Bispectral quantum Knizhnik-Zamolodchikov equations

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Citation for published version (APA):

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Chapter 3

Correspondence of BqKZ with bispectral problems

3.1 Introduction

Cherednik [7, Theorem 3.4] constructed for arbitrary root systems a correspondence between solutions of quantum KZ equations and solutions of a system of $q$-difference equations. In [7, Theorem 4.4], Cherednik made the correspondence precise for $GL_N$. It yields an explicit map $\chi_+$ from solutions of the quantum KZ equations $qKZ_\zeta$ (with fixed central character $\zeta \in T$; see (2.4.10)) to solutions of the spectral problem of Ruijsenaars’ [50] commuting trigonometric $q$-difference operators with spectral parameter $\zeta^{-1}$ (the Ruijsenaars operators are also frequently referred to as Macdonald-Ruijsenaars operators). The latter result has been generalized to arbitrary root systems in [30, Theorem 4.6] and [8]. In the classical setting ($q = 1$) it goes back to Matsuo [43]. In this chapter we analyze the map $\chi_+$ in the bispectral setting of Chapter 2. It leads to the interpretation of $\chi_+$ as an embedding of the solution space SOL of BqKZ (see Definition 2.3.7) into the space of meromorphic solutions of a bispectral problem involving the above Ruijsenaars operators as well as Ruijsenaars operators acting on the spectral parameter.

In Section 3.2 we introduce the so-called monodromy cocycle which plays a role in the construction of the correspondence, which will be established in Section 3.3. The techniques employed there are analogous to the ones for the usual correspondence (see [10, Chapter 1]). For the convenience of the reader we provide full details of the arguments involved.

In Section 3.4 we apply the correspondence to the self-dual solution $\Phi_\kappa$ (Definition 2.5.5) of BqZK to obtain a self-dual solution $\Phi_\kappa^+$ of the bispectral problem of the Ruijsenaars $q$-difference operators. For reasons which become apparent below, we call this solution the basic Harish-Chandra series solution of the bispectral problem of the Ruijsenaars $q$-difference operators.
Finally, in Section 3.5, we consider BqKZ for specialized $\gamma = \zeta \in T$ as in Subsection 2.4.4, and in this case we obtain from the bispectral correspondence the (classical) correspondence between solutions of qKZ and the spectral problem of the Ruijsenaars operators with spectral parameter $\zeta^{-1}$. Specialization of $\Phi^+_{\kappa}$ leads to a Harish-Chandra series solution of the Ruijsenaars operators with fixed spectral parameter. These Harish-Chandra solutions to the spectral problem were investigated before in, e.g., [16], [17], [31] and [36]. The present approach to Harish-Chandra series, which uses quantum KZ equations in an essential way, has the advantage that it leads to new results on the convergence and singularities of the Harish-Chandra series. These results, together with Cherednik’s recent work [11], form important building blocks in deriving the $c$-function expansion of Cherednik’s global $(q,t)$-spherical function (see [55]).

The material presented in this chapter coincides with Section 6 of the paper [45].

**Convention.** Throughout Sections 3.2-3.4, we fix $\kappa \in \mathbb{C}$. Furthermore, we use the notations and conventions of Chapter 2.

### 3.2 The monodromy cocycle

Recall the $\text{End}(H_0)$-valued meromorphic solution $U$ of BqKZ that we constructed in Subsection 2.5.6. In this section, we will define an auxiliary cocycle $T_w$ ($w \in \mathbb{W}$) of $\mathbb{W}$, which can be thought of as a family of monodromy matrices with respect to the fundamental solution $U$, as explained below.

Observe that $F \in \text{End}(H_0)^{\mathbb{W}}$ is an $\text{End}(H_0)$-valued meromorphic solution of BqKZ if and only if

$$\tau(w)F = F, \quad w \in \mathbb{Z}^N \times \mathbb{Z}^N,$$

where the $\mathbb{W}$-action $\tau$ on $\text{End}(H_0)^{\mathbb{W}}$ is defined by

$$(\tau(w)F)(t, \gamma) := C_w(t, \gamma)(wF)(t, \gamma) = C_w(t, \gamma)F(w^{-1}(t, \gamma))$$

for $w \in \mathbb{W}$ and $F \in \text{End}(H_0)^{\mathbb{W}}$, viewed as identities between $\text{End}(H_0)$-valued meromorphic functions in $(t, \gamma) \in T \times T$.

By Proposition 2.5.13(i), given an $\text{End}(H_0)$-valued meromorphic solution $F$ of BqKZ, there exists a unique $G \in \text{End}(H_0)^{\mathbb{W}}$ such that $F = UG$. Accordingly, $G$ describes the deviation of $F$ from the fundamental solution $U$ of BqKZ, and therefore can be thought of as a connection matrix (cf. [15, §12.1]). We will consider the special cases when the solutions $F$ are the $\text{End}(H_0)$-valued meromorphic solutions $\tau(w)U$ ($w \in \mathbb{W}$) of BqKZ.

For $w \in \mathbb{W}$ we set

$$T_w := U^{-1}(\tau(w)U) \in \text{End}(H_0)^{\mathbb{W}},$$

that is, $T_w$ ($w \in \mathbb{W}$) is the unique element of $\text{End}(H_0)^{\mathbb{W}}$ such that

$$\tau(w)U = UT_w.$$
Note that \( \text{End}(H_0)^F \) is a \( \mathbb{W} \)-stable subalgebra of \( \text{End}(H_0)^K \) with respect to the action 
\( (wF)(t, \gamma) = F(w^{-1}(t, \gamma)) \). The following lemma now shows that the \( \mathcal{T}_w \) \((w \in \mathbb{W})\) 
define a cocycle of \( \mathbb{W} \) with values in the group of units of \( \text{End}(H_0)^F \).

**Lemma 3.2.1.** (i) \( \mathcal{T}_w = \text{id} \) for \( w \in \mathbb{Z}^N \times \mathbb{Z}^N \). 
(ii) For \( w, w' \in \mathbb{W} \) we have the cocycle relation 
\[ \mathcal{T}_{ww'} = \mathcal{T}_w(\mathcal{T}_{w'}) \]
in \( \text{End}(H_0)^F \).

**Proof.** (i) This follows immediately from the fact that \( U \) is an \( \text{End}(H_0) \)-valued meromorphic solution of BqKZ.
(ii) Note that \( \mathcal{T}_w = U^{-1}C_ww(U) \) for \( w \in \mathbb{W} \). By the cocycle condition for \( C_w \in \text{End}(H_0)^K \), which reads in the present notations as \( C_{ww'} = C_wC_w'w \) for \( w, w' \in \mathbb{W} \), we have 
\[ \mathcal{T}_{ww'} = U^{-1}C_{ww'}w(U) = U^{-1}C_ww(C_w'w'(U)) \]
\[ = U^{-1}C_ww(U)w(U^{-1}C_w'w'(U)) = \mathcal{T}_w(\mathcal{T}_{w'}) \]
for all \( w, w' \in \mathbb{W} \).

**Definition 3.2.2.** In analogy with the terminology in [10, §1.3.3] for the quantum KZ equation, we call \( \{\mathcal{T}_w\}_{w \in \mathbb{W}} \) the monodromy cocycle of the BqKZ.

**Remark 3.2.3.** Connection matrices and Riemann-Hilbert problems for ordinary linear \( q \)-difference equations have been studied extensively; see, e.g., [3] and [52]. For quantum KZ equations, connection matrices have been computed explicitly in, e.g., [21], [15, §12], and [31].

### 3.3 The correspondence

Consider the algebra \( \mathbb{C}(T \times T)^\# \mathbb{W} \), where \( \mathbb{W} \) acts as field automorphisms on \( \mathbb{C}(T \times T) \) 
by the formula (2.3.1). Recall that \( \mathbb{C}(T \times T)^\# \mathbb{W} \) naturally acts on \( \mathbb{K} \). We write \( Df \) for the action of \( D \in \mathbb{C}(T \times T)^\# \mathbb{W} \) on \( f \in \mathbb{K} \).

We have a representation \( \vartheta : \mathbb{C}(T \times T)^\# \mathbb{W} \to \text{End}(\text{End}(H_0)^K) \) given by
\[ \vartheta(f)F = fF, \quad f \in \mathbb{C}(T \times T), \]
\[ \vartheta(w)F = w(F), \quad w \in \mathbb{W} \]
for \( F \in \text{End}(H_0)^K \). Let \( \mathbb{D} \) be the subalgebra \( \mathbb{C}(T \times T)^\#(\mathbb{Z}^N \times \mathbb{Z}^N) \) of \( \mathbb{C}(T \times T)^\# \mathbb{W} \). Under the natural action of \( \mathbb{C}(T \times T)^\# \mathbb{W} \) on \( \mathbb{C}(T \times T) \), the subalgebra \( \mathbb{D} \) identifies with the algebra of \( q \)-difference operators on \( T \times T \) with rational coefficients.

Set \( H_0^* := \text{Hom}(H_0, \mathbb{C}) \). We will regard a linear functional \( \chi \in H_0^* \) also as an element of \( \text{Hom}_\mathbb{K}(H_0^*, \mathbb{K}) \) by \( \mathbb{K} \)-linear extension. For \( F \in \text{End}(H_0)^K \), write 
\[ \phi^F_{\chi,v} := \chi(Fv) \in \mathbb{K}, \quad \chi \in H_0^*, v \in H_0 \]
for its matrix coefficients. Note that for any \( D \in \mathbb{C}(T \times T)\# \mathcal{W}, \chi \in H_0^0 \) and \( v \in H_0 \),

\[
D \phi_{\chi,v}^F = \phi_{\chi,v}^{D(F)}
\]

for all \( F \in \text{End}(H_0)^K \).

**Lemma 3.3.1.** For \( w \in \mathcal{W} \) we have

\[
\vartheta(w)U = C_w^{-1}U \tau(w).
\]

In particular, \( \vartheta(w)U = C_w^{-1}U \) for \( w \in \mathbb{Z}^N \times \mathbb{Z}^N \).

**Proof.** For \( w \in \mathcal{W} \)

\[
\vartheta(w)U = w(U) = C_w^{-1}(\tau(w)U) = C_w^{-1}U \tau(w).
\]

The second claim follows from the fact that \( \tau(w) = \text{id} \) for \( w \in \mathbb{Z}^N \times \mathbb{Z}^N \).

We are now going to look for a particular linear functional \( \chi \) such that the matrix coefficients \( \phi_{\chi,v}^F \) (\( v \in H_0^0 \)) of \( U \) solve a bispectral problem with respect to two commuting families of Ruijsenaars’ trigonometric \( q \)-difference operators (one family acting on the first torus component, the second on the second torus component). In view of (3.3.1) and the previous lemma, to obtain \( q \)-difference equations for \( \phi_{\chi,v}^F \) we have to deal with the cocycle value \( C_w \) and the monodromy matrix \( \tau_w \) in the equations \( \vartheta^F_{\chi,v} = \phi_{\chi,v}^{C_w^{-1}U \tau_w} \). It is convenient to postpone the analysis of the monodromy cocycle by initially absorbing it into the action \( \vartheta \) of \( \mathbb{C}(T \times T)\# \mathcal{W} \) via the twisted algebra homomorphism

\[
\vartheta_T : \mathbb{C}(T \times T)\# \mathcal{W} \to \text{End}(\text{End}(H_0)^K),
\]

defined by

\[
\vartheta_T(f)F = fF, \quad f \in \mathbb{C}(T \times T),
\]

\[
\vartheta_T(w)F = w(F) \tau_w^{-1}, \quad w \in \mathcal{W}
\]

for \( F \in \text{End}(H_0)^K \). Note that \( \vartheta_T \) is indeed an algebra homomorphism, thanks to the cocycle condition for \( \tau \). Moreover, \( \vartheta_T|_D = \vartheta|_D \).

For \( D \in \mathbb{C}(T \times T)\# \mathcal{W} \) we will occasionally use the notations

\[
D = \sum_{w \in \mathcal{W}} d_w w = \sum_{v \in \mathbb{S}_N} D_v v,
\]

(3.3.2)

where \( d_w \in \mathbb{C}(T \times T) \) (\( w \in \mathcal{W} \)) and \( D_v = \sum_{u \in \mathbb{Z}^N \times \mathbb{Z}^N} d_{uw} u \in D \) (\( v \in \mathbb{S}_N \)). Reformulating (3.3.1) and Lemma 3.3.1 in terms of the twisted action \( \vartheta_T \) yields the following result.
Lemma 3.3.2. (i) For $w \in \mathcal{W}$ we have
\[ \vartheta_T(w) = C_w^{-1}U. \]
(ii) For $D \in \mathbb{C}(T \times T)^\# \mathcal{W}$ we have
\[ \phi_{\chi,v}^{\vartheta_T(D)U} = \sum_{v \in SN} D_v(\phi_{\chi,v}^{C_v^{-1}U}) \]
for all $\chi \in H_0^*$ and $v \in H_0$.

Proof. (i) This is clear from Lemma 3.3.1 and the definition of $\vartheta_T$.
(ii) By Lemma 3.3.1 and (3.3.1), we obtain
\[ \phi_{\chi,v}^{\vartheta_T(D)U} = \sum_{v \in SN} \phi_{\chi,v}^{\vartheta(D)v} \vartheta_T(v)U = \sum_{v \in SN} D_v(\phi_{\chi,v}^{\vartheta_T(D)v}U). \]
The result now follows from (i). \[\square\]

We define the restriction map $\text{Res}: \mathbb{C}(T \times T)^\# \mathcal{W} \to \mathbb{D}$ to be the $\mathbb{C}(T \times T)$-linear map
\[ \text{Res}(D) := \sum_{v \in SN} D_v, \quad D \in \mathbb{D}. \]

Lemma 3.3.1(ii) implies that if we have a linear functional $\chi_+ \in H_0^*$ that satisfies
\[ \chi_+(C_v^{-1}U) = \chi_+(U) \quad \text{for all } v \in SN, \]
then the corresponding matrix coefficients $\phi_{\chi_+,v}^D$ satisfy
\[ \text{Res}(D)(\phi_{\chi_+,v}^U) = \phi_{\chi_+,v}^{\vartheta_T(D)U} \quad (3.3.3) \]
for all $D \in \mathbb{C}(T \times T)^\# \mathcal{W}$.

Lemma 3.3.3. Define $\chi_+ \in H_0^*$ by $\chi_+(T_w) = k^w$ for all $w \in SN$. Then
\[ \chi_+(C_v^{-1}F) = \chi_+(F) \]
for $F \in \text{End}(H_0)^K$ and $v \in SN$.

Proof. Since $C_{i}(T_w) = T_{w^{-1}}$ for $w \in SN$, we have $\chi_+ \circ C_i = \chi_+$. By the cocycle condition for $C_w$ ($w \in SN$) it remains to prove that $\chi_+ \circ C_{(i,e)} = \chi_+$ for $1 \leq i < N$. But this follows from the expression
\[ C_{(i,e)}(t,\gamma) = c_k(t_i/t_{i+1})^{-1}(\eta(T_i) - k) + 1 \quad \text{(see Lemma 2.4.3)}, \]
since
\[ \chi_+((T_i - k)h) = 0 \quad (3.3.4) \]
for $1 \leq i < N$ and $h \in H_0$. \[\square\]

If $D \in \mathbb{C}(T \times T)^\# \mathcal{W}$ satisfies $\vartheta_T(D)U = \lambda U$ for some $\lambda \in \mathbb{K}$, then it follows from (3.3.3) that the matrix coefficients $\phi_{\chi_+,v}^D$ ($v \in H_0$) are eigenfunctions of $\text{Res}(D)$ with eigenvalue $\lambda$. We will now construct such a commuting family of $D$’s. It leads to the interpretation of the $\phi_{\chi_+,v}^D$ ($v \in H_0$) as solutions of a bispectral problem.
The appropriate elements $D \in \mathbb{C}(T \times T)\#W$ are obtained as images of elements from the center $Z(H)$ of the affine Hecke algebra $H$ under the faithful algebra homomorphism $\rho$ from Theorem 2.2.4. Since we aim at a bispectral version, we will interpret $\rho$ as algebra map $\rho : H \to \mathbb{C}(T \times T)\#W$ in two different ways. We have, on the one hand, the algebra homomorphism

$$\rho_{k^{-1},q}^x : H(k^{-1}) \to \mathbb{C}(T \times T)\#W,$$

which is the map $\rho_{k^{-1},q}$ from Theorem 2.2.4, interpreted as algebra homomorphism from $H(k^{-1})$ to the subalgebra $\mathbb{C}(T \times \{1\})(W \times \{e\}) \subset \mathbb{C}(T \times T)\#W$. On the other hand, we have an algebra homomorphism

$$\rho_{k^{-1},q}^y : H(k) \to \mathbb{C}(T \times T)\#W,$$

defined as the map $\rho_{k,q}^{-1}$ from Theorem 2.2.4, interpreted as algebra homomorphism from $H(k)$ to the subalgebra $\mathbb{C}(\{1\} \times T)(\{e\} \times W) \subset \mathbb{C}(T \times T)\#W$. Note that they can be combined into an algebra homomorphism

$$\rho_{k^{-1},q}^x \times \rho_{k,q}^{-1} : H(k^{-1}) \otimes H(k) \to \mathbb{C}(T \times T)\#W.$$

**Definition 3.3.4.** (i) For $h \in H(k^{-1})$, define

$$D_h^x := \rho_{k^{-1},q}^x(h) \in \mathbb{C}(T \times T)\#W.$$

(ii) For $h \in H(k)$, define

$$D_h^y := \rho_{k,q}^{-1}(h) \in \mathbb{C}(T \times T)\#W.$$

**Remark 3.3.5.** Let $^{\circ} : H(k^{-1}) \to H(k)$ be the algebra isomorphism defined by $\pi^\circ = \pi$ and $T_i^\circ = T_i^{-1}$ for $1 \leq i < N$. Then

$$D_h^y = iD_h^x, \quad \forall h \in H(k^{-1}). \quad (3.3.5)$$

This follows by verifying the identity

$$\rho_{k,q}^{-1}(h^\circ) = i\rho_{k^{-1},q}^x(h)i$$

for the algebraic generators $\pi$ and $T_i (1 \leq i < N)$ of $H(k^{-1})$ using Theorem 2.2.4.

Recall the generic principal series representation, encoded by the algebra homomorphism $\eta : H(k) \to \text{End}(H_0)^\xi$ (see Subsection 2.4.1).

**Proposition 3.3.6.** (i) For $h \in H(k^{-1})$ we have

$$\vartheta_T(D_h^x)U = \eta(h^\dagger)U, \quad (3.3.6)$$

where $\dagger : H(k^{-1}) \to H(k)$ is the unique anti-algebra isomorphism satisfying

$$T_i^\dagger = T_i^{-1}, \quad \pi^\dagger = \pi^{-1}.$$
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for $1 < i < N$.

(ii) For $h \in H(k)$ we have

$$\vartheta(D^n_h)U = C_{t,1}(\eta(h^i))C,U,$$

(3.3.7)

where $\vartheta: H(k) \to H(k)$ is the unique anti-algebra involution satisfying

$$T^\vartheta_i = T_i, \quad \pi^\vartheta = \pi^{-1}$$

for $1 < i < N$ (note that $\vartheta = \vartheta \circ \vartheta$).

Proof. (i) We first show that it suffices to prove (3.3.6) for algebraic generators of $H(k^{-1})$. Indeed, since $\eta$ commutes with $\vartheta$, the action of $W$ on $\eta$ should be viewed as element in $\text{End}(\text{End}(H_{0k}))$ by left multiplication. Indeed, since $\eta(h^i)(t, \gamma)$ does not depend on the torus parameter $t \in T$, it commutes with $\vartheta(D^n_h) \in \vartheta(\mathbb{C}(T \times \{1\}) \# (W \times \{e\}))$ (which involves, besides the action of $W \times \{e\}$, only right multiplication by the monodromy cocycle).

So it remains to verify (3.3.6) for $h = \pi \in H(k^{-1})$ and for $h = T_i \in H(k^{-1})$ ($1 < i < N$). For $h = \pi \in H(k^{-1})$ we have

$$\vartheta(D^n_h)U = \vartheta((\pi, e))U = C_{(\pi, e)}^{-1}U = \eta(\pi^1)U,$$

where the last equality follows from (2.4.5). For $h = T_i \in H(k^{-1})$ ($1 < i < N$), we have

$$\vartheta(D^n_{T_i})U = (k^{-1} - c_k(x_{i+1}/x_i))U + c_k(x_{i+1}/x_i)\vartheta((s_i, e))U$$

$$= (k^{-1} - c_k(x_{i+1}/x_i))U + c_k(x_{i+1}/x_i)C_{(s_i, e)}^{-1}U$$

$$= (T_i^\vartheta)_U,$$

where we used that $c_{k-1}(z^{-1}) = c_k(z)$ in the first equality, while the second equality follows from Lemma 3.3.2(i) and the third equality from Lemma 2.4.3.

(ii) Unfortunately it is not possible to derive (ii) directly from (i) and from (3.3.5). Instead, one has to repeat the steps of the proof of (i). It again amounts to verifying (3.3.7) for $h = \pi \in H(k)$ and for $h = T_i \in H(k)$ ($1 < i < N$). We show the second case, the first case is left to the reader.

Let $1 < i < N$. Then we have for $T_i \in H(k)$,

$$\vartheta(D^n_{T_i})U = (k - c_k(y_i/y_{i+1}))U + c_k(y_i/y_{i+1})\vartheta((e, s_i))U$$

$$= (k - c_k(y_i/y_{i+1}))U + c_k(y_i/y_{i+1})C_{(e, s_i)}^{-1}U.$$
Since

\[ C_{(e,s_i)} = C_{i\ell}(C_{(s_i,e)})C_i \]

by the cocycle condition (recall Remark 2.3.5) and since \( C_i^2 = id \) and \( C_{(s_i,e)}(t, \gamma)^{-1} = C_{(s_i,e)}(s_i, \gamma) \), Lemma 2.4.3 implies that

\[ C_{(e,s_i)}^{-1} = C_{i\ell}(y_i/y_{i+1})^{-1}(C_i\ell(\eta(T_i))C_i - k) + 1. \]

Substituting in (3.3.8) gives \( \vartheta_T(D^i_{p})U = C_i\ell(\eta(T_i))C_iU \), as desired. \( \square \)

The following lemma plays an important role in the bispectral version of the correspondence. Recall that the center \( Z(H) \) of the affine Hecke algebra \( H \) is given by \( C_V[T]^{S_N} \) (Bernstein, see [38]).

**Lemma 3.3.7.** For \( p \in \mathbb{C}[T]^{S_N} \) we have

\[ p(Y)^{1} = p(Y^{-1}), \quad p(Y)^{1} = p(Y^{-1}). \]

**Proof.** By (2.2.4) it immediately follows that \( Y_i^{1} = Y_i^{-1} \) for \( 1 \leq i \leq N \). This implies the first formula.

For the second formula, it suffices to show that

\[ Y_i^{1} = T_{w_0}Y_{N-i+1}^{-1}T_{w_0}^{-1} \]

in \( H(k) \) for \( 1 \leq i \leq N \), since we then have, for \( p \in \mathbb{C}[T]^{S_N} \),

\[ p(Y)^{1} = T_{w_0}p(Y_{N}^{-1}, \ldots, Y_{1}^{-1})T_{w_0}^{-1} = T_{w_0}p(Y^{-1})T_{w_0}^{-1} = p(Y^{-1}), \]

where the last equality follows from the fact that \( p(Y^{-1}) \in Z(H(k)) \). To prove (3.3.9), note that \( T_{w_0}T_{w_0}^{-1} = T_{N-i}, T_{w_0}, \) and \( Y_{i+1} = T_{w_0}^{-1}Y_{i}T_{w_0}^{-1} \) by (2.2.4), for \( 1 \leq i < N \). Hence (3.3.9) holds for \( Y_{i+1} \) if it is true for \( Y_i \). It thus remains to prove (3.3.9) for \( i = 1 \).

We will use the following observation. Write \( \sigma_i = s_is_{i+1} \cdots s_{N-1} (1 \leq i < N) \) and \( \tau_j = s_j \cdots s_{N-2} (1 \leq j < N - 1) \), which are reduced expressions in \( S_N \). Then the longest Weyl group element \( w_0 \in S_N \) can be written as

\[ w_0 = \sigma_{N-1}\sigma_{N-2} \cdots \sigma_1 = \sigma_1(\tau_{N-2}\tau_{N-3} \cdots \tau_1), \]

(3.3.10)

and \( \ell(w_0) \) is the sum of the lengths of the factors in the respective products in (3.3.10).

By (2.2.4), formula (3.3.9) for \( i = 1 \) will be valid if

\[ T_{\sigma_1}\pi^{-1} = T_{w_0}\pi^{-1}T_{\sigma_1}T_{w_0}^{-1} \]

(3.3.11)

in \( H(k) \). By the first expression in (3.3.10) and the fact that \( \pi^{-1}T_{i+1}\pi = T_i \) for \( 1 \leq i < N - 1 \), we have in \( H(k) \),

\[ \pi^{-1}T_{\sigma_1}T_{w_0}^{-1} = \pi^{-1}T_{\sigma_1}^{-1} \cdots T_{\sigma_{N-1}}^{-1} = T_{\tau_1}\tau_2 \cdots T_{\tau_{N-2}}\pi^{-1}, \]
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so that (3.3.11) will follow from

\[ T_{\sigma_1} = T_{w_0} T_{\tau_1}^{-1} T_{\tau_2}^{-1} \cdots T_{\tau_N}^{-1} \]

in \( H(k) \). But this is a direct consequence of the second expression of \( w_0 \) in (3.3.10).

\[ \square \]

Corollary 3.3.8. For \( p \in \mathbb{C}[T]^{S_N} \), we have \( p(Y)^\circ = p(Y) \), where \( ^\circ : H(k^{-1}) \to H(k) \) is the algebra isomorphism defined in Remark 3.3.5.

**Proof.** This follows from the previous lemma and the fact that \( \dag = * \circ \).

\[ \square \]

Definition 3.3.9. (i) Define

\[ L^x_p := \text{Res}(D^x_p(Y)) \in \mathbb{D}, \quad p \in \mathbb{C}[T]^{S_N}, \]

where \( p(Y) \) is the corresponding element in \( C_Y[T]^{S_N} = Z(H(k^{-1})) \).

(ii) Define

\[ L^y_p := \text{Res}(D^y_p(Y)) \in \mathbb{D}, \quad p \in \mathbb{C}[T]^{S_N}, \]

where \( p(Y) \) is the corresponding element in \( Z(H(k)) \).

By Corollary 3.3.8 and (3.3.5) we have

\[ L^y_p = \iota L^x_p \iota, \quad \forall p \in \mathbb{C}[T]^{S_N}. \]

Furthermore, it is well-known (see [10] and [42]) that the \( L^x_p \in \mathbb{C}(\{1\} \times T) \#(\mathbb{Z}^N \times \{e\}) \subset \mathbb{D} \) are pairwise commuting and \( S_N \times S_N \)-invariant,

\[ wL^x_p w^{-1} = L^x_p, \quad \forall w \in S_N \times S_N. \]

Similarly, the \( L^y_p = \iota L^x_p \iota \in \mathbb{C}(\{1\} \times T) \#(\mathbb{Z}^N \times \{e\}) \subset \mathbb{D} \) are pairwise commuting and \( S_N \times S_N \)-invariant. Clearly also \([L^x_p, L^y_{p'}] = 0\) for all \( p, p' \in \mathbb{C}[T]^{S_N} \) in \( \mathbb{D} \).

For the elementary symmetric functions \( e_i \in \mathbb{C}[T]^{S_N} \) (1 \( \leq i \leq N \)) given by

\[ e_i(t) = \sum_{I \subseteq \{1, \ldots, N\}} \prod_{j \in I} t_j, \]

the corresponding \( L^x_{e_i} \), viewed as elements in

\[ \mathbb{C}(T) \#_q \mathbb{Z}^N = \mathbb{C}(T \times \{1\}) \#(\mathbb{Z}^N \times \{e\}) \subset \mathbb{D}, \]

are explicitly given by

\[ L^x_{e_i} = \sum_{I \subseteq \{1, \ldots, N\} \atop \#I = i} \left( \prod_{r \in I \atop s \not\in I} \frac{t^r - k^{-1}t^s}{t^r - t^s} \right) \sum_{r \in I} e_r \in \mathbb{C}(T) \#_q \mathbb{Z}^N, \quad 1 \leq i \leq N; \quad (3.3.12) \]

see, e.g., [10, §1.3.5] and [32]. Hence, the \( L^x_{e_i} \) (1 \( \leq i \leq N \)) are, under their natural interpretation as \( q \)-difference operators on \( \mathcal{M}(T) \), Ruijsenaars’ commuting, trigonometric \( q \)-difference operators from [50].
**Definition 3.3.10.** Consider the bispectral problem

\[
(L_p^x f)(t, \gamma) = p(\gamma^{-1}) f(t, \gamma), \quad \forall p \in \mathbb{C}[T]^{S_N}, \\
(L_p^y f)(t, \gamma) = p(t) f(t, \gamma), \quad \forall p \in \mathbb{C}[T]^{S_N}
\]

(3.3.13)

for \( f \in \mathbb{K} \), where the equations (3.3.13) are viewed as identities between meromorphic functions in \((t, \gamma) \in T \times T\). We write BiSP \( \subset \mathbb{K} \) for the set of solutions \( f \in \mathbb{K} \) of (3.3.13).

**Remark 3.3.11.** The bispectral problem for ordinary linear differential operators was introduced by Duistermaat and Grünbaum in [13]. Many different types of bispectral problems have since been considered. In particular, in [22] the bispectral problem for ordinary linear second-order \( q \)-difference operators is investigated. For \( N = 2 \), our bispectral problem belongs to this class.

The preceding remarks on the invariance properties of the \( L_p^x \) and \( L_p^y \) (\( p \in \mathbb{C}[T]^{S_N} \)) directly give

**Lemma 3.3.12.** BiSP is an \( S_N \)-invariant \( \mathbb{F} \)-subspace of \( \mathbb{K} \) with respect to the usual \( S_N \)-action \((w f)(t, \gamma) = f(w^{-1}(t, \gamma)) \) on \( f \in \mathbb{K} \).

We can now prove the following bispectral version of the correspondence between solutions of the quantum KZ equations and the spectral problem of the \( L_p^x \) (\( p \in \mathbb{C}[T]^{S_N} \)).

**Theorem 3.3.13.** The linear functional \( \chi_+ \in H_0^x \) (see Lemma 3.3.3) defines a \( S_N \)-equivariant \( \mathbb{F} \)-linear map

\[ \chi_+ : \text{SOL} \to \text{BiSP}. \]

**Proof.** The \( \mathbb{K} \)-linear extended linear functional \( \chi_+ \) defines an \( S_N \)-equivariant, \( \mathbb{F} \)-linear map \( \chi_+ : H_0^x \to \mathbb{K} \), since Lemma 3.3.3 implies that \( \chi_+(\tau(w)f) = w(\chi_+f) \) for \( w \in S_N \) and \( f \in H_0^x \). Hence \( \chi_+ \) restricts to an \( S_N \)-equivariant, \( \mathbb{F} \)-linear map \( \chi_+ : \text{SOL} \to \mathbb{K} \).

It remains to show that \( \chi_+(f) \in \text{BiSP} \) if \( f \in \text{SOL} \). Let \( f \in \text{SOL} \). By Proposition 2.5.13 and \( \mathbb{F} \)-linearity, it suffices only to consider \( f \) of the form \( f = Uv \) for \( v \in H_0^y \). Then \( \chi_+(f) = \chi_+(Uv) = \phi^U_{\chi_+,v} \). For \( p \in \mathbb{C}[T]^{S_N} \) we have

\[
\left( L_p^y \phi^U_{\chi_+,v} \right) (t, \gamma) = \left( \text{Res}(D_{p(y)}^y)(\phi^U_{\chi_+,v}) \right) (t, \gamma) = \phi^\tau(D_{p(y)}^y U)(t, \gamma) \\
= \phi^{q(p(y)^{-1})U}_{\chi_+,v}(t, \gamma) = p(\gamma^{-1}) \phi^U_{\chi_+,v}(t, \gamma)
\]

as meromorphic functions in \((t, \gamma) \in T \times T\), where the last equality follows from Lemma 3.3.7, (2.4.3) and the fact \( p \in \mathbb{C}[T]^{S_N} \). Similarly,

\[
\left( L_p^x \phi^U_{\chi_+,v} \right) (t, \gamma) = \left( \text{Res}(D_{p(y)}^x)(\phi^U_{\chi_+,v}) \right) (t, \gamma) = \phi^\tau(D_{p(y)}^x U)(t, \gamma) \\
= \phi^{C_{\chi_+(\tau(y))} U}_{\chi_+,v}(t, \gamma) = p(t) \phi^U_{\chi_+,v}(t, \gamma)
\]

as meromorphic functions in \((t, \gamma) \in T \times T\), hence \( f = \phi^U_{\chi_+,v} \in \text{BiSP} \). \( \square \)
3.4 Bispectral Harish-Chandra series

Definition 3.4.1. We call $\Phi^+ := \chi_\kappa + \Phi^\kappa \in \BiSP$ the basic Harish-Chandra series solution of the bispectral problem.

Corollary 3.4.2. The solution $\Phi^+ \in \BiSP$ of the bispectral problem is self-dual, i.e.,

$$\Phi^+(t, \gamma) = \Phi^+(\gamma^{-1}, t^{-1})$$

as meromorphic functions in $(t, \gamma) \in T \times T$.

Proof. By Theorem 2.5.6, we have

$$\Phi^+(t, \gamma) = \chi_\kappa(C\Phi^\kappa(\gamma^{-1}, t^{-1})).$$

But $\chi_\kappa C = \chi_\kappa$, hence the result. $\square$

Remark 3.4.3. In [18], for special values of $\kappa$, the function $\Phi^+$ is constructed as formal power series in terms of generalized characters of Verma modules over the quantum group $U_q(\mathfrak{sl}_N)$ (see also [16], [17]). The quantum group approach also leads to the self-duality of $\Phi^+$; see [18, Theorem 5.6] (see [17]).

Note that $\Phi^+ = W_\kappa \Psi^+$ with $\Psi^+ = \chi_\kappa(\Psi)$. For $\alpha \in Q_+$ set

$$\Gamma^+_\alpha(\gamma) = \chi_\kappa(\Gamma^\kappa_\alpha(\gamma))$$

as meromorphic function in $\gamma \in T$. By Lemma 2.5.7 and Theorem 2.5.10, $\Gamma^+_\alpha$ is analytic at $T \setminus S_+$ and

$$\Gamma^+_\alpha(\gamma) = k^{(\gamma)} K(\gamma)$$

with $K$ given by (2.5.17). Recall that the solution space $\BiSP$ of the bispectral problem is $S_N$-stable. In particular, we have solutions $\Phi^w \in \BiSP$ given by

$$\Phi^w(t, \gamma) := \Phi^+(t, w^{-1}\gamma).$$

These are solutions of the bispectral problem which are asymptotically free in the asymptotic sector $\{t \in T | |t^\alpha| > 0 \ \forall \ 1 \leq i < N\}$ in the following sense: by Lemma 2.5.7 we have $\Phi^w(t, \gamma) = W_\kappa(t, w^{-1}\gamma)\Psi^w(t, \gamma)$ with $\Psi^w(t, \gamma) := \Psi^+(t, w^{-1}\gamma)$ admitting, for $\epsilon > 0$ sufficiently small, the power series expansion

$$\Psi^w(t, \gamma) = \sum_{\alpha \in Q_+} \Gamma^+_\alpha(w^{-1}\gamma)t^{-\alpha}$$

for $(t, \gamma) \in B^{-1}_\epsilon \times T \setminus w(S_+)$, converging normally in compacta of $B^{-1}_\epsilon \times T \setminus w(S_+)$. 

Proposition 3.4.4. The set of asymptotic solutions $\{\Phi^w\}_{w \in S_N} \subset \BiSP$ of the bispectral problem is $\mathbb{F}$-linearly independent.
Proof. Suppose that
\[ \sum_{w \in S} a_w(t, \gamma) \Phi_w^+(t, \gamma) = 0 \]
as meromorphic functions in \((t, \gamma) \in T \times T\) with coefficients \(a_w \in \mathbb{F}\) \((w \in S_N)\). Replacing \(t\) by \(q^{-m}\delta t\) \((m \in \mathbb{N})\) and using (2.5.5) we obtain
\[ \sum_{w \in S} k^{-m(\delta, \delta)} a_w(t, \gamma) W_\kappa(t, w^{-1}\gamma) \Phi_w^+(q^{-m}\delta t, \gamma) = 0 \] 
(3.4.3)
as meromorphic functions in \((t, \gamma) \in T \times T\). Fix \(u \in S_N\). We are going to derive from (3.4.3) that \(a_u = 0\). For this we will use the fact that for \(w \neq u\),
\[ \lim_{m \to \infty} \zeta^m(uw_0(\delta) - uw_0(\delta)) = \lim_{m \to \infty} (w_0u^{-1}\zeta)^m(\delta - w_0u^{-1}uw_0(\delta)) = 0 \] 
(3.4.4)
if \(\zeta \in uw_0(B_1)\).

Recall the \(W\)-invariant subset \(S \subset T\) (see (2.4.6)), which contains \(S_+\). For generic \(\zeta \in T\) (concretely, \(\zeta \notin S\), and \(a_w(t, \gamma)\) and \(W_\kappa(t, w^{-1}\gamma)\) specializable at \(\gamma = \zeta\) for all \(w \in S_N\)), it follows from Proposition 2.5.9 and (3.4.3) that, for all \(m \in \mathbb{N}\),
\[ \sum_{w \in S_N} \zeta^m(uw_0(\delta) - uw_0(\delta)) a_w(t, \zeta) W_\kappa(t, w^{-1}\zeta) \Phi_w^+(q^{-m}\delta t, \zeta) = 0 \] 
(3.4.5)
as meromorphic function in \(t \in T\). Using (3.4.4) and the power series expansion (3.4.2), the limit \(m \to \infty\) of (3.4.5) yields, for generic \(\zeta \in uw_0(B_1)\),
\[ a_u(t, \zeta) W_\kappa(t, w^{-1}\zeta) \Gamma_0^+(u^{-1}\zeta) = 0 \]
as meromorphic function in \(t \in T\). This implies \(a_u = 0\), as desired. \(\square\)

**Corollary 3.4.5.** The map \(\chi_+ : \text{SOL} \to \text{BiSP}\) is injective.

**Proof.** Note that \(\chi_+(r(\epsilon, w) \Phi_\kappa) = \Phi_w^+ (w \in S_N)\). The statement follows now directly from Proposition 3.4.4 and Proposition 2.5.13. \(\square\)

### 3.5 Specialized central character and Harish-Chandra series

We write
\[ \text{SP}_\zeta = \{ f \in \mathcal{M}(T) \mid L_p^\zeta f = p(\zeta^{-1}) f \quad \forall p \in \mathbb{C}[T]^{S_N} \} \]
for the spectral problem of the Ruijsenaars \(q\)-difference operators with fixed spectral parameter \(\zeta \in T\). Note that \(\text{SP}_\zeta \subset H_0^{\mathcal{M}(T)}\) is \(S_N\)-stable, with \(S_N\)-action on \(H_0^{\mathcal{M}(T)}\) given by \((w f)(t) = f(w^{-1}t)\) for \(f \in H_0^{\mathcal{M}(T)}\) and \(w \in S_N\).

By [14, Proposition 5.2], the quantum KZ equations (2.4.10) are consistent for all values \(\zeta \in T\) of the central character. The arguments from Subsection 3.3, applied
to the quantum KZ equations (2.4.10) for fixed $\zeta$ and with the role of $U$ taken over by an invertible matrix solution $U_\zeta$ (2.4.10), result in the following special case of the Cherednik-Matsuo correspondence from [7, 8] (concretely, in the notations of [8], take the principal series module $V = M_\zeta$ in [8, Theorem 4.2] and let $\tau$ be the projection from $M_\zeta$, along the direct sum decomposition of $M_\zeta$ in $H_0$-isotypical components, onto the trivial component).

**Proposition 3.5.1.** Let $\zeta \in T$. Then $\chi_+$ defines an $E(T)$-linear $S_N$-equivariant map $\chi_+: \text{SOL}_\zeta \to \text{SP}_\zeta$.

For a further analysis of the map $\chi_+: \text{SOL}_\zeta \to \text{SP}_\zeta$, we refer to [8] and [10, §1.3.4].

Harish-Chandra type series solutions of the spectral problem of the Ruijsenaars q-difference operators $L^\pm_p$ ($p \in \mathbb{C}[T]^{S_N}$) with fixed spectral parameter $\zeta \in T$ were studied in, e.g., [16] and [31] (see also [36] for arbitrary root systems). The results of the previous section allow us to reobtain these solutions by specialization of the basic Harish-Chandra series $\Phi^+_\kappa$. It leads to new results on the convergence and singularities of these solutions, which we state now explicitly.

By Subsection 2.5.4, for generic $\kappa \in \mathbb{C}^\times$ the basic Harish-Chandra series $\Phi^+_\kappa(t, \gamma)$ is specializable at $\gamma = \zeta$ when $\zeta \in T \setminus S_\tau$. Concretely, for $\zeta \in T \setminus S_\tau$ and generic $\kappa \in \mathbb{C}^\times$, we can write

$$\Phi^+_\kappa(t, \zeta) = W_\kappa(t, \zeta)\Psi^+(t, \zeta)$$

as meromorphic function in $t \in T$, where $\Psi^+ = \chi_+(\Psi)$ (see Section 3.4). Due to the results in Subsection 2.5.4 (see Proposition 2.5.8) we obtain the following result.

**Corollary 3.5.2.** For $\zeta \in T \setminus S_\tau$, the meromorphic function $\Psi^+(t, \zeta)$ in $t \in T$ is analytic at $t \in T \setminus S_\tau^{-1}$.

Let $\zeta \in T \setminus S$, where $S \subseteq T$ is the $W$-invariant set (2.4.6). For $\kappa \in \mathbb{C}^\times$ such that $W_\kappa(t, w^{-1}\gamma)$ may be specialized at $\gamma = \zeta$ for all $w \in S_N$, the asymptotic solutions $\Phi^+_w(t, \gamma)$ ($w \in S_N$) of the bispectral problem (see (3.4.1)) may thus be specialized at $\gamma = \zeta$, giving rise to solutions $\Phi^+_w(; \zeta) \in \text{SP}_\zeta$ ($w \in S_N$); see Corollary 2.5.14 and Proposition 3.5.1. Observe that for $\epsilon > 0$ sufficiently small,

$$\Phi^+_w(t, \zeta) = W_\kappa(t, w^{-1}\zeta) \sum_{\alpha \in Q_+} \Gamma^+_\alpha(w^{-1}\zeta)t^{-\alpha}$$

for $t \in B_w^{-1}$, with normal convergence of the power series on compacta of $B_w^{-1}$. Since $\zeta \notin S$ we furthermore have

$$\Gamma^+_0(w^{-1}\zeta) = k(\zeta)K(w^{-1}\zeta) \neq 0,$$

with $K$ given by (2.5.17).

**Definition 3.5.3.** Let $\zeta \in T \setminus S$. The $\Phi^+_w(\cdot; \zeta) \in \text{SP}_\zeta$ ($w \in S_N$) are the Harish-Chandra series solutions of the spectral problem $L^+_p f = p(\zeta^{-1}) f (p \in \mathbb{C}[T]^{S_N})$.  

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Remark 3.5.4. In [16] (and [36]) the Harish-Chandra series are investigated as formal power series solutions to the spectral problem of the Ruijsenaars operators. The advantage of the present approach is the fact that it implies the convergence of the formal power series, basically as a consequence of a general statement about convergence of formal power series solutions of holonomic systems of \(q\)-difference equations (see the appendix). Chalykh’s [5] Baker-Akhiezer functions arise as Harish-Chandra series solutions for special values of \(k\); see [36, §4.4]. In [31], the Harish-Chandra series solutions of the Ruijsenaars operators are constructed as matrix coefficients of products of vertex operators. By this approach, one obtains an explicit integral representation of the Harish-Chandra series.

Remark 3.5.5. Observe that

\[
\lim_{\lambda \to \infty} \frac{\Phi^+_{\lambda}(t, q^\lambda k^{-\delta})}{W_{\epsilon}(t, q^\lambda k^{-\delta})} = \Gamma^+_0(t^{-1}) = k(\frac{n}{2}) K(t^{-1}) \quad (3.5.1)
\]

with \(\lambda \to \infty\) meaning \(\lambda_i - \lambda_{i+1} \to \infty\) for all \(1 \leq i < N\). Thus, \(K\) (see (2.5.17)) is a normalized limit of the asymptotic solutions \(\Phi^+_{\lambda}(\cdot, q^\lambda k^{-\delta}) \in \text{SP}_{q^\lambda k^{-\delta}}\). The solution space \(\text{SP}_{q^\lambda k^{-\delta}}\) contains the symmetric Macdonald polynomial of degree \(\lambda \in \Lambda\). It turns out though that \(\Phi^+_{\lambda}(\cdot, q^\lambda k^{-\delta})\) is not a multiple of the Macdonald polynomial of degree \(\lambda \in \Lambda\), but \(\Phi^+_{\lambda}(\cdot, q^{\omega_0(\lambda)} k^\delta)\) is (this will become apparent in the next section). On the other hand, the leading coefficient \(K\) (see (2.5.17)) also naturally appears as a normalized limit of the Macdonald polynomial when the degree \(\lambda \in \Lambda\) of the polynomial tends to infinity; see [11, Lemma 4.3] (this limit was proven in [51] in the \(L^2\)-sense).