Bispectral quantum Knizhnik-Zamolodchikov equations
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Chapter 2

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Knizhnik-Zamolodchikov
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2.1 Introduction

Let $V$ be an $N$-dimensional complex vector space and let $T$ denote the complex
torus $T := (\mathbb{C} \setminus \{0\})^N$. In this chapter we derive an explicit holonomic system of
$q$-difference equations on $V$-valued meromorphic functions on $T \times T$, which we call
the bispectral quantum Knizhnik-Zamolodchikov (BqKZ) equations. It contains as a
subsystem, quantum affine KZ equations of a particular type [10, 21]. From this per-
spective, the additional compatible equations may be thought of as the associated
quantum isomonodromy transformations. Let us briefly explain the ideas involved
and the relation of the BqKZ equations to the quantum affine KZ equations intro-
duced by Cherednik.

Let $W = S_N \ltimes \mathbb{Z}^N$ be the (extended) affine Weyl group of type GL$_N$, that is, the
semidirect product of the symmetric group $S_N$ and the lattice $\mathbb{Z}^N$. The group $S_N$
acts on $T$ by permuting the coordinates. Choose $0 < q < 1$. The action of $S_N$ on $T$
extends to an action of $W$ on $T$ by letting $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ act via

$$t = (t_1, \ldots, t_N) \mapsto q^\lambda t := (q^{\lambda_1} t_1, \ldots, q^{\lambda_N} t_N).$$

The BqKZ system which we will introduce, is a system of $q$-difference equations of
the form

$$C_{(\lambda, \mu)}(t, \gamma)f(q^{-\lambda}t, q^\mu \gamma) = f(t, \gamma), \quad \lambda, \mu \in \mathbb{P}^N,$$

for meromorphic functions $f$ on $T \times T$ with values in $V$. Here $C_{(\lambda, \mu)}$ ($\lambda, \mu \in \mathbb{Z}^N$) are
explicit $\text{End}(V)$-valued meromorphic functions on $T \times T$, satisfying the following
which reflects the fact that $B_{qKZ}$ is holonomic.

$B_{qKZ}$ contains, in some sense, two families of Cherednik’s quantum affine KZ equations associated with the principal series representation of the affine Hecke algebra $H$ of type $GL_N$. We recall [10] (for arbitrary root systems, see also Subsection 5.3.2) that the quantum affine KZ equations associated with a finite dimensional $H$-module $M$ is a consistent system of $q$-difference equations of the form

$$F_M^\lambda(t) f(q^{-\lambda} t) = f(t), \quad \lambda \in P^\vee,$$

for meromorphic functions $f$ on $T$ with values in $M$, and where $F_M^\lambda (\lambda \in \mathbb{Z}^N)$ are $\text{End}(M)$-valued meromorphic functions on $T$ satisfying cocycle conditions similar to (2.1.1). Now the first family of quantum affine KZ equations inside $B_{qKZ}$ is parameterized by $\gamma \in \mathbb{T} \cong \{1\} \times T \subset T \times T$. More precisely, if we fix $\gamma = \zeta \in \mathbb{T}$, we have

$$C_{(\lambda, \zeta)}(t, \zeta) = F_M^\lambda(t),$$

where $M_\zeta$ is the principal series representation of $H$ with central character $\zeta$, which as a vector space can be identified with $V$ via a $\zeta$-dependent isomorphism. Similarly, $B_{qKZ}$ contains a second family of quantum affine KZ equations, parameterized by $t \in T$, related to the affine Hecke algebra module $M_{t^{-1}}$. It is obtained from the first by interchanging the roles of the torus variables $t$ and $\gamma^{-1}$ and by conjugating the cocycle matrices by an explicit complex linear automorphism $C_\iota$ of $V$. Hence, it acts only on the second $T$-component of $T \times T$ and as such realizes $qKZ_{t^{-1}}$ for fixed $t \in T$. In particular, this provides a quantum isomonodromic interpretation of $qKZ$. This should be compared with the interpretation of rational KZ equations as quantizations of Schlesinger equations, see [49] and [23].

Etingof and Varchenko [19] used quantum group methods to construct systems of $q$-difference equations (so-called dynamical $q$-difference equations) that are compatible with Frenkel and Reshetikhin’s quantum KZ equations associated with evaluation representations of quantum affine algebras. It is likely that the system of dynamical $q$-difference equations associated with $qKZ_\lambda$ is equivalent to the dual $qKZ$ subsystem in $B_{qKZ}$.

Preceding the above mentioned work [19] of Etingof and Varchenko, dynamical equations for various degenerations of quantum KZ equations have been analyzed in detail; see, e.g., [20], [60], [57], [59] and [35]. An interesting aspect in, e.g., [60] and [59], is the observation that various degenerations of quantum KZ equations are the duals of their associated dynamical equations with respect to $(\mathfrak{gl}_r, \mathfrak{gl}_s)$ duality. In the present set-up (which corresponds to $r = s = N$), this duality is incorporated by the automorphism $C_\iota$, which reflects Cherednik’s duality anti-involution of the double affine Hecke algebra $H$ on the level of $B_{qKZ}$.

Related to this observation, an important virtue of the present approach is worth mentioning. The double affine Hecke algebra and its symmetry (embodied by the
duality involution) give rise to the bispectral quantum KZ equations, and thus a way to construct a system of $q$-difference equations compatible with the quantum KZ equations, which is relatively easy compared to cases considered in the above mentioned works of Etingof, Felder, Markov, Tarasov and Varchenko.

We conclude this introduction with a detailed outline of this chapter. In Section 2.2 we introduce Cherednik’s [10] double affine Hecke algebra $\mathbb{H}$ of type $GL_N$, on which the construction of BqKZ relies in the following way. As a vector space $\mathbb{H}$ is isomorphic to $\mathbb{C}[T] \otimes H \simeq \mathbb{C}[T] \otimes H_0 \otimes \mathbb{C}[T] \simeq \mathbb{C}[T \times T] \otimes H_0$ with $H_0$ the finite Hecke algebra of type $A_{N-1}$. Cherednik’s anti-algebra involution $^\ast: \mathbb{H} \to \mathbb{H}$ essentially interchanges, under the above vector space identification, the role of the two copies of $\mathbb{C}[T]$. For $w, w' \in W = S_N \rtimes \mathbb{Z}_N$ we consider the map $h \mapsto \tilde{S}_w h \tilde{S}_{w'} (h \in \mathbb{H})$, where the $\tilde{S}_w \in \mathbb{H}$ are Cherednik’s nonnormalized ($X$-)intertwiners. Restricted to $w, w' \in \mathbb{Z}_N$, suitable renormalizations of these maps become the cocycle matrices of BqKZ, with $H_0$ playing the role of $V$. The anti-involution $^\ast$ of $\mathbb{H}$ gives rise to the automorphism $C_i$ interchanging the qKZ subsystem of BqKZ with its dual subsystem in BqKZ. This construction of BqKZ is described in Section 2.3.

In Section 2.4 we make the BqKZ equations explicit by calculating the cocycle matrices $C_{(\lambda, \mu)}$ for suitable $\lambda, \mu \in \mathbb{Z}_N$. This enables us to relate qKZ $\zeta$ to Frenkel and Reshetikhin’s [21] quantum KZ equations associated with the $N$-fold tensor product of the vector representation of quantum $sl_N$ (see [10, §1.3.2]). A special case of qKZ $\zeta$ was considered earlier by Smirnov [54].

In Section 2.5 we investigate the space SOL of $M$-valued meromorphic solutions of BqKZ in detail. We first analyze BqKZ in a suitable asymptotic region. It leads to a solution $\Phi$ of BqKZ which is self-dual, in the sense that $\Phi(t, \gamma) = C_i \Phi(\gamma^{-1}, t^{-1})$ as $M$-valued meromorphic functions in $(t, \gamma) \in T \times T$. We construct a basis of SOL in terms of $\Phi$, and we give an explicit formula for the leading term of $\Phi(t, \gamma)$ as function of $t$.

The contents of this chapter agree with Sections 2–5 of [45].

### 2.2 The double affine Hecke algebra

#### 2.2.1 The extended affine Weyl group

Let $N \geq 2$ and let $D = D_N$ be the affine Dynkin diagram of affine type $\hat{A}_{N-1}$ (the cyclic graph with $N$ vertices if $N \geq 3$). The $N$ vertices are labeled by the numbers $0, 1, \ldots, N - 1$ (anticlockwise if $N \geq 3$). Occasionally we identify the set of labels with the group $\mathbb{Z}_N$ of integers modulo $N$.

Write $W_Q$ for the affine Weyl group of affine type $\hat{A}_{N-1}$. In terms of its Coxeter generators $s_i$ ($i \in \mathbb{Z}_N$), the characterizing group relations are the quadratic relations $s_i^2 = 1$ and, if $N \geq 3$, the braid relations

\begin{align}
    s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\
    s_i s_j &= s_j s_i, & i - j \neq 0, \pm 1.
\end{align}

(2.2.1)
The subgroup generated by \( s_1, \ldots, s_{N-1} \) is isomorphic to the symmetric group \( S_N \) in \( N \) letters, where \( s_i \) is identified with the simple transposition \( i \leftrightarrow i+1 \).

Let \( \text{Aut}(D) \) be the group of automorphisms of the affine Dynkin diagram of type \( \tilde{A}_{N-1} \). Let \( c \in \text{Aut}(D) \) be the element of order \( N \), acting on the label set \( \mathbb{Z}_N \) of the vertices of \( D \) by \( c(i) = i + 1 \). We view \( c \) as automorphism of \( W_Q \) by \( c(s_i) = s_{i+1} \).

Let \( \Omega = \langle \pi \rangle \) be the infinite cyclic group with cyclic generator \( \pi \). It acts by group automorphisms on \( W_Q \) by \( \pi \mapsto c \). Accordingly, we can define the semidirect product group \( W = W_Q \rtimes \Omega \), which is called the extended affine Weyl group (associated with \( \text{GL}_N \)). We write \( c \) for the identity element of \( W \).

Since \( s_0 = \pi s_{N-1} \pi^{-1} \), the subgroups \( S_N \) and \( \Omega \) already generate \( W \) as a group. Furthermore, we have \( W \simeq S_N \ltimes \mathbb{Z}^N \). The cyclic generator \( \pi \) of \( \Omega \) corresponds to \( \pi = \sigma \epsilon \), where \( \{ \epsilon_i \}_{1}^{N} \) denotes the standard \( \mathbb{Z} \)-basis of \( \mathbb{Z}^N \) and \( \sigma = s_1 s_2 \cdots s_{N-1} \in S_N \) is the “clockwise rotation” which maps \( N \) to 1 and all other \( i \) to \( i + 1 \). Conversely,

\[
\epsilon_j = s_{j-1} \cdots s_1 s_{N-1} s_{N-2} \cdots s_j
\]

for \( j = 1, \ldots, N \).

**Remark 2.2.1.** Under the identification \( W \simeq S_N \ltimes \mathbb{Z}^N \), we have \( W_Q = S_N \ltimes Q \) with \( Q \subset \mathbb{Z}^N \) the sublattice of rank \( N - 1 \) consisting of \( N \)-tuples of integers that sum up to zero (this is the (co)root lattice of the root system \( R = \{ \epsilon_i - \epsilon_j \}_{1 \leq i \neq j \leq N} \) of type \( A_{N-1} \)).

For \( w \in W \), let \( w' \in W_Q \) and \( \omega \in \Omega \) be the unique group elements such that \( w = w' \omega \). Then we define the length \( \ell(w) \) of \( w \) to be the length of \( w' \in W_Q \), i.e., it is the minimal number \( r \) such that \( w' \) can be expressed as

\[
w' = s_{i_1} \cdots s_{i_r}
\]

for some \( i_k \in \mathbb{Z}_N \) (such an expression of \( w' \), as well as the resulting expression for \( w = w' \omega \), is called a reduced expression). Thus \( \Omega \) consists of the elements of \( W \) of length zero.

A central role in this thesis is played by an action of the extended affine Weyl group \( W \) by \( q \)-difference reflection operators on suitable function spaces on \( T := (\mathbb{C}^\times)^N \), where \( \mathbb{C}^\times := \mathbb{C} \setminus \{0\} \). Here \( q \) is taken to be real and strictly between zero and one (with minor technical adjustments the condition on \( q \) may be relaxed to \( 0 < |q| < 1 \), and a parallel theory can be developed for \( |q| > 1 \)). Since \( q \) is fixed once and for all, we will in general suppress the dependence on \( q \) in notations. We start with an action of \( W \) on \( T \) by

\[
wt = (t_{w^{-1}(1)}, \ldots, t_{w^{-1}(N)}), \quad w \in S_N, \\
\lambda = (q^{\lambda_1} t_1, \ldots, q^{\lambda_N} t_N), \quad \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N,
\]

for \( t = (t_1, \ldots, t_N) \in T \). It is convenient to introduce \( \kappa^\lambda := (\kappa^{\lambda_1}, \ldots, \kappa^{\lambda_N}) \) for \( \kappa \in \mathbb{C}^\times \) and \( \lambda \in \mathbb{Z}^N \), so that the action of \( \lambda \in \mathbb{Z}^N \) on \( t \in T \) can simply be written as

\[
\lambda t = q^\lambda t
\]
in standard vector notation. Note that the action of $\pi \in \Omega$ is given by 

$$\pi(t_1, \ldots, t_N) = (qt_N, t_1, \ldots, t_{N-1}).$$

Consider the algebra $\mathbb{C}[T] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ of complex-valued regular functions on $T$, where $x_i(t) := t_i$ for $t = (t_1, \ldots, t_N) \in T$ are the standard coordinate functions. We write $x^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N} \in \mathbb{C}[T]$ for $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$, which form the monomial basis of $\mathbb{C}[T]$.

Let $\mathbb{C}(T)$ be the field of rational functions on $T$, $O(T)$ be the ring of analytic functions on $T$, and $M(T)$ be the field of meromorphic functions on $T$. Note that $M(T)$ is the quotient field of $O(T)$ (cf. [25, Theorem 7.4.6]). The $W$-action on $T$ gives rise to a left $W$-action by algebra automorphisms on $\mathbb{C}[T]$, $\mathbb{C}(T)$, $O(T)$ and $M(T)$, via 

$$(wf)(t) = f(w^{-1}t)$$

for $w \in W$, $t \in T$. We can, in particular, form the smash product algebra $\mathbb{C}(T)#W$. Recall that if $G$ is a group and $A$ is a $G$-algebra over $\mathbb{C}$ (that is, a unital associative algebra over $\mathbb{C}$ endowed with a left $G$-action by algebra automorphisms), then the smash product algebra $A#G$ is the unique complex unital associative algebra such that 

(i) $A#G = A \otimes \mathbb{C}[G]$ as a complex vector space;

(ii) the canonical linear embeddings $A \hookrightarrow A#G$, $\mathbb{C}[G] \hookrightarrow A#G$ are algebra homomorphisms; and

(iii) the cross relations 

$$(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh,$$

are satisfied for $a, b \in A$ and $g, h \in G$.

We will always write $ag := a \otimes g \in A#G$ ($a \in A, g \in G$). Observe that $A#G$ canonically acts on any $G$-algebra $B$ containing $A$ as a $G$-subalgebra.

Note that the smash product algebra $\mathbb{C}(T)#W$ depends on $q$, since the $W$-action on $\mathbb{C}(T)$ depends on $q$ (see (2.2.3)). Sometimes it is convenient to emphasize its $q$-dependence, in which case we write $\mathbb{C}(T)#_q W$ instead of $\mathbb{C}(T)#W$.

The canonical left $\mathbb{C}(T)#W$-action on $\mathbb{C}(T)$ (and $M(T)$) is faithful and realizes $\mathbb{C}(T)#W$ as the algebra of $q$-difference $S_N$-reflection operators with coefficients in $\mathbb{C}(T)$. If $f \in \mathbb{C}(T)$, then we write $f(X)$ for the associated element in $\mathbb{C}(T)#W$ (it is the operator defined as multiplication by $f$). In particular, $X_i$ is multiplication by the coordinate function $x_i$. 
2.2.2 The extended affine Hecke algebra and Cherednik’s basic representation

In this subsection we recall some constructions and results due to Cherednik (see, e.g., [10, Chapter 1] and references therein).

Fix a nonzero complex number \( k \).

**Definition 2.2.2.** The affine Hecke algebra \( H = H_Q(k) \) is the complex unital associative algebra generated by \( T_i \) (\( i \in \mathbb{Z}_N \)) and satisfying

(i) if \( N \geq 3 \), the braid relations

\[
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i, \quad i - j \neq 0, \pm 1;
\]

(ii) the quadratic relations \((T_i - k)(T_i + k^{-1}) = 0\).

The Dynkin diagram automorphism \( c \in \text{Aut}(D) \) can also be viewed as automorphism of \( H_Q \) by \( c(T_i) = T_{i+1} \). Accordingly, \( \Omega \) acts by algebra automorphisms on \( H_Q \) by \( \pi \mapsto c \). The extended affine Hecke algebra \( H \cong H(k) \) is the associated smash product algebra \( H_Q \# \Omega \).

For a reduced expression \( w = s_{i_1} \cdots s_{i_k} \), \( \omega \in \Omega \), the element

\[
T_w := T_{i_1} \cdots T_{i_k} \omega \in H
\]

is well-defined. The \( T_w \) (\( w \in W \)) form a linear basis of \( H \). For \( k = 1 \), the extended affine Hecke algebra \( H \) is isomorphic to the group algebra \( \mathbb{C}[W] \) of \( W \) via the identification \( T_w \leftrightarrow w \) (\( w \in W \)).

The finite Hecke algebra is the subalgebra \( H_0 \) of \( H \) generated by \( T_1, \ldots, T_{N-1} \). The elements \( T_w \) (\( w \in S_N \)) form a linear basis of \( H_0 \). Note that \( H \) is already generated as algebra by \( H_0 \) and \( \pi \), since \( T_0 = \pi T_{N-1} \). Thus \( H \cong H_0 \# \Omega \).

Put

\[
Y_i := T_{i-1}^{-1} \cdots T_2^{-1} T_1^{-1} \pi T_{N-1} T_{N-2} \cdots T_i \in H
\]

for \( i = 1, \ldots, N \). Note that \( Y_i \) becomes the translation element \( \epsilon_i \) in \( W \) if \( k = 1 \).

We furthermore write \( Y^\lambda := Y_1^{\lambda_1} \cdots Y_N^{\lambda_N} \) for \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \). We have the following characterization of \( H \), due to Bernstein. For details we refer to Lusztig [38] or Macdonald [42, §4.2].

**Theorem 2.2.3.** \( H \) is the unique unital complex associative algebra, such that

(i) \( H_0 \otimes \mathbb{C}[T] \cong H \) as complex vector spaces, via \( h \otimes f \mapsto hf(Y) \) for \( h \in H_0, f \in \mathbb{C}[T] \), where \( f(Y) = \sum_\lambda c_\lambda Y^\lambda \) if \( f = \sum_\lambda c_\lambda x^\lambda \in \mathbb{C}[T] \);

(ii) the canonical maps \( H_0, \mathbb{C}[T] \to H \) are algebra embeddings; we write \( \mathbb{C}_Y[T] = \text{span}_\mathbb{C}\{Y^\lambda\}_{\lambda \in \mathbb{Z}^N} \) for the image of \( \mathbb{C}[T] \) in \( H \); and
(iii) the following cross relations

\[
T_i^{-1} Y_i T_i^{-1} = Y_{i+1}, \\
Y_j T_i = T_i Y_j, \quad i \neq j - 1, j
\]

are satisfied for \(1 \leq i < N\) and \(1 \leq j \leq N\).

Cherednik realized the affine Hecke algebra \(H\) inside the algebra \(\mathbb{C}(T)^\#W\) of \(q\)-difference reflection operators as follows.

**Theorem 2.2.4.** There is a unique injective algebra homomorphism \(\rho = \rho_{k,q} : H(k) \to \mathbb{C}(T)^\#qW\) satisfying

\[
\rho(T_i) = k + c_k(X_i/X_{i+1})(s_i - 1), \\
\rho(\pi) = \pi,
\]

for \(i = 1, \ldots, N - 1\), where

\[
c_k(z) := \frac{k^{-1} - kz}{1 - z}. \quad (2.2.5)
\]

Note that for the affine Hecke algebra \(H = H(k)\) with fixed parameter \(k\), Theorem 2.2.4 yields a one-parameter family of realizations of \(H\) (the additional parameter being \(q\)).

**Remark 2.2.5.** The image \(\rho(H)\) preserves \(\mathbb{C}[T]\), viewed as a subspace of the canonical \(\mathbb{C}(T)^\#W\)-module \(\mathbb{C}(T)\). The resulting representation of \(H\) on \(\mathbb{C}[T]\) is faithful and is called the basic representation of \(H\).

We frequently identify \(H\) with its image under \(\rho\) in \(\mathbb{C}(T)^\#W\).

We now come to the definition of Cherednik’s double affine Hecke algebra which depends, besides on \(k\), on the additional parameter \(q\).

**Definition 2.2.6.** The double affine Hecke algebra \(\mathbb{H} = \mathbb{H}(k, q)\) is the subalgebra of \(\mathbb{C}(T)^\#qW\) generated by \(\rho_{k,q}(H)\) and by the multiplication operators \(f(X) (f \in \mathbb{C}[T])\).

Let \(L = \mathbb{C}[T] \otimes \mathbb{C}[T] \simeq \mathbb{C}[T \times T]\) denote the complex-valued regular functions on \(T \times T\). We view \(\mathbb{H}\) as \(L\)-module by

\[
(f \otimes g) \cdot h := f(X)h(Y) \quad (2.2.6)
\]

for \(f, g \in \mathbb{C}[T]\) and \(h \in \mathbb{H}\). The following theorem is the so-called Poincaré-Birkhoff-Witt (PBW) property of the double affine Hecke algebra.

**Theorem 2.2.7.** We have \(\mathbb{H} \simeq H_0^L = L \otimes H_0\) as \(L\)-modules.

The PBW property is an essential ingredient in deriving the characterizing relations for the double affine Hecke algebra \(\mathbb{H}\) in terms of its algebraic generators \(T_i\).
(1 ≤ i < N), π±1 and X±1 (1 ≤ j ≤ N). Since we are not going to use this presentation explicitly here, we refer the reader to [10] for further details. We use though one of its direct consequences, namely the existence of the duality anti-isomorphism (see Cherednik [10, Theorem 1.4.8]):

**Theorem 2.2.8.** There exists a unique \( \mathbb{C} \)-linear anti-algebra involution \( * : \mathbb{H} \to \mathbb{H} \) determined by

\[
T_w^* = T_{w^{-1}}, \quad w \in \mathbb{S}_N,
\]

\[
(Y^\lambda)^* = X^{-\lambda}, \quad \lambda \in \mathbb{Z}^N,
\]

\[
(X^\lambda)^* = Y^{-\lambda}, \quad \lambda \in \mathbb{Z}^N.
\]

### 2.2.3 Intertwiners

In this subsection we recall the construction of the (nonnormalized) affine intertwiners associated to the double affine Hecke algebra \( \mathbb{H} \). The intertwiners play an important role in the construction of a nontrivial \( \mathbb{W} \times \mathbb{W} \)-cocycle in the next section. Consider the elements

\[
\tilde{S}_i = (k - k^{-1}X_{i+1}/X_i)s_i, \quad 1 \leq i < N;
\]

\[
\tilde{S}_0 = (k - k^{-1}q^{-1}X_1/X_N)s_0;
\]

in \( \mathbb{C}(T)\#_q \mathbb{W} \). The following facts are well known (cf., e.g., [10, §1.3]). For the convenience of the reader, we give a short sketch of the proof.

**Proposition 2.2.9.** Let \( w \in \mathbb{W} \) and let \( w = s_{j_1} \cdots s_{j_r} \pi^m \) be a reduced expression \( (j_l \in \mathbb{Z}_N, m \in \mathbb{Z}) \).

(i) \( \tilde{S}_w := \tilde{S}_{j_1} \cdots \tilde{S}_{j_r} \tilde{S}_\pi^m \) is a well-defined element of \( \mathbb{C}(T)\# \mathbb{W} \);

(ii) \( \tilde{S}_w \in \mathbb{H} \);

(iii) the \( \tilde{S}_i \) (\( i \in \mathbb{Z}_N \)) satisfy the \( \hat{A}_{N-1} \)-type braid relations;

(iv) \( \tilde{S}_w f(X) = (w f)(X) \tilde{S}_w \) in \( \mathbb{C}(T)\# \mathbb{W} \) for all \( f \in \mathbb{C}(T) \); and

(v) \( \tilde{S}_i \tilde{S}_i = (k - k^{-1}X_{i+1}/X_i)(k - k^{-1}X_i/X_{i+1}) \) for \( 1 \leq i < N \).

**Proof.** (i) Set \( d_i := (k - k^{-1}X_{i+1}/X_i) \) (\( 1 \leq i < N \)) and \( d_0 := (k - k^{-1}q^{-1}X_1/X_N) \). We have

\[
\tilde{S}_w = d_{j_1}(s_{j_1} d_{j_2}) \cdots (s_{j_1} \cdots s_{j_{r-1}} d_{j_r}) w
\]

in \( \mathbb{C}(T)\# \mathbb{W} \). By, e.g., Macdonald [42, (2.2.9)], we know that

\[
d_w := d_{j_1}(s_{j_1} d_{j_2}) \cdots (s_{j_1} \cdots s_{j_{r-1}} d_{j_r}) \quad (2.2.7)
\]
is independent of the reduced expression of \( w \). Hence \( \tilde{S}_w \in \mathbb{C}(T)\#W \) is well defined.

(ii) Note that \( \tilde{S}_w \) can be written as

\[
\tilde{S}_w = (1 - X_{i+1}/X_i)(T_i - k) + k - k^{-1}X_{i+1}/X_i
\]

for \( 1 \leq i < N \), which shows that it lies in \( \mathbb{H} \). Furthermore, \( \pi_{\pm} \in \mathbb{H} \), hence \( \tilde{S}_{\pi_{\pm}} \in \mathbb{H} \). Consequently, \( \tilde{S}_w \in \mathbb{H} \subset \mathbb{C}(T)\#W \).

(iii) is immediate from (i), while (iv) and (v) are clear from the definition of the \( \tilde{S}_i \) and \( \tilde{S}_\pi \).

**Definition 2.2.10.** The elements \( \tilde{S}_w (w \in W) \) are called the affine intertwiners of \( \mathbb{H} \).

### 2.3 The bispectral quantum KZ equations

#### 2.3.1 Construction of the cocycle

Let \( \iota \) denote the nontrivial element of the two group \( \mathbb{Z}_2 \). We define the group \( W \) as the semidirect product

\[
W := \mathbb{Z}_2 \ltimes (W \times W),
\]

where \( \iota \in \mathbb{Z}_2 \) acts on \( W \times W \) by switching the components: \( \iota(w, w') = (w', w) \iota \) for \( w, w' \in W \). We first use the double affine Hecke algebra, its affine intertwiners, and its duality anti-isomorphism to construct a group homomorphism \( \tau_k : W \rightarrow \text{GL}_C(H_k^0) \) depending on the Hecke algebra parameter \( k \), where \( K := M(T \times T) \) is the field of meromorphic functions on \( T \times T \).

The representation \( \tau \) will be constructed from the complex linear endomorphisms \( \tilde{\sigma}_{(w, w')} (w, w' \in W) \) and \( \tilde{\sigma}_\iota \) of the double affine Hecke algebra \( \mathbb{H} \), defined by

\[
\tilde{\sigma}_{(w, w')} (h) = \tilde{S}_w h \tilde{S}_{w'}^*, \\
\tilde{\sigma}_\iota (h) = h^*
\]

for \( h \in \mathbb{H} \). In the following lemma we collect some elementary properties of the maps \( \tilde{\sigma}_{(w, w')} \) and \( \tilde{\sigma}_\iota \). First we introduce some auxiliary notations.

For a regular function \( g \in \mathbb{C}[T] \), we write \( g(x) \in \mathbb{C}[T \times T] \) (respectively \( g(y) \in \mathbb{C}[T \times T] \)) for the corresponding regular function on \( T \times T \) constant with respect to the second (respectively first) \( T \)-component. In particular, the \( x_i \) (respectively \( y_i \)) are the standard coordinate functions of the first (respectively second) copy of \( T \) in \( T \times T \). Recall the regular function \( d_w \in \mathbb{C}[T] (w \in W) \) such that

\[
\tilde{S}_w = d_w (X) w
\]

in \( \mathbb{C}(T)\#W \); see (2.2.7).

**Lemma 2.3.1.** The complex linear endomorphisms \( \tilde{\sigma}_{(w, w')} \) and \( \tilde{\sigma}_\iota \) of \( \mathbb{H} \) satisfy the following properties:
Proof. These are direct consequences of Proposition 2.2.9 and Theorem 2.2.8. \qed

To construct a $\mathcal{W}$-action from the maps $\tilde{\sigma}_{(w,w')}$ and $\tilde{\sigma}$, we need to renormalize the maps appropriately. To do so, we describe as a first step the behavior of the maps $\tilde{\sigma}_{(w,w')}$ and $\tilde{\sigma}$ with respect to the $L$-module structure (2.2.6) on $\mathbb{H}$. This will allow us to extend the maps $\tilde{\sigma}_{(w,w')}$ and $\tilde{\sigma}$ to endomorphisms of $H_0^K \simeq \mathbb{K} \otimes L \mathbb{H}$, which is a suitably flexible surrounding for the normalizations of the maps to take place in.

Consider the group involution $\circlearrowright : W \to W$ defined by $w^\circlearrowright = w$ for $w \in S_N$ and $\lambda^\circlearrowright = -\lambda$ for $\lambda \in \mathbb{Z}^N$. Then $\mathcal{W}$ acts on $T \times T$ by

\[(w, w')(t, \gamma) = (wt, w'^\circlearrowright \gamma),\]
\[(t, \gamma) = (\gamma^{-1}, t^{-1})\]

for $w, w' \in W$, where $t^{-1} := (t_1^{-1}, \ldots, t_N^{-1}) \in T$ and the action of $W$ on $T$ is by $q$-dilations and permutations; see (2.2.3). By transposition, this defines an action of $\mathcal{W}$ on $\mathbb{K} = \mathcal{M}(T \times T)$ by field automorphisms,

\[(wf)(t, \gamma) = f(w^{-1}(t, \gamma)), \quad w \in \mathcal{W}. \quad (2.3.1)\]

Note that $\mathcal{L} = \mathbb{C}[T \times T]$ is a $\mathcal{W}$-subalgebra of $\mathbb{K}$.

Lemma 2.3.2. For $h \in \mathbb{H}$ and $f \in \mathcal{L}$ we have

\[
\tilde{\sigma}_{(w,w')}(f \cdot h) = ((w, w')f) \cdot \tilde{\sigma}_{(w,w')}(h),
\]
\[
\tilde{\sigma}(f \cdot h) = (tf) \cdot \tilde{\sigma}(h)
\]

(2.3.2)

for $w, w' \in W$.

Proof. From Proposition 2.2.9 we know that $\tilde{S}_w p(X) = (wp)(X) \tilde{S}_w$ in $\mathbb{H}$ for $p \in \mathbb{C}[T]$ and $w \in W$. For $p \in \mathbb{C}[T]$, let $p^\circlearrowright \in \mathbb{C}[T]$ be defined by $p^\circlearrowright(t) = p(t^{-1})$. Then we also have

$p(Y) \tilde{S}_w = ((w^\circlearrowright p^\circlearrowright)(X) \tilde{S}_w)^* = \tilde{S}_w (w^\circlearrowright p^\circlearrowright)^\circ(Y)$

in $\mathbb{H}$. Hence for $p, r \in \mathbb{C}[T]$,

$\tilde{\sigma}_{(w,w')}(p(X)hr(Y)) = (wp)(X)\tilde{S}_w h \tilde{S}_w^*(w^\circlearrowright r^\circlearrowright)^\circ(Y)$.

The first formula of (2.3.2) now follows since $(w^\circlearrowright r^\circlearrowright)^\circ = w'^\circlearrowright r$. The second is immediate from the definition of the duality anti-involution. \qed
As a direct consequence of Lemma 2.3.2 the maps \( \tilde{\sigma}_{(w, w')} \) (\( w, w' \in W \)) and \( \tilde{\sigma} \) uniquely extend to complex linear endomorphisms of \( \tilde{H}_0^K \) such that (2.3.2) is valid for all \( f \in \mathbb{K} \) and \( h \in H^K_0 \). We keep the same notations \( \tilde{\sigma}_{(w, w')} \) and \( \tilde{\sigma} \) for these maps. Note that the properties of \( \tilde{\sigma}_{(w, w')} \) and \( \tilde{\sigma} \) as described in Lemma 2.3.1 also hold true as identities between endomorphisms of \( H^K_0 \).

**Theorem 2.3.3.** There is a unique group homomorphism

\[
\tau: \mathbb{W} \to \text{GL}_C(H^K_0)
\]

satisfying

\[
\tau(w, w')(f) = d_w(x)^{-1}d_{w'}(y)^{-1} \cdot \tilde{\sigma}_{(w, w')}(f),
\]

for \( w, w' \in W \) and \( f \in H^K_0 \). It satisfies \( \tau(w)(g \cdot f) = wg \cdot \tau(w)(f) \) for \( g \in \mathbb{K}, f \in H^K_0 \) and \( w \in \mathbb{W} \).

**Proof.** The last statement is clear.

The action \( \tau \) of \( W \times \{ e \} \) arises naturally from left multiplication by normalized affine intertwiners on a suitable localization of the double affine Hecke algebra (see Cherednik [10, §1.3]). In the present set-up, one observes that Lemma 2.3.2 and Lemma 2.3.1(i)-(ii) imply that the \( \tilde{\Lambda}_{N,i} \) braid relations and the quadratic relations \( \tau(s_i, e)^2 = \text{id}_{H^K_0} \). Since furthermore \( \tau(\pi, e) \) is a complex linear automorphism of \( H^K_0 \) with inverse \( \tau(\pi^{-1}, e) \), and \( \tau(\pi, e)\tau(s_i, e)\tau(\pi^{-1}, e) = \tau(s_{i+1}, e) \) for \( i \in \mathbb{Z}_N \) by Lemma 2.3.2 and Lemma 2.3.1(iii), we conclude that the formulas (2.3.3) for the maps \( \tau(s_i, e) \) (\( i \in \mathbb{Z}_N \)) and \( \tau(\pi, e) \) uniquely extend to a group homomorphism \( \tau: W \times \{ e \} \to \text{GL}_C(H^K_0) \). It follows from Proposition 2.2.9 and its proof that the resulting group homomorphism satisfies

\[
\tau(w, e)f = d_w(x)^{-1} \cdot \tilde{\sigma}_{(w, e)}f
\]

for \( w \in W \). This is in accordance with formula (2.3.3).

Combining Lemma 2.3.1(iv) with Lemma 2.3.2 we can relate the complex endomorphism \( \tau(e, w) \) (see (2.3.3)) of \( H^K_0 \) to \( \tau(w, e) \) by the formula

\[
\tau(e, w) = \tau(i)\tau(w, e)\tau(i), \quad w \in W;
\]

where \( \tau(i) \) is given by the second formula of (2.3.3). Since \( \tau(i)^2 = \tilde{\sigma}^2 = \text{id}_{H^K_0} \), we conclude that \( W \ni w \mapsto \tau(e, w) \) (see (2.3.3)) defines a left \( W \)-action on \( H^K_0 \).

By Lemma 2.3.1 and Lemma 2.3.2(v) we have

\[
\tau(w, e)\tau(e, w') = \tau(w, w') = \tau(e, w')\tau(w, e)
\]

for all \( w, w' \in W \). Thus \( \tau: W \times W \to \text{GL}_C(H^K_0) \), defined by the first formula of (2.3.3), is a group homomorphism. Combined with (2.3.4) and \( \tau(i)^2 = \text{id}_{H^K_0} \), we conclude that \( \tau \) (2.3.3) indeed defines a complex linear action of \( \mathbb{W} \) on \( H^K_0 \). \( \square \)
For \( w \in \mathcal{W} \) and \( f \in H^K_0 = K \otimes H_0 \), we write \( w f \) for the action of \( w \) on the \( K \)-coefficients of \( f \) in its expansion along a basis of \( H_0 \). In other words, viewing \( f(t, \gamma) \) as \( H_0 \)-valued meromorphic function in \((t, \gamma) \in T \times T\), the action is given by \((w f)(t, \gamma) = f(w^{-1}(t, \gamma))\). Consider \( \text{GL}_K(H^K_0) \) as a \( \mathcal{W} \)-group by the corresponding conjugation action
\[
(w, A) \mapsto w A w^{-1}, \quad w \in \mathcal{W}, \ A \in \text{GL}_K(H^K_0)
\]
by group automorphisms. We have the following direct consequence of the previous theorem.

**Corollary 2.3.4.** The map \( w \mapsto C_w := \tau(w)w^{-1} \) is a cocycle of \( \mathcal{W} \) with values in the \( \mathcal{W} \)-group \( \text{GL}_K(H^K_0) \). In other words, \( C_w \in \text{GL}_K(H^K_0) \) and
\[
C_{ww'} = C_w w C_{w'} w^{-1}
\]
for all \( w, w' \in \mathcal{W} \).

For more details on non-abelian group cohomology, see, e.g., the appendix in [53].

**Remark 2.3.5.** Interpreting \( A \in \text{End}_K(H^K_0) \) as \( \text{End}(H_0) \)-valued meromorphic function \( A(t, \gamma) \) in \((t, \gamma) \in T \times T\), the action \( (2.3.5) \) becomes \((wA)(t, \gamma) = A(w^{-1}(t, \gamma))\). In particular, \( w A w^{-1} = A \) for all \( w \in \mathcal{W} \) if \( A \in \text{End}_K(H^K_0) \) is the \( K \)-linear extension of a complex linear endomorphism of \( H_0 \). This is, for instance, the case for the cocycle value \( C \), (see Subsection 2.4.2).

**Remark 2.3.6.** One may replace in this subsection \( K \) by the field \( \mathbb{C}(T \times T) \) of rational functions on \( T \times T \). Consequently, the cocycle value \( C_w(t, \gamma) \) for \( w \in \mathcal{W} \) is a rational \( \text{End}(H_0) \)-valued function in \((t, \gamma) \in T \times T\). We presented the results with respect to \( K = M(T \times T) \) since this is the natural setting for the applications of the cocycle \( C \) in the analytic theory of the quantum KZ equations (to which we come at a later stage).

### 2.3.2 Bispectral quantum KZ equations

In this subsection, we use the cocycle \( C_w \in \text{GL}_K(H^K_0) \) \( (w \in \mathcal{W}) \) to define a holonomic system of \( q \)-difference equations on the space \( H^K_0 \) of \( H_0 \)-valued meromorphic functions on \( T \times T \).

The constructions thus far have led to a \( \mathbb{C} \)-linear action \( \tau \) of \( \mathcal{W} \) on \( H^K_0 \). In terms of the cocycle \( C_w \) \( (w \in \mathcal{W}) \), it is given by
\[
(\tau(w)f)(t, \gamma) = C_w(t, \gamma)f(w^{-1}(t, \gamma))
\]
for \( w \in \mathcal{W} \) and \( f \in H^K_0 \), where \( (2.3.6) \) should be read as identities between \( H_0 \)-valued meromorphic functions in \((t, \gamma) \in T \times T\). It follows that \( f \in H^K_0 \) is \( \tau(\mathbb{Z}^N \times \mathbb{Z}^N) \)-invariant if and only if
\[
C_{(\lambda, \mu)}(t, \gamma)f(q^{-\lambda t}, q^{\mu} \gamma) = f(t, \gamma) \quad \forall \lambda, \mu \in \mathbb{Z}^N,
\]
viewed as identities between \( H_0 \)-valued meromorphic functions on \( T \times T \).
§2.4. The explicit form of the bispectral quantum KZ equations

**Definition 2.3.7.** We call the $q$-difference equations (2.3.7) the bispectral quantum KZ (BqKZ) equations. We write SOL for the set of functions $f \in H_0^K$ satisfying the BqKZ equations (2.3.7).

Let $F \subset K$ denote the subfield consisting of $f \in K$ satisfying $(\lambda, \mu) f = f$ for all $\lambda, \mu \in \mathbb{Z}^N$. Let furthermore $S_N$ denote the subgroup $\mathbb{Z}_2 \times (S_N \times S_N)$ of $\mathbb{W}$.

**Corollary 2.3.8.** (i) The BqKZ equations (2.3.7) form a holonomic system of $q$-difference equations. In other words, the cocycle matrices $C_{(\lambda, \mu)} (\lambda, \mu \in \mathbb{Z}^N)$ satisfy the compatibility conditions

$$C_{(\lambda, \mu)}(t, \gamma)C_{(\nu, \xi)}(q^{-\lambda}t, q^\mu \gamma) = C_{(\nu, \xi)}(t, \gamma)C_{(\lambda, \mu)}(q^{-\nu}t, q^\xi \gamma) \quad (2.3.8)$$

for $\lambda, \mu, \nu, \xi \in \mathbb{Z}^N$, as $\text{End}(H_0)$-valued meromorphic functions in $(t, \gamma) \in T \times T$.

(ii) The solution space SOL of BqKZ is a $\tau(S_N)$-invariant $F$-subspace of $H_0^K$.

**Proof.** (i) By means of the cocycle condition, both sides of (2.3.8) can be seen to be equal to $C_{(\lambda + \nu, \mu + \xi)}(t, \gamma)$.

(ii) Clearly, SOL is an $F$-subspace of $H_0^K$. Note, furthermore, that $\mathbb{Z}^N \times \mathbb{Z}^N$ is a normal subgroup of $\mathbb{W}$ with quotient group isomorphic to $S_N$. Hence the $F$-subspace SOL of $\tau(\mathbb{Z}^N \times \mathbb{Z}^N)$-invariant elements in the $\tau(\mathbb{W})$-module $H_0^K$ is $\tau(S_N)$-invariant. \qed

### 2.4 The explicit form of the bispectral quantum KZ equations

In this section we derive explicit expressions for the cocycle values $C_w (w \in \mathbb{W})$ and, in particular, for the cocycle matrices $C_{(\lambda, \mu)} (\lambda, \mu \in \mathbb{Z}^N)$ of the BqKZ equations. It will become apparent that the $C_{(\lambda, \zeta)} (\lambda \in \mathbb{Z}^N)$ with $\zeta \in T$ fixed coincide with the cocycle matrices of Cherednik’s quantum affine KZ equation associated to the principal series module of $H(k)$ with central character $\zeta$. They also turn up as gauged cocycle matrices for a Frenkel-Reshetikhin [21] type quantum KZ equation associated to the quantum affine algebra $U_b(\hat{sl}_N)$.

#### 2.4.1 Generic principal series

View the commutative subalgebra $C_Y[T]$ of $H$ as left $C_Y[T]$-module by left multiplication. Let $M = \text{Ind}_{C_Y[T]}^H(C_Y[T])$ be the corresponding induced left $H$-module. With respect to the $C[T] \cong C[\{1\} \times T]$-module structure

$$f \cdot (h \otimes_{C_Y[T]} g(Y)) = h \otimes_{C_Y[T]} (fg)(Y) \quad f, g \in C[T], \ h \in H$$

on $M$ we have $M \cong H_0^{C[\{1\} \times T]} = C[\{1\} \times T] \otimes H_0$ as $C[\{1\} \times T]$-modules. The left $H$-action on $M$ is $C[\{1\} \times T]$-linear, hence we obtain an algebra homomorphism

$$\eta: H \to \text{End}_{C[\{1\} \times T]}(H_0^{C[\{1\} \times T]}).$$
We occasionally view $\eta(h)$ as $\text{End}(H_0)$-valued regular function in $\gamma \in T$, in which case we write it as $T \ni \gamma \mapsto \eta(h)(\gamma)$. Extending the ground ring $\mathbb{C}[\{1\} \times T]$ to $\mathbb{K} = \mathcal{M}(T \times T)$ we obtain an algebra homomorphism

$$H \to \text{End}_{\mathbb{K}}(H_0^n),$$

which we shall also denote by $\eta$. From this viewpoint, $\eta(h)(\gamma)$ is the regular $\text{End}(H_0)$-valued function in $(t, \gamma) \in T \times T$ which is constant in $t$. Note that $\eta(h)$ for $h \in H_0$ is constant as $\text{End}(H_0)$-valued function on $T \times T$ (see Remark 2.3.5).

**Lemma 2.4.1.** For $w \in S_N$ and $1 \leq i < N$ we have

$$\eta(T_i)T_w = \begin{cases} T_{s_iw} & \text{if } \ell(s_iw) = \ell(w) + 1, \\ (k - k^{-1})T_w + T_{s_iw} & \text{if } \ell(s_iw) = \ell(w) - 1, \end{cases} \tag{2.4.1}$$

and

$$\eta(\pi)(\gamma)T_w = \gamma w^{-1}(N)T_{\sigma w} \tag{2.4.2}$$

as regular $H_0$-valued functions in $\gamma \in T$.

**Proof.** The first formula follows directly from the definitions. For the second formula it suffices to verify that $\pi T_w = T_{\sigma w}Y_{w^{-1}(N)}$ in $H$.

If $w = e$ then $\pi = T_{\sigma w}Y_N$ since $\sigma = s_1s_2\cdots s_{N-1}$ is a reduced expression. If $w = s_{N-1}$ then

$$\pi T_w = \pi T_{N-1} = T_1\cdots T_{N-2}Y_{N-1} = T_{s_1\cdots s_{N-2}}Y_{w^{-1}(N)} = T_{\sigma w}Y_{w^{-1}(N)}.$$

Next, we prove that $\pi T_w = T_{\sigma w}Y_{w^{-1}(N)}$ in $H$ if $w \neq e$ and $\ell(s_iw) = \ell(w) + 1$ for all $1 \leq i < N - 2$. Then $w = s_{N-1}s_{N-2}\cdots s_j$ for some $1 \leq j < N$ (and this is a reduced expression of $w$). We find, making repetitive use of the cross relation $T_jY_{r+1}T_r = Y_r$ ($1 \leq r < N$) in $H$,

$$\pi T_w = \pi T_{N-1}T_{N-2}\cdots T_j = T_1\cdots T_{N-2}Y_{N-1}T_{N-2}\cdots T_j$$

$$= T_1\cdots T_{N-3}Y_{N-2}T_{N-3}\cdots T_j$$

$$\vdots$$

$$= T_1\cdots T_{j-1}Y_j = T_{s_1\cdots s_{j-1}}Y_j$$

$$= T_{\sigma w}Y_j = T_{\sigma w}Y_{w^{-1}(N)},$$

which is the desired relation in $H$.

The general case is now proved by induction on $\ell(w)$. Let $w \neq e$ and decompose it as $w = s_iu$ with $1 \leq i < N$ and $u \in S_N$ such that $\ell(s_iu) = \ell(u) + 1$. Suppose that $\pi T_u = T_{\sigma w}Y_{u^{-1}(N)}$ in $H$. In order to prove that $\pi T_w = T_{\sigma w}Y_{w^{-1}(N)}$ we may, in view of the previous paragraph, assume without loss of generality that $1 \leq i \leq N - 2$. Then $s_i(N) = N$ and $\ell(s_{i+1}\sigma u) = \ell(\sigma u) + 1$. (The latter equality is equivalent to
\[(\sigma u)^{-1}(i + 1) < (\sigma u)^{-1}(i + 2),\] which follows from \(u^{-1}(i) < u^{-1}(i + 1),\) which again is equivalent to the assumption \(\ell(s_t u) = \ell(u) + 1.\) Then
\[
\pi T_w = \pi T_i T_u = T_{i+1} \pi T_u = T_{i+1} T_{\sigma u} Y_{u^{-1}(N)}
= T_{s_{i+1} \sigma u} Y_{u^{-1}(s_i(N))} = T_{\sigma u} Y_{w^{-1}(N)}
= T_{\sigma w} Y_{w^{-1}(N)}
\]
in \(H,\) which completes the proof. \(\square\)

In view of the explicit expression (2.2.8) for the intertwiner \(\tilde{S}_i \in \mathbb{H} \ (1 \leq i < N)\) and the definition of the duality anti-involution, we have \(\tilde{S}_w^* \in H\) for all \(w \in S_N.\) We now set
\[
\xi_w := \eta(\tilde{S}_{w^{-1}}) T_e \in H_0^K, \quad w \in S_N.
\]
Note that \(\xi_w \in H_0^{C(1)^* T} \subset H_0^K\) for \(w \in S_N.\) We view \(\xi_w\) as regular \(H_0\)-valued function in \(\gamma \in T,\) as well as meromorphic \(H_0\)-valued function in \((t, \gamma) \in T \times T\) constant in \(t \in T.\)

**Lemma 2.4.2.** \(\{\xi_w\}_{w \in S_N}\) is a \(K\)-basis of \(H_0^K\) consisting of common eigenfunctions for the \(\eta\)-action of \(C_Y[T]\) on \(H_0^K.\) For \(p \in \mathbb{C}[T]\) and \(w \in S_N\) we have
\[
\eta(p(Y))(\gamma)\xi_w(\gamma) = (w^{-1} p)(\gamma)\xi_w(\gamma) \quad (2.4.3)
\]
as \(H_0\)-valued regular functions in \(\gamma \in T.\)

**Proof.** For \(p \in \mathbb{C}[T]\) we have \(\eta(p(Y))(\gamma)T_e = p(\gamma)T_e.\) Furthermore, observe that \(p(Y)\tilde{S}_{w^{-1}} = \tilde{S}_{w^{-1}}(w^{-1} p)(Y)\) in \(H\) for \(p \in \mathbb{C}[T]\) and \(w \in S_N;\) see the proof of Lemma 2.3.2. Combining the two observations gives (2.4.3). It follows from Proposition 2.2.9(iv)-(v) that the \(\xi_w (w \in S_N)\) are nonzero in \(H_0^K.\) The eigenvalue equations (2.4.3) then show that the \(\xi_w (w \in S_N)\) are \(K\)-linearly independent in \(H_0^K.\) \(\square\)

### 2.4.2 The cocycle values

We define
\[
R_i(z) = c_k(z)^{-1}(\eta(T_i) - k) + \text{id}, \quad 1 \leq i < N,
\]
viewed as a rational \(\text{End}(H_0)\)-valued function in \(z.\) The results of the previous subsection imply that the \(R_i(z)\) satisfy the following Yang-Baxter type equations (see Cherednik [10, §1.3.2]).

**Lemma 2.4.3.** We have
\[
C_{(s_i, s_i)}(t, \gamma) = R_i(t_i/t_{i+1}), \quad 1 \leq i < N,
\]
as rational \(\text{End}(H_0)\)-valued functions in \((t, \gamma) \in T \times T.\) In particular, the \(R_i(z)\) satisfy
\[
R_i(z) R_i(z^{-1}) = \text{id},
R_i(z) R_{i+1}(z z') R_i(z') = R_{i+1}(z') R_i(z z') R_{i+1}(z), \quad (2.4.4)
\]
for \(1 \leq i < N\) and \(1 \leq j < N - 1\) as \(\text{End}(H_0)\)-valued rational functions.
Proof. For $1 \leq i < N$ and $h \in H_0$ we have, as $H_0$-valued meromorphic functions in $(t, \gamma) \in T \times T$,

$$C(s_i, e)(t, \gamma)h = (\tau(s_i, e)h)(t, \gamma)$$

$$= d_w(t)^{-1}(\tilde{S}_i h)(t, \gamma)$$

$$= e_k(t/t_{i+1})^{-1}(\eta(T_i) - k)h + h,$$

in view of the explicit expression (2.2.8) for $\tilde{S}_i$, $c_k$ (2.2.5) and $d_w$ (see the proof of Lemma 2.2.9). For the second statement of the lemma, note that the cocycle property of $C$ implies for $1 \leq i < N$ and $1 \leq j < N - 1$ that

$$C(s_i, e)(t, \gamma)C(s_j, e)(s_it, \gamma) = \text{id},$$

$$C(s_i, e)(t, \gamma)C(s_{j+1}, e)(s_j, t, \gamma)C(s_{j+1}, e)(s_j s_j t, \gamma)$$

$$= C(s_{j+1}, e)(t, \gamma)C(s_j, e)(s_j + 1, t, \gamma)C(s_{j+1}, e)(s_j s_j + 1, t, \gamma),$$

as rational $\text{End}(H_0)$-valued functions in $(t, \gamma) \in T \times T$. Then using that $C(s_i, e)(t, \gamma) = R_i(t_i/t_{i+1})$, these formulas imply (2.4.4).

Observe that $C_i$ is the $\mathbb{K}$-linear extension of the anti-algebra involution of $H_0$ mapping $T_w$ to $T_{w^{-1}}$ for all $w \in S_N$. Note furthermore that

$$C(\pi, e) = \eta(\pi).$$

Together with the explicit description of $C(s_i, e)$ ($1 \leq i < N$) from the previous lemma, these formulas determine the values $C_w$ ($w \in W$) uniquely (cf. Corollary 2.3.4). In particular, the cocycle property of $C$ implies that

$$C(\pi, e)(t, \gamma) = C_i C(\pi, e)(\gamma^{-1}, t^{-1})C_i, \quad w \in W,$$

as $\text{End}(H_0)$-valued rational functions in $(t, \gamma) \in T \times T$.

Lemma 2.4.4. Let $w \in W$.

(i) $C(w, e) \in (\mathbb{C}(T) \otimes \mathbb{C}[T]) \otimes \text{End}(H_0)$.

(ii) The $\mathbb{C}[T] \otimes \text{End}(H_0)$-valued rational function $t \mapsto C(w, e)(t, \cdot)$ in $t \in T$ is regular at $t \in T \setminus S$, where

$$S = \{t \in T \mid t^\alpha \in k^{-2}\mathbb{Q}^2 \text{ for some } \alpha \in R\}.$$

Proof. By the cocycle condition, $C(w, e)(t, \gamma)$ can be written as a product of factors $C(s_i, e)(ut, \gamma)$ ($1 \leq i < N$, $u \in W$) and $\eta(\pi^{\pm 1})(\gamma)$. By Lemma 2.4.1 and the fact that $R_i(z)$ has a single pole at $z = k^{-2}$, we conclude (i) and (ii).

2.4.3 The cocycle matrices

Besides the standard $\mathbb{Z}$-basis $\{e_i\}_{i=1}^N$ of $\mathbb{Z}^N$, we also have the $\mathbb{Z}$-basis $\{\varpi_i\}_{i=1}^N$ consisting of the fundamental weights $\varpi_i := \sum_{j=1}^i e_j \in \mathbb{Z}^N$. Note that

$$\varpi_i = \pi^i \sigma^{-i}$$

(2.4.7)
in $W$ for all $1 \leq i \leq N$. In the following lemma, we compute the cocycle matrices $C_{(\lambda, \sigma)}$ for $\lambda \in \mathbb{Z}^N$ of BqKZ explicitly in case $\lambda$ is one of these two types of basis elements of $\mathbb{Z}^N$.

**Lemma 2.4.5.** (i) For $1 \leq j \leq N$ we have

$$C_{(\epsilon_j, \sigma)}(t, \gamma) = R_{j-1}(t_{j-1}/t_j)R_{j-2}(t_{j-2}/t_j) \cdots R_1(t_1/t_j) \times \eta(\pi)(\gamma) R_{N-1}(qt_N/t_j) \cdots R_{j+1}(qt_{j+2}/t_j) R_j(qt_{j+1}/t_j)$$

as rational $\text{End}(H_0)$-valued functions in $(t, \gamma) \in T \times T$.

(ii) For $1 \leq i < N$ we have

$$C_{(\pi, \sigma)}(t, \gamma) = \eta(\pi)(\gamma)^i (R_{N-i}(qt_N/t_1) \cdots R_2(qt_2/t_1) R_1(qt_{i+1}/t_1)) \times \cdots \times (R_{N-2}(qt_N/t_1) \cdots R_i(qt_{i+2}/t_1) R_{i+1}(qt_{i+1}/t_1))$$

as rational $\text{End}(H_0)$-valued functions in $(t, \gamma) \in T \times T$.

(iii) We have

$$C_{(\pi, \sigma)}(t, \gamma) = \gamma^{\pi s} \text{id}$$

as rational $\text{End}(H_0)$-valued functions in $(t, \gamma) \in T \times T$.

**Proof.** (i) By the cocycle property of $C$ and by the expression (2.2.2) for $\epsilon_j \in W$, we obtain an explicit expression for $C_{(\epsilon_j, \sigma)}$ in terms of the $C_{(\pi, \sigma)}$ ($1 \leq i < N$) and $C_{(\lambda, \sigma)}$. Combining (2.4.5) and the previous lemma then gives the desired expression for $C_{(\epsilon_j, \sigma)}(t, \gamma)$.

(ii) $\sigma^i$ is the permutation

$$
\begin{pmatrix}
1 & 2 & \cdots & N-i & N-i+1 & N-i+2 & \cdots & N \\
1 & 2 & \cdots & N-i & N-i+1 & N-i+2 & \cdots & i
\end{pmatrix},
$$

so we find a reduced expression

$$\sigma^i = (s_1 \cdots s_{N-1})(s_{i-1} \cdots s_{N-2}) \cdots (s_2 \cdots s_{N-i+1})(s_1 \cdots s_{N-i}). \tag{2.4.8}$$

Combined with (2.4.7) we get a reduced expression for $\sigma_i$. Using the cocycle condition for $C_w$ repeatedly, we obtain the desired result.

(iii) Since $\sigma^N = 1$, we get $C_{(\pi, \sigma)}(t, \gamma) = (\eta(\pi)(\gamma))^N$, which maps $T_w$ to $\gamma^{\pi N} T_w$ for all $w \in S_N$ in view of Lemma 2.4.1.

We end this subsection by computing the asymptotic leading terms of the cocycle matrices $C_{(\lambda, \sigma)}(t, \gamma)$ ($\lambda \in \mathbb{Z}^N$) as $|t^{-\alpha_i}| \to 0 (1 \leq i < N)$, where we take $\alpha_i := \epsilon_i - \epsilon_{i+1}$ ($1 \leq i < N$) as a base of the root system $R = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq N}$ of type $A_{N-1}$. Let $R_+ = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq N}$ denote the associated set of positive roots and $Q_+ = \bigoplus_{i=1}^{N-1} \mathbb{Z}_{\geq 0} \alpha_i$, the corresponding cone in the root lattice $Q$ of $R$. Let furthermore $\delta := (N-1, N-3, \ldots, 1 - N) \in \mathbb{Z}^N$ and write $w_0 \in S_N$ for the longest Weyl group element (mapping $i$ to $N - i + 1$ for $1 \leq i \leq N$).
Consider the subring \( \mathcal{A} := \mathbb{C}[x^{-\alpha_1}, \ldots, x^{-\alpha_N}] \) of \( \mathbb{C}[T \times \{1\}] = \mathbb{C}[x_T^{\pm 1}, \ldots, x_T^{\pm 1}] \subset \mathbb{C}[T \times T] \). We write \( Q(\mathcal{A}) \) for its quotient field and \( Q_0(\mathcal{A}) \) for the subring of \( Q(\mathcal{A}) \) consisting of rational functions which are analytic at the point \( x^{-\alpha_i} = 0 \) (1 \( \leq i < N \)). We consider \( Q_0(\mathcal{A}) \otimes \mathbb{C}[T] \) as a subring of \( \mathbb{C}(T \times T) \) in the natural way. The first part of the following corollary is a refinement of Lemma 2.4.4(i) in case \( w \in \mathbb{Z}^N \).

**Corollary 2.4.6.** Let \( \lambda \in \mathbb{Z}^N \). We have

\[
C_{(\lambda,e)}(t,\gamma) = (Q_0(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \text{End}(H_0).
\]

Writing

\[
C^{(0)}_{(\lambda,e)} = C_{(\lambda,e)}|_{x^{-\alpha_1}=\ldots=x^{-\alpha_N-1}=0} \in \mathbb{C}[T] \otimes \text{End}(H_0),
\]

we have \( C^{(0)}_{(\lambda,e)} = k^{(\delta,\lambda)}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1}) \), where \( \langle , \rangle \) is the standard scalar product on \( \mathbb{R}^N \).

**Proof.** To prove (2.4.9) it suffices, in view of the cocycle property of \( C \), to verify (2.4.9) for \( \lambda = e_i \). The statement then follows from Lemma 2.4.5(i), Lemma 2.4.1 and the explicit expression of \( R \). Observe that \( \lim_{z \to 0} R_i(z) = k\eta(T_i^{-1}) \) for \( 1 \leq i < N \). Combined with Lemma 2.4.5(i) and the explicit expression for \( Y_j \) (see (2.2.4)) we obtain for \( 1 \leq j \leq N \),

\[
C_{(e_j,e)}(t,\gamma) \to k^{2j-N-1}\eta(Y_j)(\gamma)
\]
as \( |t^{\alpha_i}| \to 0 \) for all \( 1 \leq i < N \), hence

\[
C_{(\lambda,e)}(t,\gamma) \to k^{-(\delta,\lambda)}\eta(Y^{\lambda})(\gamma)
\]
as \( |t^{\alpha_i}| \to 0 \) for all \( 1 \leq i < N \). To derive the asymptotics of \( C_{(\lambda,e)}(t,\gamma) \) as \( |t^{\alpha_i}| \to 0 \) for \( 1 \leq i < N \) we use the cocycle property to write

\[
C_{(\lambda,e)}(t,\gamma) = C_{(w_0,e)}(t,\gamma)C_{(w_0(\lambda),e)}(w_0t,\gamma)C_{(w_0,e)}(q^{-w_0(\lambda)}w_0t,\gamma).
\]

Note that \( C_{(w_0,e)}(t,\gamma) \to k^{(w_0)}\eta(T_{w_0}^{-1}) \) if \( |t^{\alpha_i}| \to 0 \) for all \( 1 \leq i < N \). Hence

\[
C^{(0)}_{(\lambda,e)} = k^{(\delta,\lambda)}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1})
\]
as desired. \( \square \)

### 2.4.4 Relation to quantum KZ equations

Fix \( \zeta \in T \) and let \( \chi_{\zeta} : \mathbb{C}[\{1\} \times T] \to \mathbb{C} \) denote the corresponding evaluation character \( \chi_{\zeta}(f) = f(\zeta) \). Recall the generic principal series \( \eta : H \to \text{End}_{\mathbb{C}(\{1\} \times T)}(M) \). The corresponding complex \( H \)-representation \( M_\zeta = \mathbb{C} \otimes_{\chi_{\zeta}} M \) of dimension \( N! \) is the principal series module of \( H \) with central character \( \zeta \). We identify \( M_\zeta \) with \( H_0 \) as a
complex vector space, and push the $H$-action on $M_\zeta$ through the linear isomorphism to $H_0$. We denote the corresponding representation map by

$$\eta_\zeta : H \to \text{End}(H_0).$$

As in Subsection 2.4.1 we have as identities in $H_0$,

$$\eta_\zeta(T_i)T_w = \begin{cases} T_{s_iw} & \text{if } \ell(s_iw) = \ell(w) + 1, \\ (k - k^{-1})T_w + T_{s_iw} & \text{if } \ell(s_iw) = \ell(w) - 1, \end{cases}$$

for $1 \leq i < N$ and $w \in S_N$,

$$\eta_\zeta(\pi)T_w = \zeta w^{-1}(N)T_{\sigma w},$$

for $w \in S_N$, as well as

$$\eta_\zeta(f(Y))\xi_w(\zeta) = (w^{-1}f)(\zeta)\xi_w(\zeta),$$

for $w \in S_N$, where $\xi_w(\zeta) \in H_0$ is the regular $H_0$-valued function $\xi_w(\gamma)$ in $\gamma \in T$ specialized at $\gamma = \zeta$. Extending the base field to $M(T)$ we get an algebra homomorphism $H \to \text{End}_{M(T)}(H_0^{M(T)})$, which is also denoted by $\eta_\zeta$.

In this subsection we consider the BqKZ for specialized values of $\gamma$. In view of Lemma 2.4.4(i) we may specialize $C_{w,e}(t,\gamma)$ at $\gamma = \zeta$. We write

$$C_\zeta^{w}(t) := C_{w,e}(t,\zeta), \quad w \in W,$$

for the resulting specialized cocycle values, viewed as $\text{End}(H_0)$-valued rational functions in $t \in T$. For $\zeta \in T$ the map $W \ni w \mapsto C_\zeta^{w}$ defines a cocycle of $W$ with values in the $W$-group $\text{GL}_C(T)(H_0^{C(T)})$. In other words,

$$C_\zeta^{w}w'(t) = C_\zeta^{w}(t)C_\zeta^{w'}(w^{-1}t), \quad w, w' \in W,$$

as rational $H_0$-valued functions in $t \in T$. Comparing the cocycle values $C_\zeta^{\lambda}$ ($\lambda \in \mathbb{Z}^N$) to the ones in [10, §1.3] we obtain the following result.

**Corollary 2.4.7.** Fix $\zeta \in T$. The holonomic system of $q$-difference equations

$$C_\zeta^{\lambda}(t)f(q^{-\lambda}t) = f(t), \quad \forall \lambda \in \mathbb{Z}^N \quad (2.4.10)$$

for $f \in H_0^{M(T)}$ is Cherednik’s quantum affine KZ equation associated to the principal $H$-module $M_\zeta$ with central character $\zeta$.

We occasionally write $qKZ_\zeta$ for the quantum KZ equations (2.4.10). Let $\text{SOL}_\zeta \subset H_0^{M(T)}$ denote the set of solutions of (2.4.10). Write $\mathcal{E}(T) \subset M(T)$ for the subfield
of meromorphic functions $f$ satisfying $f(q^\lambda t) = f(t)$ for all $\lambda \in \mathbb{Z}^N$ as meromorphic functions in $t \in T$. The set $\text{SOL}_{\zeta}$ of solutions is a $\mathcal{E}(T)$-subspace of $H_0^M(T)$. Furthermore, $\text{SOL}_{\zeta}$ is invariant for the $S_N$-action

$$(\zeta(w)f)(t) := C_{\zeta}^w(t)f(w^{-1}t), \quad w \in S_N$$

(2.4.11)
on $H_0^M(T)$ (note that $\zeta$ does not depend on $\zeta$ since $C_{\zeta}^w(t) = C_{(w,e)}(t, \zeta)$ is independent of $\zeta$ for $w \in S_N$).

**Remark 2.4.8.** The quantum KZ equations (2.4.10) are gauge equivalent to Frenkel and Reshetikhin’s [21] quantum KZ equations associated with the $N$-fold tensor product representation $C^N(t_1) \otimes \cdots \otimes C^N(t_N)$ of the quantum affine algebra $U_q(\mathfrak{sl}_N)$, where $C^N(t_i)$ is the evaluation representation of the vector representation $C^N$ of $U_q(\mathfrak{sl}_N)$ (see [49, §1.3.2] and [15] for the details).

In view of Corollary 2.4.7 the BqKZ equations (2.3.7) are a holonomic extension of the quantum KZ equations (2.4.10) by $q$-difference equations in the central character $\zeta$ of $M_\zeta$. These may be thought of as analogs of isomonodromy transformations; in fact, in view of Lemma 2.4.2 and Corollary 2.4.6 the $q$-difference equations in $\zeta$ (which are essentially the quantum KZ equations again!) are reminiscent of Schlesinger transformations. This should be compared with the quantum isomonodromic interpretation of (rational) KZ equations as quantizations of Schlesinger equations, see [49] and [23].

From a different perspective we may think of the cocycle values $C_{(e,w)}(w \in W)$ as shift operators, in the sense that they map solutions of quantum KZ equations to solutions of quantum KZ equations with respect to shifted central characters. To formulate the precise result, we view in the following proposition $\gamma \mapsto C_{(e,w)}(\cdot, \gamma)$ as $\mathbb{C}[T] \otimes \text{End}(H_0)$-valued rational function in $\gamma \in T$.

**Proposition 2.4.9.** Let $w \in W$ and $\zeta \in T$ such that $\gamma \mapsto C_{(e,w)}(\cdot, \gamma)$ is regular at $\gamma = \zeta$. Then $f \mapsto C_{(e,w)}(\cdot, \zeta)f$ defines an $S_N$-equivariant linear map $\text{SOL}_{w\cdot-\zeta} \rightarrow \text{SOL}_{\zeta}$.

**Proof.** By the cocycle property we have for $f \in \text{SOL}_{w\cdot-\zeta}$ and $\lambda \in \mathbb{Z}^N$,

$$C_{\zeta}(t)(C_{(e,w)}(q^{-\lambda}t, \zeta)f(q^{-\lambda}t)) = C_{(\lambda,w)}(t, \zeta)f(q^{-\lambda}t)$$

$$= C_{(e,w)}(t, \zeta)C_{\lambda}^{w^{-1} \zeta}(t)f(q^{-\lambda}t)$$

$$= C_{(e,w)}(t, \zeta)f(t).$$

Hence $C_{(e,w)}(\cdot, \zeta)f \in \text{SOL}_{\zeta}$. The $S_N$-equivariance of the map is again a consequence of the cocycle property of $C_w$ ($w \in W$); indeed, for $v \in S_N$ and $f \in \text{SOL}_{w\cdot-\zeta}$ we have

$$C_{\zeta}(t)(C_{(e,w)}(v^{-1}t, \zeta)f(v^{-1}t)) = C_{(v,w)}(t, \zeta)f(v^{-1}t)$$

$$= C_{(e,w)}(t, \zeta)(C_{v}^{w^{-1} \zeta}(t)f(v^{-1}t)),$$

which is the desired result. □
From the quantum group perspective (see Remark 2.4.8), Proposition 2.4.9 resembles the action of the dynamical Weyl group on solutions of quantum KZ equations from [19]. We expect that the second half of the BqKZ is closely related to the Varchenko-Etingof dynamical difference equations [19, §9]; see also [20], [60], [57], [59], [58] and [35] for detailed studies of various degenerate cases. An interesting aspect, e.g., in [60] and [59], is the observation that KZ equations are dual to the associated dynamical equations using \((\mathfrak{g}_l, \mathfrak{g}_s)\) duality (our set-up relates to \(r = s = N\)). In the present theory this duality is incorporated by the cocycle value \(C_\iota\), which relates the cocycle matrices \(C_{(\lambda, e)}(\lambda \in \mathbb{Z}^N)\) of the quantum KZ equation to the dual cocycle matrices \(C_{(e, \lambda)}\) by conjugation,

\[
C_{(e, \lambda)}(t, \gamma) = C_\iota C_{(\lambda, e)}(\gamma^{-1}, t^{-1})C_\iota
\]
as \(\text{End}(H_0)\)-valued meromorphic functions in \((t, \gamma) \in T \times T\). In turn, \(C_\iota\) is a direct reflection of (the existence of) Cherednik’s duality anti-isomorphism of the double affine Hecke algebra (see Theorem 2.2.8).

## 2.5 Solutions of the bispectral quantum KZ equations

In this section we use asymptotic analysis to construct a \(\iota\)-invariant solution \(\Phi_\kappa\) of BqKZ, which we call the basic asymptotically free solution. It depends in a mild way on an auxiliary parameter \(\kappa \in \mathbb{C}^\times\) (in fact, \(\mathbb{F}\Phi_\kappa\) is independent of \(\kappa\)). The orbit of \(\Phi_\kappa\) under the action of \(\{e\} \times S_N \subset S_N\) turns out to be an \(\mathbb{F}\)-basis of SOL consisting of asymptotically free solutions. Along the way we derive various additional properties of \(\Phi_\kappa\).

### 2.5.1 The leading term

Let \(\theta \in \mathcal{M}(T)\) denote the renormalized Jacobi theta function

\[
\theta(z) := \prod_{m \geq 0} (1 - q^m z)(1 - q^{m+1}/z)
\]
for \(z \in \mathbb{C}^\times\). It satisfies

\[
\theta(q^m z) = (-z)^{-m} q^{-\frac{1}{2}m(m-1)} \theta(z), \quad m \in \mathbb{Z}.
\]

For \(\kappa \in \mathbb{C}^\times\) we define \(W_\kappa \in \mathbb{K}\) by

\[
W_\kappa(t, \gamma) := \prod_{i=1}^N \frac{\theta(\kappa t_i \gamma^{-1}_{N-i+1})}{\theta(\kappa k^{(i, \epsilon_i)} t_i) \theta(\kappa k^{-(i, \epsilon_i)} \gamma^{-1}_{N-i+1})}.
\]

By Corollary 2.4.6, the formal asymptotic form of the quantum KZ equations

\[
C_{(\lambda, e)}(t, \gamma) f(q^{-\lambda t}, \gamma) = f(t, \gamma), \quad \lambda \in \mathbb{Z}^N
\]
in the asymptotic region $|t^\alpha|\gg 0$ ($1 \leq i < N$) is

$$k^{(\delta,\lambda)} (T_{w_0} Y^{w_0(\lambda)} T_{w_0}^{-1})(\gamma) f(q^{-\lambda t}, \gamma) = f(t, \gamma), \quad \lambda \in \mathbb{Z}^N. \quad (2.5.4)$$

**Lemma 2.5.1.** $W_\kappa \in \mathbb{K}$ enjoys the following properties.

(i) $f^{(0)}(t, \gamma) := W_\kappa(t, \gamma) T_{w_0}$ is a solution of (2.5.4).

(ii) Clearly $\iota(W_\kappa) = W_\kappa$, i.e. $W_\kappa(\gamma^{-1}, t^{-1}) = W_\kappa(t, \gamma)$. Since $C_i(T_{w_0}) = T_{w_0}$, it follows that $\iota(f^{(0)}) = f^{(0)}$.

Proof. (i) Since $\eta(T_{w_0} Y^{w_0(\lambda)} T_{w_0}^{-1})(\gamma) T_{w_0} = \gamma^{w_0(\lambda)} T_{w_0}$ for all $\lambda \in \mathbb{Z}^N$, it suffices to show that

$$W_\kappa(q^{-\lambda t}, \gamma) = k^{-(\delta,\lambda)} \gamma^{-w_0(\lambda)} W_\kappa(t, \gamma), \quad \lambda \in \mathbb{Z}^N,$$

which follows from (2.5.2).

(ii) Clearly $\iota(W_\kappa) = W_\kappa$, i.e. $W_\kappa(\gamma^{-1}, t^{-1}) = W_\kappa(t, \gamma)$. Since $C_i(T_{w_0}) = T_{w_0}$, it follows that $\iota(f^{(0)}) = f^{(0)}$.

Observe that, more generally, $W_\kappa \in \mathbb{K}$ satisfies the $q$-difference equations

$$W_\kappa(q^{-\lambda t}, q^{\mu} \gamma) = k^{-(\delta,\lambda+\mu)} t^{w_0(\mu)} \gamma^{-w_0(\lambda)} q^{-(w_0(\lambda), \mu)} W_\kappa(t, \gamma), \quad \lambda,\mu \in \mathbb{Z}^N. \quad (2.5.5)$$

**2.5.2 The basic asymptotically free solution $\Phi_\kappa$**

We now gauge BqKZ by $W_\kappa \in \mathbb{K}$. Concretely, for $\lambda,\mu \in \mathbb{Z}^N$ we write

$$D_{(\lambda,\mu)}(t, \gamma) = W_\kappa(t, \gamma)^{-1} C_{(\lambda,\mu)}(t, \gamma) W_\kappa(q^{-\lambda t}, q^{\mu} \gamma)$$

as $\text{End}(H_0)$-valued meromorphic functions in $(t, \gamma) \in T \times T$. It is independent of $\kappa$ in view of (2.5.5). For $f \in H_0^{\mathbb{K}}$ we have $f \in \text{SOL}$ if and only if $g := W_0^{-1} f \in H_0^{\mathbb{K}}$ satisfies the holonomic system of $q$-difference equations

$$D_{(\lambda,\mu)}(t, \gamma) g(q^{-\lambda t}, q^{\mu} \gamma) = g(t, \gamma), \quad \lambda,\mu \in \mathbb{Z}^N \quad (2.5.6)$$

as $H_0$-valued rational functions in $(t, \gamma) \in T \times T$.

The existence of a solution $\Psi \in H_0^{\mathbb{K}}$ of (2.5.6) admitting a convergent $H_0$-valued power series expansion

$$\Psi(t, \gamma) = \sum_{\alpha,\beta \in Q_+} K_{\alpha,\beta} t^{-\alpha} \gamma^\beta, \quad K_{0,0} = T_{w_0} \quad (2.5.7)$$

in the asymptotic region $|t^\alpha|\gg 0$ and $|\gamma^{-\alpha}|\gg 0$ ($1 \leq i < N$) is guaranteed by the following properties of the gauged cocycle matrices $D_{(\lambda,\mu)}$.

Consider the subring $B := \mathbb{C}[y^\alpha, \ldots, y^{N-1}]$ of $\mathbb{C}[\{1\} \times T] = \mathbb{C}[y_1^\pm, \ldots, y_N^\pm]$. Write $Q(B)$ for its quotient field and $Q_0(B)$ for the subring of $Q(B)$ consisting of rational functions which are analytic at the point $y^{\alpha_j} = 0$ ($1 \leq j < N$). We consider $Q_0(A) \otimes B$ and $A \otimes Q_0(B)$ as subrings of $\mathbb{C}(T \times T)$ in the natural way.
Lemma 2.5.2. Set $A_i = D_{(\varpi, e)}$ and $B_i = D_{(e, \varpi_i)}$ for $1 \leq i \leq N$.

(i) $A_N = B_N = \text{id on } H^N_0$.

(ii) $A_i \in (Q_0(A) \otimes B) \otimes \text{End}(H_0)$ and $B_j \in (A \otimes Q_0(B)) \otimes \text{End}(H_0)$.

(iii) Set $A_i^{(0,0)} \in \text{End}(H_0)$ and $B_j^{(0,0)} \in \text{End}(H_0)$ for the value of $A_i$ and $B_j$ at $x^{-\alpha_r} = 0 = y^{\sigma_s}$ ($1 \leq r, s < N$). For $w \in S_N$ we have

$$A_i^{(0,0)}(t, \gamma) = \begin{cases} 0 & \text{if } w^{-1}w_0(\varpi_i) \neq w_0(\varpi_i), \\ T_{w_0}T_w & \text{if } w^{-1}w_0(\varpi_i) = w_0(\varpi_i) \end{cases}$$

and

$$B_j^{(0,0)}(t, \gamma) = \begin{cases} 0 & \text{if } w(\varpi_i) \neq \varpi_i, \\ T_{w_0}T_w & \text{if } w(\varpi_i) = \varpi_i. \end{cases}$$

Proof. (i) We only give the proof of $A_N = \text{id}$. Since $\varpi_N = \pi^N$ in $W$, we have

$$A_N(t, \gamma) = W_{\pi}(t, \gamma)^{-1}C_{(\varpi, e)}(t, \gamma)W_{\pi}(q^{-\pi N}t, \gamma) = \gamma^{-\varpi_N}(\eta(\pi)(\gamma))^N = \text{id},$$

where we use (2.5.5) and (2.4.5) for the second equality, and Lemma 2.4.1 and $\sigma^N = e$ for the third equality.

(ii) Note that $A_i(t, \gamma) = k^{-(\delta, \varpi_i)}\gamma^{-w_0(\varpi_i)}C_{(\varpi, e)}(t, \gamma)$ by (2.5.5). Since $\varpi_i = \pi^i \sigma^{-i}$ the cocycle property of $C$ gives

$$A_i(t, \gamma) = k^{-(\delta, \varpi_i)}\gamma^{-w_0(\varpi_i)}(\eta(\pi)(\gamma)^i)C_{(\sigma^{-i}, e)}(\pi^{-i}t, \gamma).$$

It follows from the explicit expressions for the cocycle values $C_{(\sigma, e)}(1 \leq i < N)$ that the $\text{End}(H_0)$-valued rational function $C_{(\sigma^{-1}, e)}(\pi^{-i}t, \gamma)$ in $(t, \gamma) \in T \times T$ lies in $Q_0(A) \otimes \text{End}(H_0)$ (in particular, it is independent of $\gamma$). Furthermore, for $w \in S_N$

$$\gamma^{-w_0(\varpi_i)}(\eta(\pi)(\gamma)^i)(T_w) = \gamma^{-w_0(\varpi_i)}(T_{\sigma^i}) = \gamma^{-w_0(\varpi_i)}(T_{\sigma^i w})$$

by Lemma 2.4.1, hence the $\text{End}(H_0)$-valued regular function $\gamma^{-w_0(\varpi_i)}(\eta(\pi)(\gamma)^i)$ in $\gamma \in T$ lies in $B \otimes \text{End}(H_0)$. Consequently, $A_i \in (Q_0(A) \otimes B) \otimes \text{End}(H_0)$. The statement for $B_j$ follows from this result using the cocycle property $C_{(e, \varpi_j)}(t, \gamma) = C_iC_{(e, \varpi_j)}(\gamma^{-1} \sigma^{-1})C_i$.

(iii) Recall that $\xi_w = \eta(S^*_w)T_w$ with $S_w$ the intertwiners of $\mathbb{H}$ (see Proposition 2.2.9). By induction on $\ell(w)$, using the explicit expression (2.2.8) of the intertwiners $S_w$, it follows that $\xi_w \in B \otimes H_0$ and that the value of $\xi_w$ at $y^{\alpha_1} = 0 (1 \leq i < N)$ is $T_w \in H_0$. Set

$$A_i^{(0)} = A_i|_{x^{-\alpha_1}=0, \ldots, x^{-\alpha_N-1}=0} \in B \otimes \text{End}(H_0).$$
By Corollary 2.4.6 and (2.5.5),
\[ A^{(0)}_i = y^{-w_0(\varpi_i)}\eta(T_{w_0} Y^{w_0(\varpi_i)} T_{w_0}^{-1}). \]

Lemma 2.4.2 then gives
\[ A^{(0)}_i(\eta(T_{w_0})\xi_w) = y^{w^{-1}w_0(\varpi_i)-w_0(\varpi_i)}\eta(T_{w_0})\xi_w, \quad \forall w \in S_N \]  
(2.5.10)
as identities in \( B \otimes H_0 \). Specializing (2.5.10) at \( y^{\alpha_j} = 0 \) (1 \( \leq j < N \)) yields (2.5.8).

To prove (2.5.9) we consider
\[ \widetilde{B}^{(0)}_j = B_j|_{\rho^0=0,\ldots,\rho^N-1=0} \in A \otimes \text{End}(H_0). \]

It is the rational \( \text{End}(H_0) \)-valued function
\[ \widetilde{B}^{(0)}_j(t) = t^{w_0(\varpi_j)}C_i(\eta(T_{w_0} Y^{w_0(\varpi_j)} T_{w_0}^{-1})(t^{-1}))C_i \]
in \( t \in T \). Denoting \( \widetilde{\xi}_w \in A \otimes H_0 \) for the rational \( H_0 \)-valued function \( \xi_w(t^{-1}) \) in \( t \in T \), it follows that
\[ \widetilde{B}^{(0)}_j(C_i\eta(T_{w_0})\widetilde{\xi}_w) = x^{-w^{-1}w_0(\varpi_j)+w_0(\varpi_j)}C_i\eta(T_{w_0})\widetilde{\xi}_w \]  
(2.5.11)
for all \( w \in S_N \). The value of \( C_i\eta(T_{w_0})\widetilde{\xi}_w \) at \( x^{-\alpha_i} = 0 \) (1 \( \leq i < N \)) is \( C_i(T_{w_0}T_{w}) \). In addition, \( C_i \) restricts to the anti-algebra involution on \( H_0 \) mapping \( T_{w} \) to \( T_{w^{-1}} \) for \( w \in S_N \), hence
\[ C_i(T_{w_0}T_{w}) = T_{w^{-1}}T_{w_0} = T_{w_0}T_{w_0^{-1}w_0}. \]

Formula (2.5.9) then follows from specializing (2.5.11) at \( x^{-\alpha_i} = 0 \) (1 \( \leq i < N \)) and replacing \( w \) by \( w_0^{-1}w_0 \) in the resulting formula. \( \square \)

For \( \epsilon > 0 \), put \( B_\epsilon := \{ t \in T \mid |t^{\alpha_i}| < \epsilon \ \forall i \} \) and \( B_\epsilon^{-1} := \{ t \in T \mid t^{-1} \in B_\epsilon \} \).

**Theorem 2.5.3.** There exists a unique solution \( \Psi \in H^R_0 \) of the gauged equations (2.5.6) satisfying, for some \( \epsilon > 0 \),

(i) \( \Psi(t, \gamma) \) admits an \( H_0 \)-valued power series expansion
\[ \Psi(t, \gamma) = \sum_{\alpha,\beta \in Q_+} K_{\alpha,\beta} t^{-\alpha} \gamma^\beta, \quad (K_{\alpha,\beta} \in H_0) \]  
(2.5.12)
for \( (t, \gamma) \in B_\epsilon^{-1} \times B \), which is normally convergent on compacta of \( B_\epsilon^{-1} \times B \). In particular, \( \Psi(t, \gamma) \) is analytic at \( (t, \gamma) \in B_\epsilon^{-1} \times B_\epsilon \);

(ii) \( K_{0,0} = T_{w_0} \).

**Proof.** It follows from the previous lemma that the commuting endomorphisms \( A^{(0,0)}_i \), \( B^{(0,0)}_j \in \text{End}(H_0) \) (1 \( \leq i, j < N \)) are semisimple. For \( a, b \in \mathbb{C}^N \) set
\[ H_0[(a, b)] = \{ v \in H_0 \mid A^{(0,0)}_i v = a_i v \ \text{and} \ B^{(0,0)}_j v = b_j v \ (1 \leq i, j < N) \}, \]
so that $H_0 = \bigoplus_{(a,b) \in S} H_0[(a,b)]$ with $S$ the finite set of $(a, b) \in \mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ for which $H_0[(a,b)] \neq 0$. By the previous lemma, $(1^{N-1}, 1^{N-1}) \in S$ and we have $H_0[(1^{N-1}, 1^{N-1})] = \text{span}_{\mathbb{C}} \{T_{w_0}\}$. Furthermore, $a_i, b_i \notin q^{-N}$ for all $(a, b) \in S$ and $i$. Under these conditions, the holonomic system of $q$-difference equations (2.5.6) admits a unique solution $\Psi$ satisfying the desired properties; see Theorem A.6 in the appendix (to show that the gauged BqKZ falls in the class of holonomic systems of $q$-difference equations to which Theorem A.6 applies, one should take $M = 2(N-1)$, $q_i = q$ for $1 \leq i < N$ and variables $z_i = x^{-\alpha_i}$ and $z_{N-1+j} = y^{\beta_i}$ for $1 \leq i, j < N$ in the appendix).

Remark 2.5.4. In a small neighborhood of a fixed $(t', \gamma') \in T \times T$, the meromorphic solution $\Psi$ of (2.5.6) can be expressed in terms of the power series expansion (2.5.12) by the formula

$$\Psi(t, \gamma) = D_\lambda(t, \gamma) \Psi(q^{-\lambda}t, q^{\mu} \gamma),$$

where $\lambda, \mu \in \mathbb{Z}^N$ are such that $(q^{-\lambda}t', q^{\mu} \gamma') \in B_{\epsilon}^{-1} \times B_{\epsilon}$.

Definition 2.5.5. We call $\Phi_\kappa := W_\kappa \Psi \in \text{SOL}$ the basic asymptotically free solution of BqKZ.

Note that $\Phi_\kappa \in F^\times \Phi_{\kappa'}$ for $\kappa, \kappa' \in \mathbb{C}^\times$. The $\kappa$-flexibility will come in handy when we consider specializations of $\Phi_\kappa$. In the following subsections, we derive various properties of the basic asymptotically free solution $\Phi_\kappa$.

2.5.3 Duality

Theorem 2.5.6. The basic asymptotically free solution $\Phi_\kappa$ of BqKZ is self-dual, in the sense that

$$\tau(\iota) \Phi_\kappa = \Phi_\kappa.$$

Proof. SOL is $S_N$-invariant, hence $\tau(\iota) \Phi_\kappa \in \text{SOL}$. In addition,

$$\tau(\iota) \Phi_\kappa = W_\kappa(\tau(\iota) \Psi),$$

because $\iota(W_\kappa) = W_\kappa$. Hence $\tau(\iota) \Psi$ is a solution of the gauged equations (2.5.6) having a convergent $H_0$-valued power series expansion

$$(\tau(\iota) \Psi)(t, \gamma) = C_i \Psi(\gamma^{-1}, t^{-1}) = \sum_{\alpha, \beta \in Q_+} C_i(K_{\alpha, \beta}) \gamma^\alpha t^{-\beta}$$

for $(t, \gamma) \in B_{\epsilon}^{-1} \times B_{\epsilon}$. Since $C_i(K_{0,0}) = C_i(T_{w_0}) = T_{w_0}$, we conclude from Theorem 2.5.3 that $\tau(\iota) \Psi = \Psi$, hence $\tau(\iota) \Phi_\kappa = \Phi_\kappa$. \qed
2.5.4 Singularities

Define
\[ \Lambda := \{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \} = \bigoplus_{i=1}^{N-1} \mathbb{Z}_{\geq 0} x_i \oplus \mathbb{Z} x_N, \tag{2.5.13} \]
i.e., \( \Lambda \) consists of the \( \lambda \in \mathbb{Z}^N \) such that \( \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0} \) for all \( \alpha \in R_+ \). Set
\[ S_+ := \{ t \in T \mid t^\alpha \in k^{-2} q^{-N} \text{ for some } \alpha \in R_+ \}. \]

Write \( \Psi(t, \gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha} \) for \( (t, \gamma) \in B_\epsilon^{-1} \times B_\epsilon \), where \( \Gamma_\alpha \) is the \( H_0 \)-valued analytic function on \( B_\epsilon \) defined by the \( H_0 \)-valued power series
\[ \Gamma_\alpha(\gamma) := \sum_{\beta \in Q_+} K_{\alpha, \beta} \gamma^\beta. \]

**Lemma 2.5.7.** The \( \Gamma_\alpha(\alpha \in Q_+) \) extend uniquely to a meromorphic \( H_0 \)-valued function on \( T \), analytic on \( T \setminus S_+ \), such that \( \Psi(t, \gamma) \) admits an \( H_0 \)-valued power series expansion
\[ \Psi(t, \gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha} \]
for \( (t, \gamma) \in B_\epsilon^{-1} \times T \setminus S_+ \), converging normally on compacta of \( B_\epsilon^{-1} \times T \setminus S_+ \).

**Proof.** Using Lemma 2.5.2 we write for \( \mu \in \Lambda \),
\[ D_{(e, \mu)}(t, \gamma) = \sum_{\beta \in Q_+} F^\mu_\beta(\gamma) t^{-\beta}, \]
with \( F^\mu_\beta \in Q_0(B) \otimes \text{End}(H_0) \) for all \( \beta \in Q_+^\nu \). Note that \( F^\mu_\beta \equiv 0 \) for all but finitely many \( \beta \in Q_+ \).

We first show that \( F^\mu_\beta(\gamma) \) is regular at \( \gamma \in T \setminus S_+ \). By (2.5.5) and by the cocycle property, \( F^\mu_\beta(\gamma) \) is regular at \( \gamma = \zeta \) if \( C_{(e, \pi_{\mu})}(\cdot, q^\nu \gamma) \in C[T] \otimes \text{End}(H_0) \) is regular at \( \gamma = \zeta \) for all \( 1 \leq j \leq N \) and \( \nu \in \Lambda \). The latter statement follows from the fact that \( R_i(z) \) has only a (simple) pole at \( z = k^{-2} \) and from the explicit expression
\[ C_{(e, \pi_{\mu})}(t, \gamma) = C_i(\eta(\pi)(t^{-1}))^j (R_{N-1}(q_{\gamma_1}/\gamma_N) \cdots R_2(q_{\gamma_1}/\gamma_2) R_1(q_{\gamma_1}/\gamma_1)) \times \cdots \times (R_{N-2}(q_{\gamma_{j-1}}/\gamma_N) \cdots R_j(q_{\gamma_{j-1}}/\gamma_{j-2}) R_{j-1}(q_{\gamma_{j-1}}/\gamma_{j-1})) \times (R_{N-1}(q_{\gamma_j}/\gamma_N) \cdots R_{j+1}(q_{\gamma_j}/\gamma_{j+2}) R_j(q_{\gamma_j}/\gamma_{j+1})) C_i, \tag{2.5.14} \]
which follows from Lemma 2.4.5(ii) and the cocycle property of \( C \).
Let $U \subset T \setminus S_+$ be a relatively compact open subset. Choose $\mu \in \Lambda$ such that the closure of $q^\mu U$ is contained in $B_{\epsilon}$, where $q^\mu U := \{q^\mu \gamma \mid \gamma \in U\}$. As meromorphic $H_0$-valued function in $(t, \gamma) \in B_{\epsilon}^{-1} \times U$, we have

$$
\Psi(t, \gamma) = D_{(e, \mu)}(t, \gamma) \Psi(t, q^\mu \gamma)
= \sum_{\alpha, \beta \in Q_+} F^\mu_\beta(\gamma) (\Gamma_\alpha(q^\mu \gamma)) t^{-\alpha - \beta}
= \sum_{\alpha \in Q_+} \left( \sum_{\beta \in Q_+} F^\mu_\beta(\gamma) (\Gamma_{\alpha - \beta}(q^\mu \gamma)) \right) t^{-\alpha},
$$

with the sums converging normally on compacta of $B_{\epsilon}^{-1} \times U$ (note that the sums over $\beta$ are finite). It follows that $\Gamma_\alpha(\gamma \in Q_+)$ has a unique $H_0$-valued meromorphic extension to $T$ which, on $U$, is given by

$$
\Gamma_\alpha(\gamma) = \sum_{\beta \in Q_+, \alpha - \beta \in Q_+} F^\mu_\beta(\gamma) (\Gamma_{\alpha - \beta}(q^\mu \gamma)),
$$

(2.5.15)

such that $\Psi$ on $B_{\epsilon}^{-1} \times U$ admits the power series expansion

$$
\Psi(t, \gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha},
$$

which converges normally on compacta of $B_{\epsilon}^{-1} \times U$. It follows from (2.5.15) and the previous paragraph that $\Gamma_\alpha$ is analytic on $T \setminus S_+$. \hfill \qed

The arguments from the proof of Lemma 2.5.7, applied to both torus variables of $\Psi(t, \gamma)$ at the same time, directly lead to the following result.

**Proposition 2.5.8.** The $H_0$-valued meromorphic function $\Psi(t, \gamma)$ is analytic at $(t, \gamma) \in T \setminus S_+^{-1} \times T \setminus S_+$. For specialized spectral parameter, we obtain the following result.

**Proposition 2.5.9.** Let $\zeta \in T \setminus S_+$.

(i) The $H_0$-valued meromorphic function $\Psi(t, \gamma)$ in $(t, \gamma) \in T \times T$ can be specialized at $\gamma = \zeta$, giving rise to a meromorphic $H_0$-valued function $\Psi(t, \zeta)$ in $t \in T$. It has the power series expansion

$$
\Psi(t, \zeta) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\zeta) t^{-\alpha}
$$

for $t \in B_{\epsilon}^{-1}$, normally converging on compacta of $B_{\epsilon}^{-1}$.

(ii) $\Psi(t, \zeta)$ satisfies the gauged $q$-difference equations

$$
D_{(\lambda, e)}(t, \zeta) \Psi(q^{-\lambda} t, \zeta) = \Psi(t, \zeta), \quad \forall \lambda \in \mathbb{Z}^N.
$$

(2.5.16)
Proof. (i) Restricting to $t \in B_{\epsilon}^{-1}$ for $\epsilon > 0$ small enough, the statement is correct by Lemma 2.5.7. If $t' \in T$ is arbitrary then there exists a $\lambda \in \Lambda$ such that $q^{-\lambda}t' \in B_{\epsilon}^{-1}$. For $t \in T$ in a small neighborhood of $t'$ we then have

$$
\Psi(t, \gamma) = D_{(\lambda, \epsilon)}(t, \gamma)\Psi(q^{-\lambda}t, \gamma).
$$

Since $D_{(\lambda, \epsilon)} \in (Q_0(A) \otimes B) \otimes \mathrm{End}(H_0)$ by Lemma 2.5.2(ii) the statement now follows in a small open neighborhood of $t'$.

(ii) Specializing the gauged $q$-difference equations $D_{(\lambda, \epsilon)}(t, \gamma)\Psi(q^{-\lambda}t, \gamma) = \Psi(t, \gamma)$ ($\lambda \in \mathbb{Z}_N$) to $\gamma = \zeta$ yields the desired result. \hfill \Box

2.5.5 Evaluation formula

We write $(z; q)_\infty = \prod_{m=0}^\infty (1 - q^m z)$ for the $q$-shifted factorial. Recall the power series expansion $\Psi(t, \gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha}$ for $|t^{-\alpha}| \gg 0$ ($1 \leq i < N$) from Subsection 2.5.4. We call the following result the evaluation formula for the basic asymptotically free solution $\Phi_\kappa = W_\kappa \Psi$ of $BqKZ$, since it implies the celebrated evaluation formula for the Macdonald polynomials (see Subsection 4.5).

Theorem 2.5.10. We have

$$
\Gamma_0(\gamma) = K(\gamma)T_{w_0}
$$

with $K \in \mathcal{M}(T)$ explicitly given by

$$
K(\gamma) := \prod_{1 \leq i < j \leq N} \frac{(q^{\gamma_i}/\gamma_j; q)_\infty}{(q^{k_2\gamma_i}/\gamma_j; q)_\infty}. \tag{2.5.17}
$$

Proof. We use the notations of Lemma 2.5.2. Recall that $\Psi$ satisfies the gauged $q$-difference equations

$$
A_i(t, \gamma)\Psi(q^{-\varpi_i}t, \gamma) = \Psi(t, \gamma)
$$

for $1 \leq i \leq N$. In view of the proof of Lemma 2.5.2 and Lemma 2.5.7, it reduces in the limit $|t^{-\alpha}| \to 0$ ($1 \leq i < N$) to

$$
\gamma^{-\varpi_0}q(\gamma T_{w_0}^{-1}Y_{w_0}(\varpi_0)T_{w_0}^{-1})(\gamma)\Gamma_0(\gamma) = \Gamma_0(\gamma)
$$

for $1 \leq i \leq N$, as $H_0$-valued meromorphic functions in $\gamma \in T$. This forces $\Gamma_0(\gamma) = K(\gamma)\eta(T_{w_0})\zeta_\alpha(\gamma) = K(\gamma)T_{w_0}$ for some $K \in \mathcal{M}(T)$; see Lemma 2.4.2.

It remains to show that $K$ is explicitly given by (2.5.17). Write $L(\gamma)$ for the right hand side of (2.5.17). Then $L \in \mathcal{M}(T)$ is characterized by the following three properties:

(i) for some $\epsilon > 0$ we have a power series expansion

$$
L(\gamma) = \sum_{\alpha \in Q_+} l_\alpha \gamma^\alpha
$$

for $\gamma \in B_\epsilon$, converging normally on compacta of $B_\epsilon$.
(iii) $l_0 = 1$; and

(ii) $L(\gamma)$ satisfies the $q$-difference equations

$$\prod_{1 \leq r \leq j \atop j+1 \leq s \leq N} \frac{1 - q^{\gamma_r}/\gamma_s}{1 - q^{K^s}/\gamma_s} L(q^{\frac{c}{\gamma_s}}) = L(\gamma), \quad 1 \leq j \leq N.$$ 

It thus suffices to show that $K(\gamma)$ satisfies the three properties (i)--(iii). It is clear that $K \in \mathcal{M}(T)$ satisfies (i); see Subsection 2.5.4. Theorem 2.5.3(ii) implies (ii) for $K$.

What remains is the verification of the $q$-difference equations (iii) for $K$. Using the notations of Lemma 2.5.2, we write

$$B_j^{(0)} := B_j|_{x^{-\alpha_i}=0, \ldots, x^{-\alpha_{N-1}}=0} \in Q_0(B) \otimes \text{End}(H_0).$$

We view $B_j^{(0)}(\gamma)$ as an $\text{End}(H_0)$-valued meromorphic function in $\gamma \in T$. Taking the limit $|t^{-\alpha_i}| \to 0$ ($1 \leq i < N$) in the gauged $q$-difference equations

$$B_j(t, \gamma) \Psi(t, q^{c/\gamma} \gamma) = \Psi(t, \gamma), \quad 1 \leq j \leq N$$

and using $\Gamma_0(\gamma) = K(\gamma)T_{w_0}$ we obtain

$$K(q^{\frac{c}{\gamma}} \gamma)B_j^{(0)}(\gamma)T_{w_0} = K(\gamma)T_{w_0}$$

for $1 \leq j \leq N$, as meromorphic $H_0$-valued functions in $\gamma \in T$. Writing $B_j^{(0)}(\gamma)T_{w_0} = \sum_{w \in S_N} a^j_w(\gamma)T_w$ with $a^j_w \in \mathcal{M}(T)$ it thus suffices to show that

$$a^j_{w_0}(\gamma) = \prod_{1 \leq r \leq j \atop j+1 \leq s \leq N} \frac{1 - q^{\gamma_r}/\gamma_s}{1 - q^{K^s}/\gamma_s} = k^{-\langle \delta, \varpi_j \rangle} \prod_{1 \leq r \leq j \atop j+1 \leq s \leq N} c_k(q^{\gamma_r}/\gamma_s)^{-1} \tag{2.5.18}$$

for $1 \leq j \leq N$, where the second equality follows from a direct computation using the explicit expression (2.2.5) of $c_k$.

By (2.5.5) we have

$$B_j(t, \gamma) = k^{-\langle \delta, \varpi_j \rangle} \varpi_{w_0(\varpi_j)} C(c, \varpi_j)(t, \gamma)$$

and $C(c, \varpi_j)(t, \gamma)$ is given explicitly by (2.5.14). Since $R_i(z) = c_k(z)^{-1}(\eta(T_i) - k) + 1$, Lemma 2.4.1 and the reduced expression (2.4.8) for $\sigma^i$ imply that

$$B_j(t, \gamma)T_{w_0} = k^{-\langle \delta, \varpi_j \rangle} \varpi_{w_0(\varpi_j)} C(c, \varpi_j)(\eta(\pi)(t^{-1}))^j \left( \sum_{w \leq \sigma^{-j}} b^j_w(\gamma)T_{w w_0} \right)$$

with $\leq$ the Bruhat order on $S_N$ and with

$$b^j_{\sigma^{-j}}(\gamma) = \prod_{1 \leq r \leq j \atop j+1 \leq s \leq N} c_k(q^{\gamma_r}/\gamma_s)^{-1}.$$
By Lemma 2.4.1 we have
\[ t^{w_0(\varpi_j)}C_t(\eta(\pi)(t^{-1}))T_{w_0} = t^{w_0(\varpi_j) - w_0w^{-1}w_0(\varpi_j)}T_{w_0w^{-1}1,\sigma^{-j}}. \]

Hence
\[ B_j^{(0)}(\gamma)T_{w_0} = k^{-\langle \delta, \varpi_j \rangle} \sum_{w} b_w^{ij}(\gamma)T_{w_0w^{-1}1,\sigma^{-j}}, \]
with the sum running over \( w \in S_N \) satisfying \( w \leq \sigma^{-j} \) and \( w(\varpi_j) = w_0(\varpi_j) \). In particular, \( a_{w_0}^{ij}(\gamma) = k^{-\langle \delta, \varpi_j \rangle} b_{\sigma^{-j}}^{ij}(\gamma) \). This completes the proof of (2.5.18). \( \square \)

### 2.5.6 Consistency of the bispectral quantum KZ equations

In this subsection, we show that BqKZ is a consistent system of \( q \)-difference equations, i.e., \( \dim_c(SOL) = \dim_c(H_0) \), by explicitly constructing an \( \mathfrak{F} \)-basis of SOL. Since the cocycle matrices \( C_{(\lambda, \mu)}(t, \gamma) \) (\( \lambda, \mu \in \mathbb{Z}^N \)) depend rationally on \( (t, \gamma) \in T \times T \), the consistency of BqKZ follows also from the abstract arguments in [14, \S 5].

We start with a preliminary lemma on the cocycle values \( C_{(e, w)} \) for \( w \in S_N \).

**Lemma 2.5.11.** Let \( w \in S_N \). We have \( C_{(e, w)} \in Q_0(B) \otimes \text{End}(H_0) \) and
\[ C_{(e, w)}^{(0)}(h) = k^{-\ell(w)}hT_{w^{-1}}, \quad h \in H_0, \]
where
\[ C_{(e, w)}^{(0)} = C_{(e, w)}|_{y^a = 0, \ldots, y^{N-1} = 0} \in \text{End}(H_0). \]

**Proof.** Let \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \) be a reduced expression for \( w \in S_N \) (\( 1 \leq i_j < N \)) and write \( \beta_j := s_{i_j} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in R_+ \) for \( 1 \leq j \leq r \), where \( \beta_1 \) should be read as \( \alpha_{i_1} \). By Subsection 2.4.2 and the cocycle property, we have
\[ C_{(e, w)}(t, \gamma) = C_{(e, w)}(\gamma^{-1}, t^{-1})C_t = C_t(R_{i_1}(\gamma^{\beta_1}) \cdots R_{i_r}(\gamma^{\beta_r})R_{i_1}(\gamma^{\beta_1}))^{-1}C_t. \]
From the expression for \( R_i(z) \) it now follows that \( C_{(e, w)} \in Q_0(B) \otimes \text{End}(H_0) \). Since \( \lim_{z \to 0} R_i(z) = k\eta(T_i^{-1}) \) we furthermore have
\[ C_{(e, w)}^{(0)} = k^{-\ell(w)}C_t\eta(T_{w^{-1}})C_t. \]
The map \( C_t \) is the \( \mathbb{K} \)-linear extension of the anti-algebra involution of \( H_0 \) mapping \( T_w \) to \( T_{w^{-1}} \). Hence \( C_{(e, w)}^{(0)}(h) = k^{-\ell(w)}hT_{w^{-1}} \) for \( h \in H_0 \). \( \square \)

Define \( U \in \text{End}(H_0)^{\mathbb{K}} := \mathbb{K} \otimes \text{End}(H_0) \) by
\[ U(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) = \tau(e, w)\Phi_x, \quad w \in S_N. \]
(2.5.19)

Since SOL is \( S_N \)-invariant, \( U \) is an \( \text{End}(H_0) \)-valued solution of BqKZ, i.e.
\[ C_{(\lambda, \mu)}(t, \gamma)U(q^{-\lambda}t, q^{-\mu}t) = U(t, \gamma), \quad \lambda, \mu \in \mathbb{Z}^N \]
as \( \text{End}(H_0) \)-valued meromorphic functions in \( (t, \gamma) \in T \times T \).
\textbf{Lemma 2.5.12.} $U \in \text{End}(H_0^K)$ is invertible.

\textit{Proof.} Using the natural identification $\text{End}(H_0^K) \simeq \text{End}_K(H_0^K)$ as $K$-algebras, we need to verify that $U \in GL_K(H_0^K)$.

Set $\Phi_w := \tau(e, w)\Phi_\kappa$ and $\Psi_w := \tau(e, w)\Psi$ for $w \in S_N$, so that

$$\Phi_w(t, \gamma) = W_w(t, w^{-1}\gamma)\Psi_w(t, \gamma)$$

Since $C_{(e, w)}(t, \gamma)$ is independent of $t \in T$, we simply write it as $C_{(e, w)}(\gamma)$. Recall the $W$-invariant subset $S \subset T$ (see (2.4.6)), which contains $S_+$. By Lemma 2.4.4 and Lemma 2.5.7, we have for some $\epsilon > 0$ the power series expansion

$$\Psi_w(t, \gamma) = \sum_{\alpha \in Q^+} C_{(e, w)}(\gamma)(\Gamma_{\alpha}(w^{-1}\gamma))t^{-\alpha}$$

for $(t, \gamma) \in B_t^{-1} \times T \setminus S$, converging normally on compacta of $B_t^{-1} \times T \setminus S$. We write $\Gamma_{\alpha}^{w}(\gamma) := C_{(e, w)}(\gamma)(\Gamma_{\alpha}(w^{-1}\gamma))$ in the remainder of the proof. It is a meromorphic function in $\gamma \in T$, analytic on $T \setminus S$, and the power series expansion of $\Psi_w$ becomes

$$\Psi_w(t, \gamma) = \sum_{\alpha \in Q^+} \Gamma_{w, \alpha}^{w}(\gamma)t^{-\alpha}. \quad (2.5.20)$$

Observe that

$$\Gamma_{\alpha}^{w}(\gamma) \rightarrow C_{(e, w)}^{(0)}(T_{w_0}) = k^{-\ell(w)}T_{w_0}T_{w^{-1}},$$

in the limit $\gamma_{\alpha_i} \rightarrow 0$ $(1 \leq i < N)$, in view of the previous lemma.

Write $U = V\Xi$ with $V, \Xi$ the $K$-linear endomorphisms of $H_0^K$ given by

$$\Xi(k^{-\ell(w)}T_{w_0}T_{w^{-1}})(t, \gamma) = W_k(t, w^{-1}\gamma)k^{-\ell(w)}T_{w_0}T_{w^{-1}},$$

$$V(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) = \Psi_w,$$

for $w \in S_N$. Since $\Xi \in GL_K(H_0^K)$ it suffices to show that $V \in GL_K(H_0^K)$. Let $M$ be the matrix of $V$ with respect to the $K$-basis $k^{-\ell(w)}T_{w_0}T_{w^{-1}}$ $(w \in S_N)$ of $H_0^K$. Fix $\zeta \in T \setminus S$ such that $\zeta^\alpha \notin R$ for all $\alpha \in R$. The matrix $M(t, \gamma)$ may be specialized at $\gamma = \zeta$ and the limit of $M(t, \zeta)$ as $t^{-\alpha_i} \rightarrow 0$ $(1 \leq i < N)$ exists. We write $M^{(0)}(\zeta)$ for the limit and $V^{(0)}(\zeta)$ for the corresponding linear endomorphism of $H_0$. We then have

$$V^{(0)}(\zeta)(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) = \Gamma_{w}^{\alpha}(\zeta) = K(w^{-1}\zeta)C_{(e, w)}(\zeta)T_{w_0},$$

with $K(\gamma)$ given by (2.5.17). Note that $K(w^{-1}\zeta) \neq 0$ since $\zeta^\alpha \notin q^{Q}$ for all $\alpha \in R$. By the explicit expression for the cocycle value $C_{(e, w)}(\zeta) \in \text{End}(H_0)$ (see the proof of the previous lemma) we have

$$C_{(e, w)}(\zeta)(T_{w_0}) = \sum_{w \leq w} a_{w}^{w}(\zeta)T_{w_0}T_{w^{-1}},$$

with $a_w^{w}(\zeta) \neq 0$ and with $\preceq$ the Bruhat order on $S_N$. This implies that $V^{(0)}(\zeta)$ is a linear automorphism of $H_0$, hence $\det(M^{(0)}(\zeta)) \neq 0$. Consequently, $\det(M) \in K^*$ and $V \in GL_K(H_0^K)$.

\hfill $\Box$
Proposition 2.5.13. (i) \( U' \in \text{End}(H_0)^K \) is an \( \text{End}(H_0)^K \)-valued meromorphic solution of BqKZ if and only if \( U' = UF \) for some \( F \in \text{End}(H_0)^F \).

(ii) \( U \), viewed as \( K \)-linear endomorphism of \( H_0^K \), restricts to an \( F \)-linear isomorphism \( U : H_0^F \to \text{SOL} \).

(iii) \( \{ \tau(e,w)\Phi_\kappa \}_{w \in S_N} \) is an \( F \)-basis of \( \text{SOL} \).

Proof. (i) If \( U' \) is an \( \text{End}(H_0) \)-valued meromorphic solution of BqKZ then, since \( U \) is invertible, we have for all \( \lambda, \mu \in \mathbb{Z}^N \),

\[
U(q^{-\lambda}t, q^\mu \gamma) = U(t, \gamma) - 1 U'(t, \gamma).
\]

Hence \( U' = UF \) with \( F \in \text{End}(H_0)^F \). The converse implication is clear.

(ii) By the previous lemma we have \( U : H_0^F \to \text{SOL} \). It is surjective, since for \( g \in \text{SOL}, f := U^{-1}g \in H_0^K \) satisfies \( f(q^{-\lambda}t, q^\mu \gamma) = f(t, \gamma) \) for all \( \lambda, \mu \in \mathbb{Z}^N \) (cf. the proof of (i)), hence \( f \in H_0^K \).

(iii) This is clear from (ii) and from the definition of \( U \).

Corollary 2.5.14. Fix \( \zeta \in T \setminus S \) such that \( \zeta^\alpha \notin q^\mathbb{Z} \) for all \( \alpha \in R \). For generic \( \kappa \in \mathbb{C}^\times \), the \( H_0 \)-valued meromorphic functions \( \{ \tau(e,w)\Phi_\kappa \}(t, \gamma) \) in \( T \times T \) \( (w \in S_N) \) can be specialized at \( \gamma = \zeta \), giving rise to

(i) a basis \( \{ C_{(e,w)}(\zeta)\Phi_\kappa(\cdot, w^{-1} \zeta) \}_{w \in S_N} \) of \( \text{SOL}_\zeta \) over \( \mathcal{E}(T) \);

(ii) an invertible \( \text{End}(H_0) \)-valued meromorphic solution \( U_\zeta \) of the quantum KZ equations (2.4.10), where \( U_\zeta \in \text{End}(H_0)^{M(T)} \) is explicitly defined by

\[
U_\zeta(k^{-f(w)}T_{w_0}T_{w^{-1}}) := C_{(e,w)}(\zeta)\Phi_\kappa(\cdot, w^{-1} \zeta), \quad w \in S_N.
\]