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On Brauer's induction formula of characters of groups

By

ROBERT W. VAN DER WAALL

Introduction. A celebrated theorem of R. Brauer ([1], Theorem 1) states that any irreducible complex character χ of a finite group G can be expressed as a linear combination with integral rational coefficients of monomial characters λ_i^G of G , i.e.

$$(*) \quad \chi = \sum_{i=1}^{<\infty} a_i \lambda_i^G, \quad a_i \in \mathbb{Z}$$

and with λ_i linear characters of suitable subgroups of G .

Here we will show that for any non-principal irreducible character χ a summation (*) can be found in which none of the λ_i^G contains the principal character 1_G as an irreducible constituent. This answers in the affirmative a like question of G. O. Michler (Essen, Germany), as put to the author in June 1993. As Michler told him, he could not gather an immediate and positive answer to his query from the existing literature, an observation that the author is willing to share with him after loose inspection of known sources. Note, that a positive answer to Michler's question has consequences for positions of occurrence of zeroes of Dedekind ζ -functions and of Artin L -functions of algebraic number fields. To be more specific, let E/K be a finite galois extension of algebraic number fields with galois group G . Consider the Artin L -function $L_i := L(s, \lambda_i^G, E/K)$. Then if n_i is the order of the zero or pole of L_i at the point s_0 , Artin's conjecture maintains that for $\chi \neq 1_G$ $\sum a_i n_i \leq 0$ holds whenever $\chi = \sum a_i \lambda_i^G$, $a_i \in \mathbb{Z}$, is a (*)-decomposition, in which none of the λ_i^G contains 1_G as irreducible constituent; that is, $L(s, \chi, E/K)$ should be analytic at s_0 . For more information about these topics, one may consult ([2], Chapter VIII), ([3], Chapter I) and more in particular the references drawn up in [7]; for very recent developments in respect to the Aramata-Brauer theorem one is referred to [6].

Notation and conventions. We use standard notation as in [4] and [5] or else it will be self-explanatory. In particular we mention

- $\text{Irr}(G)$ the set of the complex irreducible characters of the finite group G ;
- 1_G – the principal character of G ;
- η^G – the character of G induced by the character η of a subgroup of G .

The character χ of G is *monomial* if it is induced from a *linear* character λ of a suitable subgroup of G , i.e. $\chi = \lambda^G$ and $\lambda(1) = 1$.

All groups in this paper are considered to be finite. Representation theory of groups is done over the complex number field.

A group T is called *quasi-elementary* if there exists a cyclic normal subgroup C of T for which T/C is a p -group for some prime p while also C itself is a p' -group.

We now state and establish the existence of the special kind of summation (*) as announced in the Introduction.

Theorem. *Let $\chi \in \text{Irr}(G)$, $\chi \neq \mathbf{1}_G$. Then χ is an integral linear combination of monomial characters of G each not containing $\mathbf{1}_G$ as an irreducible constituent.*

Proof. According to L. Solomon (see [5], Theorem (8.20)) it holds that $\mathbf{1}_G = \sum_H a_H (\mathbf{1}_H)^G$ for suitable $a_H \in \mathbb{Z}$, where the summation runs over the set of the quasi-elementary subgroups H of G . Therefore, as $\chi(\mathbf{1}_H)^G = (\mathbf{1}_H \chi_H)^G = (\chi_H)^G$, $\chi = \sum a_H (\chi_H)^G$ follows.

Now $\chi_H = b_H \mathbf{1}_H + T_H$, where $b_H \in \mathbb{Z}_{\geq 0}$ and where either T_H is void or else $T_H = \sum_i \xi_i$ with any ξ_i an irreducible monomial character of H not equal to $\mathbf{1}_H$ (note that each irreducible character of a quasi-elementary group is monomial). Hence $\chi = \sum a_H b_H (\mathbf{1}_H)^G + \sum a_H (T_H)^G$, where by convention either $(T_H)^G$ is void or else $(T_H)^G = \sum_i \xi_i^G$, with any ξ_i^G a monomial character of G each not containing $\mathbf{1}_G$ as irreducible constituent. Indeed, by Frobenius' reciprocity $(\mathbf{1}_G, \xi_i^G) = (\mathbf{1}_H, \xi_i) = 0$. We have

$$\sum a_H b_H (\mathbf{1}_H)^G = - \sum a_H b_H ((\mathbf{1}_{(1)})^H - \mathbf{1}_H)^G + \sum a_H b_H ((\mathbf{1}_{(1)})^H)^G.$$

Each of the quasi-elementary subgroups H is an M -group (a fact we used already), whence $(\mathbf{1}_{(1)})^H - \mathbf{1}_H = \sum_i \gamma_i$ where any such γ_i is an irreducible monomial character of H not equal to $\mathbf{1}_H$. Hence by a similar reasoning as above $((\mathbf{1}_{(1)})^H - \mathbf{1}_H)^G = \sum_i \gamma_i^G$, whereas any γ_i^G is a monomial character of G not containing $\mathbf{1}_G$ as irreducible constituent. Furthermore, it holds that $\sum a_H b_H = 0$. Namely

$$\begin{aligned} 0 = (\chi, \mathbf{1}_G) &= (\sum a_H (\chi_H)^G, \mathbf{1}_G) = \sum a_H ((\chi_H)^G, \mathbf{1}_G) \\ &= \sum a_H (\chi_H, \mathbf{1}_H) \\ &= \sum a_H b_H. \end{aligned}$$

Therefore

$$\chi = - \sum a_H b_H ((\mathbf{1}_{(1)})^H - \mathbf{1}_H)^G + \sum a_H (T_H)^G$$

is a kind of expression (*) answering Michler's question in the affirmative.

The proof of the Theorem is complete. $\square$

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