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Top tautological group of $\mathcal{M}_{g,n}$

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Abstract. We describe the structure of the top tautological group in the cohomology of the moduli space of smooth genus g curves with n marked points.

Keywords. Moduli space of curves, cohomology, tautological groups

1. Introduction

In this paper we study the cohomology groups of the moduli space of smooth genus g curves with n marked points. This moduli space is denoted by $\mathcal{M}_{g,n}$. The space $\mathcal{M}_{g,0}$ will be denoted by $\mathcal{M}_g$.

The cohomology of $\mathcal{M}_{g,n}$ has a distinguished subring of tautological classes

$$R^*(\mathcal{M}_{g,n}) \subset H^{\text{even}}(\mathcal{M}_{g,n}; \mathbb{Q})$$

studied extensively since Mumford's seminal article [Mum83].

A great step towards understanding the tautological ring of $\mathcal{M}_g$ was done by C. Faber [Fab99]. He formulated three conjectures that give a complete description of $R^*(\mathcal{M}_g)$. These conjectures are called the socle conjecture, the top intersection conjecture and the perfect pairing conjecture. The socle conjecture was proved in [Loo95], and there are several proofs of the top intersection conjecture (see [GP98, LX09, BS11]). The perfect pairing conjecture is true up to genus 23, but the accumulating evidence suggests it may be wrong for $g \geq 24$.

The socle conjecture says that $R^{>g-2}(\mathcal{M}_g) = 0$ and $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$. We will recall the other two conjectures in Section 1.2.

Analogous statements can be formulated about the tautological ring of $\mathcal{M}_{g,1}$. The socle property in this case says that $R^{>g-1}(\mathcal{M}_{g,1}) = 0$ and $R^{g-1}(\mathcal{M}_{g,1}) = \mathbb{Q}$ [Loo95].

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The top intersection property for $\mathcal{M}_{g,1}$ can be easily derived from the original top intersection statement for $\mathcal{M}_g$. The perfect pairing conjecture in this case is also open.

In this paper we formulate and prove a socle and intersection property for $\mathcal{M}_{g,n}$. In particular, the generalized socle property says that $R^{>g-1}(\mathcal{M}_{g,n}) = 0$ and $R^{g-1}(\mathcal{M}_{g,n}) = \mathbb{Q}^n$. The vanishing part was actually proved in [Ion02].

Let us say a few words about the idea of our proof. One can choose different spanning families in the tautological groups. On the one hand, the tautological groups of $\mathcal{M}_{g,n}$ are spanned by monomials in ψ-classes and κ-classes. On the other hand, the tautological groups of $\mathcal{M}_{g,n}$ are spanned by double ramification cycles. A technique for working with these cycles was developed, e.g., in [Ion02, Sha03, SZ08, BSSZ12]. In [Ion02] it was used to prove the vanishing $R^{>g-1}(\mathcal{M}_{g,n}) = 0$, and in [Sha03] to study the intersection theory of the moduli space of curves. This technique was also applied in [BS11] in order to give a simple proof of Faber's top intersection conjecture.

In this paper we develop the theory of double ramification cycles and use it for the proof of generalized socle and top intersection properties.

1.1. Tautological ring

In this section we briefly recall basic definitions related to the tautological ring of the moduli space of curves. We refer to [Vak08, Zvo12] for a more detailed introduction to this subject.

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable genus g curves with n marked points. The class $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is defined as the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the i-th marked point. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map that forgets the last marked point. The class $\kappa_k \in H^{2k}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is defined by

$$\kappa_k := \pi_*(\psi_{n+1}^{k+1}).$$

It is convenient to define multi-index kappa classes. Let $m \geq 1$ and consider the map $\pi : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ that forgets the last m marked points. Let $k_1, \ldots, k_m$ be non-negative integers. Define the class $\kappa_{k_1,\ldots,k_m} \in H^{2\sum_{i=1}^m k_i}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ by

$$\kappa_{k_1,\ldots,k_m} := \pi_*(\psi_{n+1}^{k_1+1}\psi_{n+2}^{k_2+1}\ldots\psi_{n+m}^{k_m+1}).$$

Multi-index κ-classes can be expressed as polynomials in κ-classes with one index. Conversely, any polynomial in one index κ-classes can be written as a linear combination of multi-index κ-classes.

The tautological ring $R^*(\mathcal{M}_{g,n})$ is defined as the subring of $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ generated by the classes κ_j and ψ_i. The group $R^i(\mathcal{M}_{g,n})$ is defined by $R^i(\mathcal{M}_{g,n}) := R^*(\mathcal{M}_{g,n}) \cap H^{2i}(\mathcal{M}_{g,n}; \mathbb{Q})$.

1.2. Faber's conjectures

Here we recall Faber's conjectures from [Fab99] about the structure of the tautological ring $R^*(\mathcal{M}_g)$. Let $g \geq 2$.

- *(Socle)* $R^{>g-2}(\mathcal{M}_g) = 0$ and $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$.

- (*Top intersection*) Suppose that $k_1 + \cdots + k_m = g - 2$ and $k_i \geq 0$. Then we have the following equality in $R^{g-2}(\mathcal{M}_g)$:

$$\kappa_{k_1,\ldots,k_m} = \frac{(2g-3+m)!(2g-3)!!}{(2g-2)!\prod_{i=1}^{m}(2k_i+1)!!}\kappa_{g-2}.$$

- (*Perfect pairing*) For any $0 \leq i \leq g-2$, the cup product defines a pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) = \mathbb{Q}.$$

  This pairing is non-degenerate.

It is easy to see that Faber's conjectures, if true, completely determine the structure of the tautological ring $R^*(\mathcal{M}_g)$.

1.3. Generalized Faber conjectures

In this section we formulate analogous properties of the tautological ring $R^*(\mathcal{M}_{g,n})$. Assume that $g \geq 2$ and $n \geq 1$.

- (*Generalized socle*) $R^{>g-1}(\mathcal{M}_{g,n}) = 0$ and $R^{g-1}(\mathcal{M}_{g,n}) = \mathbb{Q}^n$. The classes ψ_i^{g-1}, $i = 1, \ldots, n$, form a basis in $R^{g-1}(\mathcal{M}_{g,n})$.
- (*Generalized top intersection*) Suppose that $d_1 + \cdots + d_n + k_1 + \ldots + k_m = g - 1$ and $d_i, k_j \geq 0$. Then we have the following equality in $R^{g-1}(\mathcal{M}_{g,n})$:

$$\begin{aligned}\prod_{i=1}^{n}\psi_i^{d_i}\cdot\kappa_{k_1,\ldots,k_m} =& \frac{(2g-1)!!}{\prod_{i=1}^{n}(2d_i+1)!!\prod_{j=1}^{m}(2k_j+1)!!}\,\frac{(2g-3+n+m)!}{(2g-2+n)!}\\ &\times\sum_{i=1}^{n}\frac{(2g-2+n)d_i+\sum k_j}{g-1}\psi_i^{g-1}.\end{aligned}$$

- (*Generalized perfect pairing*) The ring $R^*(\mathcal{M}_{g,n})$ is level of type n. In other words, a polynomial in $\psi_1, \ldots, \psi_n, \kappa_1, \ldots, \kappa_{g-1}$ vanishes if and only if its products with all classes of complementary dimension vanish in $R^{g-1}(\mathcal{M}_{g,n})$.

Similarly to Faber's conjectures, these properties, if true, completely determine the structure of the ring $R^*(\mathcal{M}_{g,n})$.

In this paper we prove the generalized socle and top intersection properties. As for the perfect pairing property, it is true in many cases that can be checked on computer, but recent evidence leads to serious doubts that it is valid in general. The main result of this paper is the following theorem.

Theorem 1.1. *The generalized socle and top intersection properties are true.*

1.4. Organization of the paper

In Section 2 we prove the generalized top intersection property, assuming that the generalized socle conjecture is true. We also show that $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$ (the easy part of the socle property). The rest of the paper is devoted to the hard part of the socle property, that is, the inequality $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$.

In Section 3 we formulate three main ingredients of the proof of the generalized socle property: Lemma 3.1, Proposition 3.2 and Proposition 3.3. We show how the socle property follows from them. These statements will be proved in the subsequent sections, and the proof of the last proposition is the hardest one.

In Section 4 we introduce the main tool for proving these statements: double ramification cycles.

Section 5 contains several linear algebra lemmas that simplify the proofs of Propositions 3.2 and 3.3.

In Section 6 we prove Lemma 3.1, Proposition 3.2 and Proposition 3.3.

2. Generalized top intersection

In this section we show that the classes ψ_i^{g-1} are linearly independent in $R^{g-1}(\mathcal{M}_{g,n})$, and thus $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$. Then we prove the generalized top intersection property assuming that $\dim R^{g-1}(\mathcal{M}_{g,n}) = n$. This equality will be proved in the subsequent sections.

Proposition 2.1. *The classes $\alpha_s := \lambda_g\lambda_{g-1}\psi_1 \cdots \widehat{\psi_s} \cdots \psi_n$ (where the hat means omission) vanish on the boundary of $\overline{\mathcal{M}}_{g,n}$.*

Proof. It is well-known (see [Fab97, Lemma 1]) that $\lambda_g\lambda_{g-1}$ vanishes on $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}^{\mathrm{rt}}_{g,n}$, where $\mathcal{M}^{\mathrm{rt}}_{g,n}$ is the space of stable curves with one genus g component and possibly several "rational tails" composed of genus 0 components. Thus, it remains to show that the classes α_s also vanish on $\mathcal{M}^{\mathrm{rt}}_{g,n} \setminus \mathcal{M}_{g,n}$. Every boundary divisor in $\mathcal{M}^{\mathrm{rt}}_{g,n}$ is isomorphic to a product $\mathcal{M}^{\mathrm{rt}}_{g,n_1+1} \times \overline{\mathcal{M}}_{0,n_2+1}$, where $n_1 + n_2 = n$. Among the ψ-classes that make part of α_s, at least $n_2 - 1$ are sitting on the second factor. Since the dimension of $\overline{\mathcal{M}}_{0,n_2+1}$ equals $n_2 - 2$, we see that the class α_s vanishes on our boundary divisor for dimensional reasons. □

Let $\pi : \mathcal{M}^{\mathrm{rt}}_{g,N} \to \mathcal{M}_g$ be the forgetful map. Let $l_1, \ldots, l_N$ be non-negative integers such that $l_1 + \ldots + l_N = g - 2$. Recall that Faber's top intersection conjecture says that

$$\pi_*(\psi_1^{l_1+1} \ldots \psi_N^{l_N+1}) = \frac{(2g-3+N)!(2g-3)!!}{(2g-2)! \prod_{i=1}^{N}(2l_i+1)!!}\kappa_{g-2}. \tag{2.1}$$

The following small generalization of this formula will be useful for us. Define $(-1)!! := 1$.

Lemma 2.2. *Let $l_1, \ldots, l_N$ be integers such that $l_1 + \cdots + l_N = g-2$. Suppose that at most one of $l_1, \ldots, l_N$ is equal to -1 and the others are non-negative. Then formula (2.1) holds.*

Proof. If all l_i's are non-negative, then (2.1) is exactly Faber's top intersection conjecture.

Suppose one of l_i's is equal to -1. We proceed by induction on N. If $N = 1$, then $l_1 = g-2 \geq 0$, so the formula is true. Suppose $N \geq 2$. Without loss of generality we can assume that $l_1 = -1$. Using the string equation and the induction assumption, we get

$$\begin{aligned}\pi_*(\psi_2^{l_2+1} \ldots \psi_N^{l_N+1}) &= \sum_{i=2}^{N} \frac{(2g-4+N)!(2g-3)!!}{(2g-2)!(2l_i-1)!! \prod_{\substack{2\leq j\leq N\\ j\neq i}}(2l_j+1)!!}\kappa_{g-2} \\ &= \frac{(2g-3+N)!(2g-3)!!}{(2g-2)! \prod_{i=1}^{N}(2l_i+1)!!}\kappa_{g-2}. \qquad \square\end{aligned}$$

Let $d_1, \ldots, d_n$ and $k_1, \ldots, k_m$ be non-negative integers. Assume that $\sum d_i + \sum k_i = g-1$. These integers will be fixed for the rest of the section. Denote

$$C := \frac{(2g-3+n+m)!(2g-3)!!}{(2g-2)! \prod_{i=1}^{n}(2d_i+1)!! \prod_{j=1}^{m}(2k_j+1)!!}.$$

Lemma 2.3. *Let $\pi \colon \mathcal{M}_{g,n}^{\mathrm{rt}} \to \mathcal{M}_g$ be the forgetful map. Then in $R^{g-2}(\mathcal{M}_g)$ we have*

$$\pi_*(\psi_1^{d_1+1} \cdots \psi_s^{d_s} \cdots \psi_n^{d_n+1}\kappa_{k_1,\ldots,k_m}) = C \cdot (2d_s+1) \cdot \kappa_{g-2}.$$

Proof. Let $\pi' \colon \mathcal{M}_{g,n+m}^{\mathrm{rt}} \to \mathcal{M}_g$ be the forgetful map. We have

$$\begin{aligned}\pi_*(\psi_1^{d_1+1} \cdots \psi_s^{d_s} \cdots \psi_n^{d_n+1}\kappa_{k_1,\ldots,k_m}) &= \pi'_*(\psi_1^{d_1+1} \cdots \psi_s^{d_s} \cdots \psi_n^{d_n+1}\psi_{n+1}^{k_1+1} \cdots \psi_{n+m}^{k_m+1}) \\ &\overset{\text{Lemma 2.2}}{=} C \cdot (2d_s+1) \cdot \kappa_{g-2}. \qquad \square\end{aligned}$$

Denote by A_g the non-zero intersection number

$$A_g := \int_{\overline{\mathcal{M}}_g} \kappa_{g-2}\lambda_g\lambda_{g-1} = \frac{(-1)^{g-1}B_{2g}(g-1)!}{2^g(2g)!},$$

where B_{2g} is the Bernoulli number (see [Fab97, Lemma 2]).

Corollary 2.4. *We have $\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i}\kappa_{k_1,\ldots,k_m} \cdot \alpha_s = C \cdot (2d_s+1) \cdot A_g$.*

Proof. Compute the integral by first projecting on $\overline{\mathcal{M}}_g$ and use Lemma 2.3. $\square$

Proposition 2.5. *The $n \times n$ matrix $M_{is} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_i^{g-1}\alpha_s$ is non-degenerate.*

Proof. Denote by U the $n \times n$ matrix given by $U_{is} = 1$ for all i, s. It has exactly two eigenvalues: 0 and n. Therefore $U + a\,\mathrm{Id}$ is non-degenerate whenever $a \neq 0, -n$.

According to the corollary, we have $M_{ii} = C \cdot A_g \cdot (2g-1)$ and $M_{is} = C \cdot A_g$ for $i \neq s$. Thus,

$$M = C \cdot A_g \cdot (U + (2g-2)\,\mathrm{Id}),$$

so it is non-degenerate. □

Proposition 2.6. *The classes*

$$\prod_{i=1}^{n} \psi_i^{d_i} \kappa_{k_1,\dots,k_m} \quad \text{and} \quad C\frac{(2g-1)!}{(2g-2+n)!}\sum_{i=1}^{n} \frac{(2g-2+n)d_i + \sum k_j}{g-1}\psi_i^{g-1}$$

have the same intersection number with every α_s.

Proof. Apply Corollary 2.4 and divide both intersection numbers by the common factor CA_g. We obtain the following equality that needs to be checked:

$$\begin{aligned}
2d_s+1 &\stackrel{?}{=} \frac{2g-1}{(2g-2+n)!}\sum_{i=1}^{n}\left[\frac{(2g-2+n)d_i+\sum k_j}{g-1}\cdot\frac{(2g-3+n)!}{2g-1}(1+(2g-2)\delta_{is})\right]\\
&= \frac{1}{(2g-2+n)(g-1)}\sum_{i=1}^{n}\Big((2g-2+n)d_i+\sum k_j\Big)(1+(2g-2)\delta_{is})\\
&= \frac{1}{(2g-2+n)(g-1)}\Big[(2g-2+n)\Big(\sum d_i+\sum k_j\Big)+(2g-2+n)(2g-2)d_s\Big]\\
&= \frac{1}{g-1}[(g-1)+(2g-2)d_s] = 2d_s+1.
\end{aligned}$$

Thus, the equality is indeed true. □

Let us sum up the results of our computations. We have found n classes α_s, $1 \le s \le n$, of degree $2g+n-2$ that vanish on the boundary of $\overline{\mathcal{M}}_{g,n}$, and thus can be used as linear forms on the group $R^{g-1}(\mathcal{M}_{g,n})$. We have proved that the intersection matrix of the classes ψ_i^{g-1} and α_s is non-degenerate, and therefore $\dim R^{g-1}(\mathcal{M}_{g,n}) \ge n$. Finally, we have computed the intersection numbers of all tautological classes in $R^{g-1}(\mathcal{M}_{g,n})$ with the classes α_s. *Assuming* that (ψ_i^{g-1}) is a basis of $R^{g-1}(\mathcal{M}_{g,n})$, this allows us to decompose any class in this basis, thus proving the generalized top intersection property.

3. Generalized socle property: three statements

In this section we formulate three statements and show how to use them to prove the generalized socle property. We will prove these statements in the next sections.

3.1. Three statements

Denote by $\pi_k\colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ the map that forgets the k-th marked point. Let $i_{k,l}\colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ be the map that interchanges the k-th and the l-th marked points.

Let $S^j_{k,l}(\mathcal{M}_{g,n})$ be the subspace of $R^j(\mathcal{M}_{g,n})$ defined by

$$S^j_{k,l}(\mathcal{M}_{g,n}) := \{\alpha \in R^j(\mathcal{M}_{g,n}) \mid i^*_{k,l}\alpha = \alpha\}.$$

Lemma 3.1. *Let $n \geq 2$ and $1 \leq k < l \leq n$. Then*

$$S^{g-1}_{k,l}(\mathcal{M}_{g,n}) \subset \pi^*_k(R^{g-1}(\mathcal{M}_{g,n-1})) + \pi^*_l(R^{g-1}(\mathcal{M}_{g,n-1})).$$

Proposition 3.2. *For any $n \geq 1$, we have*

$$R^{g-1}(\mathcal{M}_{g,n}) = \pi^*_1(R^{g-1}(\mathcal{M}_{g,n-1})) + \psi_1\pi^*_1(R^{g-2}(\mathcal{M}_{g,n-1})).$$

Proposition 3.3. *For any $n \geq 1$, we have*

$$R^{g-2}(\mathcal{M}_{g,n}) = \pi^*_1(R^{g-2}(\mathcal{M}_{g,n-1})) + \psi_1\pi^*_1(R^{g-3}(\mathcal{M}_{g,n-1})) + \sum_{1\leq k<l\leq n} S^{g-2}_{k,l}(\mathcal{M}_{g,n}).$$

3.2. *Proof of the generalized socle conjecture*

In Section 2 we have proved $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$. It remains to prove the opposite inequality.

Let $p_i\colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,1}$ be the map that forgets all marked points except the i-th. Since $\dim R^{g-1}(\mathcal{M}_{g,1}) = 1$ [Loo95], it is sufficient to prove that, for any $n \geq 1$, the group $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by the pull-backs $p^*_i(R^{g-1}(\mathcal{M}_{g,1}))$, where $1 \leq i \leq n$. Equivalently, we have to prove that

$$R^{g-1}(\mathcal{M}_{g,n}) = \sum_{i=1}^{n} \pi^*_i(R^{g-1}(\mathcal{M}_{g,n-1})) \tag{3.1}$$

for any $n \geq 2$.

From Propositions 3.2 and 3.3 it follows that

$$\begin{aligned} R^{g-1}(\mathcal{M}_{g,n}) = \pi^*_1(R^{g-1}(\mathcal{M}_{g,n-1})) + \psi_1(\pi_2\circ\pi_1)^*(R^{g-2}(\mathcal{M}_{g,n-2})) \\ + \psi_1\psi_2(\pi_2\circ\pi_1)^*(R^{g-3}(\mathcal{M}_{g,n-2})) + \psi_1\pi^*_1\Big(\sum_{2\leq k<l\leq n} S^{g-2}_{k,l}(\mathcal{M}_{g,n-1})\Big). \end{aligned}$$

Obviously, we have

$$\begin{aligned} &\psi_1(\pi_2\circ\pi_1)^*(R^{g-2}(\mathcal{M}_{g,n-2})) \subset \pi^*_2(R^{g-1}(\mathcal{M}_{g,n-1})),\\ &\psi_1\psi_2(\pi_2\circ\pi_1)^*(R^{g-3}(\mathcal{M}_{g,n-2})) \subset S^{g-1}_{1,2}(\mathcal{M}_{g,n}),\\ &\psi_1\pi^*_1(S^{g-2}_{k,l}(\mathcal{M}_{g,n-1})) \subset S^{g-1}_{k,l}(\mathcal{M}_{g,n}), \qquad \text{where } 2\leq k<l\leq n. \end{aligned}$$

Applying Lemma 3.1 to the last two formulas, we get (3.1).

4. Double ramification cycles

Here we give the definition of a particular type of double ramification cycles that we need, and formulate the main formulas that we will use.

In Section 4.1 we define double ramification cycles. In Section 4.2 we prove the main formulas for them. In Section 4.3 we formulate Hain's result. In Section 4.4 we explain how double ramification cycles span the tautological groups of $\mathcal{M}_{g,n}$.

4.1. Definition of double ramification cycles

Let $a_1, \dots, a_n$, $n \geq 1$, be integers satisfying $\sum a_i = 0$. Suppose that not all of them are zero. Denote by n_+ the number of positive integers among the a_i's. They form a partition $\mu = (\mu_1, \dots, \mu_{n_+})$. Similarly, denote by n_- the number of negative integers among the a_i's. After a change of sign they form another partition $\nu = (\nu_1, \dots, \nu_{n_-})$. Both μ and ν are partitions of the same integer

$$d := \frac{1}{2}\sum_{i=1}^{n} |a_i|.$$

Finally, let n_0 be the number of vanishing a_i's.

Let $\overline{\mathcal{M}}_{g,n_0}(\mu,\nu)$ be the moduli space of "rubber" stable maps to $\mathbb{CP}^1$ relative to 0 and ∞ (see e.g. [GJV11, OP06]). The partitions μ and ν correspond to ramification profiles over 0 and ∞. Denote by ρ the forgetful map $\overline{\mathcal{M}}_{g,n_0}(\mu,\nu) \to \overline{\mathcal{M}}_{g,n}$.

The double ramification cycle without forgotten points is defined by

$$DR_g\Big(\prod_{i=1}^{n} m_{a_i}\Big) := \rho_*([\overline{\mathcal{M}}_{g,n_0}(\mu,\nu)]^{\mathrm{virt}}),$$

where $[\overline{\mathcal{M}}_{g,n_0}(\mu,\nu)]^{\mathrm{virt}}$ is the virtual fundamental class in the homology of $\overline{\mathcal{M}}_{g,n_0}(\mu,\nu)$ (see e.g. [GJV11]).

General double ramification cycles are defined as follows. Let $k \geq 0$ and $a_1, \dots, a_n$, $b_1, \dots, b_k$ be integers, not all zero, with $\sum_{i=1}^{n} a_i + \sum_{j=1}^{k} b_j = 0$. Let $\pi : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ be the map that forgets the last k marked points. Then

$$DR_g\Big(\prod_{i=1}^{n} m_{a_i}\prod_{j=1}^{k}\widetilde{m}_{b_j}\Big) := \pi_*\Big(DR_g\Big(\prod_{i=1}^{n} m_{a_i}\prod_{j=1}^{k} m_{b_j}\Big)\Big).$$

In this notation the i-th marked point corresponds to the ramification of order a_i, so the order of the symbols m_{a_i} is important. On the other hand, the order of the symbols $\widetilde{m}_{b_j}$ is irrelevant. Sometimes we will place them in different positions in the bracket. For example, $DR_g(m_a\widetilde{m}_b)$ and $DR_g(\widetilde{m}_b m_a)$ are, by definition, the same class.

The Poincaré dual of $DR_g(\prod_{i=1}^{n} m_{a_i}\prod_{j=1}^{k}\widetilde{m}_{b_j})$ in the cohomology of $\overline{\mathcal{M}}_{g,n}$ will be denoted by the same symbol. We have (see e.g. [GJV11])

$$DR_g\Big(\prod_{i=1}^{n} m_{a_i}\prod_{j=1}^{k}\widetilde{m}_{b_j}\Big)\begin{cases} = 0 & \text{if } k \geq g+1,\\ \in H^{2(g-k)}(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) & \text{if } k \leq g.\end{cases}$$

In [FP05] it is proved that double ramification cycles belong to the tautological ring of $\overline{\mathcal{M}}_{g,n}$. Below we often use the restrictions of the classes $DR_g(\prod_{i=1}^n m_{a_i} \prod_{j=1}^k \widetilde{m}_{b_j})$ to the open moduli space $\mathcal{M}_{g,n}$. Abusing notation we denote the restrictions by the same symbol.

4.2. *Main formulas for double ramification cycles*

Here we list the main properties of double ramification cycles.

Lemma 4.1. *We have*

$$DR_g\Big(\prod_{i=1}^n m_{a_i} \prod_{i=1}^k \widetilde{m}_{b_i}\Big) = DR_g\Big(\prod_{i=1}^n m_{-a_i} \prod_{i=1}^k \widetilde{m}_{-b_i}\Big). \tag{4.1}$$

Proof. The proof is obvious from the definition of double ramification cycles. □

Lemma 4.2. *Let $a_1, \dots, a_n$, $n \geq 1$, and $b_1, \dots, b_k$, $1 \leq k \leq g+1$, be non-zero integers such that $\sum_{i=1}^n a_i + \sum_{j=1}^k b_j = 0$. Then we have the following equality in $R^{g-k+1}(\mathcal{M}_{g,n})$:*

$$\begin{aligned}\psi_1 \cdot DR_g\Big(\prod_{i=1}^n m_{a_i} \prod_{j=1}^k \widetilde{m}_{b_j}\Big) = &- \sum_{1 \leq i < j \leq k} \frac{b_i + b_j}{r a_1} DR_g\Big(\prod_{l=1}^n m_{a_l} \widetilde{m}_{b_i+b_j} \prod_{l \neq i,j} \widetilde{m}_{b_l}\Big) \\ &- \sum_{i=2}^n \sum_{j=1}^k \frac{a_i + b_j}{r a_1} DR_g\Big(\prod_{l=1}^n m_{a_l + \delta_{l,i} b_j} \prod_{l \neq j} \widetilde{m}_{b_l}\Big) \\ &+ \sum_{j=1}^k \frac{-a_1 + (r-1) b_j}{r a_1} DR_g\Big(m_{a_1+b_j} \prod_{i=2}^n m_{a_i} \prod_{l \neq j} \widetilde{m}_{b_l}\Big),\end{aligned} \tag{4.2}$$

where $r := 2g - 2 + n + k$.

Proof. First, observe that for $k = 1$ we have a trivial identity, because $R^g(\mathcal{M}_{g,n}) = 0$. For $k = g+1$ the left-hand side of the equation is zero, while the vanishing of the right-hand side follows from the substitution

$$DR_g\Big(\prod_{i=1}^n m_{a_i} \prod_{j=1}^g \widetilde{m}_{b_j}\Big) = g! \prod_{j=1}^g b_j^2.$$

The proof in the general case is based on [BSSZ12, Theorem 4]. Indeed, by definition, $DR_g(\prod_{i=1}^n m_{a_i} \prod_{j=1}^k \widetilde{m}_{b_j})$ is the restriction to $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ of the push-forward of $DR_g(\prod_{i=1}^n m_{a_i} \prod_{j=1}^k m_{b_j})$ defined in the tautological ring of $\overline{\mathcal{M}}_{g,n+k}$. To prove the identity we will use the projection formula. Namely, we lift ψ_1 to $\overline{\mathcal{M}}_{g,n+k}$, intersect it there with $DR_g(\prod_{i=1}^n m_{a_i} \prod_{j=1}^k m_{b_j})$ using [BSSZ12, Theorem 4], and then push forward this intersection to $\overline{\mathcal{M}}_{g,n}$ and restrict the result to $\mathcal{M}_{g,n}$.

The class $\pi^*\psi_1$ for $\pi : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is equal to $\psi_1 - D$, where D is the class of the divisor on which the points labeled by a_1 and b_i, $i \in I \subset \{1, \dots, k\}$, lie on a rational tail. The choice of $I \neq \emptyset$ here corresponds to a choice of an irreducible component of D. For dimensional reasons, the only non-trivial contribution under the push-forward is given by the irreducible components of $-D$ that correspond to $|I| = 1$, that is, the point a_1 lie on a rational tail with exactly one point b_i, $i = 1, \dots, k$. The push-forward π_* of this intersection is equal to the class

$$-\sum_{j=1}^{k} DR_g\Big(m_{a_1+b_j} \prod_{i=2}^{n} m_{a_i} \prod_{l \neq j} \widetilde{m}_{b_l}\Big). \tag{4.3}$$

(See e.g. the local computation of multiplicity in [SZ08, Lemma 3.3]. Here we have a different global geometry of the space, but the local multiplicity is computed in exactly the same way.)

Now we use the formula for $\psi_1 \cdot DR_g(\prod_{i=1}^{n} m_{a_i} \prod_{j=1}^{k} m_{b_j})$ in [BSSZ12, Theorem 4]. We have there a sum over all possible degenerations of the DR-cycle into two DR-cycles,

$$DR_{g_1}\Big(\prod_{i \in I} m_{a_i} \prod_{j \in J} m_{b_j} \prod_{l=1}^{p} m_{-c_k}\Big) \boxtimes DR_{g_2}\Big(\prod_{i \in I'} m_{a_i} \prod_{j \in J'} m_{b_j} \prod_{l=1}^{p} m_{c_k}\Big),$$

where $g_1 + g_2 + p - 1 = g$, $I \sqcup I' = \{1, \dots, n\}$, and $J \sqcup J' = \{1, \dots, k\}$, with some combinatorially defined coefficients. Here the symbol $\boxtimes$ indicates the gluing of the points $-c_k$ and c_k to a node, $k = 1, \dots, p$, and for the full definition we refer to [BSSZ12, Section 2.1]. Now we claim that the only degenerations of the DR-cycle that do not vanish under the restriction of the push-forward π_* to the open moduli space $\mathcal{M}_{g,n}$ can be described in the following way:

1. We have $p = 1$.
2. Either $g_1 = 0$ or $g_2 = 0$.
3. The genus zero component has only three special points, where one point is $m_{\pm c_1}$, and the other two are either a pair m_{a_i}, m_{b_j}, or a pair m_{b_i}, m_{b_j}.

Indeed, if $p > 1$, or $g_1, g_2 \geq 1$, or the genus 0 component contains at least two points a_i, a_j, then the push-forward of this class lies on the boundary of $\overline{\mathcal{M}}_{g,n}$. So, we have the genus zero component with at most one point a_i. Then this component should contain precisely two points from the list $a_1, \dots, a_n, b_1, \dots, b_k$ for dimensional reasons. This way we come to the description above.

Thus, we have three essentially different cases: the two marked points on the genus 0 component can be either m_{b_i}, m_{b_j}, or m_{a_i}, m_{b_j}, $i \neq 1$, or m_{a_1}, m_{b_j}. In the first case, the class (including the coefficient) that we get is

$$-\frac{b_i + b_j}{ra_1} DR_g\Big(\prod_{l=1}^{n} m_{a_l} \widetilde{m}_{b_i+b_j} \prod_{l \neq i,j} \widetilde{m}_{b_l}\Big).$$

The sum of these classes is exactly the first summand on the right-hand side of (4.2). In the second case, after the push-forward and restriction to the open part, we obtain

$$-\frac{a_i + b_j}{ra_1} DR_g\Big(\prod_{l=1}^{n} m_{a_l+\delta_{l,i} b_j} \prod_{l \neq j} \widetilde{m}_{b_l}\Big),$$

and these classes add up to the second summand of (4.2). In the last case, we have the class

$$\frac{(r-1)(a_1+b_j)}{ra_1} DR_g\Big(m_{a_1+b_j}\prod_{i=2}^n m_{a_i}\prod_{l\neq j}\widetilde{m}_{b_l}\Big). \tag{4.4}$$

Recall that we already got a contribution of the same class with a different coefficient in (4.3). The sum of the classes (4.4) and the corresponding summand in (4.3) is equal to

$$\frac{-a_1+(r-1)b_j}{ra_1} DR_g\Big(m_{a_1+b_j}\prod_{i=2}^n m_{a_i}\prod_{l\neq j}\widetilde{m}_{b_l}\Big).$$

The sum of these terms is precisely the third summand in (4.2). □

Lemma 4.3. *Let $a_1,\dots,a_n$, $n\geq 0$, and $b_1,\dots,b_k$, $1\leq k\leq g$, be non-zero integers such that $\sum_{i=1}^n a_i+\sum_{j=1}^k b_j=0$. Then we have the following relation in $R^{g-k+1}(\mathcal{M}_{g,n+1})$:*

$$\sum_{i=1}^k b_i DR_g\Big(m_{b_i}\prod_{j\neq i}\widetilde{m}_{b_j}\prod_{l=1}^n m_{a_l}\Big)\in \pi_1^*(R^{g-k+1}(\mathcal{M}_{g,n})). \tag{4.5}$$

Proof. Several proofs of this lemma are possible. In the case $k=1$, the lemma is obvious, since the class (4.5) is zero. If we assume that $n\geq 1$, then we can argue in the following way. We consider the class ψ_2 (the ψ-class at the point labeled by a_1) multiplied by $\pi_1^*DR_g(\prod_{j=1}^k\widetilde{m}_{b_j}\prod_{i=1}^n m_{a_i})$. On the open part it is equal to $\pi_1^*\psi_2\cdot\pi_1^*DR_g(\prod_{j=1}^k\widetilde{m}_{b_j}\prod_{i=1}^n m_{a_i})$, so it is a pull-back. On the other hand, we can compute this class using [BSSZ12, Theorem 5] (we should assign multiplicity zero to x_1), and, via the same argument as in the proof of Lemma 4.2, we see that the only terms that contribute to the restriction of the class obtained to the open moduli space are the class (4.5) and further classes in $\pi_1^*(R^{g-k+1}(\mathcal{M}_{g,n}))$. The general argument is very close to the one in [Ion02, Proposition 2.8].

Consider the space $\overline{\mathcal{M}}_{g,1}(\mu,\nu)$ of rubber maps to $\mathbb{C}\mathrm{P}^1$ with one marked point x_1. We assume that the collection of multiplicities $\{\mu_1,\dots,\mu_{m_+},-\nu_1,\dots,-\nu_{m_-}\}$ is equal to $\{a_1,\dots,a_n,b_1,\dots,b_k\}$, up to order. The branching morphism σ takes $\overline{\mathcal{M}}_{g,1}(\mu,\nu)$ to LM_{r+1}/S_r, the quotient of the Losev–Manin moduli space [LM00] by the action of the symmetric group that permutes the branch points of the rubber maps. We refer to [BSSZ12, Section 2.2] for a full discussion of this branching morphism in this setting.

We consider the following divisors in LM_{r+1}. Let $p_1,\dots,p_r$ be the branch points and let q be the image of x_1. We consider the S_r-symmetrization of the divisor $D_{0,p_1|q,\infty}-D_{0,q|p_1,\infty}$, where $D_{a,b|c,d}$ denotes the divisor of two-component curves, where the pairs a,b and c,d lie on different components. Denote this symmetrized divisor by D. The class of $D_{0,p_1|q,\infty}-D_{0,q|p_1,\infty}$ is zero, therefore the class σ^*D of the pull-back of the symmetrization of this divisor to $\overline{\mathcal{M}}_{g,1}(\mu,\nu)$ is also zero.

We consider the maps $\rho\colon\overline{\mathcal{M}}_{g,1}(\mu,\nu)\to\overline{\mathcal{M}}_{g,n+k+1}$ and $\pi:\overline{\mathcal{M}}_{g,n+k+1}\to\overline{\mathcal{M}}_{g,n+1}$. Our goal now is to compute the restriction of the class $\pi_*\rho_*\sigma^*D$ to the open moduli space $\mathcal{M}_{g,n+1}$. We claim that this class is equal to the sum of the class in (4.5) and a class in

$\pi_1^*(R^{g-k+1}(\mathcal{M}_{g,n}))$. Since, on the other hand, we know that $\pi_*\rho_*\sigma^*D = 0$, this proves the lemma.

So let us compute $\pi_*\rho_*\sigma^*D$. We can do it for each component separately along the lines of the analogous computation in [BSSZ12, Lemma 2.4]. Using the same argument as in the proof of Lemma 4.2, we obtain a non-trivial contribution only from the divisors

$$DR_{g_1}\Big(\prod_{i\in I} m_{a_i} \prod_{j\in J} m_{b_j} \prod_{l=1}^{p} m_{-c_l}\Big) \boxtimes DR_{g_2}\Big(\prod_{i\in I'} m_{a_i} \prod_{j\in J'} m_{b_j} \prod_{l=1}^{p} m_{c_l}\Big)$$

of two possible types. One type is exactly the divisors described in Lemma 4.2, that is, $p=1$, one component is of genus 0, it contains exactly two marked points that are labeled either by a_i, b_j or b_i, b_j, and the marked point x_1 lies on the other component. The last condition is for dimensional reasons. Since there are no restrictions on the position of x_1, all classes of this type project to $\pi_1^*(R^{g-k+1}(\mathcal{M}_{g,n}))$.

Let us describe the other possible type. We still have $p=1$ and one of the components is of genus 0 (otherwise, the restriction to the open moduli space would be trivial), and on the component of genus 0 we have only two marked points (for dimensional reasons), but now these two marked points will be x_1 and b_i. This class projects to the i-th term in (4.5), and its coefficient, according to [BSSZ12, Lemma 2.4], is precisely b_i. This completes the proof of the lemma. □

4.3. Hain's formula

Consider the moduli space $\mathcal{M}_{g,n}^{\mathrm{rt}}$ of stable curves with rational tails. In this section we work in the cohomology of $\mathcal{M}_{g,n}^{\mathrm{rt}}$. Let $\psi_i^\dagger = p_i^*\psi_i$, where $p_i\colon \mathcal{M}_{g,n}^{\mathrm{rt}} \to \mathcal{M}_{g,1}^{\mathrm{rt}}$ is the map that forgets all marked points except the i-th. Let J be any subset of $\{1,\ldots,n\}$ such that $|J|\geq 2$. Denote by D_J the divisor in $\mathcal{M}_{g,n}^{\mathrm{rt}}$ that is formed by stable curves with a rational component that contains exactly the marked points numbered by J.

The following formula was discovered by Hain [Hain11]:

$$DR_g\Big(\prod_{i=1}^n m_{a_i}\Big) = \frac{1}{g!}\Bigg(\sum_{i=1}^n \frac{a_i^2\psi_i^\dagger}{2} - \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|\geq 2}} \Big(\sum_{i,j\in J,\, i<j} a_i a_j\Big) D_J\Bigg)^g. \tag{4.6}$$

To be precise, in [Hain11] this formula was proved for a version of double ramification cycles that is constructed using the universal Jacobian over the moduli space of curves. Luckily, in [CMW11] it is proved that this version coincides with ours when we restrict both to $\mathcal{M}_{g,n}^{\mathrm{rt}}$.

From (4.6) it follows that, as a cohomology class in $H^{2g}(\mathcal{M}_{g,n}^{\mathrm{rt}};\mathbb{Q})$, the class

$$DR_g\Big(m_{-\sum_{i=2}^n a_i} \prod_{i=2}^n m_{a_i}\Big)$$

is a homogeneous polynomial of degree $2g$ in the variables $a_2,\ldots,a_n$. Let us prove the following simple lemma.

Lemma 4.4. *As a cohomology class in $H^{2g-2}(\mathcal{M}_{g,n}^{\mathrm{rt}};\mathbb{Q})$, the class*

$$DR_g\Big(m_{-\sum_{i=2}^n a_i}\prod_{i=2}^{n-1} m_{a_i}\widetilde{m}_{a_n}\Big)$$

is a homogeneous polynomial of degree $2g$ in the variables $a_2,\dots,a_n$. Moreover, this polynomial is divisible by a_n.

Proof. Let $a_1 := -\sum_{i=2}^n a_i$. We have $DR_g(\prod_{i=1}^{n-1} m_{a_i}\widetilde{m}_{a_n}) = \pi_*(DR_g(\prod_{i=1}^n m_{a_i}))$, where $\pi\colon \mathcal{M}_{g,n}^{\mathrm{rt}} \to \mathcal{M}_{g,n-1}^{\mathrm{rt}}$ is the morphism that forgets the last marked point. Thus, the first statement is clear.

It is easy to see that

$$\Big(\sum_{i=1}^n \frac{a_i^2\psi_i^\dagger}{2} - \sum_{\substack{J\subset\{1,\dots,n\}\\|J|\ge 2}}\Big(\sum_{i,j\in J,\, i<j} a_i a_j\Big) D_J\Big)\Big|_{a_n=0}$$

$$= \pi_n^*\Big(\sum_{i=1}^{n-1} \frac{a_i^2\psi_i^\dagger}{2} - \sum_{\substack{J\subset\{1,\dots,n-1\}\\|J|\ge 2}}\Big(\sum_{i,j\in J,\, i<j} a_i a_j\Big) D_J\Big).$$

Therefore, if we set $a_n = 0$ on the right-hand side of (4.6) and push it forward to $\mathcal{M}_{g,n-1}^{\mathrm{rt}}$, we get zero. Hence, the second statement of the lemma is also proved. □

4.4. DR-cycles and the tautological ring of $\mathcal{M}_{g,n}$

Lemma 4.5. *Let $n \ge 1$ and $1 \le k \le g$. The group $R^{g-k}(\mathcal{M}_{g,n})$ is spanned by the double ramification cycles of the form*

$$DR_g\Big(\prod_{i=1}^{n-1} m_{a_i} m_{-d}\prod_{j=1}^{k}\widetilde{m}_{b_j}\Big), \tag{4.7}$$

where $a_1,\dots,a_{n-1},b_1,\dots,b_k$ and d are positive integers such that $a_1+\dots+a_{n-1}+b_1+\dots+b_k = d$.

Proof. This lemma is a version of [Ion02, Corollary 2.5] adapted to our situation: we are using the DR-cycles defined by "rubber" stable relative maps rather than the admissible coverings, we want to have the DR-cycles with exactly one negative multiplicity, and we consider the restriction of the DR-cycles to the open part of the moduli space.

We want to show that the push-forward of any monomial of ψ-classes on $\overline{\mathcal{M}}_{g,n+l}$ to the space $\overline{\mathcal{M}}_{g,n}$, and further restricted to its open part, can be expressed in terms of the DR-cycles (4.7).

We represent the degree zero class $g!\prod_{i=1}^{g} b_i^2$ on $\overline{\mathcal{M}}_{g,n+l}$ as a DR-cycle

$$DR_g\Big(\prod_{i=1}^{n+l} m_{a_i}\prod_{j=1}^{g}\widetilde{m}_{b_j}\Big),$$

where a_n is the unique negative index. Obviously, we can choose the multiplicities in this way. Then we lift the monomial of ψ-classes to the moduli space $\overline{\mathcal{M}}_{g,n+l+g}$

(as we did in the proof of Lemma 4.2 above) and intersect it there with the DR-cycle $DR_g(\prod_{i=1}^{n+l} m_{a_i} \prod_{j=1}^{g} m_{b_j})$ using [BSSZ12, Theorem 4]. Then we apply the push-forward to $\overline{\mathcal{M}}_{g,n}$ and restrict the result to $\mathcal{M}_{g,n}$.

Let us discuss the structure of the class we obtain. First of all, since we are interested only in the non-degenerate restrictions to the open moduli space, one of the components in the degeneration formula

$$DR_{g_1}\Big(\prod_{i\in I} m_{a_i} \prod_{j\in J} m_{b_j} \prod_{l=1}^{p} m_{-c_k}\Big) \boxtimes DR_{g_2}\Big(\prod_{i\in I'} m_{a_i} \prod_{j\in J'} m_{b_j} \prod_{l=1}^{p} m_{c_k}\Big)$$

must be of genus g. This implies, exactly as in the proofs of Lemmas 4.2 and 4.3, that $p = 0$ and the other component has genus 0. This means that on the component of genus g we again have the structure of a DR-cycle with exactly one negative index. So, if we are interested only in the terms that can be non-trivially restricted to the open part of the moduli space, then this property of a DR-cycle is preserved throughout the computation of the monomial of ψ-classes. It is also preserved under the push-forward that forgets some of the marked points. This implies that the expression we obtain for the push-forward of a monomial of ψ-classes is a linear combination of DR-cycles with one negative index. It is also easy to see that the only negative index in these cycles corresponds to the n-th marked point. This is precisely the statement of the lemma. □

5. Technical lemmas

Here we collect several linear algebra lemmas that we will use in Section 6.

Lemma 5.1. *Let V be a vector space and $x_1, \dots, x_{p-2} \in V$, where $p \geq 3$. Suppose that for any positive integers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = p$ we have*

$$x_{\lambda_1} + x_{\lambda_2} + x_{\lambda_3} = 0.$$

Then there exists a vector $\alpha \in V$ such that $x_i = (i/p - 1/3)\alpha$ for all i.

Proof. If $p = 3$ the lemma is obvious. Suppose $p \geq 4$. We have

$$x_i + x_{p-1-i} = -x_1 \qquad \text{for } i = 1, \dots, p-2.$$

On the other hand,

$$x_i + x_{p-2-i} = -x_2 \qquad \text{for } i = 1, \dots, p-3.$$

Subtracting the second equation from the first, we obtain

$$x_{p-1-i} - x_{p-2-i} = x_2 - x_1 \qquad \text{for } i = 1, \dots, p-3.$$

Hence, $x_i = x_1 + (i-1)(x_2 - x_1)$. Inserting this in $2x_1 + x_{p-2} = 0$, we get $x_1 = -(p-3)(x_2 - x_1)/3$. Therefore, $x_i = (i - p/3)(x_2 - x_1)$. □

Lemma 5.2. *Let V be a vector space and let $v_{i,j} \in V$ for all $i, j \geq 1$. Define*

$$z_{i,j} := -v_{i,j} - v_{j,i}.$$

Suppose that

$$\begin{cases} \sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} (v_{a_k,a_i+a_j} + z_{a_i,a_j}) = 0 & \text{if } a_1, a_2, a_3 \geq 1, \\ z_{a_1,a_2} - z_{c_1,c_2} - \sum_{a_i > c_j} v_{c_j,a_i-c_j} + \sum_{c_i > a_j} v_{a_j,c_i-a_j} = 0 & \text{if } a_1 + a_2 = c_1 + c_2. \end{cases} \tag{5.1}$$

Then there exist $\alpha_2, \alpha_3, \ldots \in V$ such that

$$v_{i,j} = \left(\frac{i}{i+j} - \frac{1}{3}\right)\alpha_{i+j} + \left(\frac{1}{3} - \frac{i+j}{i}\right)\alpha_i + \frac{1}{3}\alpha_j. \tag{5.2}$$

Here, by definition, $\alpha_1 := 0$.

Proof. Obviously, $\alpha_2 = 6v_{1,1}$. Suppose $d \geq 3$ and we have found vectors $\alpha_2, \ldots, \alpha_{d-1}$ such that (5.2) holds for $i + j \leq d - 1$. Let us construct α_d such that (5.2) holds for $i + j = d$. For $i + j = d$, define

$$\widetilde{v}_{i,j} := v_{i,j} - \left(\frac{1}{3} - \frac{d}{i}\right)\alpha_i - \frac{1}{3}\alpha_j, \quad \widetilde{z}_{i,j} := -\widetilde{v}_{i,j} - \widetilde{v}_{j,i}.$$

It is easy to check that (5.1) implies

$$\begin{cases} \sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} \widetilde{v}_{a_k,a_i+a_j} = 0 & \text{if } a_1 + a_2 + a_3 = d, \\ \widetilde{z}_{a_1,a_2} - \widetilde{z}_{c_1,c_2} = 0 & \text{if } a_1 + a_2 = c_1 + c_2 = d. \end{cases}$$

If $d = 3$, then we can take $\alpha_3 = 3\widetilde{v}_{2,1}$. Suppose $d \geq 4$. By Lemma 5.1, there exists a vector α_d such that $\widetilde{v}_{i,d-i} = (i/d - 1/3)\alpha_d$ for $i \leq d-2$. Since $\widetilde{z}_{1,d-1} = \widetilde{z}_{2,d-2}$, we also have $\widetilde{v}_{d-1,1} = ((d-1)/d - 1/3)\alpha_d$. □

Lemma 5.3. *Let V be a vector space and $d \geq 3$. Suppose that vectors $v_{i,j,k} \in V$, where $i + j + k = d$ and $i, j, k \geq 1$, satisfy*

$$\sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} v_{b_k,b_i+b_j,a} = 0 \quad \text{if } b_1 + b_2 + b_3 + a = d \text{ and } a, b_i \geq 1, \tag{5.3}$$

$$k v_{i,j,k} + j v_{i,k,j} = 0. \tag{5.4}$$

If $d = 3, 4$, then $v_{i,j,k} = 0$. If $d \geq 5$, then there exists a vector $\alpha \in V$ such that

$$v_{i,j,k} = \left(\frac{i}{d-1} - \frac{1}{3}\right)\left(\delta_{k,1} - \frac{1}{k}\delta_{j,1}\right)\alpha.$$

Proof. The cases $d = 3, 4, 5$ can be easily checked by a direct computation.

Suppose that $d \geq 6$. By Lemma 5.1, there exist $\alpha_3, \ldots, \alpha_{d-1} \in V$ such that

$$v_{i,j,k} = \left(\frac{i}{i+j} - \frac{1}{3}\right)\alpha_{i+j} \quad \text{for } j \geq 2.$$

Let us prove that

$$\alpha_{d-k} = 0 \quad \text{for } 2 \leq k \leq (d-1)/2. \tag{5.5}$$

From (5.4) it follows that $v_{i,j,k} = 0$ if $j = k$. Therefore, if $2 \leq k \leq (d-1)/2$, then $\left(\frac{d-2k}{d-k} - \frac{1}{3}\right)\alpha_{d-k} = 0$. If $d \neq \frac{5}{2}k$, then $\alpha_{d-k} = 0$. Suppose that $d = \frac{5}{2}k$. We have $v_{k/2+1,k-1,k} = -\frac{k-1}{k} v_{k/2+1,k,k-1} = 0$. On the other hand, we obtain $v_{k/2+1,k-1,k} = \left(\frac{k/2+1}{d-k} - \frac{1}{3}\right)\alpha_{d-k}$. The coefficient of α_{d-k} here is not zero, therefore $\alpha_{d-k} = 0$. Thus, (5.5) is proved.

We see that $v_{i,j,k} = 0$ if $j \geq 2$ and $2 \leq k \leq (d-1)/2$. If we apply (5.4), we find that $v_{i,j,k} = 0$ if $k \geq 2$ and $2 \leq j \leq (d-1)/2$, thus $v_{i,j,k} = 0$ if $j \geq 2$ or $k \geq 2$.

If $2 \leq j \leq d-2$, then $v_{d-j-1,j,1} = \left(\frac{d-j-1}{d-1} - \frac{1}{3}\right)\alpha_{d-1}$ and $v_{d-j-1,1,j} = -\frac{1}{j} v_{d-j-1,1,j} = -\frac{1}{j}\left(\frac{d-j-1}{d-1} - \frac{1}{3}\right)\alpha_{d-1}$. Also, $v_{d-2,1,1} = 0$. This completes the proof. $\square$

6. Proofs of Lemma 3.1 and Propositions 3.2 and 3.3

6.1. Proof of Lemma 3.1

Without loss of generality, we can assume that $k = 1$ and $l = 2$. It is sufficient to prove

$$\alpha + i_{1,2}^*\alpha \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})) + \pi_2^*(R^{g-1}(\mathcal{M}_{g,n-1})) \tag{6.1}$$

for any $\alpha \in R^{g-1}(\mathcal{M}_{g,n})$. By Lemma 4.5, we can assume that $\alpha = DR_g(\prod_{i=1}^n m_{a_i}\widetilde{m}_b)$. From Lemma 4.3 it follows that

$$DR_g\left(m_{a_1}\widetilde{m}_b m_{a_2}\prod_{i=3}^n m_{a_i}\right) + \frac{b}{a_1}DR_g\left(m_b\widetilde{m}_{a_1}m_{a_2}\prod_{i=3}^n m_{a_i}\right) \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})),$$

$$\frac{b}{a_1}DR_g\left(m_b m_{a_2}\widetilde{m}_{a_1}\prod_{i=3}^n m_{a_i}\right) + \frac{b}{a_2}DR_g\left(m_b m_{a_1}\widetilde{m}_{a_2}\prod_{i=3}^n m_{a_i}\right) \in \pi_2^*(R^{g-1}(\mathcal{M}_{g,n-1})),$$

$$\frac{b}{a_2}DR_g\left(m_b\widetilde{m}_{a_2}m_{a_1}\prod_{i=3}^n m_{a_i}\right) + DR_g\left(m_{a_2}\widetilde{m}_b m_{a_1}\prod_{i=3}^n m_{a_i}\right) \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})).$$

If we sum the first and third rows and subtract the second row, we get

$$DR_g\left(m_{a_1}m_{a_2}\widetilde{m}_b\prod_{i=3}^n m_{a_i}\right) + DR_g\left(m_{a_2}m_{a_1}\widetilde{m}_b\prod_{i=3}^n m_{a_i}\right) \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})) + \pi_2^*(R^{g-1}(\mathcal{M}_{g,n-1})),$$

as desired.

6.2. Proof of Proposition 3.2

Suppose $n = 1$. We know that $R^{g-1}(\mathcal{M}_{g,1}) = \mathbb{Q}$, $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$ and $R^{g-2}(\mathcal{M}_g)$ is spanned by κ_{g-2} (see [Loo95]). Therefore, it is sufficient to prove that $\psi_1\pi_1^*(\kappa_{g-2}) \neq 0$. But this is true because $(\pi_1)_*(\psi_1\pi_1^*(\kappa_{g-2})) = (2g-2)\kappa_{g-2}$.

Suppose that $n \geq 2$. We denote by K the subspace of $R^{g-1}(\mathcal{M}_{g,n})$ defined by

$$K := \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})) + \psi_1\pi_1^*(R^{g-2}(\mathcal{M}_{g,n-1})).$$

By Lemma 4.5, it is sufficient to prove that

$$DR_g\Big(\widetilde{m}_b\prod_{i=1}^{n-1} m_{a_i}m_{-d}\Big) \in K, \tag{6.2}$$

where $a_1, \ldots, a_{n-1}, b, d$ are positive integers such that $a_1 + \cdots + a_{n-1} + b = d$. We use double induction on d and $\widetilde{d} := b + a_1$. If $\widetilde{d} = 2$, then $b = a_1 = 1$, and from Lemma 4.3 it immediately follows that

$$DR_g\Big(\widetilde{m}_1 m_1\prod_{i=2}^{n} m_{a_n}m_{-d}\Big) \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})). \tag{6.3}$$

Suppose that $\widetilde{d} \geq 3$. Consider positive integers b_1, b_2, b_3 such that $b_1 + b_2 + b_3 = \widetilde{d}$. By Lemma 4.3, we have

$$\sum_{i=1}^{3} b_i DR_g\Big(m_{b_i}\prod_{j\neq i}\widetilde{m}_{b_j}\prod_{l=2}^{n-1} m_{a_l}m_{-d}\Big) \in \pi_1^*(R^{g-2}(\mathcal{M}_{g,n-1})).$$

Let us multiply both sides of this formula by ψ_1. Using Lemmas 4.2 and 4.3 and the induction assumption we get

$$\sum_{i=1}^{3}\frac{\widetilde{d} + (r-3)b_i}{r} DR_g\Big(\widetilde{m}_{\widetilde{d}-b_i}m_{b_i}\prod_{j=2}^{n-1} m_{a_j}m_{-d}\Big) \in K, \tag{6.4}$$

where $r = 2g + n$.

Let us fix $a_2, \ldots, a_n$ and analyze relation (6.4). Let

$$u_i := DR_g\Big(\widetilde{m}_{\widetilde{d}-i}m_i\prod_{j=2}^{n-1} m_{a_j}m_{-d}\Big) \quad \text{for } i = 1, \ldots, \widetilde{d}-1.$$

For any two classes $\alpha, \beta \in R^{g-1}(\mathcal{M}_{g,n})$, we will write $\alpha \overset{\text{mod } K}{=} \beta$ if $\alpha - \beta \in K$.

From (6.4) and Lemma 5.1 it follows that there exists $\alpha \in R^{g-1}(\mathcal{M}_{g,n})$ such that

$$\frac{\widetilde{d} + (r-3)i}{r} u_i \overset{\mathrm{mod}\, K}{=} \left(\frac{i}{\widetilde{d}} - \frac{1}{3}\right)\alpha \quad \text{for } i = 1, \ldots, \widetilde{d} - 2. \tag{6.5}$$

By Lemma 4.3, we have

$$i u_i + (\widetilde{d} - i) u_{\widetilde{d}-i} \in \pi_1^*(R^{g-1}(\mathcal{M}_{g,n-1})) \quad \text{for } i = 1, \ldots, \widetilde{d} - 1. \tag{6.6}$$

Let us prove that (6.5) and (6.6) imply that $u_i \in K$ for $i = 1, \ldots, \widetilde{d} - 1$. This will complete the proof of the proposition.

Suppose $\widetilde{d} = 3$. Then from (6.5) it follows that $u_1 \in K$, and (6.6) yields $u_2 \in K$.

Suppose $\widetilde{d} \geq 4$. From (6.5) and (6.6) we see that it is sufficient to prove that $\alpha \in K$. Let $2 \leq i \leq \widetilde{d} - 2$. From (6.5) it follows that

$$i u_i + (\widetilde{d} - i) u_{\widetilde{d}-i} \overset{\mathrm{mod}\, K}{=} \frac{\left(i - \frac{\widetilde{d}}{2}\right)^2 \left(3 - \frac{r}{3}\right) + \frac{\widetilde{d}^2(r-1)}{12}}{\left(\widetilde{d} + (r-3)i\right)\left(\widetilde{d} + (r-3)(\widetilde{d} - i)\right)} r\alpha. \tag{6.7}$$

If $\widetilde{d} = 4$ or $\widetilde{d} = 5$, then, for $i = 2$, the numerator in (6.7) is non-zero. Thus, $\alpha \in K$. Suppose $\widetilde{d} \geq 6$. It is clear that the numerator cannot be zero for two different $i \geq \widetilde{d}/2$. Hence, $\alpha \in K$, as claimed.

6.3. *Proof of Proposition 3.3: main relation*

In this section we construct a relation between double ramification cycles that is the main ingredient in the proof of Proposition 3.3.

In Section 6.3.1 we construct a basic relation using the same idea as in the proof of Proposition 3.2. The problem is that this relation is too complicated to work with. In Section 6.3.2 we introduce new variables that are linear combinations of double ramification cycles. This change of variables allows us to simplify the basic relation considerably. This is done in Section 6.3.3.

6.3.1. Basic relation. We denote by K the subspace of $R^{g-2}(\mathcal{M}_{g,n})$ defined by

$$K = \pi_1^*(R^{g-2}(\mathcal{M}_{g,n-1})) + \psi_1 \pi_1^*(R^{g-3}(\mathcal{M}_{g,n-1})) + \sum_{1 \leq i < j \leq n} S_{i,j}^{g-2}(\mathcal{M}_{g,n}).$$

Let us fix a triple $((b_1, b_2, b_3), b_4, (a_1, \ldots, a_{n-1}))$, where (b_1, b_2, b_3) is an unordered triple of non-zero integers, b_4 is a non-zero integer, $(a_1, \ldots, a_{n-1})$ is sequence of non-zero integers and $\sum b_i + \sum a_j = 0$. By Lemma 4.3, we have

$$\sum_{i=1}^{4} b_i DR_g\left(m_{b_i} \prod_{j \neq i} \widetilde{m}_{b_j} \prod_{p=1}^{n-1} m_{a_p}\right) \in \pi_1^*(R^{g-3}(\mathcal{M}_{g,n-1})). \tag{6.8}$$

Let us multiply both sides of (6.8) by ψ_1. Using Lemmas 4.2 and 4.3 we get

$$\begin{aligned}
&\sum_{\substack{\{i,j,k\}=\{1,2,3\}\\ i<j}} \frac{b_i+b_j}{r}\left(r-2+\frac{b_i+b_j}{b_4}\right)DR_g\Big(\widetilde{m}_{b_4}\widetilde{m}_{b_k}m_{b_i+b_j}\prod_{l=1}^{n-1}m_{a_l}\Big)\\
&\quad-\sum_{\substack{\{i,j,k\}=\{1,2,3\}\\ i<j}} \frac{b_i+b_j}{r}\left(1-\frac{b_k}{b_4}\right)DR_g\Big(\widetilde{m}_{b_4}\widetilde{m}_{b_i+b_j}m_{b_k}\prod_{l=1}^{n-1}m_{a_l}\Big)\\
&\quad-\sum_{l=1}^{n-1}\sum_{\{i,j,k\}=\{1,2,3\}} \frac{b_i+a_l}{r}\left(1-\frac{b_j}{b_4}\right)DR_g\Big(\widetilde{m}_{b_4}\widetilde{m}_{b_k}m_{b_j}\prod_{p=1}^{n-1}m_{a_p+\delta_{p,l}b_i}\Big)\\
&\quad+\sum_{l=1}^{n-1}\sum_{\substack{\{i,j,k\}=\{1,2,3\}\\ i<j}} \frac{b_j-b_i}{r}DR_g\Big(\widetilde{m}_{b_4+a_l}\widetilde{m}_{b_k}m_{b_i}\prod_{p=1}^{n-1}m_{a_p+\delta_{p,l}(b_j-a_l)}\Big)\\
&\quad-\sum_{\{i,j,k\}=\{1,2,3\}} \frac{b_4+b_k+(r-2)b_i}{r}DR_g\Big(\widetilde{m}_{b_4+b_k}\widetilde{m}_{b_j}m_{b_i}\prod_{l=1}^{n-1}m_{a_l}\Big)\in K, \qquad (6.9)
\end{aligned}$$

where $r=2g+n+1$. This relation will be called *the basic relation*.

6.3.2. New variables. Let $f_1,\dots,f_{n+1}$ be arbitrary integers, not all zero. We introduce a cycle $Z_g(\prod_{i=1}^{n+1}m_{f_i})\in R^{g-2}(\mathcal{M}_{g,n})$ as follows:

$$\begin{aligned}
Z_g\Big(\prod_{i=1}^{n+1}m_{f_i}\Big) :=&\ \frac{f_1-(r-n-2)f_2}{r}DR_g\Big(\widetilde{m}_{-d}\widetilde{m}_{f_1}m_{f_2}\prod_{l=3}^{n+1}m_{f_l}\Big)\\
&+\frac{f_2-(r-n-2)f_1}{r}DR_g\Big(\widetilde{m}_{-d}\widetilde{m}_{f_2}m_{f_1}\prod_{l=3}^{n+1}m_{f_l}\Big)\\
&+\frac{f_2-f_1}{r}\sum_{l=3}^{n+1}DR_g\Big(\widetilde{m}_{-d}\widetilde{m}_{f_l}m_{f_2}\prod_{p=3}^{n+1}m_{f_p+\delta_{p,l}(f_1-f_p)}\Big),
\end{aligned}$$

where $r:=2g+n+1$ and $d:=\sum_{i=1}^{n+1}f_i$. Suppose that $d\neq 0$. Define a cycle $V_g(\prod_{i=1}^{n+1}m_{f_i})\in R^{g-2}(\mathcal{M}_{g,n})$ by

$$\begin{aligned}
V_g\Big(\prod_{i=1}^{n+1}m_{f_i}\Big) :=&\ \frac{f_2}{r}\bigg[\Big(r-n-2+\frac{f_1}{d}\Big)DR_g\Big(\widetilde{m}_{-d}\widetilde{m}_{f_1}\prod_{l=2}^{n+1}m_{f_l}\Big)\\
&-\Big(1+\frac{f_1}{d}\Big)\sum_{l=2}^{n+1}DR_g\Big(\widetilde{m}_{-d}\widetilde{m}_{f_l}\prod_{p=2}^{n+1}m_{f_p+\delta_{p,l}(f_1-f_p)}\Big)\bigg].
\end{aligned}$$

Note that Z_g is a linear combination of double ramification cycles of the same degree. Thus, the degree of Z_g is well defined. The same is true for V_g.

From the definition it immediately follows that

$$V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) = 0 \quad \text{if } f_2 = 0. \tag{6.10}$$

Lemma 4.4 implies that

$$Z_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) = 0 \quad \text{if } \sum f_i = 0. \tag{6.11}$$

It is not hard to show that

$$V_g\Big(m_{f_1} m_{f_2} \prod_{i=3}^{n+1} m_{f_i}\Big) + V_g\Big(m_{f_2} m_{f_1} \prod_{i=3}^{n+1} m_{f_i}\Big) \overset{\text{mod } K}{=} -Z_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big).$$

Using Lemma 4.3 we can easily derive the following relations:

$$f_i V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) + f_2 V_g\Big(m_{f_1} m_{f_i} \prod_{k=3}^{n+1} m_{f_{k+\delta_{i,k}(2-k)}}\Big) \in K \qquad \text{if } 3 \le i \le n+1; \tag{6.12}$$

$$V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) + V_g\Big(m_{f_1} m_{f_2} \prod_{k=3}^{n+1} m_{f_{k+\delta_{i,k}(j-k)+\delta_{j,k}(i-k)}}\Big) \in K \quad \text{if } 3 \le i < j \le n+1; \tag{6.13}$$

$$V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) \overset{\text{mod } K}{=} V_g\Big(m_{-d} \prod_{i=2}^{n+1} m_{f_i}\Big) \qquad \text{if } f_1 \ne 0; \tag{6.14}$$

$$V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) \overset{\text{mod } K}{=} Z_g\Big(m_{f_1} m_{-d} \prod_{i=3}^{n+1} m_{f_i}\Big). \tag{6.15}$$

Also from (4.1) it follows that

$$V_g\Big(\prod_{i=1}^{n+1} m_{f_i}\Big) = -V_g\Big(\prod_{i=1}^{n+1} m_{-f_i}\Big). \tag{6.16}$$

Lemma 6.1. *The space $R^{g-2}(\mathcal{M}_{g,n})/K$ is spanned by the cycles $V_g(\prod_{i=1}^{n+1} m_{f_i})$ with positive f_i's.*

Proof. Lemmas 4.3 and 4.5 imply that $R^{g-2}(\mathcal{M}_{g,n})/K$ is spanned by double ramification cycles of the form

$$DR_g\Big(\widetilde{m}_{-d} \widetilde{m}_b \prod_{i=1}^{n} m_{a_i}\Big),$$

where $a_1, \ldots, a_n$ and b are positive integers and $d = b + \sum a_i$. For $1 \le k \le n+1$, let

$$\alpha_k := DR_g\Big(\widetilde{m}_{-d} \widetilde{m}_{f_k} \prod_{p=2}^{n+1} m_{f_p+\delta_{p,k}(f_1-f_p)}\Big), \qquad \beta_k := V_g\Big(\prod_{i=k}^{n+1} m_{f_i} \prod_{j=1}^{k-1} m_{f_j}\Big).$$

From the definition of V_g it is easy to compute that $\beta_i \overset{\text{mod}\, K}{=} \sum_{j=1}^{n+1} g_{i,j}\alpha_j$, where

$$g_{i,j} = \begin{cases} (-1)^{(i-1)n+1+\delta_{j,1}} \frac{f_{i+1}}{r}\left(r-n-2+\frac{f_i}{d}\right) & \text{if } i = j, \\ (-1)^{(i-1)n+1+\delta_{j,1}} \frac{f_{i+1}}{r}\left(1+\frac{f_i}{d}\right) & \text{if } i \neq j. \end{cases}$$

Here, by definition, $f_{n+2} := f_1$. We see that it is sufficient to prove that the matrix $G := (g_{i,j})$ is non-degenerate. Consider the matrix $\widetilde{G} := (\tilde{g}_{i,j})$ defined by

$$\tilde{g}_{i,j} = \begin{cases} r-n-2+f_i/d & \text{if } i = j, \\ 1+f_i/d & \text{if } i \neq j. \end{cases}$$

Denote by D the diagonal matrix with diagonal entries $(-1)^{(i-1)n+1} f_{i+1}/r$. It is clear that the matrices G and $\widetilde{G}$ are related by

$$G = D \cdot \widetilde{G} \cdot \operatorname{diag}(-1, 1, 1, \ldots, 1).$$

Therefore, it is sufficient to prove that $\widetilde{G}$ is non-degenerate. It is easy to compute that $\det \widetilde{G} = (r-n-3)^n(r-1) \neq 0$, so the lemma is proved. $\square$

6.3.3. Main relation. Let us consider the same triple $((b_1, b_2, b_3), b_4, (a_1, \ldots, a_{n-1}))$, as in Section 6.3.1. Choose an arbitrary $1 \leq p \leq n-1$ and consider four basic relations corresponding to the following triples:

$$\begin{aligned} &((b_1, b_2, b_3), b_4, (a_1, \ldots, a_{n-1})), \\ &((b_1, b_2, a_p), b_4, (a_1, \ldots, a_{p-1}, b_3, a_{p+1}, \ldots, a_{n-1})), \\ &((b_1, a_p, b_3), b_4, (a_1, \ldots, a_{p-1}, b_2, a_{p+1}, \ldots, a_{n-1})), \\ &((a_p, b_2, a_3), b_4, (a_1, \ldots, a_{p-1}, b_1, a_{p+1}, \ldots, a_{n-1})). \end{aligned}$$

Let us sum these relations with the coefficients $1 - a_p/b_4$, $1 - b_3/b_4$, $1 - b_2/b_4$ and $1 - b_1/b_4$ respectively. We get

$$\begin{aligned} \sum_{\substack{\{i,j,k,l\}=\{1,2,3,4\} \\ i<j}} \Bigg[&(c_i + c_j)\left(1 - \frac{c_k}{b_4}\right) DR_g\left(\widetilde{m}_{b_4}\widetilde{m}_{c_l} m_{c_i+c_j} \prod_{q=1}^{n-1} m_{a_q+\delta_{q,p}(c_k-a_q)}\right) \\ &+ (c_j - c_i) DR_g\left(\widetilde{m}_{b_4+c_l}\widetilde{m}_{c_k} m_{c_i} \prod_{q=1}^{n-1} m_{a_q+\delta_{q,p}(c_j-a_q)}\right)\Bigg] \in K, \end{aligned} \tag{6.17}$$

where $c_i = b_i$ for $1 \leq i \leq 3$, and $c_4 = a_p$. Consider the sum of relations (6.17) for $p = 1, \ldots, n-1$. If we subtract this sum, divided by r, from (6.9), we get

$$\sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} \left[V_g\left(m_{b_k} m_{b_i+b_j} \prod_{l=1}^{n-1} m_{a_l}\right) + Z_g\left(m_{b_i} m_{b_j} \prod_{l=1}^{n-1} m_{a_l}\right)\right] \in K. \tag{6.18}$$

This relation will be called *the main relation*.

Remark 6.2. In the subsequent sections we want to work with relations between the cycles $V_g(\prod_{i=1}^{n+1} m_{f_i})$ and $Z_g(\prod_{i=1}^{n+1} m_{f_i})$, where $f_i \neq 0$ and $\sum f_i \neq 0$. At first glance, relation (6.18) contains other cycles as well, because in the first summand the multiplicity $b_i + b_j$ may be to zero, and in the second summand the sum of the multiplicities $b_i + b_j + \sum a_l = -b_4 - b_k$ may be zero. Happily, these extra terms vanish due to properties (6.10) and (6.11).

Remark 6.3. In the case $n \geq 2$ our proof of Proposition 3.3 is based only on the main relation (6.18) and relations (6.12)–(6.16). The case $n = 1$ is exceptional, because we also have to use Lemma 4.4.

6.3.4. Proof of Proposition 3.3: the case n = 1. Let $v_{i,j} := V_g(m_i m_j)$ and $z_{i,j} := Z_g(m_i m_j)$. By Lemma 6.1, it is sufficient to prove that $v_{i,j} \in K$ for $i, j \geq 1$.

First of all let us write relations (6.14)–(6.16) in this case:

$$v_{i,j} \overset{\mathrm{mod}\,K}{=} v_{-(i+j),j}, \qquad v_{i,j} \overset{\mathrm{mod}\,K}{=} z_{i,-(i+j)}, \qquad v_{i,j} \overset{\mathrm{mod}\,K}{=} -v_{-i,-j}. \tag{6.19}$$

Let a_1, a_2, a_3 be positive integers and $d = a_1+a_2+a_3$. Let us write (6.18) for $b_1 = a_1$, $b_2 = a_2$, $b_3 = a_3$ and $b_4 = -d$:

$$\sum_{i=1}^{3} v_{a_i,d-a_i} + \sum_{i<j} z_{a_i,a_j} \in K. \tag{6.20}$$

Let a_1, a_2 and c_1, c_2 be positive integers such that $a_1 + a_2 = c_1 + c_2$. Let us write relation (6.18) for $b_1 = a_1$, $b_2 = a_2$, $b_3 = -c_1$ and $b_4 = -c_2$. We get a linear combination of cycles $z_{i,j}$ and $v_{k,l}$, where the indices i, j, k, l may be negative. If we apply relations (6.19) in order to make all indices positive, we get

$$z_{a_1,a_2} - z_{c_1,c_2} - \sum_{a_i>c_j} v_{c_j,a_i-c_j} + \sum_{c_i>a_j} v_{a_j,c_i-a_j} \in K. \tag{6.21}$$

From Lemma 5.2 it follows that, for any $d \geq 2$, there is a class $\alpha_d \in R^{g-2}(\mathcal{M}_{g,1})$ such that

$$v_{i,j} \overset{\mathrm{mod}\,K}{=} \left(\frac{i}{i+j} - \frac{1}{3}\right)\alpha_{i+j} + \left(\frac{1}{3} - \frac{i+j}{i}\right)\alpha_i + \frac{1}{3}\alpha_j. \tag{6.22}$$

Here, by definition, $\alpha_1 := 0$.

From (4.6) it follows that $v_{ai,aj} = a^{2g+1} v_{i,j}$. Hence, $v_{d,d} = d^{2g+1} v_{1,1}$. Using (6.22) we obtain

$$\alpha_{2d} \overset{\mathrm{mod}\,K}{=} 8\alpha_d + d^{2g+1}\alpha_2. \tag{6.23}$$

We have $v_{2d-2,2} = 2^{2g+1} v_{d-1,1}$. Using (6.22) we get

$$\begin{aligned}&\left(\frac{d-1}{d} - \frac{1}{3}\right)\alpha_{2d} + \left(\frac{1}{3} - \frac{d}{d-1}\right)\alpha_{2d-2} + \frac{\alpha_2}{3} \\ &\qquad \overset{\mathrm{mod}\,K}{=} 2^{2g+1}\left(\left(\frac{d-1}{d} - \frac{1}{3}\right)\alpha_d + \left(\frac{1}{3} - \frac{d}{d-1}\right)\alpha_{d-1}\right).\end{aligned}$$

If we combine this equation with (6.23), we get

$$\begin{aligned}&\left(\frac{d-1}{d}-\frac{1}{3}\right)(8-2^{2g+1})\alpha_d+\left(\frac{1}{3}-\frac{d}{d-1}\right)(8-2^{2g+1})\alpha_{d-1}\\&\quad+\left(\left(\frac{d-1}{d}-\frac{1}{3}\right)d^{2g+1}+\left(\frac{1}{3}-\frac{d}{d-1}\right)(d-1)^{2g+1}+\frac{1}{3}\right)\alpha_2\in K.\end{aligned}\tag{6.24}$$

Relation (6.24) allows one to compute all α_d, for $d\geq 3$, in terms of α_2. It is not hard to check that this recursion has the following solution:

$$\alpha_d\overset{\mathrm{mod}\,K}{=}\frac{d^3-d^{2g+1}}{2^3-2^{2g+1}}\alpha_2.$$

Therefore,

$$v_{i,j}\overset{\mathrm{mod}\,K}{=}\left[\left(\frac{i}{i+j}-\frac{1}{3}\right)(i+j)^{2g+1}+\left(\frac{1}{3}-\frac{i+j}{i}\right)i^{2g+1}+\frac{j^{2g+1}}{3}\right]\alpha_2.\tag{6.25}$$

From Lemma 4.4 it follows that the class $DR_g(\widetilde{m}_a\widetilde{m}_b m_{-a-b})$ is a homogeneous polynomial of degree $2g$ in the variables a and b. Moreover, this polynomial is divisible by ab. Hence,

$$\lim_{a\to\infty}\frac{1}{a^{2g}}DR_g(\widetilde{m}_a\widetilde{m}_1 m_{-a-1})=0.\tag{6.26}$$

It is easy to compute that

$$\frac{(r-1)(r-4)}{r-3}DR_g(\widetilde{m}_i\widetilde{m}_j m_{-i-j})\overset{\mathrm{mod}\,K}{=}\frac{r}{r-3}\frac{iv_{i,j}+jv_{j,i}}{ij}+\frac{r}{i+j}(v_{i,j}+v_{j,i}).$$

If we substitute here the expression (6.25) for $v_{i,j}$, we get

$$\lim_{a\to\infty}\frac{1}{a^{2g}}DR_g(\widetilde{m}_a\widetilde{m}_1 m_{-a-1})\overset{\mathrm{mod}\,K}{=}\frac{r}{(r-1)(r-4)}\frac{4g-4}{3}\alpha_2.$$

From (6.26) it now follows that $\alpha_2\in K$. This completes the proof of the proposition in the case $n=1$.

6.3.5. Proof of Proposition 3.3: the case $n=2$. Let $v_{i,j,k}:=V_g(m_im_km_k)$ and $z_{i,j,k}:=Z_g(m_im_km_k)$. By Lemma 6.1, it is sufficient to prove that $v_{i,j,k}\in K$ for $i,j,k\geq 1$. The problem here is that relations (6.18), which involve only those terms, are not enough. So we also have to consider the classes $v_{i,j,k}$ with negative indices.

In this section we consider the cycles $v_{i,j,k}$ and $z_{i,j,k}$ as elements of $R^{g-2}(\mathcal{M}_{g,n})/K$. So, instead of writing $v_{i,j,k}\overset{\mathrm{mod}\,K}{=}0$, we will simply write $v_{i,j,k}=0$.

Let us write relations (6.12), (6.14), (6.15) and (6.16) in this case:

$$kv_{i,j,k}+jv_{i,k,j}=0,\tag{6.27}$$

$$v_{i,j,k}=v_{-(i+j+k),j,k},\tag{6.28}$$

$$v_{i,j,k} = z_{i,-(i+j+k),k}, \tag{6.29}$$
$$v_{i,j,k} = -v_{-i,-j,-k}. \tag{6.30}$$

From these relations it follows that any class $v_{a,b,c}$, where $a, b, c \neq 0$ and $a+b+c \neq 0$, can be expressed in terms of $v_{i,j,k}$ with $i, j, k \geq 1$ or $v_{i,j,-k}$ with $i, j, k \geq 1$ and $k < i+j$. Therefore, we have to prove that

$$v_{i,j,k} = 0 \qquad \text{whenever } i, j, k \geq 1, \tag{6.31}$$
$$v_{i,j,-k} = 0 \qquad \text{whenever } i, j, k \geq 1 \text{ and } k < i+j. \tag{6.32}$$

We use by induction on the degree d. In (6.31) the degree of $v_{i,j,k}$ is $i+j+k$ and in (6.32) the degree of $v_{i,j,-k}$ is $i+j$. The smallest possible degree is 2 and $v_{1,1,-1}$ is the only class of degree 2. We have $v_{1,1,-1} \overset{(6.27)}{=} v_{1,-1,1} \overset{(6.28)}{=} v_{-1,-1,1} \overset{(6.30)}{=} -v_{1,1,-1}$. Thus, $v_{1,1,-1} = 0$.

Suppose that $d \geq 3$. In the main relation (6.18) the numbers b_i and a_j may be positive or negative. In Section 6.3.6 we write explicitly the relations that we have, depending on the numbers of positive b_i's and a_j's. Using the induction assumption we ignore the terms of smaller degree. Then in Section 6.3.7 we prove (6.31) and in Section 6.3.8 we prove (6.32).

6.3.6. Relations. Let c_1, c_2, c_3, a be positive integers such that $c_1 + c_2 + c_3 + a = d$. Then, if we set $b_1 = c_1$, $b_2 = c_2$, $b_3 = c_3$, $b_4 = -d$ and $a_1 = a$ in (6.18), we get

$$\sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} v_{c_k, c_i+c_j, a} = 0. \tag{6.33}$$

Let c_1, c_2, c_3, c_4, a be positive integers such that $c_1 + c_2 + a = c_3 + c_4 = d$. Setting $b_1 = c_1$, $b_2 = c_2$, $b_3 = -c_3$, $b_4 = -c_4$ and $a_1 = a$ in (6.18), we get

$$v_{-c_3, c_1+c_2, a} + z_{c_1, c_2, a} = 0.$$

Applying relations (6.29) and (6.30), we get

$$z_{c_1,c_2,a} - z_{c_3,c_4,-a} = 0. \tag{6.34}$$

Let c_1, c_2, c_3, c_4, a be positive integers such that $c_1 + c_2 + c_3 = a + c_4 = d$. Then, if we set $b_1 = c_1$, $b_2 = c_2$, $b_3 = c_3$, $b_4 = -c_4$ and $a_1 = -a$ in (6.18), we get

$$\sum_{\substack{\{i,j,k\}=\{1,2,3\} \\ i<j}} v_{c_k, c_i+c_j, -a} = 0. \tag{6.35}$$

6.3.7. Proof of (6.31). Since we have relations (6.27) and (6.33), we can apply Lemma 5.3. By that lemma, the cases $d = 3, 4$ are done. Suppose $d \geq 5$. Then there exists a class $\alpha \in R^{g-2}(\mathcal{M}_{g,n})/K$ such that

$$v_{i,j,k} = \left(\frac{i}{d-1} - \frac{1}{3}\right)\left(\delta_{k,1} - \frac{1}{k}\delta_{j,1}\right)\alpha.$$

By (6.34), we have $z_{d-2,1,1} = z_{d-3,2,1}$. Therefore,

$$0 = z_{d-2,1,1} - z_{d-3,2,1} = -\left(\frac{1}{d-1} - \frac{1}{3}\right)\alpha + \frac{\alpha}{3} = \left(-\frac{1}{d-1} + \frac{2}{3}\right)\alpha.$$

Since the coefficient $-\frac{1}{d-1} + \frac{2}{3}$ is not equal to zero, we have $\alpha = 0$. This completes the proof of (6.31).

6.3.8. Proof of (6.32). We have

$$v_{i,j,-k} \overset{(6.28)}{=} v_{-(d-k),j,-k} \overset{(6.27)}{=} \frac{j}{k} v_{-(d-k),-k,j} \overset{(6.30)}{=} -\frac{j}{k} v_{d-k,k,-j}. \tag{6.36}$$

As a consequence,

$$v_{i,j,-j} = 0. \tag{6.37}$$

Suppose that $d = 3$. Then we have only four classes $v_{1,2,-1}$, $v_{1,2,-2}$, $v_{2,1,-1}$ and $v_{2,1,-2}$. By (6.37), $v_{1,2,-2} = v_{2,1,-1} = 0$. By (6.36), $v_{2,1,-2} = -\frac{1}{2} v_{1,2,-1}$. Finally, by (6.35), $v_{1,2,-1} = 0$.

Suppose that $d \geq 4$. From relations (6.35) and Lemma 5.1 it follows that, for any $1 \leq k \leq d-1$, there exists a class $\alpha_k \in R^{g-2}(\mathcal{M}_{g,n})/K$ such that

$$v_{i,j,-k} = \left(\frac{i}{d} - \frac{1}{3}\right)\alpha_k \quad \text{if } 2 \leq j \leq d-1.$$

On the other hand, from (6.34) and (6.31) it follows that $z_{i,j,-k} = 0$ if $k \leq d-2$. If $i, j \geq 2$, then $z_{i,j,-k} = -\alpha_k/3$. Thus, $\alpha_k = 0$ if $1 \leq k \leq d-2$. Therefore $v_{i,j,-k} = 0$ if $j \geq 2$ and $k \leq d-2$. Since $v_{d-1,1,-k} = -z_{d-1,1,-k} - v_{1,d-1,-k}$, we conclude that $v_{i,j,-k} = 0$ if $k \leq d-2$.

It remains to prove that $v_{i,j,-(d-1)} = 0$. Applying (6.36), we get $v_{i,j,-(d-1)} = 0$ if $j \leq d-2$. Finally, by (6.37), $v_{1,d-1,-(d-1)} = 0$. This completes the proof of Proposition 3.3 in the case $n = 2$.

6.3.9. Proof of Proposition 3.3: the case $n \geq 3$. From Lemma 6.1 it follows that it is sufficient to prove that $V_g(\prod_{i=1}^{n+1} m_{a_i}) \in K$ if $a_i \geq 1$.

We proceed by induction on the degree $d = \sum_{i=1}^{n+1} a_i$. The smallest possible degree is $n+1$; then $V_g(\prod_{i=1}^{n+1} m_1) \in K$, because of (6.12).

Suppose that $d \geq n+2$. From relations (6.18), (6.12) and Lemma 5.3 it follows that $V_g(\prod_{i=1}^{n+1} m_{a_i}) \in K$ if $a_2 \geq 2$ and $a_3 \geq 2$. Applying (6.12) and (6.13), we find that $V_g(\prod_{i=1}^{n+1} m_{a_i}) \in K$ if there exist $n+1 \geq j > i \geq 2$ such that $a_i, a_j \geq 2$. Suppose there exists at most one $i \geq 2$ such that $a_i \geq 2$. Since $n \geq 3$, there exist $n+1 \geq j > k \geq 2$ such that $a_j = a_k = 1$. If we again apply (6.12) or (6.13), we get $V_g(\prod_{i=1}^{n+1} m_{a_i}) \in K$. This completes the proof of Proposition 3.3 in the case $n \geq 3$.

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