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A dynamic logic for learning theory

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ABSTRACT

Building on previous work [4,5] that bridged Formal Learning Theory and Dynamic Epistemic Logic in a topological setting, we introduce a *Dynamic Logic for Learning Theory* (DLLT), extending Subset Space Logics [18,10] with dynamic *observation modalities* $[o]\varphi$, as well as with a *learning operator* $L(\vec{o})$, which encodes the learner's *conjecture* after observing a finite sequence of data $\vec{o}$. We completely axiomatise DLLT, study its expressivity and use it to characterise various notions of knowledge, belief, and learning.

1. Introduction

The process of learning consists of incorporating new information into one's prior information state. Dynamic epistemic logic (DEL) studies such one-step information changes from a logical perspective [6,21,25], but the general concept of *learning* encompasses not only one-step revisions, but also their *long-term horizon*. In the long run, learning should lead to *knowledge*—an epistemic state of a particular value. Examples include language learning (inferring the underlying grammar from examples of correct sentences), and scientific inquiry (inferring a theory of a phenomenon on the basis of observations). Our goal in this paper is to provide a simple logic for reasoning about this process of *inductive learning* from successful observations. Understanding inductive inference is of course an infamously difficult open problem, and there are many different approaches in the literature, from probabilistic and statistical formalisms based on Bayesian reasoning, Popper-style measures of corroboration, through default and non-monotonic logics, Carnap-style 'inductive logic', to AGM-style rational belief revision and theory change. However, in this paper we *do not try to solve* the problem of induction, but only to *reason about* a (rational) inductive learner. For this, we adopt the more flexible and open-ended approach of Formal Learning Theory (FLT). While most other approaches adopt a normative stance, aimed at prescribing 'the' correct algorithm for forming and changing rational beliefs from observations (e.g., Bayesian conditioning), or at least at prescribing some general rational constraints that any such algorithm should obey (e.g., the AGM postulates for belief revision), FLT gives the learner a high degree of freedom, allowing the choice of *any learning method* that produces conjectures *based on the data*

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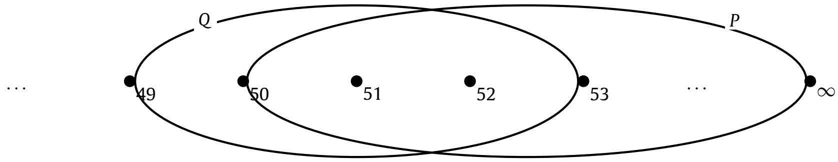

Fig. 1. Example 1; $P :=$ 'the car is speeding', $Q :=$ 'the reading of the radar is 51 km/h'.

(no matter how 'crazy' or unjustified are these conjectures, or how erratic is the process of belief change). In FLT the only criterion of success is... success: tracking the truth in the limit. In other words, the only thing that matters is whether or not the iterated belief revision process will eventually stabilise on a conjecture which matches the truth (about some given issue). We are of course not interested in cases of convergence to the truth 'by accident', but in determining whether or not a given learner is *guaranteed* to eventually track the truth; hence, the focus is on 'The Logic of *Reliable* Inquiry'.[1]

We propose a formalism that combines ideas from: Subset Space Logics, as introduced by Moss and Parikh [18], investigated further by Dabrowski et al. [10] and already merged with the DEL tradition in prior work [27,2,23,22,8,7]; the topological approach to FLT in [17,5]; and the general agenda of bridging DEL and FLT in [14]. Semantically, we take *intersection spaces* (a type of subset spaces that are closed under finite non-empty intersections), with points interpreted as possible worlds and neighbourhoods interpreted as *observations* (or *information states*) (see, e.g., [19] for a survey on subset space logics). We enhance these structures with a *learner* $\mathbb{L}$, mapping every information state to a *conjecture*, representing the learner's strongest belief in this state. As in Subset Space Logics, our language features an S5-type *'knowledge-with-certainty' modality*, capturing the learner's *hard information*, as well as the so-called *'effort' modality*, which we interpret as 'stable truth' (i.e., truth immune to further observations). We add to this *observation modalities* $[o]\varphi$, analogous to the dynamic modalities in Public Announcement Logic (PAL), as well as a *learning operator* $L(\overrightarrow{o})$, which encodes the learner's conjecture after having observed a finite sequence of pieces of evidence $\overrightarrow{o}$. This can be used to give a natural definition of *belief*: a learner believes P iff she knows that P is entailed by her current conjecture.

We present a sound and complete axiomatisation of DLLT with respect to our learning models. The completeness uses a neighbourhood version of the standard canonical model construction. We use this logic to characterise various learnability notions. In particular, we are able to model inductive learning as *coming to stably believe*[2] *a true fact after observing an incoming sequence of true data*. The possibility of such learning corresponds to a key concept in FLT, namely *identifiability in the limit* first introduced and studied by Gold in [16]. Finally, we discuss the expressivity of DLLT, showing that the dynamic observation modalities are in principle eliminable via reduction laws.

This work can be seen as an extension of [3]: while the main results in [3] and the current paper coincide, the proofs—which were omitted in the shorter version—are presented in this version in full detail.

1.1. Effort modality and knowledge

In [26], Vickers reconstructed general topology as a logic of observation, in which the points of the space represent possible states of the world, while basic open neighbourhoods of a point are interpreted as *information states* produced by accumulating finitely many observations. Moss and Parikh [18] gave an account of learning in terms of *observational* effort. Making the epistemic effort to obtain more information about a possible world has a natural topological interpretation—it can be seen as *shrinking* the open neighbourhood (representing the current information state), thus providing a more accurate approximation of the actual state of the world [18,10,12,13,19]. A similar line was proposed in Formal Epistemology [17,5], where it was combined with more sophisticated notions of learning borrowed from FLT. The following example relates the effort modality with knowledge.

Example 1 *([19]).* Let us consider some *measurement*, say of a vehicle's velocity. Suppose a policeman uses radar to determine whether a car is speeding in a 50-mile speed-limit zone. The property *speeding* can be identified with the interval $(50, \infty)$. Suppose the radar shows 51 mph, but the radar's accuracy is ± 2 mph. The intuitive meaning of a speed measurement of 51 ± 2 is that the car's true speed v is in the *open interval* $(49, 53)$. According to [19], 'anything which we *know* about v must hold not only of v itself, but also of any v' in the same interval' [19, p. 300]. Since the interval $(49, 53)$ is not fully included in the 'speeding' interval $(50, \infty)$, the policeman does *not know* that the car is speeding. But suppose that he does another measurement, using a more accurate radar with an accuracy of ± 1 mph, which shows 51.5 mph. Then he will *come to know* that the car is speeding: the open interval $(50.5, 52.5)$ *is* included in $(50, \infty)$ (see Fig. 1).

[1] 'The Logic of Reliable Inquiry' is the title of a classic text in FLT-based epistemology [17].

[2] By "*stably* believing (a proposition)" we here mean that the learner believes that proposition and her belief in that proposition cannot be defeated by further truthful observations.

1.2. Infallible knowledge versus inductive knowledge

Let us now extend this picture with learning as understood in FLT. We start by setting the stage—briefly introducing *learning frames*, the underlying structures of learning. We will return to it, with complete definitions, later in the paper. Our DLLT is interpreted over such frames and, using them, we will be able to explain and model various epistemic notions.

First, consider a pair $(X, \mathcal{O})$, where X is a non-empty set of *possible worlds*; $\mathcal{O} \subseteq \mathcal{P}(X)$ is a non-empty set of *information states* (or 'observables', or 'evidence'). We take $\mathcal{O}$ to be closed under finite intersections, i.e., for any $O_1, O_2 \in \mathcal{O}$, we have $O_1 \cap O_2 \in \mathcal{O}$. The resulting $(X, \mathcal{O})$ is called an intersection space. A *learning frame* is a triplet $(X, \mathcal{O}, \mathbb{L})$, where $\mathbb{L} : \mathcal{O} \to \mathcal{P}(X)$ is a *learner*, i.e., a map associating to every $O \in \mathcal{O}$ some 'conjecture' $\mathbb{L}(O) \subseteq X$ (see Definition 3 for the full description of $\mathbb{L}$ and how it can be extended to range over finite sequences of observations).

Let us now reconstruct Example 1 as a learning frame. We take $X = (0, \infty)$ as the set of possible worlds (representing possible velocities of the car, where we assume the car is known to be *moving*); $\mathcal{O} = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : 0 < a < b < \infty\}$ is the set of all open intervals with positive rational endpoints (representing possible measurement results by arbitrarily accurate radars). The pair $(X, \mathcal{O})$ is an intersection frame, and the topology generated by $\mathcal{O}$ is the standard topology on real numbers (restricted to X).

Certain (Infallible) Knowledge. In an information state $U \in \mathcal{O}$, the learner is said to *infallibly know a proposition* $P \subseteq X$ *conditional on observation* O if her conditional information state entails P, i.e., if $U \cap O \subseteq P$. The learner *(unconditionally) knows* P if $U \subseteq P$. The possibility of achieving certain knowledge about a proposition $P \subseteq X$ in a possible world $x \in X$ by a learner $\mathbb{L}$ if given enough evidence (true at x) is called *learnability with certainty*. In other worlds P is learnable with certainty if there exists some observable property $O \in \mathcal{O}$ (with $x \in O$) such that the learner infallibly knows P in information state O. Learnability can be used to define verifiability and falsifiability. A proposition $P \subseteq X$ is *verifiable with certainty (by* $\mathbb{L}$*)* if it is learnable with certainty by $\mathbb{L}$ whenever it is true; i.e., if P is learnable with certainty at all worlds $x \in P$. Dually, a proposition $P \subseteq X$ is *falsifiable with certainty (by* $\mathbb{L}$*)* if its negation $X - P$ is learnable with certainty by $\mathbb{L}$ whenever P is false; i.e., if $X - P$ is learnable with certainty at all worlds $x \notin P$. Finally, a proposition $P \subseteq X$ is *decidable with certainty (by* $\mathbb{L}$*)* if it is both verifiable and falsifiable with certainty (by $\mathbb{L}$).

In the context of Example 1, let us consider the certain knowledge of the policeman. In the information state $U = (49, 53)$, the learner/policeman does not know the proposition $P = (50, \infty)$, so he cannot be certain that the car is speeding. However, the speeding property P is verifiable with certainty: whenever P is actually true, he could perform a more accurate speed measurement, by which he can get to an information state in which P is infallibly known. In our example, the policeman refined his measurement getting to the information state $O = (50.5, 52.5)$, thus coming to know P. In contrast, the property $X - P = (0, 50]$ ('not speeding') is not verifiable with certainty: if by some kind of miraculous coincidence, the speed of the car is exactly 50 mph, then the car is not speeding, but the policeman will never know that for certain (since every speed measurement, of any degree of accuracy, will be consistent both with P and with $X - P$). Nevertheless, $X - P$ is always falsifiable with certainty: if false (i.e., if the speed is in P, so that car is speeding), then as we saw the policeman will come to infallibly know that (by some more accurate measurement).

Inductive (Defeasible) Knowledge. Before we proceed to Inductive Knowledge let us consider epistemic states weaker than certainty, namely *belief*. In an information state $U \in \mathcal{O}$, the learner $\mathbb{L}$ is said to:

- *un-conditionally believe* $P \subseteq X$ if $\mathbb{L}(U) \subseteq P$.[3]
- *believe a proposition* $P \subseteq X$ *conditional on observation* O if $\mathbb{L}(U \cap O) \subseteq P$;
- *have undefeated belief in a proposition* $P \subseteq X$ *at world* x if she believes P in every information state $O \in \mathcal{O}$ that is true at x (i.e., $x \in O$) and is at least as strong as U (i.e., $O \subseteq U$). This means that, once she reaches information state U, no further evidence can defeat the learner's belief in P.

One of the central problems in epistemology is to define a realistic notion of knowledge that fits the needs of empirical sciences. It should allow fallibility, while requiring higher standards of evidence and robustness than simple belief. One of the main contenders is the so-called Defeasibility Theory of Knowledge, which defines *defeasible (fallible) knowledge* as true undefeated belief. In the learning-theoretic context, this gives us an evidence-based notion of 'inductive knowledge': in an information state U, P is *inductively known* at world x if it is true at x (i.e., $x \in P$) and it is undefeated belief (in the sense defined above). This is the kind of knowledge that can be gained by empirical (incomplete) induction, based on experimental evidence.

As in the case of learnability with certainty, achieving inductive knowledge is defined as learnability. A proposition $P \subseteq X$ is *inductively learnable* (or 'learnable in the limit') *by the learner* $\mathbb{L}$ at world x if $\mathbb{L}$ will come to inductively know P if given enough evidence (true at x); i.e., if there exists some observable property $O \in \mathcal{O}$ of world x (i.e., with $x \in O$) such that $\mathbb{L}$ inductively knows P in information state O. Inductive verifiability and falsifiability are defined in terms of learnability. A proposition $P \subseteq X$ is *inductively verifiable by the learner* $\mathbb{L}$, if it is inductively learnable whenever it is true; i.e., if P is inductively learnable at all worlds $x \in P$. Dually, a proposition $P \subseteq X$ is *inductively falsifiable by the learner* $\mathbb{L}$, if its negation

[3] In the tautological information state X, the learner believes P iff $\mathbb{L}(X) \subseteq P$.

$X - P$ is inductively learnable whenever P is false; i.e., if $X - P$ is inductively learnable at all worlds $x \notin P$. A proposition $P \subseteq X$ is *inductively decidable by* $\mathbb{L}$ if it is both inductively verifiable and inductively falsifiable by $\mathbb{L}$.

In the context of Example 1, let us now turn to inductive knowledge of the policeman. Both speeding (P) and non-speeding ($X - P$) are inductively decidable (and thus both inductively verifiable and inductively falsifiable): for instance, they are inductively decidable by the learner $\mathbb{L}$, defined by putting

$$\mathbb{L}((a,b)) = \begin{cases} (a,b), & \text{if } (a,b) \subseteq P \\ (a,b) \cap (X-P), & \text{otherwise (i.e., } (a,b) \cap (X-P) \neq \emptyset) \end{cases}$$

Intuitively, such a learner is like a fair 'judge' who assumes innocence until proven guilty: she conjectures that the car is not speeding as along as her measurement is consistent with $(X - P)$. The dual learner, the 'suspicious cop',

$$\mathbb{L}((a,b)) = \begin{cases} (a,b) \cap P, & \text{if } (a,b) \cap P \neq \emptyset \\ (a,b), & \text{otherwise (i.e., if } (a,b) \subseteq X-P). \end{cases}$$

on the other hand can not inductively decide the speeding issue. Intuitively, this learner believes the car to be *speeding* whenever the available evidence cannot settle the issue, and keeps this conjecture until it is disproven by some more accurate measurement. In some cases, this policeman will be right 'in the limit': after doing enough accurate measurements, he will *eventually settle on the correct belief* (about speeding or not); though of course (in case the car's speed is exactly 50 mph) he will never be certain of this. This is because any measurement (a, b) that contains 50 will intersect P, therefore the policeman will believe that the car is speeding when in reality it is not. An example of a property which is *inductively decidable but neither verifiable with certainty nor falsifiable with certainty* is the proposition $S = [50, 51)$. It is not verifiable with certainty, since if the car's speed is exactly 50 mph, then S is true but the learner will never be certain of this; and it is not falsifiable with certainty, since if the car's speed is exactly 51 mph, then S is false but the learner will never be certain of that. Nevertheless, S is *inductively decidable*, e.g., by the learner defined by:

$$\mathbb{L}((a,b)) = \begin{cases} (a,b) \cap S, & \text{if } a < 50 < b \\ (a,b), & \text{if } (a,b) \subseteq S \text{ or } (a,b) \cap S = \emptyset \\ [51,b), & \text{if } 50 < a < 51 < b. \end{cases}$$

Dependence on the Learner. It is easy to see that learnability (verifiability, falsifiability, decidability) with certainty are *learner-independent* notions (since they are directed towards achieving infallible knowledge), so they do not depend on $\mathbb{L}$, but only on the underlying intersection model. In contrast, *the corresponding inductive notions above are learner-dependent*. As a consequence, the interesting concepts in Learning Theory are obtained from them by *quantifying existentially over learners*: a proposition P is *inductively learnable (verifiable, falsifiable, decidable)* if there exists *some* learner $\mathbb{L}$ such that P is respectively inductively learnable (verifiable, falsifiable, decidable) by $\mathbb{L}$. This property of a learning frame is called *generic inductive learnability*.

Topological Characterisations. As it is well-known in learning theory and formal epistemology [26,17], the above notions are *topological* in nature: P is learnable with certainty at world x iff x is in the *interior* of P with respect to the topology generated by $\mathcal{O}$; P is verifiable with certainty iff it is *open* in the same topology; P is falsifiable with certainty iff it is *closed* in this topology; finally, P is decidable with certainty iff it is *clopen*. The corresponding inductive notions can be easily characterised (see [17]), in the case when the topology generated by $\mathcal{O}$ satisfies the separation condition[4] $T1$: in this case, P is inductively verifiable iff it is Σ_2 in the Borel hierarchy for this topology (i.e., a countable union of closed sets); in the same conditions, P is inductively falsifiable iff it is Π_2 (a countable intersection of open sets), and it is inductively decidable iff it is Δ_2 (i.e., Σ_2 and Π_2). More recently, in [5], these characterisations were generalised to arbitrary topologies satisfying the weaker separation condition[5] $T0$; in particular, P is inductively verifiable iff it is a countable union of locally closed sets.[6]

2. Dynamic logic for learning theory

In this section we introduce our 'dynamic logic for learning theory' DLLT. As already mentioned, this is obtained by adding two ingredients to the language of Subset Space Logics: *dynamic observation modalities* $[o]\varphi$ and a *learning operator* $L(\vec{o})$.

[4] This topology is $T1$ iff for every two distinct points $x \neq y$ there exist an observation $O \in \mathcal{O}$ with $x \in O$ and $y \notin O$.

[5] The observational topology is $T0$ iff points can be distinguished by observations; i.e., if x and y satisfy the same observable properties in $\mathcal{O}$, then $x = y$. Obviously, $T0$ is a minimally necessary condition for any kind of learnability of the real world from observations.

[6] A set is locally closed if it is the intersection of a closed and an open set.

2.1. Syntax and semantics of DLLT

Let $\mathrm{Prop} = \{p, q, \dots\}$ be a countable set of *propositional variables*, denoting arbitrary 'ontic' (i.e., non-epistemic) facts that might hold in a world (even if they might never be observed), and let $\mathrm{Prop}_{\mathcal{O}} = \{o, u, v, \dots\}$ be a countable set of *observational variables*, denoting 'observable facts' (which, if true, will eventually be observed).

Definition 2. The syntax of our language $\mathfrak{L}$ is defined by the grammar:

$$\varphi ::= p \mid o \mid L(\vec{o}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid \Box\varphi \mid [o]\varphi$$

where $p \in \mathrm{Prop}$ and $o \in \mathrm{Prop}_{\mathcal{O}}$, and $\vec{o} = (o_1, \dots, o_n) \in \mathrm{Prop}^*_{\mathcal{O}}$ is a finite sequence of observational variables (in particular, the empty sequence is denoted by λ). We employ the usual abbreviations for propositional connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ and for the dual modalities $\langle K\rangle, \Diamond, \langle o\rangle$. We will follow the usual rules for the elimination of the parentheses.

The informal meaning of our formulas is as follows. Propositional variables denote *ontic facts* (i.e., factual, non-epistemic features of a world), while observational variables o denote *observable facts* (i.e., facts that, if true, will eventually be observed). We read $K\varphi$ as 'the learner *knows* φ (with absolute certainty)'. $\Box\varphi$ is the so-called 'effort modality' from Subset Space Logic; we read $\Box\varphi$ as 'φ is *stably true*'. Indeed, $\Box\varphi$ holds iff φ is true and will stay true *no matter what new (true) evidence is observed*. The operator $[o]\varphi$ is similar to the operator $[\psi]\varphi$ in Public Announcement Logic, but it is restricted to the cases when ψ is a particular kind of atomic formula, namely an observational variable $o \in \mathrm{Prop}_{\mathcal{O}}$. So we read $[o]\varphi$ as 'after o is observed, φ will hold'. Finally, $L(\vec{o})$ denotes the learner's *conjecture* given observations $\vec{o}$; i.e., her strongest belief (or the set of worlds considered to be most plausible) after observing $\vec{o}$.

Definition 3 *(Intersection frame/model and learning frame/model).* An *intersection frame* [18,10] is a pair $(X, \mathcal{O})$, where: X is a non-empty set of *possible worlds* (or 'ontic states'); $\mathcal{O} \subseteq \mathcal{P}(X)$ is a non-empty set of subsets, called *information states* (or 'observables', or 'evidence'), which is assumed to be *closed under finite intersections*: if $\mathcal{F} \subseteq \mathcal{O}$ is finite then $\bigcap\mathcal{F} \in \mathcal{O}$. An *intersection model* $(X, \mathcal{O}, \|\cdot\|)$ is an intersection frame $(X, \mathcal{O})$ together with a valuation map $\|\cdot\| : \mathrm{Prop} \cup \mathrm{Prop}_{\mathcal{O}} \rightarrow \mathcal{P}(X)$, that maps propositional variables p into arbitrary sets $\|p\| \subseteq X$ and observational variables o into observable properties $\|o\| \in \mathcal{O}$.

A *learning frame* is a triplet $(X, \mathcal{O}, \mathbb{L})$, where $(X, \mathcal{O})$ is an intersection frame and $\mathbb{L} : \mathcal{O} \rightarrow \mathcal{P}(X)$ is a *learner*, i.e., a map associating to every information state $O \in \mathcal{O}$ some 'conjecture' $\mathbb{L}(O) \subseteq X$, and satisfying two properties:

1. $\mathbb{L}(O) \subseteq O$ (*conjectures fit the evidence*), and
2. if $O \neq \emptyset$ then $\mathbb{L}(O) \neq \emptyset$ (*consistency of conjectures* based on consistent evidence).

We can extend $\mathbb{L}$ to range over *strings* of information states $\vec{O} = (O_1, \dots, O_n) \in \mathcal{O}^*$ in a natural way, by putting $\mathbb{L}(\vec{O}) := \mathbb{L}(\bigcap\vec{O})$, where $\bigcap\vec{O} := O_1 \cap \dots \cap O_n$. A *learning model* $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ is a learning frame $(X, \mathcal{O}, \mathbb{L})$ together with a valuation map $\|\cdot\| : \mathrm{Prop} \cup \mathrm{Prop}_{\mathcal{O}} \rightarrow \mathcal{P}(X)$ as above; equivalently, it consists of an intersection model $(X, \mathcal{O}, \|\cdot\|)$ together with a learner $\mathbb{L}$, as defined above.

Intuitively, the states in X represent possible worlds. The tautological evidence $X = \bigcap\emptyset$ represents the state of 'no information' (before anything is observed), while the contradictory evidence $\emptyset$ represents inconsistent information. Finally, $\mathbb{L}(O)$ represents the learner's conjecture after observing O, while $\mathbb{L}(O_1, \dots, O_n) = \mathbb{L}(O_1 \cap \dots \cap O_n)$ represents the conjecture after observing a finite sequence of observations $O_1, \dots, O_n$. The fact that $\mathcal{O}$ is closed under finite intersections is important here for identifying any finite sequence $O_1, \dots, O_n$ with a single observation $O = O_1 \cap \dots \cap O_n \in \mathcal{O}$.

Epistemic Scenarios. As in Subset Space Semantics, the formulas of our logic are not interpreted at possible worlds, but at so-called *epistemic scenarios*, i.e., pairs (x, U) of an ontic state $x \in X$ and an information state $U \in \mathcal{O}$ such that $x \in U$. Therefore, only the truthful observations about the actual state play a role in the evaluation of formulas. We denote by $ES(\mathcal{M}) := \{(x, U) \mid x \in U \in \mathcal{O}\}$ the set of all epistemic scenarios of a learning model $\mathcal{M}$.

Definition 4 *(Semantics).* Given a learning model $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and an epistemic scenario (x, U), the semantics of the language $\mathfrak{L}$ is given by a binary relation $(x, U) \models_{\mathcal{M}} \varphi$ between epistemic scenario and formulas, called the *satisfaction relation*, as well as a *truth set* (interpretation) $[\![\varphi]\!]^U_{\mathcal{M}} := \{x \in U \mid (x, U) \models_{\mathcal{M}} \varphi\}$, for all formulas φ. We typically omit the subscript, simply writing $(x, U) \models \varphi$ and $[\![\varphi]\!]^U$, whenever the model $\mathcal{M}$ is understood. The satisfaction relation is defined by the following recursive clauses:

$$
\begin{array}{lll}
(x, U) \models p & \text{iff} & x \in \|p\| \\
(x, U) \models o & \text{iff} & x \in \|o\| \\
(x, U) \models L(o_1, \ldots, o_n) & \text{iff} & x \in \mathbb{L}(U, \|o_1\|, \ldots, \|o_n\|) \\
(x, U) \models \neg\varphi & \text{iff} & (x, U) \not\models \varphi \\
(x, U) \models \varphi \wedge \psi & \text{iff} & (x, U) \models \varphi \text{ and } (x, U) \models \psi \\
(x, U) \models K\varphi & \text{iff} & (\forall y \in U)((y, U) \models \varphi) \\
(x, U) \models \Box\varphi & \text{iff} & (\forall O \in \mathscr{O})(x \in O \subseteq U \text{ implies } (x, O) \models \varphi) \\
 & \text{i.e.} & (\forall O \in \mathscr{O})(x \in O \text{ implies } (x, U \cap O) \models \varphi) \\
(x, U) \models [o]\varphi & \text{iff} & x \in \|o\| \text{ implies } (x, U \cap \|o\|) \models \varphi
\end{array}
$$

where $p \in \text{Prop}$, $o, o_1, \ldots, o_n \in \text{Prop}_{\mathscr{O}}$, and, as discussed before, $\mathbb{L}(O_1, \ldots, O_n) := \mathbb{L}(O_1 \cap \ldots \cap O_n)$. We say that a formula φ is *valid in a learning model* $\mathcal{M}$, and write $\mathcal{M} \models \varphi$, if $(x, U) \models_{\mathcal{M}} \varphi$ for all epistemic scenarios $(x, U) \in ES(\mathcal{M})$. We say φ is *validable in an intersection model* $(X, \mathscr{O}, \|\cdot\|)$, and write $(X, \mathscr{O}, \|\cdot\|) \models \varphi$, if there exists some learner $\mathbb{L} : \mathscr{O} \to \mathcal{P}(X)$ such that φ is valid in the learning model $(X, \mathscr{O}, \mathbb{L}, \|\cdot\|)$. We say φ is *valid*, and write $\models \varphi$, if it is valid in *all* learning models.

Abbreviations: For any string $\vec{o} = (o_1, \ldots, o_n) \in \text{Prop}^*_{\mathscr{O}}$ of observational variables, and any formula φ we set:

$$
\begin{aligned}
\bigwedge \vec{o} &:= o_1 \wedge \ldots \wedge o_n \text{ (with the convention that } \bigwedge \lambda := \top) \\
\vec{o} \Leftrightarrow \vec{u} &:= K\left((\bigwedge \vec{o}) \leftrightarrow (\bigwedge \vec{u})\right) \text{ (extensional equivalence of observations)} \\
[\vec{o}]\varphi &:= [o_1] \ldots [o_n]\varphi \text{ (with the convention that } [\lambda]\varphi := \varphi\text{); similarly for } \langle \vec{o} \rangle) \\
B^{\vec{o}}\varphi &:= K(L(\vec{o}) \to \varphi) \\
B\varphi &:= B^{\lambda}\varphi,
\end{aligned}
$$

where λ is the empty string. We read $B\varphi$ as the 'observer *believes* φ' (given no observations), and $B^{\vec{o}}\varphi$ as 'the observer *believes* φ *conditional on evidence* $\vec{o}$'.

Lemma 5. *Given a learning model* $\mathcal{M} = (X, \mathscr{O}, \mathbb{L}, \|\cdot\|)$ *and* $(x, U) \in ES(\mathcal{M})$,

$$(x, U) \models [\vec{o}]\varphi \textit{ iff } x \in [\![\bigwedge \vec{o}]\!] \textit{ implies } (x, U \cap [\![\bigwedge \vec{o}]\!]) \models \varphi.$$

Proof. Let $\mathcal{M} = (X, \mathscr{O}, \mathbb{L}, \|\cdot\|)$, $(x, U) \in ES(\mathcal{M})$ and $\vec{o} = (o_1, \ldots, o_n)$.

Observe that $(*)$ $[\![\bigwedge \vec{o}]\!] = \bigcap_{1 \le i \le n} \|o_i\|$. Therefore, we obtain the following:

$$
\begin{aligned}
(x, U) \models [\vec{o}]\varphi \text{ iff } & (x, U) \models [o_1] \ldots [o_n]\varphi && \text{(by the Definition of } [\vec{o}]) \\
\text{iff } & x \in \|o_1\| \text{ implies } (x, U \cap \|o_1\|) \models [o_2] \ldots [o_n]\varphi && \text{(by the semantics of } [o]) \\
\text{iff } & x \in \|o_1\| \text{ implies } (x \in \|o_2\| \text{ implies } (x, U \cap \|o_1\| \cap \|o_2\|) \models [o_3] \ldots [o_n]\varphi) && \text{(by the semantics of } [o]) \\
\text{iff } & x \in \|o_1\| \cap \|o_2\| \text{ implies } (x, U \cap \|o_1\| \cap \|o_2\|) \models [o_3] \ldots [o_n]\varphi \\
\text{iff } & x \in [\![\bigwedge \vec{o}]\!] \text{ implies } (x, U \cap [\![\bigwedge \vec{o}]\!]) \models \varphi && \text{(by repetitive applications of the same steps and by } (*))
\end{aligned}
$$

□

2.2. Axiomatisation and proof system

We will now provide the formal definition of our proposed system **L** of the Dynamic Logic for Learning Theory (DLLT) by listing the axioms and derivation rules, see Table 1 below. Given a formula $\varphi \in \mathfrak{L}$, we denote by P_φ and O_φ the set of all propositional variables and observational variables respectively occurring in φ (we will use the same notation for the necessity and possibility forms defined below).

The $S5$ axioms for epistemic modality K expresses that our notion of certain, infallible knowledge is factive and (positively and negatively) introspective. Given that observational variables behave like the propositional variables, the reduction axioms are as in Public Announcement Logic [20]. The learning axioms (CC), (EC) and (SP) express pre-conditions in formal learning theory on *observations*, namely that they are truthful observations about the world (CC); that the history of observations is irrelevant for the learner, what is relevant is the extensional evidence provided by observations (EC); and that conjectures fit what is observed (SP). Since the effort modality □ quantifies over possible observations, we could think of the Effort axiom (□-Ax) and the Effort rule (□-Rule) as elimination and introduction rules for □. The former one expresses

Table 1
The axiom schemas for the Dynamic Logic of Learning Theory, **L**.

	Basic axioms:	
(P)	all instantiations of propositional tautologies	
(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	
(T_K)	$K\varphi \to \varphi$	
(4_K)	$K\varphi \to KK\varphi$	
(5_K)	$\neg K\varphi \to K\neg K\varphi$	
($K_{[o]}$)	$[o](\psi \to \chi) \to ([o]\psi \to [o]\chi)$	
	Basic rules:	
(MP)	From φ and $\varphi \to \psi$, infer ψ	
(Nec_K)	From φ, infer $K\varphi$	
($Nec_{[o]}$)	From φ, infer $[o]\varphi$	
	Learning axioms:	
(CC)	$(\bigwedge \vec{o}) \to \langle K \rangle L(\vec{o})$	Consistency of Conjectures
(EC)	$(\vec{o} \Leftrightarrow \vec{u}) \to (L(\vec{o}) \leftrightarrow L(\vec{u}))$	Extensionality of Conjectures
(SP)	$L(\vec{o}) \to \bigwedge \vec{o}$	Success Postulate
	Reduction axioms:	
(R_p)	$[o]p \leftrightarrow (o \to p)$	
(R_u)	$[o]u \leftrightarrow (o \to u)$	
(R_L)	$[o]L(\vec{u}) \leftrightarrow (o \to L(o, \vec{u}))$	
($R_\neg$)	$[o]\neg\psi \leftrightarrow (o \to \neg[o]\psi)$	
(R_K)	$[o]K\psi \leftrightarrow (o \to K[o]\psi)$	
($R_\Box$)	$[o]\Box\psi \leftrightarrow \Box[o]\psi$	
	Effort axiom and rule:	
($\Box$-Ax)	$\Box\varphi \to [\vec{o}]\varphi$, for $\vec{o} \in \mathrm{Prop}^*_{\mathcal{O}}$	
($\Box$-Rule)	From $\psi \to [o]\varphi$, infer $\psi \to \Box\varphi$, where $o \notin O_\psi \cup O_\varphi$	

the fact that if a property is stably true then it holds after any observations. Finally, the latter states that if a property holds after any arbitrary observation, it is stably true.

So each of our axioms is easily readable and has a transparent and intuitive interpretation, in contrast to other axiomatisations of (the less expressive) Subset Space Logic over intersection spaces (i.e., the $\mathbb{L}$-free analogues of our models). Having such a simple axiomatisation is one of the advantages brought by the addition of dynamic observation modalities. See more discussion of this issue in Section 5.

Before moving on to further results, we briefly recall the following usual definitions from basic modal logic (see, e.g., [9]). An **L**-*derivation/proof* is a finite sequence of formulas such that each element of the sequence is either an axiom of **L**, or obtained from the previous formulas in the sequence by one of the inference rules. We call a formula $\varphi \in \mathfrak{L}$ *provable* in **L**, or, equivalently, a *theorem* of **L**, if it is the last formula of some **L**-proof. In this case, we write $\vdash \varphi$ (or, equivalently, $\varphi \in \mathbf{L}$). For any set of formulas $\Gamma \subseteq \mathfrak{L}$ and any formula $\varphi \in \mathfrak{L}$, we write $\Gamma \vdash \varphi$ if there exist finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi$. We say that Γ is **L**-*consistent* if $\Gamma \nvdash \bot$, and **L**-*inconsistent* otherwise. A formula φ is *consistent with* Γ if $\Gamma \cup \{\varphi\}$ is **L**-consistent (or, equivalently, if $\Gamma \nvdash \neg\varphi$). We drop mention of **L** when it is contextually clear.

We now give derivable reduction laws for conjunction ($R_\wedge$) and strings of observational variables in $\mathrm{Prop}^*_{\mathcal{O}}$.

Proposition 6 *(Reduction laws for strings of observational variables). The following reduction laws are provable in* **L** *for all* $\varphi \in \mathfrak{L}$*:*

1. $[u](\varphi \wedge \psi) \leftrightarrow ([u]\varphi \wedge [u]\psi)$ *(denoted by $R_\wedge$)*
2. $[u](\varphi \to \psi) \leftrightarrow ([u]\varphi \to [u]\psi)$
3. $\langle o \rangle \psi \leftrightarrow (o \wedge [o]\psi)$
4. $[\vec{u}]p \leftrightarrow (\bigwedge \vec{u} \to p)$
5. $[\vec{u}]o \leftrightarrow (\bigwedge \vec{u} \to o)$
6. $[\vec{u}]L(\vec{o}) \leftrightarrow (\bigwedge \vec{u} \to L(\vec{u}, \vec{o}))$
7. $[\vec{u}]\neg\varphi \leftrightarrow (\bigwedge \vec{u} \to \neg[\vec{u}]\varphi)$
8. $[\vec{u}]K\varphi \leftrightarrow (\bigwedge \vec{u} \to K[\vec{u}]\varphi)$
9. $[\vec{u}]\Box\varphi \leftrightarrow \Box[\vec{u}]\varphi$
10. $[\vec{u}](\varphi \wedge \psi) \leftrightarrow ([\vec{u}]\varphi \wedge [\vec{u}]\psi)$
11. $[\vec{u}](\varphi \to \psi) \leftrightarrow ([\vec{u}]\varphi \to [\vec{u}]\psi)$
12. $\langle \vec{o} \rangle \psi \leftrightarrow \left(\bigwedge \vec{o} \wedge [\vec{o}]\psi\right)$

Proof. (1) follows from ($K_{[o]}$) and ($Nec_{[o]}$) in a standard way. (2) follows from ($K_{[o]}$), ($R_\neg$), ($R_\wedge$) similarly to the proof of [25, Exercise 4.50, p. 252]. (3) follows from the definition $\langle o \rangle \psi := \neg[o]\neg\psi$ and ($R_\neg$).

The proofs of (4)-(9) follow by induction on the length of $\vec{u}$. For the base case $n = 1$, we simply use the corresponding reduction axiom for one observational variable. We provide the proof of the inductive step only for $[\vec{u}]p \leftrightarrow (\bigwedge \vec{u} \to p)$. For the other equivalences the inductive step follows exactly with the same procedure by using the corresponding reduction axiom.

$\vdash [\vec{u}]p \leftrightarrow (\bigwedge \vec{u} \to p)$
Suppose that $\vec{u} := u_{n+1}, u_n, \ldots, u_2, u_1$ and consider the following *induction hypothesis*, (**IH**) : for every $\vec{u}$ of length less than $n + 1$, it follows that $\vdash [\vec{u}]p \leftrightarrow (\bigwedge \vec{u} \to p)$.
We will prove that the conclusion holds for an arbitrary $\vec{u}$ of length $n + 1$. Let $\vec{u} = u_{n+1}, u_n, \ldots, u_1$ and we denote $\vec{v} := u_n, \ldots, u_1$.

From left to right:

1. $\vdash [\vec{u}]p \to [u_{n+1}][\vec{v}]p$ (by the definition of $[\vec{u}]$)
2. $\vdash [u_{n+1}][\vec{v}]p \to [u_{n+1}](\bigwedge \vec{v} \to p)$ (IH on $\vec{v}$)
3. $\vdash [u_{n+1}, \vec{v}]p \to ([u_{n+1}] \bigwedge \vec{v} \to [u_{n+1}]p)$ ($K_{[o]}$)
4. $\vdash ([u_{n+1}] \bigwedge \vec{v} \to [u_{n+1}]p) \to ([u_{n+1}] \bigwedge \vec{v} \to (u_{n+1} \to p))$ (R_p, P)
5. $\vdash ([u_{n+1}] \bigwedge \vec{v} \to (u_{n+1} \to p)) \to (([u_{n+1}] \bigwedge \vec{v} \wedge u_{n+1}) \to p)$ (P)
6. $\vdash (([u_{n+1}] \bigwedge \vec{v} \wedge u_{n+1}) \to p) \to ((\bigwedge \vec{v} \wedge u_{n+1}) \to p)$ ($R_\wedge$ and R_u)
7. $\vdash [\vec{u}]p \to (\bigwedge \vec{u} \to p)$ (1 to 6)

From right to left:

$\vdash (\bigwedge \vec{u} \to p) \to (u_{n+1} \wedge \bigwedge \vec{v} \to p)$ (P)

$\vdash (u_{n+1} \wedge \bigwedge \vec{v} \to p) \to (u_{n+1} \to (\bigwedge \vec{v} \to p))$ (P)

$\vdash (u_{n+1} \to (\bigwedge \vec{v} \to p)) \to (u_{n+1} \to [\vec{v}]p)$ (IH on $\vec{v}$)

$\vdash [u_{n+1}]((u_{n+1} \to (\bigwedge \vec{v} \to p)) \to (u_{n+1} \to [\vec{v}]p))$ ($Nec_{[o]}$)

$\vdash [u_{n+1}](u_{n+1} \to (\bigwedge \vec{v} \to p)) \to [u_{n+1}](u_{n+1} \to [\vec{v}]p)$ ($K_{[o]}$)

$\vdash ([u_{n+1}]u_{n+1} \to [u_{n+1}](\bigwedge \vec{v} \to p)) \to ([u_{n+1}]u_{n+1} \to [u_{n+1}][\vec{v}]p)$ (Proposition 6.2)

$\vdash (u_{n+1} \to u_{n+1}) \to ([u_{n+1}] \bigwedge \vec{v} \to (u_{n+1} \to p)) \to ((u_{n+1} \to u_{n+1}) \to [u_{n+1}][\vec{v}]p)$ ($K_{[o]}$, R_u and R_p)

$\vdash (((u_{n+1} \to u_{n+1}) \wedge ([u_{n+1}] \bigwedge \vec{v}) \wedge u_{n+1}) \to p) \to ((u_{n+1} \to u_{n+1}) \to [u_{n+1}][\vec{v}]p)$ (P)

$\vdash (([u_{n+1}] \bigwedge \vec{v}) \wedge u_{n+1}) \to p) \to [u_{n+1}][\vec{v}]p$ (P)

$\vdash ((\bigwedge \vec{v} \wedge u_{n+1}) \to p) \to [u_{n+1}][\vec{v}]p$ ($R_\wedge$, R_u, and P)

$\vdash (\bigwedge \vec{u} \to p) \to [\vec{u}]p$ (P)

Similarly, the proofs of (10), (11), and (12) follow from (1), (2), and (1), respectively, and the definition of strings of observations for $\vec{o}$ and $\vec{u}$. □

In our framework, belief (B) and conditional beliefs ($B^{\vec{o}}\varphi$) are defined in terms of the operators K and L. The axiomatic system **L** given in Table 1 over the language $\mathfrak{L}$ can therefore derive the properties describing the type of belief and conditional belief modalities we intend to formalise in this paper. More precisely, as stated in Proposition 7, the system **L** yields the standard belief system KD45 for B. More generally, if we replace the D axiom for a 'weaker' version $D' := \langle K \rangle (\bigwedge \vec{o}) \to \neg B^{\vec{o}} \bot$ then we have a weak version of the system KD45, denoted by wKD45, for conditional beliefs $B^{\vec{o}}$.

Proposition 7 *(wKD45 axioms and rules for conditional belief). The standard axioms and rules of the doxastic logic KD45 are derivable for our belief operator B in the system* **L**. *More generally, the following axioms and rules of the weaker system wKD45 are derivable for our conditional belief operator $B^{\vec{o}}$ in the system* **L**:

($Nec_{B^{\vec{o}}}$) *From φ infer $B^{\vec{o}}\varphi$.*
($K_{B^{\vec{o}}}$) $B^{\vec{o}}(\varphi \to \psi) \to (B^{\vec{o}}\varphi \to B^{\vec{o}}\psi)$
($D'_{B^{\vec{o}}}$) $\langle K \rangle (\bigwedge \vec{o}) \to \neg B^{\vec{o}} \bot$
($4_{B^{\vec{o}}}$) $B^{\vec{o}}\varphi \to B^{\vec{o}}(B^{\vec{o}}\varphi)$
($5_{B^{\vec{o}}}$) $\neg B^{\vec{o}}\varphi \to B^{\vec{o}}(\neg B^{\vec{o}}\varphi)$

Proof. We prove ($Nec_{B^{\vec{o}}}$), ($K_{B^{\vec{o}}}$), ($D'_{B^{\vec{o}}}$), and ($4_{B^{\vec{o}}}$). The derivation of ($5_{B^{\vec{o}}}$) follows similarly to the case for ($4_{B^{\vec{o}}}$). Recall that $B^{\vec{o}}\varphi := K(L(\vec{o}) \to \varphi)$ and let $\vec{o} \in Prop^*_{\mathcal{O}}$.

($Nec_{B^{\vec{o}}}$): From $\vdash \varphi$ infer $\vdash B^{\vec{o}}\varphi$.

1. $\vdash \varphi$ (assumption)
2. $\vdash L(\vec{o}) \to \varphi$ (1, classical propositional logic (P))
3. $\vdash K(L(\vec{o}) \to \varphi)$ (Nec_K)

($K_{B^{\vec{o}}}$): we need to show that

$$\vdash K(L(\vec{o}) \to (\varphi \to \psi)) \to (K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to \psi)).$$

1. $\vdash (L(\vec{o}) \to (\varphi \to \psi)) \to ((L(\vec{o}) \to \varphi) \to (L(\vec{o}) \to \psi))$ (P)
2. $\vdash K((L(\vec{o}) \to (\varphi \to \psi)) \to ((L(\vec{o}) \to \varphi) \to (L(\vec{o}) \to \psi)))$ (Nec_K)
3. $\vdash K(L(\vec{o}) \to (\varphi \to \psi)) \to K((L(\vec{o}) \to \varphi) \to (L(\vec{o}) \to \psi))$ (K_K)
4. $\vdash K(L(\vec{o}) \to (\varphi \to \psi)) \to (K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to \psi))$ (K_K, 3, MP)

($D'_{B^{\vec{o}}}$): we need to show that

$$\vdash \langle K\rangle(\bigwedge \vec{o}) \to \langle K\rangle L(\vec{o}).$$

1. $\vdash (\bigwedge \vec{o}) \to \langle K\rangle L(\vec{o})$ (CC)
2. $\vdash K\neg L(\vec{o}) \to \neg(\bigwedge \vec{o})$ (contraposition of CC)
3. $\vdash K(K\neg L(\vec{o}) \to \neg(\bigwedge \vec{o}))$ (Nec_K)
4. $\vdash KK\neg L(\vec{o}) \to K\neg(\bigwedge \vec{o})$ (K_K, 3, MP)
5. $\vdash K\neg L(\vec{o}) \to K\neg(\bigwedge \vec{o})$ (4_K)
6. $\vdash \langle K\rangle(\bigwedge \vec{o}) \to \langle K\rangle L(\vec{o})$ (contraposition of 5)

($4_{B^{\vec{o}}}$): we need to show that

$$\vdash K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to K(L(\vec{o}) \to \varphi)).$$

1. $\vdash K(L(\vec{o}) \to \varphi) \to (L(\vec{o}) \to K(L(\vec{o}) \to \varphi))$ (P)
2. $\vdash KK(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to K(L(\vec{o}) \to \varphi))$ (Nec_K, K_K, 1, MP)
3. $\vdash K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to K(L(\vec{o}) \to \varphi))$ (4_K)

($5_{B^{\vec{o}}}$): we need to show that

$$\vdash \neg K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to \neg K(L(\vec{o}) \to \varphi)).$$

1. $\vdash \neg K(L(\vec{o}) \to \varphi) \to (L(\vec{o}) \to \neg K(L(\vec{o}) \to \varphi))$ (P)
2. $\vdash K\neg K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to \neg K(L(\vec{o}) \to \varphi))$ (Nec_K, K_K, 1, MP)
3. $\vdash \neg K(L(\vec{o}) \to \varphi) \to K(L(\vec{o}) \to \neg K(L(\vec{o}) \to \varphi))$ (5_K)

The KD45 axioms and rules for B follows from the system wKD45 as a special case when $\vec{o} = \lambda$. □

Proposition 8. *The S4 axioms for the effort modality □ are derivable in* **L***.*

Proof. The derivation of ($Nec_\Box$) easily follows from ($Nec_{[o]}$) and (□-Rule). The T-axiom for □ follows from (□-Ax) for $\vec{o} = \lambda$. For the K-axiom:

1. $\vdash (\Box(\varphi \to \psi) \land \Box\varphi) \to ([o](\varphi \to \psi) \land [o]\varphi)$ (□-Ax, for some $o \notin O_\varphi \cup O_\psi$)
2. $\vdash ([o](\varphi \to \psi) \land [o]\varphi) \to [o]\psi$ ($K_{[o]}$)
3. $\vdash (\Box(\varphi \to \psi) \land \Box\varphi) \to [o]\psi$ (P, 1, 2)
4. $\vdash (\Box(\varphi \to \psi) \land \Box\varphi) \to \Box\psi$ (□-Rule, $o \notin O_\varphi \cup O_\psi$)

For the 4-axiom:

1. $\vdash \Box\varphi \to [o,u]\varphi$ ($\Box$-Ax, for some $o, u \notin O_\varphi$)
2. $\vdash \Box\varphi \to [o][u]\varphi$ (by the definition of $[o,u]$)
3. $\vdash \Box\varphi \to \Box[u]\varphi$ ($\Box$-Rule)
4. $\vdash \Box\varphi \to [u]\Box\varphi$ ($R_\Box$)
5. $\vdash \Box\varphi \to \Box\Box\varphi$ ($\Box$-Rule)

□

3. Soundness and completeness

In this section we prove soundness and completeness. Note that, although our logic is more expressive than Subset Space Logic (interpreted on intersection spaces), our completeness proof is *much simpler*, via a *canonical construction*: this is one of the advantages of having the (expressively redundant) dynamic observation modalities.

3.1. Soundness

We first prove soundness, for which we need the following lemma. Note that by the definition of the valuation $||\cdot||$ in a learning model $\mathcal{M}$, we have that for all $U \in \mathcal{O}$, $U \cap ||p|| = [\![p]\!]^U_{\mathcal{M}}$ and $U \cap ||o|| = [\![o]\!]^U_{\mathcal{M}}$, for all $p \in \text{Prop}$ and $o \in \text{Prop}_{\mathcal{O}}$.

Lemma 9. *Let $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and $\mathcal{M}' = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|')$ be two learning models and $\varphi \in \mathfrak{L}$, such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some $o \notin O_\varphi$. Then, for all $U \in \mathcal{O}$, we have $[\![\varphi]\!]^U_{\mathcal{M}} = [\![\varphi]\!]^U_{\mathcal{M}'}$.*

Proof. The proof follows by subformula induction on φ. Let $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ and $\mathcal{M}' = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|')$ be two learning models such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some $o \notin O_\varphi$, and let $U \in \mathcal{O}$.

Base case, $\varphi := q \in \text{Prop}$, follows by the fact that q is not an observational variable, thus, for all $q \in P_\varphi$ we have that $||q|| = ||q||'$. Since $\mathcal{M}$ and $\mathcal{M}'$ have the same set of opens $\mathcal{O}$, for all $U \in \mathcal{O}$ we have that $[\![q]\!]^U_{\mathcal{M}} = U \cap ||q|| = U \cap ||q||' = [\![q]\!]^U_{\mathcal{M}'}$.
Base case $\varphi := o \in \text{Prop}_{\mathcal{O}}$. Since $o \in O_\varphi$, we have that $||o|| = ||o||'$. By the same reasoning as above, $[\![o]\!]^U_{\mathcal{M}} = [\![o]\!]^U_{\mathcal{M}'}$.
Case $\varphi := L(\vec{o})$
Let $\vec{o} = (o_1, \dots, o_n)$ and note that $O_{L(\vec{o})} = O_{\vec{o}} = \{o_1, \dots, o_n\}$. Thus, $||o_i|| = ||o_i||'$ for every $1 \le i \le n$. Therefore, since $\mathbb{L}$ and $\mathcal{O}$ are the same in both models we have that $[\![L(\vec{o})]\!]^U_{\mathcal{M}} = \mathbb{L}(U, ||o_1||, \dots ||o_n||) = \mathbb{L}(U, ||o_1||', \dots ||o_n||') = [\![L(\vec{o})]\!]^U_{\mathcal{M}'}$.
The cases for Booleans $\varphi := \neg\psi$ and $\varphi := \psi \wedge \chi$ are straightforward.
Case $\varphi := K\psi$
Note that $O_{K\psi} = O_\psi$. Then, by induction hypothesis (IH), we have that $[\![\psi]\!]^U_{\mathcal{M}} = [\![\psi]\!]^U_{\mathcal{M}'}$. We have two cases (1) if $U = [\![\psi]\!]^U_{\mathcal{M}} = [\![\psi]\!]^U_{\mathcal{M}'}$, then $[\![K\psi]\!]^U_{\mathcal{M}} = [\![K\psi]\!]^U_{\mathcal{M}'} = U$, and (2) if $[\![\psi]\!]^U_{\mathcal{M}} = [\![\psi]\!]^U_{\mathcal{M}'} \neq U$, then $[\![K\psi]\!]^U_{\mathcal{M}} = [\![K\psi]\!]^U_{\mathcal{M}'} = \emptyset$.
Case $\varphi := [o]\psi$
Note that $O_{[o]\psi} = \{o\} \cup O_\psi$. Suppose $x \in [\![[o]\psi]\!]^U_{\mathcal{M}}$ and $x \in ||o||'$. Since $o \in O_{[o]\psi}$, we have $||o|| = ||o||'$. Therefore, from $x \in [\![[o]\psi]\!]^U_{\mathcal{M}}$ and $x \in ||o||$, we have $(x, U \cap ||o||) \models_{\mathcal{M}} \psi$. Since $U \cap ||o|| = U \cap ||o||'$ and $U \cap ||o||' \in \mathcal{O}$ (as $\mathcal{O}$ is closed under finite intersections), by IH on ψ, we obtain $(x, U \cap ||o||') \models_{\mathcal{M}'} \psi$. We then conclude that $x \in [\![[o]\psi]\!]^U_{\mathcal{M}'}$. The other direction follows similarly.
Case $\varphi := \Box\psi$
Suppose $x \in [\![\Box\psi]\!]^U_{\mathcal{M}}$. This means, by the semantics of $\Box$, that for all $V \in \mathcal{O}$ with $x \in V \subseteq U$ we have that $(x, V) \models_{\mathcal{M}} \psi$, i.e., that $x \in [\![\psi]\!]^V_{\mathcal{M}}$. Therefore, by IH and the fact that $O_{\Box\psi} = O_\psi$, we obtain $x \in [\![\psi]\!]^V_{\mathcal{M}'}$ for all $V \subseteq U$ with $x \in V$. Since $\mathcal{M}$ and $\mathcal{M}'$ carry exactly the same collection $\mathcal{O}$, we conclude that $x \in [\![\Box\psi]\!]^U_{\mathcal{M}'}$. The opposite direction follows similarly. □

Theorem 1. *The system* **L** *in Table 1 is sound wrt the class of learning models.*

Proof. The soundness proof follows standardly via validity check. We here only present the validity proofs for the Learning axioms (all), the Reduction axioms R_L, $R_\Box$, as well as the Effort-axiom ($\Box$-Ax) and Effort-rule ($\Box$-Rule). The other cases follow standardly.

Let $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \|\cdot\|)$ be a learning model and $(x, U) \in ES(\mathcal{M})$.

Learning axioms:

(CC): Suppose $(x, U) \models \bigwedge \vec{o}$ with $\vec{o} = (o_1, \ldots, o_n)$. We want to show that $(x, U) \models \langle K \rangle L(\vec{o})$, i.e., that there is a $y \in U$ s.t. $(y, U) \models L(\vec{o})$, i.e., by the semantic definition of L, that there is a $y \in U$ s.t. $y \in \mathbb{L}(U, ||o_1||, \ldots, ||o_n||)$. Recall that $\mathbb{L}(U, ||o_1||, \ldots, ||o_n||) := \mathbb{L}(U \cap \bigcap_{1 \le i \le n} ||o_i||)$, $\mathbb{L}(O) \subseteq O$ for every $O \in \mathcal{O}$ (Definition 3.1), and $\mathbb{L}(O) \neq \emptyset$ if $O \neq \emptyset$ (Definition 3.2). Since $x \in [\![o_i]\!]^U$ for every $1 \le i \le n$ and $x \in U$, we have that $\mathbb{L}(U, ||o_1||, \ldots, ||o_n||) \neq \emptyset$, i.e., there is $y \in \mathbb{L}(U, ||o_1||, \ldots, ||o_n||)$. Then, by the semantics of L, we obtain that $(y, U) \models L(\vec{o})$. Thus $(x, U) \models \langle K \rangle L(\vec{o})$.
(EC): Suppose $(x, U) \models \vec{o} \Leftrightarrow \vec{u}$. This means that $x \in [\![K(\bigwedge \vec{o} \leftrightarrow \bigwedge \vec{u})]\!]^U$. Therefore, by the semantics of K, we obtain that $[\![\bigwedge \vec{o}]\!]^U = [\![\bigwedge \vec{u}]\!]^U$. This means that $[\![\bigwedge \vec{o}]\!]^U = U \cap ||o_1|| \cap \ldots ||o_n|| = [\![\bigwedge \vec{u}]\!]^U = U \cap ||u_1|| \cap \ldots ||u_k||$, where $\vec{o} = (o_1, \ldots, o_n)$ and $\vec{u} = (u_1, \ldots, u_k)$. Recall that $\mathbb{L}(U, ||o_1||, \ldots, ||o_n||) := \mathbb{L}(U \cap \bigcap_{1 \le i \le n} ||o_i||)$ and $\mathbb{L}(U, ||u_1||, \ldots, ||u_k||) := \mathbb{L}(U \cap \bigcap_{1 \le i \le k} ||u_i||)$. Hence, since $\mathbb{L}$ is a function, we obtain $\mathbb{L}(U, ||o_1||, \ldots, ||o_n||) = \mathbb{L}(U, ||u_1||, \ldots, ||u_k||)$. Therefore, by the semantics of L, we conclude that $[\![L(\vec{o})]\!]^U = [\![L(\vec{u})]\!]^U$, i.e., that $(x, U) \models L(\vec{o}) \leftrightarrow L(\vec{u})$.
(SP): Suppose $(x, U) \models L(\vec{o})$. By the semantics and the fact that $\mathbb{L}(U, ||o_1||, \ldots, ||o_n||) := \mathbb{L}(U \cap \bigcap_{1 \le i \le n} ||o_i||)$ where $\vec{o} = (o_1, \ldots, o_n)$, we have $x \in \mathbb{L}(U \cap \bigcap_{1 \le i \le n} ||o_i||) = \mathbb{L}(U \cap [\![\bigwedge \vec{o}]\!]^U)$. Since we have that $\mathbb{L}(U \cap [\![\bigwedge \vec{o}]\!]^U) \subseteq U \cap [\![\bigwedge \vec{o}]\!]^U \subseteq [\![\bigwedge \vec{o}]\!]^U$ (Definition 3.1), we obtain that $x \in [\![\bigwedge \vec{o}]\!]^U$, i.e., that $(x, U) \models \bigwedge \vec{o}$.
Reduction Axioms:
(R_L): From left-to-right: Suppose $(x, U) \models [o]L(\vec{u})$. This means that $x \in ||o||$ implies $(x, U \cap ||o||) \models L(\vec{u})$, i.e., that $x \in ||o||$ implies $x \in \mathbb{L}(U \cap ||o|| \cap \bigcap_{1 \le i \le k} ||u_i||)$ where $\vec{u} = (u_1, \ldots, u_k)$. Then, by the semantics of L, we obtain that $(x, U) \models o \to L(o, \vec{u})$. The opposite direction follows similarly.
($R_\Box$):

$$
\begin{aligned}
&(x, U) \models [o]\Box\varphi \\
&\text{iff } x \in ||o|| \text{ implies } (x, U \cap ||o||) \models \Box\varphi && \text{(by the semantics of } [o]) \\
&\text{iff } x \in ||o|| \text{ implies } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x, (U \cap ||o||) \cap O) \models \varphi) && \text{(by the semantics of } \Box) \\
&\text{iff } (\forall O \in \mathcal{O})((x \in ||o|| \text{ and } x \in O) \text{ implies } (x, (U \cap ||o||) \cap O) \models \varphi) \\
&\text{iff } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x \in ||o|| \text{ implies } (x, (U \cap O) \cap ||o||) \models \varphi)) \\
&\text{iff } (\forall O \in \mathcal{O})(x \in O \text{ implies } (x, U \cap O) \models [o]\varphi) && \text{(by the semantics of } [o]) \\
&\text{iff } (x, U) \models \Box[o]\varphi && \text{(by the semantics of } \Box)
\end{aligned}
$$

Effort Axiom and Rule:
($\Box$-Ax): Suppose $(x, U) \models \Box\varphi$. This mean, by the semantics of $\Box$, that for all $O \in \mathcal{O}$ with $x \in O$ we have $(x, U \cap O) \models \varphi$. In particular, since $||o|| \in \mathcal{O}$ for all $o \in Prop_{\mathcal{O}}$, we obtain that $x \in ||o||$ implies $(x, U \cap ||o||) \models \varphi$. Therefore, since $\mathcal{O}$ is closed under finite intersections, $x \in [\![\bigwedge \vec{o}]\!]$ implies $(x, U \cap [\![\bigwedge \vec{o}]\!]) \models \varphi$ for every $\vec{o} \in Prop^*_{\mathcal{O}}$. Therefore, by Lemma 5, we obtain that $(x, U) \models [\vec{o}]\varphi$ for every finite string $\vec{o}$ of observational variables.
($\Box$-Rule): Suppose towards a contradiction that $\models \psi \to [o]\varphi$ and $\not\models \psi \to \Box\varphi$ where $o \notin O_\psi \cup O_\varphi$. The latter means that there is a learning model $\mathcal{M}$ and $(x, U) \in ES(\mathcal{M})$ such that $(x, U) \not\models_{\mathcal{M}} \psi \to \Box\varphi$, i.e., $(x, U) \models_{\mathcal{M}} \psi$ and $(x, U) \not\models_{\mathcal{M}} \Box\varphi$. Thus, $(x, U) \models_{\mathcal{M}} \psi$ and $(x, U) \models_{\mathcal{M}} \Diamond\neg\varphi$. By the semantics of $\Box$, there exists a $U_0 \subseteq U$ s.t. $x \in U_0$ and $(x, U_0) \models_{\mathcal{M}} \neg\varphi$. Now consider the model $\mathcal{M}' = (X, \mathcal{O}, \mathbb{L}, ||\cdot||')$ where $||o||' = U_0$, $||u||' = ||u||$ for all $u \in Prop_{\mathcal{O}}$ such that $u \neq o$, and $||p||' = ||p||$ for all $p \in Prop$. It is easy to see that $\mathcal{M}' = (X, \mathcal{O}, \mathbb{L}, ||\cdot||')$ is a learning model. By Lemma 9, we have that $[\![\psi]\!]^U_{\mathcal{M}'} = [\![\psi]\!]^U_{\mathcal{M}}$ and $[\![\neg\varphi]\!]^{U_0}_{\mathcal{M}'} = [\![\neg\varphi]\!]^{U_0}_{\mathcal{M}}$. Therefore $(x, U) \models_{\mathcal{M}'} \psi$ and $(x, U_0) \models_{\mathcal{M}'} \neg\varphi$. Since $||o||' = U_0$, we have $(x, U) \models_{\mathcal{M}'} \psi$ and $(x, ||o||') \models_{\mathcal{M}'} \neg\varphi$. By the semantics of $\langle o \rangle$, we obtain $(x, U) \models_{\mathcal{M}'} \psi \wedge \langle o \rangle \neg\varphi$. Thus $(x, U) \not\models_{\mathcal{M}'} \psi \to [o]\varphi$, contradicting the validity of $\models \psi \to [o]\varphi$. $\Box$

3.2. Completeness

We now move to the completeness proof for our logic **L**, which will be shown via a 'simple' canonical model construction. But its simplicity is deceiving, due to two main technical differences between our construction and the standard canonical model from Modal Logic. First, this is *not* a relational (Kripke) model, but a *neighbourhood model*; so the closest analogue is the type of canonical construction used in Topological Modal Logic or Neighbourhood Semantics [1]. Second, the standard notion of maximally consistent theory (a set of formulas $\Gamma \subseteq \mathfrak{L}$ is consistent if Γ does not derive a contradiction, and it is maximally consistent if any consistent theory $\Gamma' \supseteq \Gamma$, $\Gamma' = \Gamma$) is *not* very useful for our logic, since such theories do not 'internalise' the $\Box$-Rule. To do this, we need instead to consider *'witnessed' (maximally consistent) theories*, in which every occurrence of a $\Diamond\varphi$ in any 'existential context' is 'witnessed' by some $\langle o \rangle\varphi$ (with o observational variable). The appropriate notion of 'existential contexts' is represented by *possibility forms*, in the following sense:

Definition 10 *('Pseudo-modalities': necessity and possibility forms).* The *set of necessity-form expressions* of our language is given by $NF_{\mathfrak{L}} := (\{\varphi \to \ \mid \varphi \in \mathfrak{L}\} \cup \{K\} \cup Prop_{\mathcal{O}})^*$. For any finite string $s \in NF_{\mathfrak{L}}$, we define pseudo-modalities $[s]$ (called *necessity form*) and $\langle s \rangle$ (called *possibility form*) that generalize our dynamic modalities $[o]$ and $\langle o \rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathfrak{L}$ to another formula $[s]\varphi \in \mathfrak{L}$, and respectively $\langle s \rangle\varphi \in \mathfrak{L}$. Necessity forms are defined recursively, by putting: $[\lambda]\varphi := \varphi$, $[s, \varphi \to]\varphi := [s](\varphi \to \varphi)$, $[s, K]\varphi := [s]K\varphi$, $[s, o]\varphi := [s][o]\varphi$. As for possibility forms, we put $\langle s\rangle\varphi := \neg[s]\neg\varphi$.

To illustrate, expression $[K, o, \Diamond p \to, u]$ constitutes a necessity form such that $[K, o, \Diamond p \to, u]\varphi = K[o](\Diamond p \to [u]\varphi)$.

Lemma 11. *For every necessity form* $[s]$*, there exist a string of observational variables* $\vec{o} \in \text{Prop}^*_{\mathcal{O}}$ *and a formula* $\psi \in \mathfrak{L}$*, with* $O_\psi \cup O_{\vec{o}} \subseteq O_s$*, such that for all* $\varphi \in \mathfrak{L}$*, we have*

$$\vdash [s]\varphi \text{ iff } \vdash \psi \to [\vec{o}]\varphi.$$

Proof. We proceed by induction on the structure of necessity forms. For $s := \lambda$, take $\psi := \top$ and $\vec{o} := \lambda$, then it follows from classical propositional logic.

$s := s', \eta \to$

$$\begin{aligned}
&\vdash [s', \eta \to]\varphi \text{ iff } \vdash [s'](\eta \to \varphi) && \text{(Definition 10)}\\
&\qquad \text{iff } \vdash \psi' \to [\vec{o}](\eta \to \varphi) && \text{(for some } \psi' \in \mathfrak{L} \text{ and } \vec{o} \in \text{Prop}^*_{\mathcal{O}} \text{ with } O_{\psi'} \cup O_{\vec{o}} \subseteq O_{s'}\text{, by IH)}\\
&\qquad \text{iff } \vdash \psi' \to ([\vec{o}]\eta \to [\vec{o}]\varphi) && \text{(Proposition 6.11)}\\
&\qquad \text{iff } \vdash (\psi' \wedge [\vec{o}]\eta) \to [\vec{o}]\varphi && \text{(P)}\\
&\qquad \text{iff } \vdash \psi \to [\vec{o}]\varphi && (\psi := \psi' \wedge [\vec{o}]\eta\text{, thus, } O_\psi \cup O_{\vec{o}} \subseteq O_s)
\end{aligned}$$

$s := s', K$

$$\begin{aligned}
&\vdash [s', K]\varphi \text{ iff } \vdash [s']K\varphi && \text{(Definition 10)}\\
&\qquad \text{iff } \vdash \psi' \to [\vec{o}]K\varphi && \text{(for some } \psi' \in \mathfrak{L} \text{ and } \vec{o} \in \text{Prop}^*_{\mathcal{O}} \text{ with } O_{\psi'} \cup O_{\vec{o}} \subseteq O_{s'}\text{, by IH)}\\
&\qquad \text{iff } \vdash \psi' \to (\bigwedge \vec{o} \to K[\vec{o}]\varphi) && \text{(Proposition 6.8)}\\
&\qquad \text{iff } \vdash (\psi' \wedge \bigwedge \vec{o}) \to K[\vec{o}]\varphi && \text{(P)}\\
&\qquad \text{iff } \vdash \langle K\rangle(\psi' \wedge \bigwedge \vec{o}) \to [\vec{o}]\varphi && \text{(pushing } K \text{ back with its dual } \langle K\rangle\text{, since } K \text{ is an S5 modality)}\\
&\qquad \text{iff } \vdash \psi \to [\vec{o}]\varphi && (\psi := \langle K\rangle(\psi' \wedge \bigwedge \vec{o}) \in \mathfrak{L}\text{, thus, } O_\psi \cup O_{\vec{o}} \subseteq O_s)
\end{aligned}$$

$s := s', u$

$$\begin{aligned}
&\vdash [s', u]\varphi \text{ iff } \vdash [s'][u]\varphi && \text{(Definition 10)}\\
&\qquad \text{iff } \vdash \psi' \to [\vec{o}][u]\varphi && \text{(for some } \psi' \in \mathfrak{L} \text{ and } \vec{o} \in \text{Prop}^*_{\mathcal{O}} \text{ with } O_{\psi'} \cup O_{\vec{o}} \subseteq O_{s'}\text{, by IH)}\\
&\qquad \text{iff } \vdash \psi' \to [\vec{o}, u]\varphi && \text{(by the definition of } [\vec{o}][u]\text{, and } (\vec{o}, u) \in \text{Prop}^*_{\mathcal{O}})
\end{aligned}$$

□

Lemma 12. *For all* $\vec{u}, \vec{o} \in \text{Prop}^*_{\mathcal{O}}$ *and* $\varphi \in \mathfrak{L}$*, we have* $\vdash [\vec{u}, \vec{o}]\varphi \leftrightarrow [\vec{o}, \vec{u}]\varphi$.

Proof. The proof follows by subformula induction on φ. Along the proof we will make use of the following observation: (a) $\vdash (\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \leftrightarrow (\bigwedge \vec{o} \wedge \bigwedge \vec{u})$, and the following *induction hypothesis* (**IH**): for all subformula ψ of φ, and all $\vec{u}, \vec{o} \in \text{Prop}^*_{\mathcal{O}}$, $\vdash [\vec{u}, \vec{o}]\psi \leftrightarrow [\vec{o}, \vec{u}]\psi$.

Base case $\varphi := v \in \text{Prop}_{\mathcal{O}}$ ($\varphi := p \in \text{Prop}$ follows analogously).

$$\begin{aligned}
&1. \vdash [\vec{u}, \vec{o}]v \leftrightarrow ((\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \to v) && \text{(Proposition 6.5)}\\
&2. \vdash ((\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \to v) \leftrightarrow ((\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \to v) && \text{(by obs. (a) above)}\\
&3. \vdash ((\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \to v) \leftrightarrow [\vec{o}, \vec{u}]v && \text{(Proposition 6.5)}\\
&4. \vdash [\vec{u}, \vec{o}]v \leftrightarrow [\vec{o}, \vec{u}]v && \text{(P, 1-3)}
\end{aligned}$$

Cases for Booleans and K follow from observation (a) and the corresponding reduction laws for strings of observations presented in Proposition 6 in a straightforward way. We here present the inductive cases for $\varphi := L(\vec{v})$, $\varphi := [\vec{v}]\psi$, and $\varphi := \Box\psi$.

Case $\varphi := L(\vec{v})$

1. $\vdash [\vec{u}, \vec{o}]L(\vec{v}) \leftrightarrow ((\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \to L(\vec{u}, \vec{o}, \vec{v}))$	(Proposition 6.6)
2. $\vdash \Big((\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \to L(\vec{u}, \vec{o}, \vec{v})\Big) \leftrightarrow \Big(\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \to L(\vec{u}, \vec{o}, \vec{v})\Big)$	(by obs. (a))
3. $\vdash (\bigwedge \vec{u} \wedge \bigwedge \vec{o} \wedge \bigwedge \vec{v}) \leftrightarrow (\bigwedge \vec{o} \wedge \bigwedge \vec{u} \wedge \bigwedge \vec{v})$	(by obs. (a))
4. $\vdash K\Big((\bigwedge \vec{u} \wedge \bigwedge \vec{o} \wedge \bigwedge \vec{v}) \leftrightarrow (\bigwedge \vec{o} \wedge \bigwedge \vec{u} \wedge \bigwedge \vec{v})\Big)$	(by Nec$_K$)
5. $\vdash L(\vec{u}, \vec{o}, \vec{v}) \leftrightarrow L(\vec{o}, \vec{u}, \vec{v})$	(EC and MP with 4)
6. $\vdash \Big((\bigwedge \vec{u} \wedge \bigwedge \vec{o}) \to L(\vec{u}, \vec{o}, \vec{v})\Big) \leftrightarrow \Big((\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \to L(\vec{o}, \vec{u}, \vec{v})\Big)$	(P, 2, 5)
7. $\vdash \Big((\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \to L(\vec{o}, \vec{u}, \vec{v})\Big) \leftrightarrow [\vec{o}, \vec{u}]L(\vec{v})$	(Proposition 6.6)
8. $\vdash [\vec{u}, \vec{o}]L(\vec{v}) \leftrightarrow [\vec{o}, \vec{u}]L(\vec{v})$	(P, 1, 6, 7)

Case $\varphi := [\vec{v}]\psi$

1. $\vdash [\vec{u}, \vec{o}][\vec{v}]\psi \leftrightarrow [\vec{u}, \vec{o}, \vec{v}]\psi$	(by the definition of $[\vec{o}]\varphi$)
2. $\vdash [\vec{u}, \vec{o}, \vec{v}]\psi \leftrightarrow [\vec{o}, \vec{v}, \vec{u}]\psi$	(IH, reorder $\vec{u}$ and $(\vec{o}, \vec{v})$)
3. $\vdash [\vec{o}, \vec{v}, \vec{u}]\psi \leftrightarrow [\vec{o}][\vec{v}, \vec{u}]\psi$	(by definition)
4. $\vdash [\vec{o}][\vec{v}, \vec{u}]\psi \leftrightarrow [\vec{o}][\vec{u}, \vec{v}]\psi$	(IH, reorder $\vec{v}$ and $\vec{u}$)
5. $\vdash [\vec{o}][\vec{u}, \vec{v}]\psi \leftrightarrow [\vec{o}, \vec{u}, \vec{v}]\psi$	(by the definition of $[\vec{o}]\varphi$)
6. $\vdash [\vec{o}, \vec{u}, \vec{v}]\psi \leftrightarrow [\vec{o}, \vec{u}][\vec{v}]\psi$	(by the definition of $[\vec{o}]\varphi$)
7. $\vdash [\vec{u}, \vec{o}][\vec{v}]\psi \leftrightarrow [\vec{o}, \vec{u}][\vec{v}]\psi$	(P, 1-6)

Case $\varphi := \Box\psi$

1. $\vdash [\vec{u}, \vec{o}]\Box\psi \leftrightarrow \Box[\vec{u}, \vec{o}]\psi$	(Proposition 6.9)
2. $\vdash \Box[\vec{u}, \vec{o}]\psi \leftrightarrow \Box[\vec{o}, \vec{u}]\psi$	(IH)
3. $\vdash \Box[\vec{o}, \vec{u}]\psi \leftrightarrow [\vec{o}, \vec{u}]\Box\psi$	(Proposition 6.9)
4. $\vdash [\vec{u}, \vec{o}]\Box\psi \leftrightarrow [\vec{o}, \vec{u}]\Box\psi$	(P, 1-3)

□

Lemma 13. *The following rule is admissible in* **L***:*

if $\vdash [s][o]\varphi$ *then* $\vdash [s]\Box\varphi$, *where* $o \notin O_s \cup O_\varphi$.

Proof. Suppose $\vdash [s][o]\varphi$ where $o \notin O_s \cup O_\varphi$. Then, by Lemma 11, there exist $\vec{u} \in \mathrm{Prop}^*_{\mathcal{O}}$ and $\psi \in \mathfrak{L}$ with $O_\psi \cup O_{\vec{u}} \subseteq O_s$ such that $\vdash \psi \to [\vec{u}][o]\varphi$. Thus we get $\vdash \psi \to [\vec{u}, o]\varphi$. By Lemma 12, we know that $\vdash [\vec{u}, o]\varphi \leftrightarrow [o, \vec{u}]\varphi$. Therefore, $\vdash \psi \to [\vec{u}, o]\varphi$ iff $\vdash \psi \to [o, \vec{u}]\varphi$. Hence we obtain $\vdash \psi \to [o, \vec{u}]\varphi$, i.e., $\vdash \psi \to [o][\vec{u}]\varphi$. Since $O_\psi \cup O_{\vec{u}} \subseteq O_s$, and so $o \notin O_\psi \cup O_{\vec{u}} \cup O_\varphi$. Therefore, by the Effort rule (□-Rule) we have $\vdash \psi \to \Box[\vec{u}]\varphi$, implying, by the reduction axiom ($R_\Box$), that $\vdash \psi \to [\vec{u}]\Box\varphi$. Applying again Lemma 11, we obtain $\vdash [s]\Box\varphi$. □

Definition 14. For every countable set O, let $\mathfrak{L}^{\mathrm{O}}$ be the language of the logic $\mathbf{L}^{\mathrm{O}}$ based only on the observational variables in O (i.e., having as set of observational variables $\mathrm{Prop}_{\mathcal{O}} := \mathrm{O}$). Let $NF^{\mathrm{O}}_{\mathfrak{L}}$ denote the set of necessity-form expressions of $\mathbf{L}^{\mathrm{O}}$ (i.e., necessity forms involving only observational variables in O). An O-*theory* is a *consistent* set of formulas in $\mathfrak{L}^{\mathrm{O}}$. Here, 'consistent' means consistent with respect to the axiomatisation **L** formulated for $\mathfrak{L}^{\mathrm{O}}$. A *maximal* O-*theory* is an O-theory Γ that is maximal with respect to ⊆ among all O-theories; in other words, Γ cannot be extended to another O-theory. An O-*witnessed theory* is an O-theory Γ such that, for every $s \in NF^{\mathrm{O}}_{\mathfrak{L}}$ and $\varphi \in \mathfrak{L}^{\mathrm{O}}$, if $\langle s\rangle\Diamond\varphi$ is consistent with Γ then there is $o \in \mathrm{O}$ such that $\langle s\rangle\langle o\rangle\varphi$ is consistent with Γ. A *maximal* O-*witnessed theory* Γ is an O-witnessed theory that is not a proper subset of any O-witnessed theory.

Lemma 15. *For every maximal* O*-witnessed theory* Γ, *and any* $\varphi, \psi \in \mathfrak{L}^{\mathrm{O}}$,

1. *either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$,

2. $\varphi \wedge \psi \in \Gamma$ *iff* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$,
3. $\varphi \in \Gamma$ *and* $\varphi \to \psi \in \Gamma$ *implies* $\psi \in \Gamma$.

Lemma 16. *For every* $\Gamma \subseteq \mathfrak{L}^{\mathrm{O}}$, *if* Γ *is an* O*-theory and* $\Gamma \nvdash \neg\varphi$ *for some* $\varphi \in \mathfrak{L}^{\mathrm{O}}$, *then* $\Gamma \cup \{\varphi\}$ *is an* O*-theory. Moreover, if* Γ *is* O*-witnessed, then* $\Gamma \cup \{\varphi\}$ *is also* O*-witnessed.*

Proof. Let $\Gamma \subseteq \mathfrak{L}^{\mathrm{O}}$ be an O-theory and $\varphi \in \mathfrak{L}^{\mathrm{O}}$ such that $\Gamma \nvdash \neg\varphi$. We first show that $\Gamma \cup \{\varphi\}$ is an O-theory. Suppose, toward a contradiction, that $\Gamma \cup \{\varphi\}$ is not an O-theory, i.e., that $\Gamma \cup \{\varphi\} \vdash \bot$. Thus, there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \neg\varphi$ and therefore $\Gamma \vdash \neg\varphi$, which contradicts the assumption that $\Gamma \nvdash \neg\varphi$.

Now suppose that Γ is O-witnessed but $\Gamma \cup \{\varphi\}$ is not O-witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ consistent. Therefore, the latter means that there is $s \in NF^{\mathrm{O}}_{\mathfrak{L}}$ and $\psi \in \mathfrak{L}^{\mathrm{O}}$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle o\rangle\psi$ for all $o \in \mathrm{O}$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][o]\neg\psi$ for all $o \in \mathrm{O}$. Therefore, $\Gamma \vdash \varphi \to [s][o]\neg\psi$ for all $o \in \mathrm{O}$. Note that $\varphi \to [s][o]\neg\psi := [\varphi \to, s][o]\neg\psi$, and $[\varphi \to, s] \in NF^{\mathrm{O}}_{\mathfrak{L}}$. We thus have $\Gamma \vdash [\varphi \to, s][o]\neg\psi$ for all $o \in \mathrm{O}$. Since Γ is O-witnessed, we obtain $\Gamma \vdash [\varphi \to, s]\Box\neg\psi$. By unravelling the necessity form $[\varphi \to, s]$, we get $\Gamma \vdash \varphi \to [s]\Box\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\Box\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\Diamond\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$. $\square$

Lemma 17. *If* $\{\Gamma_i\}_{i\in\mathbb{N}}$ *an increasing chain of* O*-theories such that* $\Gamma_i \subseteq \Gamma_{i+1}$, *then* $\bigcup_{n\in\mathbb{N}} \Gamma_n$ *is an* O*-theory.*

Proof. Let $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \dots$ be an increasing chain of O-theories and suppose, toward contradiction, that $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is not an O-theory, i.e., suppose that $\bigcup_{n\in\mathbb{N}} \Gamma_n \vdash \bot$. This means that there exists a finite $\Delta \subseteq \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that $\Delta \vdash \bot$. Then, since $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is a union of an increasing chain of O-theories, there is some $m \in \mathbb{N}$ such that $\Delta \subseteq \Gamma_m$. Therefore, $\Gamma_m \vdash \bot$ contradicting the fact that Γ_m is an O-theory. $\square$

Lemma 18 *(Lindenbaum's Lemma). Every* O*-witnessed theory* Γ *can be extended to a maximal* O*-witnessed theory* T_Γ.

Proof. The proof follows by constructing an increasing chain

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_n \subseteq \dots,$$

of O-witnessed theories where $\Gamma_0 := \Gamma$, and each Γ_i will be recursively defined. We have to guarantee that each Γ_i is O-witnessed and, in order to do so, we follow a two-fold construction, where $\Gamma_0 = \Gamma'_0 := \Gamma$. Let $\gamma_n := (s_n, \varphi_n)$ be the nth-pair in the enumeration $\mathcal{A}$ of all pairs of the form (s, φ) consisting of a necessity form expression $s \in NF^{\mathrm{O}}_{\mathfrak{L}}$ and a formula $\varphi \in \mathfrak{L}^{\mathrm{O}}$. Note that all pairs of the form (λ, φ), for every formula $\varphi \in \mathfrak{L}^{\mathrm{O}}$ and the empty string λ of observational variables, are in $\mathcal{A}$. By the definition of necessity forms and possibility forms we have that every formula $\varphi \in \mathfrak{L}^{\mathrm{O}}$ can be written as $\varphi := \langle\lambda\rangle\varphi = [\lambda]\varphi$. Therefore, every formula $\varphi \in \mathfrak{L}^{\mathrm{O}}$ is considered in the form of (λ, φ) in the enumeration $\mathcal{A}$. We then set

$$\Gamma'_n = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\varphi_n\} & \text{if } \Gamma \nvdash \neg\langle s_n\rangle\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases}$$

By Lemma 16, each Γ'_n is O-witnessed. Therefore, if φ_n is of the form $\varphi_n := \Diamond\theta$ for some $\theta \in \mathfrak{L}^{\mathrm{O}}$, there must exist an $o \in \mathrm{O}$ such that Γ'_n is consistent with $\langle s\rangle\langle o\rangle\theta$ (since Γ'_n is O-witnessed). We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma'_n & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n \text{ is not of the form } \Diamond\theta \\ \Gamma'_n \cup \{\langle s_n\rangle\langle o\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \Diamond\theta \text{ for some } \theta \in \mathfrak{L}^{\mathrm{O}} \\ \Gamma_n & \text{otherwise} \end{cases}$$

where $o \in \mathrm{O}$ such that Γ'_n is consistent with $\langle s\rangle\langle o\rangle\theta$. Again by Lemma 16, it is guaranteed that each Γ_n is O-witnessed. Now consider the union $T_\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. By Lemma 17, we know that T_Γ is an O-theory. To show that T_Γ is O-witnessed, let $s \in NF^{\mathrm{O}}_{\mathfrak{L}}$ and $\theta \in \mathfrak{L}^{\mathrm{O}}$ and suppose $\langle s\rangle\Diamond\theta$ is consistent with T_Γ. The pair $(s, \Diamond\theta)$ appears in the enumeration $\mathcal{A}$, thus $\gamma_m := (s_m, \varphi_m) = (s, \Diamond\theta)$ with $s_m := s$ and $\varphi_m := \Diamond\theta$, for some $\gamma_m \in \mathcal{A}$. Since $\langle s_m\rangle\varphi_m$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s_m\rangle\Diamond\theta$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle o\rangle\theta \in \Gamma_{m+1}$ for some $o \in \mathrm{O}$ such that Γ'_m is consistent with $\langle s\rangle\langle o\rangle\theta$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle o\rangle\theta$ is also consistent with T_Γ, moreover $\langle s\rangle\langle o\rangle\theta \in T_\Gamma$. Hence, we conclude that T_Γ is O-witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be an O-witnessed theory T such that $T_\Gamma \subset T$. This implies that there exists $\varphi \in \mathfrak{L}^{\mathrm{O}}$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ, we obtain $\Gamma_i \vdash \neg\langle\lambda\rangle\varphi$ for all $i \in \mathbb{N}$. Thus $\Gamma_i \vdash \neg\varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg\varphi$. Hence, since $\varphi \in T$, we obtain $T \vdash \bot$ (contradicting T being consistent). $\square$

Lemma 19 *(Extension Lemma). Let* O *be a set of observational variables and* O′ *be a countable set of* fresh *observational variables, i.e.,* $\mathrm{O} \cap \mathrm{O}' = \emptyset$. *Let* $\widetilde{\mathrm{O}} = \mathrm{O} \cup \mathrm{O}'$. *Then, every* O*-theory* Γ *can be extended to a* $\widetilde{\mathrm{O}}$*-witnessed theory* $\widetilde{\Gamma} \supseteq \Gamma$, *and hence to a maximal* $\widetilde{\mathrm{O}}$*-witnessed theory* $T_\Gamma \supseteq \Gamma$.

Proof. Let $\mathcal{A} = \{\gamma_0, \gamma_1, \dots, \gamma_n, \dots\}$ be an enumeration of all pairs of the form $\gamma_i := (s_i, \varphi_i)$ consisting of any necessity form $s_i \in NF^{\tilde{O}}_{\tilde{\mathfrak{L}}}$ and every formula $\varphi_i \in \mathfrak{L}^{\tilde{O}}$. We will recursively construct a chain of $\tilde{O}$-theories $\Gamma_0 \subseteq \dots \subseteq \Gamma_n \subseteq \dots$ such that:

1. $\Gamma_0 = \Gamma$,
2. $O'_n = \{o \in O' : o \text{ occurs in } \Gamma_n\}$ is finite for every $n \in \mathbb{N}$, and,
3. for every $\gamma_n := (s_n, \varphi_n)$ with $s_n \in NF^{\tilde{O}}_{\tilde{\mathfrak{L}}}$ and $\varphi_n \in \mathfrak{L}^{\tilde{O}}$:
   If $\Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\varphi_n$ then there is o_m 'fresh' such that $\langle s_n\rangle\langle o_m\rangle\varphi_n \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every $\gamma_n \in \mathcal{A}$, let $O'(n) = \{o \in O' : o \text{ occurs either in } s_n \text{ or } \varphi_n\}$. Clearly every $O'(n)$ is always finite. We now construct an increasing chain of $\tilde{O}$-theories recursively: We fix $\Gamma_0 := \Gamma$ and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\langle o_m\rangle\varphi_n\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases}$$

where m is the least natural number bigger than the indices in $O'_n \cup O'(n)$, i.e., o_m is fresh. We now need to show that $\tilde{\Gamma} := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is an $\tilde{O}$-witnessed theory.

We first show that $\tilde{\Gamma}$ is an $\tilde{O}$-theory. By Lemma 17, it suffices to show by induction that every Γ_n is an $\tilde{O}$-theory. Clearly Γ_0 is an $\tilde{O}$-theory. For the inductive step suppose that Γ_n is consistent but Γ_{n+1} is not. Hence $\Gamma_{n+1} \neq \Gamma_n$, and moreover $\Gamma_{n+1} \vdash \bot$. Then, since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle o_m\rangle\varphi_n\}$, we have $\Gamma_n \vdash [s_n][o_m]\neg\varphi_n$. Therefore there exists $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \dots, \theta_k\} \vdash [s_n][o_m]\neg\varphi_n$. Let $\theta = \bigwedge_{1\le i\le k}\theta_i$. Then $\vdash \theta \to [s_n][o_m]\neg\varphi_n$, so $\vdash [\theta \to, s_n][o_m]\neg\varphi_n$ with $o_m \notin O_{\Gamma_n} \cup O_{s_n} \cup O_{\varphi_n}$.

By the admissible rule in Lemma 13 we obtain that $\vdash [\theta \to, s_n]\Box\neg\varphi_n$, thus $\vdash \theta \to [s_n]\Box\neg\varphi_n$. It follows $\theta \vdash [s_n]\Box\neg\varphi_n$ and therefore $\theta \vdash \neg\langle s_n\rangle\Diamond\varphi_n$. Since $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$, we obtain $\Gamma_n \vdash \neg\langle s_n\rangle\Diamond\varphi_n$, contradicting our assumption. Therefore Γ_{n+1} is consistent and thus an $\tilde{O}$-theory. Therefore, by Lemma 17, $\tilde{\Gamma}$ is an $\tilde{O}$-theory. Condition (3) above implies that $\tilde{\Gamma}$ is also $\tilde{O}$-witnessed. Then, by Lindenbaum's Lemma, there is a maximal O-witnessed theory T_Γ such that $T_\Gamma \supseteq \tilde{\Gamma} \supseteq \Gamma$. □

We are now ready to build the canonical model.

Canonical Model for T_0. For any consistent set of formulas Φ, consider a maximally consistent O-witnessed extension $T_0 \supseteq \Phi$. As our canonical set of worlds, we take the set $\mathcal{X}^c := \{T : T \text{ maximally consistent O-witnessed theory with } T \sim_K T_0\}$, where we put

$$T \sim_K T' \text{ iff } \forall\varphi \in \mathfrak{L}_O\left((K\varphi) \in T \to \varphi \in T'\right).$$

As usual, it is easy to see (given the $S5$ axioms for K) that $\sim_K$ is an equivalence relation. For any formula φ, we use the notation $\widehat{\varphi} := \{T \in \mathcal{X}^c : \varphi \in T\}$. In particular, for any observational variable $o \in O$, we have $\widehat{o} = \{T \in \mathcal{X}^c : o \in T\}$. We can generalize this notation to *finite sequences* $\vec{o} = (o_1, \dots, o_n) \in O^*$ of observational variables, by putting: $\widehat{\vec{o}} := \{T \in \mathcal{X}^c : o_1, \dots, o_n \in T\}$.

As canonical set of information states, we take $\mathcal{O}^c := \{\widehat{\vec{o}} : \vec{o} \in O^*\}$. Finally, our canonical learner is given by $\mathbb{L}^c(\widehat{\vec{o}}) := \widehat{L(\vec{o})}$, and the canonical valuation $\|\cdot\|_c$ is given as $||p||_c = \widehat{p}$ and $||o||_c = \widehat{o}$. The learning model $\mathcal{M}^c = (\mathcal{X}^c, \mathcal{O}^c, \mathbb{L}^c, \|\cdot\|_c)$ is called the *canonical model*. Note that we use c as a subindex instead of a superindex for the canonical valuation $\|\cdot\|_c$, this is in order to avoid confusion with our 'open-restriction' notation for the truth set of a formula $[\![\varphi]\!]^U$.

Before proving that the canonical model is well-defined, we need the following.

Lemma 20. *For every maximal O-witnessed theory T, the set $\{\theta : K\theta \in T\}$ is an O-witnessed theory.*

Proof. Observe that, by axiom (T_K), $\{\theta : K\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta : K\theta \in T\}$ is consistent. Let $s \in NF^O_{\mathfrak{L}}$ and $\varphi \in \mathfrak{L}^O$ such that $\{\theta : K\theta \in T\} \vdash [s][o]\neg\varphi$ for all $o \in O$. Then, by normality of K, $T \vdash K[s][o]\neg\varphi$ for all $o \in O$. Since $K[s][o]\neg\varphi := [K, s][o]\neg\varphi$ is a necessity form and T is O-witnessed, we obtain $T \vdash [K, s]\Box\neg\varphi$, i.e., $T \vdash K[s]\Box\neg\varphi$. As T is maximal, we have $K[s]\Box\neg\varphi \in T$, thus $[s]\Box\neg\varphi \in \{\theta : K\theta \in T\}$. □

Lemma 21. *Let $T \in \mathcal{X}^c$. Then, $K\varphi \in T$ iff $\varphi \in S$ for all $S \in \mathcal{X}^c$.*

Proof. From left-to-right follows directly from the definition of $\mathcal{X}^c$ and $\sim_K$. For the right-to-left direction, we prove the contrapositive: Let $\varphi \in \mathfrak{L}$ such that $K\varphi \notin T$. Then, by Lemma 20 and Lemma 16, we obtain that $\{\psi : K\psi \in T\} \cup \{\neg\varphi\}$ is an O-witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 18) and extend it to a maximal O-witnessed theory S such that $\varphi \notin S$. □

Corollary 22. *Let $T \in \mathcal{X}^c$. Then, $\langle K\rangle\varphi \in T$ iff there is $S \in \mathcal{X}^c$ such that $\varphi \in S$.*

Proposition 23. *The canonical model is well-defined.*

Proof. We need to show that the following properties hold:

1. *If $F = \{\widehat{\vec{o_1}}, \ldots, \widehat{\vec{o_m}}\} \subseteq \mathcal{O}^c$ is finite then $\bigcap F \in \mathcal{O}^c$*: Let $F = \{\widehat{\vec{o_1}}, \ldots, \widehat{\vec{o_m}}\} \subseteq \mathcal{O}^c$. It is easy to see that $\bigcap\{\widehat{\vec{o_1}}, \ldots, \widehat{\vec{o_m}}\} = \widehat{\vec{o}}$, where $\vec{o}$ is the concatenation of all the $\vec{o_i}$'s with $1 \le i \le m$. Since each $\vec{o_i}$ is finite, $\vec{o}$ is also finite. By the definition of $\mathcal{O}^c$ in the canonical model we obtain $\bigcap\{\widehat{\vec{o_1}}, \ldots, \widehat{\vec{o_m}}\} = \widehat{\vec{o}} \in \mathcal{O}^c$.
2. *$\mathbb{L}^c$ is a well-defined function and a learner*: For this, note that $\mathbb{L}^c(\widehat{\vec{o}}) := \widehat{L(\vec{o})} \subseteq X^c$. We will first prove that:
(2a) *if $\widehat{\vec{o}} = \widehat{\vec{u}}$ then $\mathbb{L}^c(\widehat{\vec{o}}) = \mathbb{L}^c(\widehat{\vec{u}})$*: Suppose $\widehat{\vec{o}} = \widehat{\vec{u}}$. This means that $(\forall T \in X^c)(\bigwedge \vec{o} \in T$ iff $\bigwedge \vec{u} \in T)$. Therefore, we obtain $\vdash \bigwedge \vec{o} \leftrightarrow \bigwedge \vec{u}$. Then, by (Nec$_K$), we have $\vdash K\left(\bigwedge \vec{o} \leftrightarrow \bigwedge \vec{u}\right)$, i.e., $\vdash \vec{o} \Leftrightarrow \vec{u}$. Since $\mathbb{L}^c(\widehat{\vec{o}}) := \widehat{L(\vec{o})}$, showing $\mathbb{L}^c(\widehat{\vec{o}}) = \mathbb{L}^c(\widehat{\vec{u}})$ boils down to showing that $\widehat{L(\vec{o})} = \widehat{L(\vec{u})}$, i.e., that $\vdash L(\vec{o}) \leftrightarrow L(\vec{u})$, which follows from axiom (EC) and that $\vdash \vec{o} \Leftrightarrow \vec{u}$.
Next, we must prove that
(2b) *$\mathbb{L}^c$ is a learner*, i.e., $\mathbb{L}^c$ satisfies the properties of a learner given in Definition 3. To show this, we first check that $\mathbb{L}^c(\widehat{\vec{o}}) \subseteq \widehat{\vec{o}}$ holds. Let $T \in \mathbb{L}^c(\widehat{\vec{o}})$. This means, by the definition of $\mathbb{L}^c(\widehat{\vec{o}})$, that $L(\vec{o}) \in T$. Since $(L(\vec{o}) \to \bigwedge \vec{o}) \in T$ (by the axiom (SP)), we have that $\bigwedge \vec{o} \in T$. Therefore, as T is maximally consistent, we obtain $o_1, \ldots, o_m \in T$ for $\vec{o} = (o_1, \ldots, o_m)$, meaning that $\vec{o} \in T$. Thus, $T \in \widehat{\vec{o}}$. Finally we show that if $\widehat{\vec{o}} \neq \emptyset$ then $\mathbb{L}^c(\widehat{\vec{o}}) \neq \emptyset$. Suppose $\widehat{\vec{o}} \neq \emptyset$, i.e., there is $T \in X^c$ with $T \in \widehat{\vec{o}}$. This means, by the definition of $\widehat{\vec{o}}$, that $o_1, \ldots, o_m \in T$ for $\vec{o} = (o_1, \ldots, o_m)$. Then, since T is a maximal consistent theory, we have $\bigwedge \vec{o} \in T$. Therefore, by $((\bigwedge \vec{o}) \to \langle K \rangle L(\vec{o})) \in T$ (the axiom (CC)), we obtain that $\langle K \rangle L(\vec{o}) \in T$. Then, by Corollary 22, there is $S \in X^c$ such that $L(\vec{o}) \in S$. Thus, by the definition of $\widehat{L(\vec{o})}$, we have $S \in \widehat{L(\vec{o})}$ meaning that $\widehat{L(\vec{o})} = \mathbb{L}^c(\widehat{\vec{o}}) \neq \emptyset$. □

Our aim is to prove a Truth Lemma for the canonical model, that will immediately imply completeness, as usual. For this we need the following result.

Lemma 24. *Let $T \in \mathcal{X}^c$. Then, $\Box\varphi \in T$ iff $[\vec{u}]\varphi \in T$ for all $\vec{u} \in \mathrm{Prop}_O^*$.*

Proof. The direction from left-to-right follows by the axiom ($\Box$-Ax). For the direction from right-to-left, suppose, toward a contradiction, that for all $\vec{u} \in \mathrm{Prop}_O^*$, $[\vec{u}]\varphi \in T$ and $\Box\varphi \notin T$. Then, since T is a maximally consistent theory, $\Diamond\neg\varphi \in T$. Since T is an O-witnessed theory, there is $v \in \mathrm{O}$ such that $\langle v \rangle \neg\varphi$ is consistent with T. Since T is also maximally consistent, we obtain that $\langle v \rangle \neg\varphi \in T$, i.e., that $\neg[v]\varphi \in T$, contradicting our initial assumption. □

We now proceed to our key result:

Lemma 25 *(Truth Lemma). For all formulas φ, all $T \in X^c$ and all $\widehat{\vec{o}} \in \mathcal{O}^c$, we have:*

$$\langle \vec{o} \rangle \varphi \in T \;\; \textit{iff} \;\; (T, \widehat{\vec{o}}) \models_{\mathcal{M}^c} \varphi.$$

Proof. The proof is by induction over subformulas using the following *induction hypothesis* (**IH**): for all ψ subformula of φ, and all $\widehat{\vec{o}} \in \mathcal{O}^c$, $\langle \vec{o} \rangle \psi \in T$ iff $(T, \widehat{\vec{o}}) \models_{\mathcal{M}^c} \psi$.

The base cases for propositional and observational variables, as well as for Boolean formulas are straightforward. We only verify the remaining inductive cases. At each step of the proof, $\bigwedge \vec{o} \in T$ guarantees that the pair $(T, \widehat{\vec{o}})$ is a well-defined epistemic scenario of the canonical model since $\widehat{\bigwedge \vec{o}} = \widehat{\vec{o}}$.

- Case $\varphi := L(\vec{u})$.

$$\begin{aligned}
\langle \vec{o} \rangle L(\vec{u}) \in T &\text{ iff } (\textstyle\bigwedge \vec{o} \wedge [\vec{o}]L(\vec{u})) \in T && \text{(Proposition 6.12)}\\
&\text{ iff } (\textstyle\bigwedge \vec{o} \wedge L(\vec{o}, \vec{u})) \in T && \text{(Proposition 6.6)}\\
&\text{ iff } \textstyle\bigwedge \vec{o} \in T \text{ and } L(\vec{o}, \vec{u}) \in T && \\
&\text{ iff } T \in \widehat{\vec{o}} \text{ and } T \in \widehat{L(\vec{o}, \vec{u})} = \mathbb{L}^c(\widehat{\vec{o}}, \widehat{\vec{u}}) && (\text{since } \widehat{\textstyle\bigwedge \vec{o}} = \widehat{\vec{o}})\\
&\text{ iff } (T, \widehat{\vec{o}}) \models_{\mathcal{M}^c} L(\vec{u}) && \text{(the semantics of } L)
\end{aligned}$$

- Case $\varphi := K\psi$.

$$\langle \vec{o} \rangle K\psi \in T \text{ iff } (\textstyle\bigwedge \vec{o} \wedge K[\vec{o}]\psi) \in T \qquad \text{(Propositions 6.12 and 6.8)}$$

$$\begin{aligned}
&\text{iff } \bigwedge \vec{o} \in T \text{ and } K[\vec{o}]\psi \in T \\
&\text{iff } \bigwedge \vec{o} \in T \text{ and } (\forall S \sim_K T)([\vec{o}]\psi \in S) && \text{(Lemma 21)} \\
&\text{iff } \bigwedge \vec{o} \in T \text{ and } (\forall S \sim_K T \text{ s.t. } \bigwedge \vec{o} \in S)(\langle \vec{o} \rangle \psi \in S) && \text{(Proposition 6.12)} \\
&\text{iff } \widehat{\vec{o}} \in T \text{ and } (\forall S \in \widehat{\vec{o}})((S, \widehat{\vec{o}}) \models \psi) && (\widehat{\bigwedge \vec{o}} = \widehat{\vec{o}} \text{ and I.H)} \\
&\text{iff } (T, \widehat{\vec{o}}) \models_{\mathcal{M}^c} K\psi && \text{(Semantics of } K)
\end{aligned}$$

- Case $\varphi := \langle \vec{u} \rangle \psi$.

$$\begin{aligned}
\langle \vec{o} \rangle \langle \vec{u} \rangle \psi \in T &\text{ iff } \langle \vec{o}, \vec{u} \rangle \psi \in T && (\text{since } [\vec{o}, \vec{u}] := [\vec{o}][\vec{u}]) \\
&\text{iff } (\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \wedge \langle \vec{o}, \vec{u} \rangle \psi \in T && \text{(Proposition 6.12)} \\
&\text{iff } (\bigwedge \vec{o} \wedge \bigwedge \vec{u}) \in T \text{ and } \langle \vec{o}, \vec{u} \rangle \psi \in T \\
&\text{iff } T \in \widehat{\vec{o}} \cap \widehat{\vec{u}} \text{ and } \langle \vec{o}, \vec{u} \rangle \psi \in T && (\text{since } \widehat{\bigwedge \vec{o} \wedge \bigwedge \vec{u}} = \widehat{\vec{o}} \cap \widehat{\vec{u}}) \\
&\text{iff } T \in \widehat{\vec{o}} \cap \widehat{\vec{u}} \text{ and } (T, \widehat{(\vec{o}, \vec{u})}) \models \psi && \text{(I.H)} \\
&\text{iff } T \in \widehat{\vec{u}} \text{ and } (T, \widehat{\vec{o}} \cap \widehat{\vec{u}}) \models \psi && (\text{since } \widehat{(\vec{o}, \vec{u})} = \widehat{\vec{o}} \cap \widehat{\vec{u}}, \text{ and } T \in \widehat{\vec{o}}) \\
&\text{iff } T \in [\![\bigwedge \vec{u}]\!]_c \text{ and } (T, \widehat{\vec{o}} \cap [\![\bigwedge \vec{u}]\!]_c) \models \psi \\
& && (\text{since } [\![\bigwedge \vec{u}]\!]_c = \widehat{\vec{u}}, \text{ by the base case for } o \text{ and the case for } \wedge) \\
&\text{iff } (T, \vec{o}) \models \langle \vec{u} \rangle \psi && \text{(Lemma 5)}
\end{aligned}$$

- Case $\varphi := \Box\psi$.
($\Leftarrow$) Suppose $\langle \vec{o} \rangle \Box\psi \in T$. Then, by Proposition 6.12 and Proposition 6.9, we obtain that (1) $\bigwedge \vec{o} \in T$, i.e., $T \in \widehat{\vec{o}}$, and (2) $\Box[\vec{o}]\psi \in T$. Thus, by Lemma 24 and (2), we have $[\vec{u}][\vec{o}]\psi \in T$, i.e., $[\vec{u}, \vec{o}]\psi \in T$, for all $\vec{u} \in \mathrm{Prop}_O^*$. Now let $O \in \mathscr{O}^c$ such that $T \in O$. By the construction of $\mathscr{O}^c$, we know that $O = \widehat{\vec{v}}$ for some $\vec{v} \in \mathrm{Prop}_O^*$. We want to show that $(T, \widehat{\vec{o}} \cap \widehat{\vec{v}}) \models \psi$. Since $T \in \widehat{\vec{o}} \cap \widehat{\vec{v}}$ and $\mathcal{M}^c$ is an intersection space, we know that $(T, \widehat{\vec{o}} \cap \widehat{\vec{v}})$ is a well-defined epistemic scenario. $T \in \widehat{\vec{o}} \cap \widehat{\vec{v}}$ also implies that $(\bigwedge \vec{o} \wedge \bigwedge \vec{v}) \in T$ as in the above case. Hence, by Proposition 6.12 and the fact that $[\vec{v}, \vec{o}]\psi \in T$, we obtain $\langle \vec{v}, \vec{o} \rangle \psi \in T$. Then, by I.H., we obtain $(T, \widehat{\vec{v}} \cap \widehat{\vec{o}}) \models \psi$. Therefore, by the semantics of $\Box$, we obtain $(T, \widehat{\vec{o}}) \models \Box\psi$.
($\Rightarrow$) Suppose $(T, \widehat{\vec{o}}) \models \Box\psi$. This means, by the definition of $\mathscr{O}^c$, that for all $\vec{u} \in \mathrm{Prop}_O^*$, if $T \in \widehat{\vec{u}}$ then $(T, \widehat{\vec{o}} \cap \widehat{\vec{u}}) \models \psi$. Now let $\vec{v} \in \mathrm{Prop}_O^*$ such that $T \in \widehat{\vec{v}}$. Therefore, $T \in \widehat{\vec{v}} \cap \widehat{\vec{o}}$. Since $(\vec{v}, \vec{o}) \in \mathrm{Prop}_O^*$ and $\widehat{\vec{v}} \cap \widehat{\vec{o}} = \widehat{(\vec{v}, \vec{o})}$, we obtain, by the assumption, that $(T, \widehat{\vec{v}} \cap \widehat{\vec{o}}) \models \psi$. Thus, by I.H., we have $\langle \vec{v}, \vec{o} \rangle \psi \in T$. As $\vdash \langle \vec{v}, \vec{o} \rangle \psi \to [\vec{v}, \vec{o}]\psi$ and T is maximal, we obtain $[\vec{v}, \vec{o}]\psi \in T$, i.e., $[\vec{v}][\vec{o}]\psi \in T$. Since $\vec{v}$ has been chosen from Prop_O^* arbitrarily, by Lemma 24, we have $\Box[\vec{o}]\psi \in T$. Then, by Proposition 6.9, the fact that $\bigwedge \vec{o} \in T$ and Proposition 6.12, we obtain $\langle \vec{o} \rangle \Box\psi \in T$. □

Theorem 2. **L** *is complete with respect to the class of all learning models.*

Proof. Let φ be an **L**-consistent formula, i.e., it is an O_φ-theory. Then, by Lemma 19, it can be extended to some maximal O-witnessed theory T. Then, we have $\langle \lambda \rangle \varphi \in T$ where λ is the empty string, i.e., $T \in \widehat{\langle \lambda \rangle \varphi}$. Note that $\widehat{\bigwedge \lambda} = \bigcap \emptyset = X^c$. Then, by Truth Lemma (Lemma 25), we obtain that $(T, X^c) \models_{\mathcal{M}^c} \varphi$, where $\mathcal{M}^c = (X^c, \mathscr{O}^c, \mathbb{L}^c, \| \cdot \|_c)$ is the canonical model for T. This proves completeness. □

4. Expressivity

We first investigate how various notions of learnability can be expressed in our language. In fact, the following was already noticed in [18].

Proposition 26. *$\Diamond Kp$ is true at (x, U) in a model $\mathcal{M}$ iff $\|p\|$ is* learnable with certainty *at state x. Similarly, $p \to \Diamond Kp$ is valid (i.e., true at all epistemic scenarios) in a model $\mathcal{M}$ iff $\|p\|$ is* verifiable with certainty *(i.e., 'finitely identifiable' in the sense of FLT [11,15]). A similar statement holds for* falsifiability with certainty.

Proof. As we know from Section 1.2, $\|p\|$ is learnable with certainty x iff $x \in Int\|p\|$, and $\|p\|$ is verifiable with certainty iff it is open in the topology generated by $\mathscr{O}$. It is well-known [18] that these properties are expressible in SSL via the above validities. □

In particular, the following validity of our logic expresses the fact that *all observable properties are verifiable with certainty*:

$$o \to \Diamond Ko.$$

By adding the learning operator to subset space logic, DLLT can capture, not only belief, but also the various inductive notions of knowledge and learnability:

Proposition 27 *(Inductive notions of knowledge and learnability).*

- *$\Box Bp$ holds at (x, U) in a model $\mathcal{M}$ iff the learner $\mathbb{L}$ has* undefeated belief *in $\|p\|$ (at world x in information state U). Hence, $p \wedge \Box Bp$ captures* inductive knowledge *of p, and so $p \wedge \Diamond\Box Bp$ captures* inductive learnability *of p by learner $\mathbb{L}$.*
- *Similarly, $p \to \Diamond\Box Bp$ is valid in a model $\mathcal{M}$ iff $\|p\|$ is* inductively verifiable by $\mathbb{L}$. *For the corresponding generic notion: $\|p\|$* is inductively verifiable *(by* some *learner) iff* $p \to \Diamond\Box Bp$ is validable in the intersection space $(X, \mathcal{O})$. *Similar statements hold for* inductive fasifiability.
- *Finally, $\Diamond L(\lambda)$ is true if (given enough observations) the observer* will eventually reach a true conjecture *(though he might later fall again into false ones); and similarly, $\Diamond\Box L(\lambda)$ is true if (given enough observations) the observer will* eventually produce only true conjectures thereafter.

Proof. Easy to verify, given the relevant definitions and our semantics. □

As usual in Dynamic Epistemic Logic, the dynamic 'observation' modalities $[u]\varphi$ are only a convenient way to express complex properties in a succinct manner, but they can in principle be eliminated. To show this, we first need the following lemma.

Lemma 28. *There is a well-founded strict partial order $<$ on formulas (called 'complexity order'), satisfying the following conditions:*

- *if φ is a (proper) subformula of ψ then $\varphi < \psi$*
- $(u \to p) < [u]p$
- $(u \to o) < [u]o$
- $(u \to L(u, \vec{o})) < [u]L(\vec{o})$
- $([u]\varphi \wedge [u]\psi) < [u](\varphi \wedge \psi)$
- $(u \to K[u]\varphi) < [u]K\varphi$
- $\Box[u]\varphi < [u]\Box\varphi$

Proof. A complexity measure $c : \mathfrak{L} \to \mathbb{N}$ that gives such a strict partial order on $\mathfrak{L}$ can be defined by extending and, in a sense, simplifying the one in [25, Definition 7.21] as follows:

$$\begin{aligned} c(p) = c(o) &= 1 \\ c(L(o_1, \ldots, o_n)) &= 1 + n \\ c(\varphi \wedge \psi) &= 1 + max(c(\varphi), c(\psi)) \\ c(\neg\varphi) = c(K\varphi) = c(\Box\varphi) &= c(\varphi) + 1, \\ c([o]\varphi) &= 5 \cdot c(\varphi). \end{aligned}$$

We then define $< \subseteq \mathfrak{L} \times \mathfrak{L}$ as $\varphi < \psi$ iff $c(\varphi) < c(\psi)$. The lemma then follows via easy calculations using c. □

Proposition 29. *(Expressivity) The above language is co-expressive with the one obtained by removing all dynamic modalities $[u]\varphi$. Moreover, this can be proved in the above proof system: for every formula φ there exists some formula φ' free of any dynamic modalities, such that $\varphi \leftrightarrow \varphi'$ is a theorem in the above proof system. Furthermore, if φ contains dynamic modalities then φ' can be chosen such that $\varphi' < \varphi$.*

Proof. Suppose, towards contradiction, that φ is a formula which is *not* free of dynamic modalities, and moreover that φ is not provably equivalent to any formula of lower complexity (in the sense of $<$) that is free of dynamic modalities. We construct an infinite descending sequence

$$\varphi_0 > \varphi_1 > \ldots > \varphi_n > \ldots$$

of provably equivalent formulas, none of which is free of dynamic modalities. The construction goes as follows. We first put $\varphi_0 := \varphi$. Then, at step n, assuming given φ_n not free of dynamic modalities, and provably equivalent to all the previous formulas, we chose the first dynamic modality occurring in φ_n, and apply once to it the relevant Reduction Axiom (from left to right), obtaining a provably equivalent formula φ_{n+1}, which by Lemma 28 has the property that $\varphi_{n+1} < \varphi_n$. By transitivity of provable equivalence, φ_{n+1} is provably equivalent to $\varphi_0 = \varphi$, and (by transitivity of $<$) it is of lower complexity than $\varphi_0 = \varphi$; so, by our assumption above, φ_{n+1} is still not free of dynamic modalities. But the existence of this infinite descending sequence contradicts the well-foundedness of $<$. $\square$

5. Conclusion and comparison with other work

In this paper we proposed a dynamic logic which allows reasoning about inductive inference. Our Dynamic Logic of Learning Theory (DLLT) is an extension of previously studied Subset Space Logics, and a natural continuation of the work bridging Dynamic Epistemic Logic and Formal Learning Theory. Together with a syntax, featuring dynamic observation operators, and a topological semantics, we give a sound and complete axiomatisation of this logic. We show how natural learnability properties, such as learnability in the limit and learnability with certainty, can be expressed in DLLT.

Our technical results (the complete axiomatisation and expressivity results), as well as the methods used to prove them (the canonical neighbourhood model and the reduction laws), may look deceivingly simple. But in fact, achieving this simplicity is one of the major contributions of our paper. The most well-known relative to our logic is *Subset Space Logic (SSL) over intersection spaces*, completely axiomatised by Weiss and Parikh [28] (and, indeed, our operator $\Box$ originates in the 'effort modality' of the SSL formalism introduced in [18,10]). Although less expressive than our logic (since it has no notion of belief B or conjecture L), the Weiss-Parikh axiomatisation of SSL over intersection spaces is in a sense more complex and less transparent (such as is their completeness proof, which is non-canonical). That axiomatisation consists of the following list:

$S5_K$	The $S5$ axioms and rules for K
$S4_\Box$	The $S4$ axioms and rules for $\Box$
Cross Axiom	$K\Box\varphi \to \Box K\varphi$
Weak Directedness	$\Diamond\Box\varphi \to \Box\Diamond\varphi$
M_n (for all n)	$(\Box\langle K\rangle\varphi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n) \to \langle K\rangle(\Diamond\varphi \wedge \Diamond K\psi_1 \wedge \ldots \wedge \Diamond K\psi_n)$

Although this list looks shorter than our list in Table 1, each of our axioms is simple and readable and has a transparent intuitive interpretation. In contrast, note the complexity and opaqueness of the last axiom schemata M_n above (having one schema for each natural number n). Our completeness result implies that all these complex validities are provable in our simple system (and in fact in the even simpler system that omits all the axioms that refer to the learner L). This shows the usefulness of adding the (expressively redundant) dynamic observation modalities: they help to describe the behaviour of the effort modality $\Box$ in a much simpler and natural manner, by combining the Effort axiom and the Effort rule (which together capture the meaning of $\Box$ as universally quantifying over observation modalities).

Moreover, our completeness proof is also much simpler (though with some technical twists). Traditionally, the use of canonical models has been considered impossible for Subset Space Logics, and so authors had to use other, more *ad-hoc* methods (e.g., step-by-step constructions). The fact that in this paper we can get away with a canonical construction is again due to the addition of the dynamic modalities.

More recent papers, closely related to our contribution, are Bjorndahl [8], van Ditmarsch et al. [23,24], and Baltag et al. [7]. Bjorndahl [8] introduces dynamic modalities $[\varphi]$ for arbitrary formulas (rather than restricting to observational variables $[o]$, as we do), though with a different semantics (according to which $[\varphi]$ restricts the space to the interior of φ, in contrast to our simpler semantics, that follows the standard definition of update or 'public announcement'). His syntax does *not* contain the effort modality, or any other form of quantifying over observations. The work of van Ditmarsch et al. [23,24] uses Bjorndahl-style dynamic modalities in combination with a topological version of the so-called 'arbitrary public announcement' operator, which is a more syntactic-driven relative of the effort modality. This syntactic nature comes with a price: the logic of arbitrary public announcements is much less well-behaved than SSL (or our logic), in particular it has non-compositional features (the meaning of a formula may depend on the meaning of *all* atomic variables, including the ones that do not occur in that formula). As a consequence, the soundness of (the arbitrary-public announcement analogue of) our Effort Rule is not at all obvious for their logic, which instead relies on an infinitary inference rule. Since that rule makes use of infinitely many premises, their complete axiomatisation is truly infinitary, and impossible to automatise: indeed, it does not even necessarily imply that the set of their validities is recursively enumerable (in contrast with our finitary axiomatisation, which immediately implies such a result). In [7] these problems are solved by replacing the arbitrary announcement modality with the effort modality (or equivalently, extending SSL with Bjorndahl-style dynamic modalities). But note that, in contrast to the work presented here, *all the above papers are concerned only with axiomatisations over topological spaces* (rather than the wider class of intersection spaces), and that *none of them has any belief B or conjecture operators L*. Hence, none of them can be used to capture any learning-theoretic notions going beyond finite identifiability.

Declaration of competing interest

There is no competing interest.

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