$ is closed in $Z$, we have that $Y$ is a normal $F$-space by [19, Lem. 1.2.2(d)]. We require the following definition.

Definition 4.8. (See, e.g., [20, p. 37]) A point $x$ of a space $X$ is called a $P$-point provided for any $G_\delta$-set $S$ in $X$ we have that $x \in S$ implies $x \in iS$.

Remark 4.9. By taking complements we obtain that $x \in X$ is a $P$-point iff for each $F_\sigma$-set $S$ in $X$ we have that $x \not\in S$ implies $x \not\in cS$. This will be utilized in Lemma 4.16(5).

Lemma 4.10. Either $a$ is a $P$-point in the subspace $B_0 \cup \{a\}$ or a $P$-point in the subspace $B_1 \cup \{a\}$.

Proof. Suppose not. Then we show that there are disjoint open $F_\sigma$-sets $U_0$ and $U_1$ of $Y$ whose closures have nonempty intersection, which is a contradiction since $Y$ is a normal $F$-space. We only show how to construct $U_0$ because $U_1$ is constructed similarly. Since $a$ is not a $P$-point in $B_0 \cup \{a\}$, for each $n \in \omega$, there is $W_n$ open in $B_0 \cup \{a\}$ such that $a \in \bigcap_{n \in \omega} W_n$ but $a \not\in i(\bigcap_{n \in \omega} W_n)$, where $i$ is taken in $B_0 \cup \{a\}$. As $Z$ is zero-dimensional, $B_0 \cup \{a\}$ is zero-dimensional. Thus, for each $n \in \omega$, there is $V_n$ clopen in $B_n \cup \{a\}$ such that $a \in V_n \subseteq W_n$. Clearly, $a \in V := \bigcap_{n \in \omega} V_n$ and $V$ is a closed $G_\delta$-set in $B_0 \cup \{a\}$. Moreover, $a \not\in iV$ since $V \subseteq \bigcap_{n \in \omega} W_n$ and $a \not\in i(\bigcap_{n \in \omega} W_n)$. Put $U_0 = (B_0 \cup \{a\}) \setminus V$. Then $U_0$ is an open $F_\sigma$-set in $B_0 \cup \{a\}$ such that $a \not\in U_0$ and $a \in cU_0$. Clearly $U_0 \subseteq B_0$, and so $U_0$ is open in $B_0$. As $B_0 = Y \cap f^{-1} w_0$ is open in $Y$, it follows that $U_0$ is open in $Y$. Because $B_0 \cup \{a\}$ is closed in $Y$ and $U_0$ is an $F_\sigma$-set in $B_0 \cup \{a\}$, we have that $U_0$ is an $F_\sigma$-set in $Y$. Thus, $U_0$ is an open $F_\sigma$-set in $Y$ such that $a \in cU_0$. Analogously, there is an open $F_\sigma$-set $U_1$ in $Y$ such that $a \in cU_1$. By construction, $U_0 \subseteq B_0$ and $U_1 \subseteq B_1$, so $U_0$ and $U_1$ are disjoint. On the other hand, $a \in cU_0 \cap cU_1$, yielding the desired contradiction. □
Convention 4.11. Without loss of generality we assume that $a$ is a $P$-point in $X := B_0 \cup \{a\}$.

Remark 4.12. Since $X$ is closed in $Z$, the closure in $X$ of any subset $S$ of $X$ coincides with the closure of $S$ in $Z$. Therefore, there is no ambiguity in writing $cS$ whenever $S \subseteq X$.

The following lemma is an easy consequence of Zorn’s lemma, and we skip its proof.

Lemma 4.13. There is a family $\mathcal{F}$ of subsets of $X$ that is maximal with respect to the following two properties:

1. Each $F \in \mathcal{F}$ is a nonempty clopen in $X$ such that $a \notin F$;
2. The family $\mathcal{F}$ is pairwise disjoint.

Lemma 4.14. Let $N = B_0 \setminus \bigcup \mathcal{F}$. Then we have:

1. $\bigcup \mathcal{F}$ is open in both $X$ and $B_0$.
2. $\bigcup \mathcal{F}$ is dense in both $B_0$ and $X$.
3. $N$ is closed in $Z$.
4. There is a clopen subspace $U$ of $Z$ such that $U \cap N = \emptyset$ and $L(U) = L$.

Proof. (1) Since $\bigcup \mathcal{F}$ is a union of clopen subsets of $X$, it is open in $X$. Also, since $a \notin F$ for each $F \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \subseteq B_0$, and hence it is also open in $B_0$.

(2) Let $z \in B_0$. If $z \notin c(\bigcup \mathcal{F})$, then as $X$ is zero-dimensional, there is clopen $V$ in $X$ such that $z \in V$ and $V \cap \bigcup \mathcal{F} = \emptyset$. Since $z \neq a$, we may assume that $a \notin V$ (by shrinking $V$ further if necessary). But this contradicts the maximality of $\mathcal{F}$ because the family $\{V\} \cup \mathcal{F}$ satisfies the conditions of Lemma 4.13. Thus, $z \in c(\bigcup \mathcal{F})$, and so $\bigcup \mathcal{F}$ is dense in $B_0$.

Finally, since $a \in cB_0$, we conclude that $\bigcup \mathcal{F}$ is dense in $X$.

(3) It suffices to show that $N$ is closed in $X$. For any $z \in B_0 \setminus N$, we have that $\bigcup \mathcal{F}$ is open in $X$ and $z \in \bigcup \mathcal{F}$. Since $N \cap \bigcup \mathcal{F} = \emptyset$, it follows that $z \notin cN$. Because $\{B_0 \setminus N, N, \{a\}\}$ is a partition of $X$, it remains to show that $a \notin cN$. But (1) and (2) imply that $N$ is nowhere dense in $B_0$, hence nowhere dense in $B_0 \cup B_1$. This yields that $a \notin cN$ by Lemma 4.6.

(4) Since $\{a\}$ and $N$ are closed in the zero-dimensional normal space $Z$, there is $U$ clopen in $Z$ such that $a \in U$ and $U \cap N = \emptyset$. Because $U$ is open, the restriction of $f$ as defined in Definition 4.3 is an interior map from $U$ to $\mathfrak{D}$. To see that it is onto, observe that $U \cap M \neq \emptyset$ since $M$ is dense in $Z$, and both $U \cap B_0$ and $U \cap B_1$ are nonempty because $a \in cB_0, cB_1$ and $a \in U$. Therefore, $\mathfrak{D}$ is an interior image of $Z$, and so $L(U) \subseteq L = L(Z) \subseteq L(U)$ by [4, Prop. 2.9]. Thus, $L(U) = L$. □

Let $U$ be the clopen subspace of $Z$ constructed in the proof of Lemma 4.14(4). Then $U$ is normal since it is a closed subspace of a normal space. In addition, $a$ remains a $P$-point of $X \cap U$ because $X \cap U$ is an open subspace of $X$ and $a$ is a $P$-point of $X$. Therefore, without loss of generality we may assume that $Z = U$. Thus, $B_0 = \bigcup \mathcal{F}$ and $N = \emptyset$.

Definition 4.15.

(1) Let $\kappa$ be the cardinality of $\mathcal{F}$, and let $\varphi : \kappa \to \mathcal{F}$ be a bijection. Denoting $\varphi(\alpha)$ by $F_\alpha$, we may write $\mathcal{F} = \{F_\alpha \mid \alpha \in \kappa\}$.

(2) Let

$$\mathcal{G} = \left\{ \Gamma \subseteq \kappa \mid a \in c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \right\}.$$ 

We are ready to prove the main lemma of this section.

Lemma 4.16.

1. If $\Gamma \in \mathcal{G}$ and $\Gamma \subseteq \Lambda$, then $\Lambda \in \mathcal{G}$.
2. For any $\Gamma \subseteq \kappa$, exactly one of $\Gamma, \kappa \setminus \Gamma$ belongs to $\mathcal{G}$. 


3. If $\Gamma, \Lambda \in \mathcal{G}$, then $\Gamma \cap \Lambda \in \mathcal{G}$.
4. $\mathcal{G}$ is a free ultrafilter on $\kappa$.
5. $\mathcal{G}$ is countably complete.

Proof. (1) Let $\Gamma \in \mathcal{G}$ and $\Gamma \subseteq \Lambda$. Then $\bigcup_{\alpha \in \Gamma} F_\alpha \subseteq \bigcup_{\alpha \in \Lambda} F_\alpha$, yielding

$$a \in c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \subseteq c \left( \bigcup_{\alpha \in \Lambda} F_\alpha \right).$$

Thus, $\Lambda \in \mathcal{G}$.

(2) Let $\Gamma \subseteq \kappa$. We have that

$$a \in c B_0 = c \left( \bigcup_{\alpha \in \kappa} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \cup \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \cup c \left( \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \right).$$

Therefore, $\Gamma \in \mathcal{G}$ or $\kappa \setminus \Gamma \in \mathcal{G}$. Suppose that both $\Gamma$ and $\kappa \setminus \Gamma$ belong to $\mathcal{G}$. Then the frame $\mathcal{F}$ depicted in Figure 5 with $m = 2$ is an interior image of $Z$ via the mapping $g : Z \to W$ given by

$$g(z) = \begin{cases} 
1 & \text{if } z \in M \\
v_0 & \text{if } z \in \bigcup_{\alpha \in \Gamma} F_\alpha \\
v_1 & \text{if } z \in \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \\
v_2 & \text{if } z \in B_1 \\
0 & \text{if } z = a
\end{cases}$$

The function $g$ is depicted in Figure 8 where each fiber of $g$ is labeled to the right by its image in $W$.

Figure 8. The function $g : Z \to W$.

This yields that $\mathcal{F} \models L(Z) = L$, which is a contradiction since $\mathcal{F} \not\models L$. Thus, exactly one of $\Gamma$ or $\kappa \setminus \Gamma$ is a member of $\mathcal{G}$.

(3) If $\Gamma \cap \Lambda \not\in \mathcal{G}$, then $a \not\in c \left( \bigcup_{\alpha \in \Gamma \cap \Lambda} F_\alpha \right)$. On the other hand,

$$a \in c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma \cap \Lambda} F_\alpha \cup \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma \cap \Lambda} F_\alpha \right) \cup c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right).$$

Therefore, $a \in c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right)$. Thus, $\Gamma \setminus \Lambda \not\in \mathcal{G}$. Since $\Gamma \setminus \Lambda \subseteq \kappa \setminus \Lambda$, (1) implies that $\kappa \setminus \Lambda \in \mathcal{G}$. However, as $\Lambda \in \mathcal{G}$, (2) implies that $\kappa \setminus \Lambda \not\in \mathcal{G}$. The obtained contradiction proves that $\Gamma \cap \Lambda \in \mathcal{G}$.

(4) That $\mathcal{G}$ is an ultrafilter follows from (1), (2), and (3). To see that $\mathcal{G}$ is free, let $\alpha \in \kappa$. Then $F_\alpha$ is clopen in $X$ and $a \not\in F_\alpha$. Therefore, $a \not\in c F_\alpha$, yielding that $\{\alpha\} \not\in \mathcal{G}$. Thus, $\mathcal{G}$ is a free ultrafilter.
(5) Let $\Lambda_n \in \mathcal{G}$ for each $n \in \omega$ and let $\Gamma := \bigcap_{n \in \omega} \Lambda_n \notin \mathcal{G}$. For $n \in \omega$ set $\Gamma_n = \bigcap_{i=0}^{n} \Lambda_i$. Then $\Gamma_n \in \mathcal{G}$ by (3), $\Gamma_{n+1} \subseteq \Gamma_n$, and $\Gamma = \bigcap_{n \in \omega} \Gamma_n$. For $n \in \omega$ set $\Delta_n = \Gamma_n \setminus \Gamma_{n+1}$. Since $\mathcal{G}$ is an ultrafilter, $\Delta_n \notin \mathcal{G}$ for each $n \in \omega$.

Claim 4.17. The set $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is clopen in $X$.

Proof. Clearly $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is open in $X$ since each $F \in \mathcal{F}$ is clopen in $X$. To see that $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is closed in $X$ we show that $c\left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right) = \bigcup_{\alpha \in \Delta_n} F_\alpha$. As $X$ is closed in $Z$, we have that $c\left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right) \subseteq X$. Let $z \in X \setminus \bigcup_{\alpha \in \Delta_n} F_\alpha$. We show that $z \notin c\left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$. Either $z = a$ or $z \in B_0$. The former case is clear since $\Delta_n \notin \mathcal{G}$ implies that $z = a \notin \bigcup_{\alpha \in \Delta_n} F_\alpha$. Suppose $z \in B_0$. Then there is $\beta \in \kappa$ such that $z \in F_\beta$. Since $z \notin \bigcup_{\alpha \in \Delta_n} F_\alpha$, it follows that $\beta \notin \Delta_n$. Because $F_\beta$ is clopen in $X$, there is $U$ open in $Z$ such that $F_\beta = U \cap X$. Clearly $z \in U$. As $\mathcal{F}$ is pairwise disjoint, we have that

$$U \cap \bigcup_{\alpha \in \Delta_n} F_\alpha = U \cap \bigcup_{\alpha \in \Delta_n} (X \cap F_\alpha) = \bigcup_{\alpha \in \Delta_n} (U \cap X \cap F_\alpha) = \bigcup_{\alpha \in \Delta_n} (F_\beta \cap F_\alpha) = \emptyset.$$ 

Therefore, $z \notin c\left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$.

As $\Gamma_0 \setminus \Gamma = \bigcup_{n \in \omega} \Delta_n$, it follows from Claim 4.17 that

$$\bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha = \bigcup_{n \in \omega} \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$$

is an open $F_\sigma$-set in $X$. Moreover, $a \in c\left( \bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha \right)$ because $\Gamma_0 \setminus \Gamma \in \mathcal{G}$. But $a \notin \bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha$ since $a \notin F_\alpha$ for each $\alpha \in \kappa$. This implies that $a$ is not a $P$-point of $X$ (see Remark 4.9). The obtained contradiction proves that $\mathcal{G}$ is countably complete. \hfill $\square$

As a consequence of Lemma 4.16 and Section 2.3, we obtain:

Lemma 4.18. The cardinal $\kappa$ is Ulam-measurable, and hence there exists a measurable cardinal.

Consequently, we have proved the following result.

Theorem 4.19. If there exists a normal space $Z$ such that $\mathcal{L}(Z) = \mathcal{L}$, then there exists a measurable cardinal.

Putting Theorems 3.15 and 4.19 together yields the main result of the paper:

Theorem 4.20. There exists a measurable cardinal iff there exists a normal space $Z$ such that $\mathcal{L}(Z) = \mathcal{L}$.

We conclude the paper by the following open problem:

Problem 4.21. In Theorem 4.20 can ‘normal’ be replaced by ‘Tychonoff’?

Clearly the interesting implication is to prove that the existence of a Tychonoff space whose logic is $\mathcal{L}$ implies the existence of a measurable cardinal.

References


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