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DOI
10.1063/1.530888

Publication date
1994

Published in
Journal of Mathematical Physics

Link to publication

Citation for published version (APA):
Bongaarts, P. J. M., & Pijls, H. G. J. (1994). Almost commutative algebras and differential calculus on the quantum hyperplane. *Journal of Mathematical Physics*, *35*, 959-970. https://doi.org/10.1063/1.530888

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Download date:17 Jun 2024

Almost commutative algebra and differential calculus on the quantum hyperplane

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(Received 6 July 1992; accepted for publication 12 October 1993)

A notion of almost commutative algebra is given that makes it possible to extend differential geometric ideas associated with commutative algebras in a simple manner to certain classes of noncommutative algebras. As an example differential calculus on the N-dimensional quantum hyperplane is discussed.

I. INTRODUCTION

Let M be a C^∞ manifold (Hausdorff and second countable) and $C^\infty(M)$ the algebra of C^∞ functions on M. The properties of M are to a large extent determined by the purely algebraic properties of $C^\infty(M)$: The derivations of $C^\infty(M)$ are precisely the vector fields on M. They form a $C^\infty(M)$ module from which objects like forms and tensor fields together with the usual operations can be obtained within the framework of the multilinear algebra of $C^\infty(M)$ modules. This point of view was championed at an early stage by Koszul.[1] The algebraic skeleton of differential geometry suggests a piece of mathematics that can stand on its own and that can be associated with an arbitrary commutative algebra. The next step is then to try to generalize it to the case of noncommutative algebras and to obtain in this way what may be called a noncommutative version of differential geometry. This idea has been developed along a broad front by Connes.[2]

In this article we will discuss a class of noncommutative algebras which we will call almost commutative and to which the pseudodifferential geometric ideas connected with commutative algebras can be extended in a simple and straightforward manner. Well-known examples of almost commutative algebras are the supercommutative algebras that play a basic role in the theory of supermanifolds. There are other examples. We will show in particular that the noncommutative algebra that characterizes the so-called N-dimensional quantum hyperplane is an almost commutative algebra. This point of view makes its properties very transparent. There is a differential calculus which is just the de Rham complex associated with it in a unique and rather obvious way.

In Sec. II we review the commutative case and give the definition of the de Rham complex for an arbitrary commutative algebra. In Sec. III we introduce algebras that are graded by a group G. A choice of a two-cocycle ρ on G leads to modified definitions of derivation, commutator, Lie algebra and finally to a definition of almost commutativity or ρ-commutativity. A ρ-commutative algebra has a de Rham complex, there is a Lie derivative and among other things a suitably adapted definition of a Poisson structure. We discuss the case of superalgebras briefly in Sec. IV, and in Sec. V we treat our main example, the quantum hyperplane algebra. There are intimate relations between this algebra and quantum groups and Hopf algebra ideas. We comment on these aspects in our final Sec. V.

II. COMMUTATIVE ALGEBRA

Let A be a commutative algebra (algebra without further specification will in this article mean associative algebra with unit element; the basic field will be $k=\mathbb{R}$ or $\mathbb{C}$). Let Der A be the

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linear space of derivations of A, i.e., linear maps $X: A\to A$ with $X(fg)=(Xf)g+f(Xg)$, $(f,g\in A)$. Der A is a Lie algebra with as bracket the commutator $[X, Y]:=XY-YX$ for X and Y as linear operators on A. For this the commutativity of A is not needed. It is needed however for the next statement: Der A is an A module with the action of A on Der A defined as $(fX)g:=f(Xg)$ $(X\in\text{Der}\,A; f,g\in A)$. This allows us to use concepts of linear and multilinear algebra in the context of modules over a commutative algebra. For basic textbooks on linear and multilinear algebra in this spirit see Ref. 3.

We define a sequence of vector spaces $\Omega^p(A)$ $(p=0,1,2,...,)$. First $\Omega^0(A):=A$. For $p=1,2,...$, $\Omega^p(A)$ will be the space of all p-linear alternating maps α: $\times^p\,\text{Der}\,A\to A$, which are moreover A-linear in each factor. Each $\Omega^p(A)$ is an A-module in a natural way. The direct sum $\Omega(A):=\oplus_{p=0}^{\infty}\Omega^p(A)$ is also an A-module. There is a linear map d: $\Omega(A)\to\Omega(A)$, which maps $\Omega^p(A)$ into $\Omega^{p+1}(A)$, and which is defined as $(df)(X):=Xf$, for $f\in\Omega^0(A)=A$, and for $\alpha_p\in\Omega^p(A)$ $(p=1,2,...,)$ as

$$(d\alpha_p)(X_1,...,X_{p+1}):=\sum_{j=1}^{p+1}(-1)^{j-1}X_j(\alpha_p(X_1,...,\hat{X}_j,...,X_{p+1}))$$
$$+\sum_{1<j<k<p+1}(-1)^{j+k}\alpha_p([X_j,X_k],X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{p+1}) \tag{2.1}$$

for $X_1,...,X_{p+1}\in\text{Der}\,A$, and with $\hat{X}_j,\hat{X}_k$ meaning that X_j,X_k are omitted. One has $d^2=0$, so $(\Omega(A),d)$ is a cochain complex determined by A. $\Omega(A)$ also has the structure of an algebra: There is an obvious antisymmetrization procedure which when applied to elements α_p of $\Omega^p(A)$ and β_q of $\Omega^q(A)$ will give a product element in $\Omega^{p+q}(A)$, denoted as $\alpha_p\wedge\beta_q$.

An important special case is the situation where A possesses elements $x^1,...,x^n$ such that $dx^1,...,dx^n$ form a basis of $\Omega^1(A)$ as an A-module. Der A will then have a dual basis consisting of elements $E_1,...,E_n$ determined by the requirement $(dx^j)(E_k)=\delta^j_k$ for $j,k=1,...,n$. An arbitrary $\alpha_p\in\Omega^p(A)$ can be written as $(1/p!)\alpha_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ (summation convention) with $\alpha_{i_1\cdots i_p}\in A$, and antisymmetric in the indices $i_1,...,i_p$. An appropriate notation for the basis elements E_j of Der A is $\partial/\partial x^j$; every $X\in\text{Der}\,A$ can be written as $X=X^j(\partial/\partial x^j)$ with $X^j\in A$ for $j=1,2,...,n$.

The geometrical meaning or origin of all this is clear: If M is a (finite dimensional) C^∞ manifold and $A=C^\infty(M)$, then Der A consists precisely of all (smooth) vector fields on M, $\Omega^p(A)$ is the space of p forms on M, d is exterior differentiation, and $(\Omega(A),d)$ is the de Rham complex. A special case is where $A=C^\infty(U)$ with $U\subset M$ a coordinate neighborhood with coordinate functions $x^1,...,x^n$.

All sorts of other geometric notions can be introduced in this purely algebraic fashion: general tensor fields, Lie derivatives, connections, etc.

Two important aspects of this formalism should be stressed:

(1) The formalism is suggested by differential geometry, but it can stand on its own. The only input is a commutative algebra A; all the rest follows from this.

(2) For a commutative algebra A there is no need to distinguish between left and right A-modules. This together with the commutativity of A itself makes multilinear algebra possible.

Note finally that $\Omega(A)$ is well defined for every commutative algebra A, but may be trivial. Take, for instance, A to be the algebra of continuous functions from X to k, with X a compact Hausdorff space. Then Der $A=\{0\}$, so $\Omega(A)=\{0\}$ (see Chap. 4 in Ref. 4).

III. ALMOST COMMUTATIVE ALGEBRA

In this section we will introduce noncommutative algebras that are almost commutative in the sense that the commutation properties allow a pseudogeometric scheme such as described in the preceding section.

Let G be an Abelian group, additively written, and let A be a G-graded algebra. This means that as a vector space A has a G-grading $A=\oplus_{a\in G}A_a$, and that one has moreover $A_aA_b\subset A_{a+b}(a,b\in G)$. The G-degree of a (nonzero) homogeneous element f of A is denoted as $|f|$. Furthermore let ρ: $G\times G\rightarrow k$ be a map which satisfies

$$\rho(a,b)=\rho(b,a)^{-1}, \quad (a,b\in G), \tag{3.1}$$

$$\rho(a+b,c)=\rho(a,c)\rho(b,c), \quad (a,b,c\in G). \tag{3.2}$$

This implies $\rho(a,b)\neq 0$, $\rho(0,b)=1$, and $\rho(c,c)=\pm 1$, for all $a,b,c\in G, c\neq 0$. The function ρ is in fact a special type of two-cocycle on G. We define for homogeneous elements f and g in A an expression, which we will call the *ρ-commutator* of f and g as

$$[f,g]_\rho:=fg-\rho(|f|,|g|)gf. \tag{3.3}$$

This expression as it stands makes sense only for homogeneous elements f and g, but can be extended linearly to general elements. In similar situations we usually will not mention this explicitly, but it will be tacitly understood whenever necessary. The ρ-commutator has the following properties, which can be easily verified:

$$[A_a,A_b]_\rho\subset A_{a+b}, \quad (a,b\in G), \tag{3.4}$$

$$[f,g]_\rho=-\rho(|f|,|g|)[g,f]_\rho, \tag{3.5}$$

$$\rho(|f|,|h|)^{-1}[f,[g,h]_\rho]_\rho+\rho(|g|,|f|)^{-1}[g,[h,f]_\rho]_\rho+\rho(|h|,|g|)^{-1}[h,[f,g]_\rho]_\rho$$
$$=0 \quad (f,g,h\in A). \tag{3.6}$$

Formula (3.4) means that $[\cdot,\cdot]_\rho$, as a bilinear map $A\times A\rightarrow A$ has itself G-degree 0; Eq. (3.5) may be called ρ-antisymmetric and Eq. (3.6) the ρ-Jacobi identity.

The definition of a ρ-commutator can of course be used more generally for linear transformations of a G-graded vector space into itself. (Remember that a linear map T between G-graded vector spaces V and W has G-degree $|T|$ if $TV_a\subset W_{a+|T|}$). $[\cdot,\cdot]_\rho$ is again first defined for homogeneous elements, and then extended linearly to general elements. These form a G-graded algebra. The properties (3.4)–(3.6) suggest the definition of a *ρ-Lie algebra* as a G-graded vector space L with a bilinear map $L\times L\rightarrow L$, $(u,v)\mapsto[u,v]_\rho$, in which the bracket $[\cdot,\cdot]_\rho$ satisfies Eqs. (3.4)–(3.6). Later on in this section we will have occasion to use a slightly more general notion of ρ-Lie algebra in which the G degree of the bracket $[\cdot,\cdot]_\rho$ need not be 0.

The derivations of a G-graded algebra A form a Lie algebra, as they do for any algebra, commutative or not. In this context one obtains however more interesting results by not using ordinary derivations, but derivations of an appropriately modified type: A *ρ-derivation* X of A, of G-degree $|X|$, is a linear map $X:A\rightarrow A$, of G-degree $|X|$, such that one has for all homogeneous elements f and g in A

$$X(fg)=(Xf)g+\rho(|X|,|f|)f(Xg). \tag{3.7}$$

It is not hard to verify that the ρ-commutator of two ρ-derivations is again a ρ-derivation, i.e., the linear space of all ρ-derivations of A, suitably extended from homogeneous to general elements X, is a ρ-Lie algebra. It will be denoted by ρ-Der A.

The material collected so far leads us now to introduce in a rather obvious manner the particular version of almost commutativity that is the subject of this section. A G-graded algebra A with given cocycle ρ will be called *ρ-commutative* iff $fg=\rho(|f|,|g|)gf$ for homogeneous elements f and g in A. Note the role of the properties (3.1) and (3.2) in this definition. One verifies immediately that for such an A ρ-Der A is not only a ρ-Lie algebra but also a left A-module with the action of A on ρ-Der A defined by

$$(fX)g:=f(Xg) \quad (f,g\in A, X\in \rho\text{-Der}\,A). \tag{3.8}$$

This is one fact that justifies the definition of ρ-commutativity. The second fact is the following. Let M be a (G-graded) left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi|=|f|+|\psi|$ for $f\in A$, $\psi\in M$. Then M is also a right A-module with the right action on M defined by

$$\psi f:=\rho(|\psi|,|f|)f\psi. \tag{3.9}$$

In fact M is a bimodule over A, i.e.,

$$f(\psi g)=(f\psi)g \quad (f,g\in A;X\in M). \tag{3.10}$$

[The right A-action on ρ-Der A defined by Eq. (3.9) will be written as $((Xf))$, to distinguish it from the left action of ρ-Der A on A, so $((Xf))\in$ ρ-Der A and Xf or $(Xf)\in A$.] For the remainder of this section A will be a G-graded algebra, which is ρ-commutative for a given cocycle ρ. Because there is no essential difference between left and right modules over a ρ-commutative algebra and because of the properties of ρ-commutativity itself, we have the possibility of doing multilinear algebra in terms of A modules, starting from ρ-Der A, and that is all that the general pseudogeometric formalism that we are about to develop amounts to. We shall employ a notation—especially for pairing formulas—in which the ordering of various symbols is such that most of the relevant properties can be read off immediately. We write, for instance, the (left) action of ρ-Der A on A as $(X,f)\mapsto\langle X;f\rangle$, and a p-linear map as

$$\alpha_p:\times^p(\rho\text{-Der}\,A)\to A \quad \text{as} \quad (X_1,\dots,X_p,\alpha_p)\mapsto\langle X_1,\dots,X_p;\alpha_p\rangle.$$

We introduce p-forms in this context: Define $\Omega^0(A):=A$, and $\Omega^p(A)$, for $p=1,2,\dots$, as the G-graded vector space of p-linear maps α_p: $\times^p(\rho\text{-Der}\,A)\to A$, p-linear in the sense of A modules

$$\langle fX_1,\dots X_p;\alpha_p\rangle=f\langle X_1,\dots,X_p;\alpha_p\rangle, \tag{3.11}$$

$$\langle X_1,\dots,((X_jf)),X_{j+1},\dots X_p\rangle=\langle X_1,\dots,X_j,fX_{j+1},\dots,X_p;\alpha_p\rangle \tag{3.12}$$

and ρ-alternating

$$\langle X_1,\dots,X_j,X_{j+1},\dots,X_p;\alpha\rangle=-\rho(|X_j|,|X_{j+1}|)\langle X_1,\dots,X_{j+1},X_j,\dots,X_p;\alpha_p\rangle$$
$$\text{for} \quad j=1,\dots,p-1; \quad X_k\in\rho\text{-Der}\,A, \quad k=1,\dots,p; f\in A. \tag{3.13}$$

$\Omega^p(A)$ is in a natural way a G-graded right A-module with

$$|\alpha_p| = |\langle X_1,...,X_p;\alpha_p\rangle| - (|X_1| + \cdots + |X_p|) \tag{3.14}$$

and with the right action of A defined as

$$\langle X_1,...,X_p;\alpha_p f\rangle := \langle X_1,...,X_p;\alpha_p\rangle f. \tag{3.15}$$

The direct sum $\Omega(A) := \oplus_{p=0}^{\infty}\Omega^p(A)$ is again a G-graded A-module.

One defines *exterior differentiation* as a linear map $d:\Omega(A)\to\Omega(A)$, carrying $\Omega^p(A)$ into $\Omega^{p+1}(A)$, as $\langle X, d\alpha_0\rangle = X\alpha_0$, for $\alpha_0 \in A = \Omega^0(A)$, and for $p=1,2,...,$ as

$$\begin{aligned}\langle X_1,...,X_{p+1};d\alpha_p\rangle :=& \sum_{j=1}^{p+1} (-1)^{j-1}\rho\left(\sum_{l=1}^{j-1} |X_l|,|X_j|\right) X_j(\langle X_1,...,\hat{X}_j,...,X_{p+1};\alpha_p\rangle)\\ &+ \sum_{1\leqslant j<k\leqslant p+1} (-1)^{j+k}\rho\left(\sum_{l=1}^{j-1} |X_l|,|X_j|\right)\rho\left(\sum_{l=1}^{j-1} |X_l|,|X_k|\right)\\ &\times\rho\left(\sum_{l=j+1}^{k+1} |X_l|,|X_k|\right)\langle [X_j, X_k]_\rho, X_1,...,\hat{X}_j,...,\hat{X}_k,...,X_{p+1};\alpha_p\rangle.\end{aligned} \tag{3.16}$$

One shows in a straightforward manner that this indeed defines a linear map from $\Omega^p(A)\to\Omega^{p+1}(A)$, that d has G-degree 0, and of course that $d^2=0$.

In addition to d there are other linear operators in $\Omega(A)$. One has, for $X\in\rho$-Der A, a *contraction* i_X defined as

$$\langle X_1,...,X_{p-1};i_X\alpha_p\rangle := \rho\left(\sum_{l=1}^{p-1} |X_l|,|X|\right)\langle X, X_1,...,X_{p-1};\alpha_p\rangle \tag{3.17}$$

and a *Lie derivative* L_X given by

$$\begin{aligned}\langle X_1,...,X_p;L_X\alpha_p\rangle :=& \rho\left(\sum_{l=1}^{p} |X_l|,|X|\right)X(\langle X_1,...,X_p;\alpha_p\rangle) - \sum_{j=1}^{p}\rho\left(\sum_{l=j}^{p} |X_l|,|X|\right)\\ &\times\langle X_1,...,[X, X_j]_\rho,...X_p;\alpha_p\rangle,\end{aligned} \tag{3.18}$$

with of course $i_X\,\alpha_0=0$, $\alpha_0\in\Omega^0(A)$. Note that $|i_X| = |L_X| = |X|$.

$\Omega(A)$ carries a natural algebraic structure: There is an exterior product $\Omega^p(A) \times\Omega^q(A)\to\Omega^{p+q}(A)$, $(\alpha_p,\beta_q)\mapsto\alpha_p\wedge\beta_q$, defined by an obvious ρ-antisymmetrization formula

$$\langle X_1,...,X_{p+q};\alpha_p\wedge\beta_q\rangle := \sum_\sigma \text{sign}\,\sigma(\rho\ \text{factor})\langle X_{\sigma(1)},...,X_{\sigma(p)};\alpha_p\rangle\langle X_{\sigma(p+1)},...,X_{\sigma(p+q)};\beta_q\rangle. \tag{3.19}$$

The sum is over all permutations σ of the indices $1,2,...,p+q$, such that $\sigma(1)<\sigma(2)<\cdots<\sigma(p)$ and $\sigma(p+1)<\sigma(p+2)<\cdots<\sigma(p+q)$. The ρ factor consists of all $\rho(|X_{\sigma(j)}|,|\alpha_p|)$ for $p+1\leqslant j\leqslant p+q$ and all $\rho(|X_{\sigma(j)}|,|X_{\sigma(k)}|)^{-1}$ for $j<k$ and $\sigma(j)>\sigma(k)$. One has of course $\alpha_0\wedge\beta_0=\alpha_0\beta_0$, $\alpha_0\wedge\beta_q=\alpha_0\beta_q$ and $\alpha_p\wedge\beta_0=\alpha_p\beta_0$. $\Omega(A)$ has an additional **Z** grading, in fact it is a G'-graded algebra with $G'=\mathbf{Z}\times G$. Denote the G' degree of α_p as $|\alpha_p|'=(p,|\alpha_p|)$. Define a cocycle ρ': $G'\times G'\to k$ as $\rho'((p,a),(q,b))=(-1)^{pq}\rho(a,b)$. Then $\Omega(A)$ is a ρ'-commutative G'-graded algebra. The maps d, i_X, and L_X are ρ'-derivations of

$\Omega(A)$ with G'-degrees $|d|'=(+1,0)$, $|i_X|'=(-1,|X|)$, and $|L_X|'=(0,|X|)$. Relations of the usual type hold between these operators, and can neatly be written in terms of ρ'-commutators as

$$[d,d]_{\rho'}=0,\quad [d,i_X]_{\rho'}=L_X,\quad [d,L_X]_{\rho'}=0,$$
$$[i_{X_1},i_{X_2}]_{\rho'}=0,\quad [i_{X_1},L_{X_2}]_{\rho'}=i_{[X_1,X_2]_\rho},\tag{3.20}$$
$$[L_{X_1},L_{X_2}]_{\rho'}=L_{[X_1,X_2]_\rho}.$$

Verification of all these properties is straightforward but tedious. This completes the discussion of the basic elements of the de Rham complex as it appears naturally in this almost commutative algebraic setting. The remark about the two important aspects of the algebraic formalism, made at the end of the preceding sections, is even more pertinent here: (1) The only input for the formalism is a particular algebra A, now ρ-commutative instead of commutative. This determines everything else: The space of forms $\Omega(A)$, the exterior differentiation d, and in particular the structure of $\Omega(A)$ as a ρ'-commutative algebra. (2) For the purpose of developing multilinear algebra, ρ-commutativity is as good as commutativity. Left and right A-modules are different objects but very simply related.

It will be clear that various other pseudogeometric ideas can be pursued in this context. We give one more example:

Let A be a ρ-commutative algebra as before. A will be called a *ρ-Poisson algebra*, with bracket of degree $|P|$, if it has a bilinear map $P\colon A\times A\to A$, $(f,g)\mapsto\{f,g\}_\rho$, with the following properties:

$$|\{f,g\}_\rho|=|P|+|f|+|g|,\tag{3.21}$$

$$\{f,g\}_\rho=-\rho(|P|,|P|)\rho(|f|,|g|)\{g,f\}_\rho,\tag{3.22}$$

$$\rho(|f|,|h|)^{-1}\{\{f,g\}_\rho,h\}_\rho+\mathrm{cycl}(f,g,h)=0,\tag{3.23}$$

$$\{f,gh\}_\rho=\{f,g\}_\rho h+\rho(|P|+|f|,|g|)g\{f,h\}_\rho\quad (f,g,h\in A).\tag{3.24}$$

Formula (3.21) states that P has G-degree $|P|$ as a bilinear map; Eqs. (3.21)–(3.23) make $(A,\{\cdot,\cdot\}_\rho)$ into a ρ-Lie algebra, with bracket of G-degree $|P|$, generalizing the definition of the ρ-Lie algebra given in the beginning of this section. The Jacobi identity (3.23) is equivalent to the statement that, for every fixed $f\in A$, the map $\{f,\cdot\}_\rho$ is a ρ-derivation of G-degree $|P|+|f|$, of A in the sense of ρ-Lie algebras, and Eq. (3.24) says that it is also a ρ-derivation of the same G-degree of A as a G-graded (associative) algebra. All this is fairly obvious if one would use a notation for the bracket which exhibits its G-degree as a bilinear map, i.e., as $(f,g)\mapsto\{P;f,g\}_\rho$. Two other versions of the brackets are suggested by writing $\{f;P,g\}_\rho$ and $\{f;g,P\}_\rho$. These have slightly different properties but are equivalent to the one above.

IV. SUPERCOMMUTATIVE ALGEBRAS

A well-known example of what we have called an almost commutative algebra is a *supercommutative algebra*. A superalgebra is a $\mathbf{Z}_2$-graded algebra. It is called supercommutative if it is ρ-commutative with respect to $\rho(a,b)=(-1)^{ab}$, for $a,b\in\mathbf{Z}_2$. Notions such as superderivation, supercommutator, and super-Lie algebra are standard and have in fact suggested the more general definitions of ρ-derivation, ρ-commutator, and ρ-Lie algebra in the preceding section.

Λ_n, the Grassmann algebra over n generators, has an obvious $\mathbf{Z}_2$-grading and is supercommutative. It can be seen as the algebra of polynomials in n anticommuting variables. Anticommuting or Grassmann variables were introduced in physics independently by Martin,[5] Schwinger,[6] and Berezin[7] and have become extremely useful in quantum field theory, in particular in problems connected with the quantization of fermion systems. Λ_n gives also a precise meaning to the idea of an n-dimensional superplane. More general supercommutative algebras lead to supermanifolds and to superdifferential geometry. In the approach of Kostant[8] and Leites[9] an (m,n)-dimensional supermanifold is an ordinary m-dimensional manifold with the structure sheaf of C^∞ functions extended to a sheaf of supercommutative algebras which are locally isomorphic to the tensor product of Λ_n and a commutative algebra of local C^∞ functions. Superderivations of these algebras play the role of vector fields on the supermanifold. They are used to define, in accordance with the ideas of the preceding section, further geometrical objects, in particular differential forms. This was worked out in detail by Kostant. The concept of a super-Poisson algebra arises quite naturally in this context. A super-Poisson bracket of odd degree, in the version suggested by the notation $\{f;1;g\}$ appears as "antibracket" in the Batalin–Vilkovisky formalism for quantum gauge field theory, see Ref. 10.

It is easy to see that for a $\mathbf{Z}_2$-grading one has apart from the trivial cocycle only the cocycle that leads to supercommutative algebras. The same is true for a $\mathbf{Z}$-grading. There are gradings for which one has only the trivial cocycle, an obvious example being $\mathbf{Z}_3$. In the next section we shall discuss gradings which have more interesting cocycles.

V. THE *N*-DIMENSIONAL QUANTUM HYPERPLANE

The N-dimensional quantun hyperplane is characterized by an algebra S_N^q generated by the unit element and N linearly independent elements $x_1,...,x_N$ satisfying the relations

$$x_j x_k = q x_k x_j, \quad j<k \tag{5.1}$$

for some fixed $q \in k$, $q \neq 0$. There are no further relations between the generators. S_N^q is of course a deformation of the symmetric algebra on N generators and can intuitively be seen as the algebra of noncommuting polynomials on the quantum hyperplane. Manin[11] has given a more precise meaning to the notion of quantum hyperplane itself, using his approach to noncommutative algebraic geometry. Here it will be enough to regard it as a heuristic idea behind the purely algebraic discussion of S_N^q and its properties. This is similar to the superalgebra case in the preceding section.

S_N^q is a $\mathbf{Z}^N$-graded algebra

$$S_N^q = \bigoplus_{n_1,...,n_N}^{\infty} (S_N^q)_{n_1 \cdots n_N}, \tag{5.2}$$

with $(S_N^q)_{n_1 \cdots n_N}$ the one-dimensional subspace spanned by the products $x_1^{n_1} \cdots x_N^{n_N}$. The $\mathbf{Z}^N$ degree of these elements is denoted by $|x_1^{n_1} \cdots x_N^{n_N}| = \mathbf{n} = (n_1,...,n_N)$. Define a function ρ: $\mathbf{N}^N \times \mathbf{N}^N \to k$ as

$$\rho(\mathbf{n},\mathbf{n}') = q^{\sum_{j,k=1}^N n_j n_k' \alpha_{jk}}, \tag{5.3}$$

with $\alpha_{jk}=1$ for $j<k$, 0 for $j=k$, -1 for $j>k$. The function ρ is a cocycle on $\mathbf{Z}^N$. It can easily be checked that Eq. (5.1) is equivalent to the general relation

$$(x_1^{n_1} \cdots x_N^{n_N})(x_1^{n_1'} \cdots x_N^{n_N'}) = \rho(\mathbf{n},\mathbf{n}')(x_1^{n_1'} \cdots x_N^{n_N'})(x_1^{n_1} \cdots x_N^{n_N}). \tag{5.4}$$

This means that S_N^q is a ρ-commutative algebra and that therefore the differential geometric ideas of Sec. III can be applied to it. The basic objects are the ρ-derivations of S_N^q, the vector fields on the quantum hyperplane. We are in the special situation where we have coordinate vector fields, the ρ derivations $\partial/\partial x_j$, $j=1,...,N$, of $\mathbf{Z}^N$-degree $|\partial/\partial x_j|$, with $|\partial/\partial x_j|=-|x_j|$ and defined by $(\partial/\partial x_j)x_k=\delta_{jk}$. One has

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}=q\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j},\quad j<k \tag{5.5}$$

and

$$\frac{\partial}{\partial x_j}(x_1^{n_1}\cdots x_N^{n_N})=n_j q^{(n_1+\cdots+n_{j-1})}(x_1^{n_1}\cdots x_j^{n_j-1}\cdots x_N^{n_N}), \tag{5.6}$$

which follows from the Leibniz relation (3.7)

$$\frac{\partial}{\partial x_j}(fg)=\left(\frac{\partial}{\partial x_j}f\right)g+\rho\left(\left|\frac{\partial}{\partial x_j}\right|,|f|\right)f\left(\frac{\partial}{\partial x_j}g\right). \tag{5.7}$$

ρ-Der S_N^q is a free S_N^q-module of rank (dimension) N with $\partial/\partial x_1,...,\partial/\partial x_N$ as the basis. An arbitrary ρ-derivation X can be written as

$$X=\sum_{j=1}^{N}F_j\frac{\partial}{\partial x_j}, \tag{5.8}$$

with $F_j\in S_N^q$, $F_j=X(x_j)$. $\Omega^1(S_N^q)$, the S_N^q-module of one-forms, is also free of rank N. The coordinate one-forms $dx_1,...,dx_N$, defined by $\langle X;dx_j\rangle=X(x_j)$ or $\langle\partial/\partial x_k;dx_j\rangle=\delta_{kj}$, form a basis in $\Omega^1(S_N^q)$, dual to the basis $\partial/\partial x_1,...,\partial/\partial x_N$ in ρ-Der(S_N^q). Note that $|dx_j|=|x_j|$. For $f\in S_N^q$ one has $\langle X;df\rangle=X(f)$ or $\langle\partial/\partial x_k;df\rangle=(\partial/\partial x_k)f$ and therefore

$$df=\sum_{j=1}^{N}(dx_j)\frac{\partial}{\partial x_j}f. \tag{5.9}$$

An arbitrary one-form α_1 can be written as

$$\alpha_1=\sum_{j=1}^{N}(dx_j)A_j, \tag{5.10}$$

with

$$A_j=\left\langle\frac{\partial}{\partial x_j};\alpha_1\right\rangle\in S_N^q. \tag{5.11}$$

Because $\Omega^1(S_N^q)$ is of finite rank N, $\Omega^p(S_N^q)$ is the pth exterior power of $\Omega^1(S_N^q)$, in the sense of S_N^q-modules, and as such is again free and has rank $\binom{p}{N}$. An arbitrary p-form α_p can be written as

$$\alpha_p=\frac{1}{p!}(-1)^{(1/2)p(p-1)}\sum_{j_1,...,j_p=1}^{N}(dx_{j_p}\wedge\cdots\wedge dx_{j_1})A_{j_1\cdots j_p}, \tag{5.12}$$

with

$$A_{j_1\cdots j_p}=\left\langle\frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial}{\partial x_{j_p}};\alpha_p\right\rangle\in S_N^q. \tag{5.13}$$

Formula (5.12) is the same as for the ordinary commutative case. Note however that $dx_j \wedge dx_k = -q^{\alpha_{jk}} dx_k \wedge dx_j$ and that therefore the "coordinate functions" $A_{j_1 \cdots j_p}$ are ρ-alternating, i.e.,

$$A_{j_1 \cdots j_k j_{k+1} \cdots j_p} = -q^{\alpha_{j_k j_{k+1}}} A_{j_1 \cdots j_{k+1} j_k \cdots j_p}. \tag{5.14}$$

Because of the particular choice of ordering there are no q factors in Eq. (5.12). Such factors can however not be avoided in the explicit expressions for the exterior derivative. One has, for instance, for $\alpha_1 \in \Omega^1(S_N^q)$

$$(d\alpha_1)_{jk} = \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}; d\alpha_1 \right\rangle = \frac{\partial}{\partial x_j} A_k - q^{\alpha_{jk}} \frac{\partial}{\partial x_k} A_j. \tag{5.15}$$

All this can be obtained by the straightforward application of the general formulas of Sec. III, in particular Eqs. (3.11)–(3.13) for the definition of p-forms, Eq. (3.16) for the definition of exterior derivative, and Eq. (3.19) for the exterior multiplication in $\Omega(S_N^q)$. $\Omega(S_N^q)$ as an algebra is generated by the elements $x_1,...,x_N$ and $y_1 = dx_1,...,y_N = dx_N$ with the relations

$$x_j x_k = q^{\alpha_{jk}} x_k x_j, \quad y_j y_k = -q^{\alpha_{jk}} y_k y_j, \quad x_j y_k = q^{\alpha_{jk}} y_k x_j. \tag{5.16}$$

This algebra is different from the algebra associated with the differential calculus of Wess and Zumino.[12] In our approach here the differential calculus on the quantum hyperplane is just the de Rham complex $(\Omega(S_N^q), d)$ uniquely defined by the structure of S_N^q as a ρ-commutative algebra.

For $N=2$, S_N^q can be given the additional structure of a ρ-Poisson algebra: for a $\lambda \in k$, $\lambda \neq 0$, there is a unique bracket $\{\cdot,\cdot\}_\rho$, satisfying Eqs. (3.21),...,(3.24), of $\mathbf{Z}_2$-degree $(-1, -1)$, such that $\{x_1, x_2\}_\rho = \lambda$. It is given by the explicit expression

$$\{f,g\}_\rho = \lambda \left[q^{-|f|_1+1} \left(\frac{\partial}{\partial x_1} f \right) \left(\frac{\partial}{\partial x_2} g \right) - q^{|f|_2} \left(\frac{\partial}{\partial x_2} f \right) \left(\frac{\partial}{\partial x_1} g \right) \right] \tag{5.17}$$

for homogeneous f,g in S_N^q.

Other somewhat different Poisson brackets for the quantum hyperplane are known in the literature, see Ref. 13, but in the present context this is the natural one.

S_N^q has generalizations in various directions:

(1) The $N \times N$ matrix α_{jk} in Eq. (5.3) is rather special. One may use an arbitrary antisymmetric matrix α_{jk} with matrix elements in $\mathbf{Z}$, or even in $\mathbf{C}$ when q is real and positive.

(2) S_N^q can be embedded in the larger ρ-commutative algebra of what in this context can be called Laurent polynomials by adding to the generators $x_1,...,x_N$ their inverses $x_1^{-1},...,x_N^{-1}$, which have obviously degree $|x_j^{-1}| = (0,...,-1,...,0)$.

(3) S_N^q has a superversion: consider generators $x_1,...,x_N$ with $\mathbf{Z}_2$-degree $\epsilon(x_j) = 0$ or 1, and with the basic relations

$$x_j x_k = (-1)^{\epsilon(x_j)\epsilon(x_k)} q^{\alpha_{jk}} x_k x_j \tag{5.18}$$

implying $x_j^2 = 0$ for $\epsilon(x_j) = 1$. A monomial $x_1^{n_1} \cdots x_N^{n_N}$ has a $G_s = \mathbf{Z}_2 \times \mathbf{Z}^N$-degree given by

$$|x_1^{n_1} \cdots x_N^{n_N}| = \left(\sum_{j=1}^{N} n_j \epsilon(x_j), \mathbf{n} \right). \tag{5.19}$$

Let $N_1 = \Sigma_{j=1}^N \epsilon(x_j)$ and $N_0 = N - N_1$. The algebra S_{N_0,N_1}^q generated by $x_1,...,x_N$ is G_s-graded and ρ_s-commutative with ρ_s the cocycle on G_s given by

$$\rho_s((\nu,\mathbf{n}),(\nu',\mathbf{n}'))=(-1)^{\nu\nu'}\rho(\mathbf{n},\mathbf{n}')=(-1)^{\nu\nu'}q^{\sum_{j,k=1}^{N} n_j n_k' \alpha_{jk}} \tag{5.20}$$

for ν, $\nu'\in\mathbf{Z}_2$, $\mathbf{n},\mathbf{n}'\in\mathbf{Z}^N$. $S^q_{N_0,N_1}$ may—not very elegantly—be called the algebra of the super-quantum hyperplane of mixed dimension (N_0,N_1). It is a deformation of the supercommutative algebra of the ordinary (N_0,N_1)-dimensional superplane. Of course $S^q_{N_0,0}=S^q_{N_0}$.

One may finally observe that new generators can be introduced by a linear transformation

$$y_j=\sum_{l=1}^{N} W_{jl}x_l, \tag{5.21}$$

with W_{jl} a nonsingular $N\times N$ matrix. The basic multiplication relations become

$$y_jy_k=\sum_{r,s=1}^{N} B_{jk,rs}y_ry_s, \tag{5.22}$$

with

$$B_{jk,rs}=\sum_{l,m=1}^{N} W_{jl}W_{km}q^{\alpha_{lm}}(W^{-1})_{mr}(W^{-1})_{ls}. \tag{5.23}$$

This suggests the question under which conditions an algebra determined by general quadratic relations of the form (5.22) is in fact an almost commutative algebra in disguise.

VI. CONCLUDING REMARKS

Let M be a group of $m\times m$ matrices and A the algebra of functions $f:M\rightarrow k$ that are polynomial in the matrix elements β_{jk} of elements β in M. A is generated by the functions e_{jk} defined by $e_{jk}(\beta)=\beta_{jk}$ for $\beta\in M$. The m^2 generators are not independent. There are relations and these characterize M as a subspace of k^{2m}. The group multiplication in M defines a comultiplication Δ: $A\rightarrow A\otimes A$ by $(\Delta f)(\beta_1,\beta_2)=f(\beta_1,\beta_2)$. Together with the counit ε: $A\rightarrow k$ and the antipode s: $A\rightarrow A$ defined, respectively, as $\varepsilon f=f(1)$ and $(sf)(\beta)=f(\beta^{-1})$ this makes A into a commutative but in general not cocommutative Hopf algebra. In terms of the generators the comultiplication is defined by

$$\Delta e_{jk}=\sum_{l=1}^{m} e_{jl}\otimes e_{lk} \tag{6.1}$$

and the requirement that Δ is an algebra homomorphism $A\rightarrow A$. Similarly one has

$$\varepsilon e_{jk}=\delta_{jk}, \tag{6.2}$$

$$(se_{jk})(\beta)=(\beta^{-1})_{jk}. \tag{6.3}$$

A quantum group or a matrix pseudogroup is a suitable deformation of this Hopf algebra. This is basically—at least as far as algebraic aspects are concerned—the approach to quantum groups developed by Woronowicz.[14]

Let $M=k^N$, the N-dimensional translation group. Elements of M can be written as $(N+1)\times(N+1)$ matrices β by putting $\beta_{jk}=1$, for $j=k$, $j=1,2,...,N+1$, $\beta_{jk}=\lambda_j\in k$, for $j=1,...,N$, $k=N+1$, and $\beta_{jk}=0$ otherwise. A, the algebra of polynomials on M, has generators e_{jk}: For $j=1,...,N$, $k=N+1$, e_{jk} may be called x_j. One has $e_{jk}=1$ for $j=k$, $j=1,...,N+1$ and

$e_{jk}=0$ for the remaining values of j and k. The basic multiplication rule in A is $x_j x_k = x_k x_j$; hence $A=S_N$, the symmetric algebra over N generators. It is a Hopf algebra with Eqs. (6.1)–(6.3) reducing to

$$\Delta x_j = 1 \otimes x_j + x_j \otimes 1, \tag{6.4}$$

$$\varepsilon x_j = 0, \tag{6.5}$$

$$s x_j = -x_j. \tag{6.6}$$

The algebra S_N^q discussed in the preceding section is a noncommutative deformation of S_N. It has the same set of generators e_{jk} or equivalently $x_1,\dots,x_N$ but the basic multiplication rule is now $x_j x_k = q x_k x_j$ for $j<k$. Formulas (6.4)–(6.6) again define a Hopf algebra structure, however in a slightly modified sense. The comultiplication Δ is defined by Eq. (6.4) and the requirement that $\Delta: S_N^q \rightarrow S_N^q \times S_N^q$ is a homomorphism of algebras. In general the tensor product $A \otimes B$ of two algebras A and B is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2), \quad (a_1, a_2 \in A, b_1, b_2 \in B). \tag{6.7}$$

This would lead to inconsistencies here, e.g., $\Delta(x_j x_k) \neq \Delta(x_k x_j)$ for $j>k$. Fortunately there is a modified definition for the tensor product of two G-graded algebras in the presence of a cocycle ρ. It is quite natural and is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = \rho(|b_1|,|a_2|)(a_1 a_2 \otimes b_1 b_2). \tag{6.8}$$

Note that this definition ensures—among other things—that the tensor product of ρ-commutative algebras is again ρ-commutative. The idea for such a modified tensor product is well known for the special case of superalgebras. Recently it has been developed in a more general way by Schürmann[15] in the context of q-deformed quantum stochastic processes. His commutation factors correspond to our cocycles. S_N^q, provided with this particular comultiplication is a Hopf algebra (or rather a ρ-Hopf algebra) and represents what may be called the N-dimensional quantum translation group. It is neither commutative nor cocommutative, but ρ-commutative and—in an obvious sense—ρ-cocommutative. Other comultiplications are known, in particular for $N=2$, leading to alternative interpretations of S_N^q as a Hopf algebra.

Woronowicz[16] defines a (first order) differential calculus over an algebra A as a pair (d,Γ), where Γ is an A bimodule and d is a derivation from A to Γ, i.e., a linear map with $d(fg) = (df)g + f(dg)$, for all f and g in A, and such that $\alpha \in \Gamma$ can be written as $\alpha = \Sigma_k f_k\, dg_k$. For a ρ-commutative algebra the pair $(d,\Omega_0^1(A))$ with $\Omega_0^1(A)$ the space of one-forms that can be written as $\Sigma_k f_k dg_k$ is a differential calculus in this sense. For $A=S_N^q$ one has $\Omega_0^1(A)=\Omega^1(A)$. When A is moreover a Hopf algebra with comultiplication Δ, the differential calculus is called left, respectively, right invariant if $\Sigma_k f_k\, dg_k=0$ implies $\Sigma_k(\Delta f_k)(\mathrm{id}\otimes d)\Delta g_k=0$, respectively, $\Sigma_k(\Delta f_k)(d\otimes \mathrm{id})\Delta g_k=0$, and bicovariant when it is left and right covariant. For $A=S_N^q$ and with the comultiplication defined above, i.e., with the interpretation of S_N^q as quantum translation group, $(d,\Omega^1(A))$ is bicovariant. We prove the left covariance. One needs for this the following identity:

$$(\mathrm{id}\otimes d)\Delta g = \sum_{j=1}^{N} (1 \otimes dx_j) \frac{\partial g}{\partial x_j} \quad (g \in S_N^q). \tag{6.9}$$

This is clearly true for $g=1$ and $g=x_l$, $l=1,2,\dots,N$. If it is true for g_1 and g_2, then it is true for linear combinations $\lambda_1 g_1 + \lambda_2 g_2$ and moreover for the product $g_1 g_2$. To prove the last statement one uses the fact that $\mathrm{id}\otimes d$ is a derivation from $S_N^q \otimes S_N^q$ to $S_N^q \otimes \Omega^1(S_N^q)$ (all tensor products are of the modified type) and furthermore formula (5.7) and the identities $|\Delta g_1|=|g_1|$, $|1\otimes dx_j|=|dx_j|=-|\partial/\partial x_j|$. This establishes formula (6.9) for all g in S^q_N. Suppose now that $\Sigma_k f_k\, dg_k=0$. One has, using Eq. (5.9)

$$\sum_k f_k\, dg_k=\sum_k f_k \sum_{j=1}^N (dx_j)\frac{\partial g_k}{\partial x_j}=\sum_{j=1}^N (dx_j)\sum_k \rho(|f_k|,|dx_j|)f_k\frac{\partial g_k}{\partial x_j} \tag{6.10}$$

and therefore

$$\sum_k \rho(|f_k|,|dx_j|)f_k\frac{\partial g_k}{\partial x_j}=0,\quad j=1,2,\ldots,N. \tag{6.11}$$

Then

$$\begin{aligned}\sum_k (\Delta f_k)(\mathrm{id}\otimes d)\Delta g_k&=\sum_k (\Delta f_k)\sum_{j=1}^N (1\otimes dx_j)\Delta\frac{\partial g_k}{\partial x_j}\\&=\sum_{j=1}^N (1\otimes dx_j)\Delta\left(\sum_k \rho(|f_k|,|dx_j|)f_k\frac{\partial g_k}{\partial x_j}\right)=0.\end{aligned} \tag{6.12}$$

This proves the left covariance of d. The proof of the right covariance is similar, so d is bicovariant. The result means that the de Rham complex $(\Omega(S^q_N),d)$, uniquely determined by the structure of S^q_N as a ρ-commutative algebra, can be regarded as the differential calculus on the N-dimensional quantum hyperplane, covariant with respect to the associated quantum translation group.

ACKNOWLEDGMENTS

We are indebted to I. V. Volovich for valuable discussions and to G. M. Tuynman for drawing our attention to an error in an earlier version of one of the bracket formulas in Sec. III.

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