Chapter 4

Polynomial solutions of quantum KZ equations and Macdonald polynomials

4.1 Introduction

In view of Proposition 2.4.9, the cocycle matrices corresponding to the dual quantum KZ equations of the bispectral quantum KZ equations can be viewed as shift operators, mapping solutions of $qKZ_{\zeta}$ (for fixed central character $\zeta \in T$; see (2.4.10)) to solutions of $qKZ_{\zeta'}$ where $\zeta' \in T$ is a certain shift of $\zeta$. In this chapter we use this fact to create Laurent polynomial solutions of quantum KZ equations, starting from a constant solution of the quantum KZ equations. Exploiting the correspondence with the spectral problem of the Ruijsenaars $q$-difference operators (see Proposition 3.5.1), this leads to a new construction of the symmetric self-dual Macdonald polynomials. From the opposite perspective, we could say that we have found analogs of the symmetric Macdonald polynomials as solutions of quantum KZ equations. Anyway, together with the results from the previous chapters, these observations yield a new approach to the well-known duality and evaluation formulas for the symmetric Macdonald polynomials ([41, Chapter VI]).

Let us give a detailed description of the contents of this chapter. In Section 4.2, we start with constructing a constant solution of the quantum KZ equations. From this constant solution we obtain Laurent polynomial solutions $Q_{\lambda}$ (with $\lambda$ running over $\Lambda \subset \mathbb{Z}^N$; see (2.5.13)) by means of the cocycle matrices of the dual quantum KZ equations.

In Section 4.3, we prove that the $Q_{\lambda}$ satisfy a certain duality property between $\lambda \in \Lambda$ and specific points of evaluation. The Laurent polynomial $Q_{\lambda}$ can be related to the basic asymptotic solution $\Phi_{\kappa}$ of $BqKZ$ (see Subsection 2.5.2) and this happens in Section 4.4.
In the final section, we use the correspondence (Proposition 3.5.1) between the quantum KZ equations and the spectral problem of the Ruijsenaars \(q\)-difference operators to derive from \(Q_\lambda\) a symmetric self-dual Laurent polynomial eigenfunction of the Ruijsenaars operators, which we prove to be the normalized symmetric Macdonald polynomial of degree \(\lambda\). We reobtain the well-known duality and evaluation formulas for the Macdonald polynomials as consequences of the properties of the Laurent polynomials \(Q_\lambda\).

This chapter agrees with Section 7 of [45].

Convention

We adopt the notations from the previous chapters. In particular, we still have fixed \(0 < q < 1\). For various reasons, which we address specifically when appropriate, we need to impose some generic conditions on the Hecke algebra parameter \(k\). Concretely, we assume throughout this chapter that

\[
k^2j \not\in q\mathbb{Z}, \quad \forall 1 \leq j \leq N,
\]

\[
k^{(\delta, w_j - w((w_j)))} \not\in q\mathbb{Z}, \quad \forall 1 \leq j < N, \forall w \in S_N : w(w_j) \neq w_j.
\]

(4.1.1)

4.2 Constructing polynomial solutions of qKZ

We are going to use a special case of Proposition 2.4.9 to create \(S_N\)-invariant (with respect to the \(S_N\)-action \(\varsigma\) on \(\text{SOL}_q\); see (2.4.11)) polynomial solutions of the quantum KZ equations.

**Lemma 4.2.1.** Let \(\lambda \in \Lambda\). The possible poles of the \(\mathbb{C}[T]\otimes \text{End}(H_0)\)-valued rational function

\[
\gamma \mapsto C_{(e, -\lambda)}(\cdot, q^\lambda \gamma) = C_{(e, \lambda)}(\cdot, \gamma)^{-1}
\]

in \(\gamma \in T\) are at \(\gamma^\alpha \in k^2q^{-N}\) for some \(\alpha \in R^+\). The possible poles of

\[
\gamma \mapsto C_{(e, \lambda)}(\cdot, \gamma)
\]

are at \(\gamma^\alpha \in k^{-2}q^{-N}\) for some \(\alpha \in R^+\).

**Proof.** Since \(R_\lambda(z)\) has only a (simple) pole at \(z = k^{-2}\), this follows from (2.5.14) and the cocycle property of \(C\); see Lemma 2.5.7. \(\square\)

Since \(k\) satisfies \(k^{2j} \notin q \mathbb{Z}\) for \(1 \leq j \leq N\) by (4.1.1), the spectrum of \(\eta_{q^\lambda k^{-2}}(\mathbb{C}Y[T])\) is simple and the \(\xi_w(q^{\lambda k^{-2}}) (w \in S_N)\) form a \(\mathbb{C}\)-basis of \(H_0\) for all \(\lambda \in \Lambda\). Furthermore, for such \(k\) we have that \(\gamma \mapsto C_{(e, \lambda)}(\cdot, \gamma)^{\pm 1}\) is regular at \(\gamma = k^{-}\delta\) for all \(\lambda \in \Lambda\); see Lemma 4.2.1. The additional conditions on \(k\) in (4.1.1) will play a role in Sections 4.4 and 4.5.

Proposition 2.4.9 now immediately implies the following result.
Corollary 4.2.2. Let \( \lambda \in \Lambda \). Then \( f \mapsto C_{(e, \lambda)}(\cdot, k^{-\delta})^{-1} f \) defines an \( S_N \)-equivariant isomorphism \( \text{SOL}_{k^{-\delta}} \to \text{SOL}_{q^{-1}k^{-\delta}} \).

The special interest in the quantum KZ equations for the particular central character \( \gamma = q^k k^{-\delta} \) \( (\lambda \in \Lambda) \) comes from the fact that it admits \( S_N \)-invariant polynomial solutions. The key step in deriving this result is the following lemma.

Lemma 4.2.3. The element \( v_+ := \sum_{w \in S_N} k^{\ell(w)} T_w \in H_0 \) is a constant \( S_N \)-invariant solution of the quantum KZ equation with central character \( k^{-\delta} \). In other words,

\[
C^\delta_{\Lambda}(t) v_+ = v_+, \quad \forall \lambda \in \mathbb{Z}^N.
\]

Proof. Note that \( R_t(z) v_+ = v_+ \), so by Lemma 2.4.1 and Lemma 2.4.5(ii), for any \( \zeta \in T \),

\[
C^\delta_{\zeta}(t)v_+ = \eta(\pi)^i v_+ = \sum_{w \in S_N} k^{\ell(w)} \zeta^{-1} w_0 \pi^i T_{\sigma^i w} = \sum_{w \in S_N} k^{\ell(\sigma^{-1} i w)} \zeta^{-1} w_0 \pi^i T_{w}.
\]

Then use \( \ell(\sigma^{-1} i w) - \ell(w) = \langle \delta, w_0^{-1} \pi^i \rangle \) for \( 1 \leq i \leq N \) (for the proof of this formula, it suffices to prove it for \( i = 1 \)). In that case, look at the positive roots that are mapped to negative roots by \( \sigma^{-1} w \). It implies that

\[
C^\delta_{\zeta}(t)v_+ = \sum_{w \in S_N} k^{\ell(w)} (k^\delta \zeta)^{-1} w_0 \pi^i T_{w}.
\]

In particular, \( C^\delta_{\zeta}(t)v_+ = v_+ \) for all \( i \). Note, furthermore, that \( R_t(z) v_+ = v_+ \) implies that \( \varsigma(s_i)v_+ = C^\delta_{\varsigma_i}(t)v_+ = v_+ \) for all \( 1 \leq i < N \) (with \( \varsigma \) given by (2.4.11)). Hence, \( v_+ \in \text{SOL}_{k^{-\delta}} \) is \( S_N \)-invariant.

Proposition 4.2.4. For \( \lambda \in \Lambda \), the nonzero \( S_N \)-invariant solution

\[
Q_\lambda := C_{(e, \lambda)}(\cdot, k^{-\delta})^{-1} v_+ \in \text{SOL}_{q^{-1}k^{-\delta}}
\]

of the quantum KZ equation is an \( H_0 \)-valued Laurent polynomial on \( T \) satisfying

\[
Q_\lambda(t) = \sum_{\alpha \in Q_+} K_\alpha(\lambda) t^{\lambda - \alpha}, \quad \text{(4.2.1)}
\]

with \( K_\alpha(\lambda) \in H_0 \) (all but finitely many terms zero).

Proof. Note that Corollary 4.2.2 and Lemma 4.2.3 imply that \( 0 \neq Q_\lambda \in \text{SOL}_{q^{-1}k^{-\delta}} \) and that \( Q_\lambda \) is \( S_N \)-invariant. The triangularity property (4.2.1) follows from the cocycle property, (2.5.14), the explicit form of the \( R_t(z) \) and the fact that

\[
\eta(\pi)(t^{-1})^{-1} T_w = t^{w_0 \pi^i} T_{\sigma^{-1} w}, \quad w \in S_N,
\]

which is a direct consequence of Lemma 2.4.1.
4.3 Duality

**Lemma 4.3.1.** For \( \lambda \in \Lambda \), we have \( Q_\lambda(k^\delta) = v_+ \).

**Proof.** Since \( v_+ \in \text{SOL}_{k^{-\delta}} \) and \( C_i v_+ = v_+ \), we obtain

\[
Q_\lambda(k^\delta) = C_{(e,-\lambda)}(k^\delta, q^{\lambda} k^{-\delta}) v_+
\]

\[
= C_{(-\lambda,e)}(q^{\lambda} k^\delta, k^{-\delta}) C_i v_+ = v_+,
\]

for \( \lambda \in \Lambda \). \( \square \)

The polynomial solutions \( Q_\lambda \) of the quantum KZ equations are self-dual in the following sense.

**Proposition 4.3.2.** For \( \lambda, \mu \in \Lambda \), we have

\[
Q_\lambda(q^{-\mu} k^\delta) = C_i Q_\mu(q^{-\lambda} k^\delta).
\]

**Proof.** For \( \lambda, \mu \in \Lambda \) we have, using \( v_+ \in \text{SOL}_{k^{-\delta}} \) and the previous lemma,

\[
Q_\lambda(q^{-\mu} k^\delta) = C_{(e,-\lambda)}(q^{-\mu} k^\delta, q^{\lambda} k^{-\delta}) v_+
\]

\[
= C_{(e,-\lambda)}(q^{-\mu} k^\delta, q^{\lambda} k^{-\delta}) C_{(-\mu,e)}(q^{-\mu} k^\delta, k^{-\delta}) v_+ \tag{4.3.1}
\]

\[
= C_{(-\mu,-\lambda)}(q^{-\mu} k^\delta, q^{\lambda} k^{-\delta}) v_+.
\]

Since \( C_{(-\mu,-\lambda)}(q^{-\mu} k^\delta, q^{\lambda} k^{-\delta}) = C_i C_{(-\lambda,-\mu)}(q^{-\lambda} k^\delta, q^{\mu} k^{-\delta}) C_i \) and \( C_i v_+ = v_+ \), we conclude from (4.3.1) that \( Q_\lambda(q^{-\mu} k^\delta) = C_i Q_\mu(q^{-\lambda} k^\delta) \). \( \square \)

4.4 Relation to the basic asymptotically free solution

In this section, we relate the polynomial solutions \( Q_\lambda (\lambda \in \Lambda) \) of the quantum KZ equations to the basic asymptotic solution \( \Phi_\kappa \). Some care is needed though: it is not possible to specialize all the asymptotic solutions \( C_{(e,w)}(t,\gamma) \Phi_\kappa(t,w_0 \gamma) \) \( (w \in S_N) \) to \( \gamma = q^{\lambda} k^{-\delta} \) \( (\lambda \in \Lambda) \) since \( q^{\lambda} k^{-\delta} \in S \); see Corollary 2.5.14. We shall see that \( C_{(e,w)}(t,\gamma) \Phi_\kappa(t,w_0 \gamma) \) can be specialized at \( \gamma = q^{\lambda} k^{-\delta} \), which is sufficient for our purposes.

**Lemma 4.4.1.** Let \( \lambda \in \Lambda \). There exists a unique \( \Xi_\lambda \in \text{SOL}_{q^{\lambda} k^{-\delta}} \) such that, for \( \epsilon > 0 \) sufficiently small, we have an \( H_0 \)-valued power series expansion

\[
\Xi_\lambda(t) = \sum_{\alpha \in Q^+} \bar{\Gamma}_\alpha(\lambda) t^{\lambda-\alpha}
\]

converging normally on compacta of \( B^{-1}_\epsilon \) and with leading coefficient

\[
\bar{\Gamma}_0(\lambda) = \eta_{q^{\lambda} k^{-\delta}}(T_{w_0}) \xi_{w_0}(q^{\lambda} k^{-\delta}).
\]
Lemma 2.5.2. Observe that \( \tilde{\text{eigenvectors}} \) of \( H \) are such that

\[
\tilde{\lambda}_i(t) \Xi(t) = \Xi(t), \quad \Xi(t) \in H^M_0(T),
\]

with cocycle matrices \( \tilde{\lambda}_i(t) = q^{-(\lambda,\pi_i)} C_{(\pi_i,\nu)}(t, q^\lambda k^{-\delta}) \). Note that \( \tilde{\lambda}_N(t) = \text{id} \); see Lemma 2.5.2. Observe that \( \Xi(t) \) is a solution of the holonomic system (4.4.1) of q-difference equations if and only if \( x^\lambda \Xi(t) \in \text{SOL}_{q^\lambda k^{-\delta}} \). By Corollary 2.4.6, we have \( \tilde{\lambda}_i \in Q_0(A) \otimes \text{End}(H_0) \) and

\[
\tilde{\lambda}_i^{(0)}(q^{-(\lambda,\pi_i)} k^{(\delta,\pi_i)}) \eta_{q^\lambda k^{-\delta}} \left( T_{w_0} V_{w_0(\pi_i)} T_{w_0}^{-1} \right). \tag{4.4.2}
\]

The \( \tilde{\lambda}_i^{(0)}(1 \leq i < N) \) are semisimple endomorphisms of \( H_0 \). A basis of simultaneous eigenvectors of \( H_0 \) is given by \( \eta_{q^\lambda k^{-\delta}}(T_{w_0}) \xi_w(q^\lambda k^{-\delta}) \) \((w \in S_N)\). In fact,

\[
\tilde{\lambda}_i^{(0)}(\eta_{q^\lambda k^{-\delta}}(T_{w_0}) \xi_w(q^\lambda k^{-\delta})) = \gamma_{w,i} \eta_{q^\lambda k^{-\delta}}(T_{w_0}) \xi_w(q^\lambda k^{-\delta})
\]

for all \( 1 \leq i < N \) and \( w \in S_N \) with

\[
\gamma_{w,i} = q^{(\lambda, w^{-1} w_0(\pi_i) - \pi_i) k^{(\delta, \pi_i - w^{-1} w_0(\pi_i))}};
\]

see (2.5.10). Note, in particular, that \( \gamma_{w,i} \notin q^{-N} \) for all \( w \in S_N \) and all \( 1 \leq i < N \) by the generic conditions (4.1.1) on \( k \), and that \( \gamma_{w,i} = 1 \) for all \( 1 \leq i < N \). Hence, Theorem A.6 in the appendix, applied to (4.4.1) by taking \( M = N - 1 \), \( q_i = q \) and variables \( z_i = x^{-\alpha_i} \) \((1 \leq i < N)\) shows that there exists a unique \( \Xi(t) \in \mathcal{M}(T) \otimes H_0 \) satisfying (4.4.1) and admitting an \( H_0 \)-valued power series expansion

\[
\Xi(t) = \sum_{\alpha \in Q^+} \hat{\Gamma}_\alpha(\lambda)t^{-\alpha}
\]

converging normally on compacta of \( B^{-1}_\epsilon \) for some \( \epsilon > 0 \) small enough, and having as leading coefficient \( \hat{\Gamma}_0(\lambda) = \eta_{q^\lambda k^{-\delta}}(T_{w_0}) \xi_{w_0}(q^\lambda k^{-\delta}) \). This directly implies the lemma.

Recall that the cocycle values \( C_{(e, w)}(t, \gamma)(w \in S_N) \) are independent of \( t \in T \). We suppress \( t \) from the notation and simply write \( C_{(e, w)}(\gamma) \). Recall that \( C_{(e, w)}(\gamma) \) for \( w \in S_N \) is an \( \text{End}(H_0) \)-valued regular function in \( \gamma \in T \).

Theorem 4.4.2. Fix \( \lambda \in \Lambda \). For \( \kappa \notin q^2 \), the basic asymptotic solution \( \Phi_{\kappa}(t, \gamma) \) of \( BqKZ \)

\[ Q_\lambda(t) = r_\kappa C_{(e, w_0)}(q^\lambda k^{-\delta}) \Phi_{\kappa}(t, q^{w_0(\lambda) k^\delta}) \tag{4.4.3} \]

with

\[
r_\kappa = \theta(\kappa)^N k^{-\gamma(\frac{j}{2})} \prod_{1 \leq i < j \leq N} \frac{(k^{2(j-i+1)}; q)_{\infty}}{(k^{2j-i}; q)_{\infty}} \tag{4.4.4} \]

where

\[ Q_\lambda(t) = \prod_{\alpha \in Q^+} \hat{\Gamma}_\alpha(\lambda)t^{-\alpha} \]


Proof. We first show that both $Q_\lambda$ and the right-hand side of (4.4.3) are nonzero scalar multiples of $\Xi_\lambda$.

We start with the right-hand side of (4.4.3). Since $\Phi$ is $S_N$-stable, we have

$$\Phi_{w_0} := \tau(e, w_0)\Phi_\kappa \in \text{SOL}.$$ 

Concretely, it is given by

$$\Phi_{w_0}(t, \gamma) = C_{(e, w_0)}(\gamma)\Phi_\kappa(t, w_0(\gamma)) = W_\kappa(t, w_0(\gamma))C_{(e, w_0)}(\gamma)\Psi(t, w_0(\gamma)).$$

Since $w_0(q^\lambda k^{-\delta}) = q^{w_0(\lambda)k^\delta} \not\in S_+$ by (4.1.1), we may, in view of Proposition 2.5.9, specialize $\Phi_{w_0}(t, \gamma)$ at $\gamma = q^\lambda k^{-\delta}$, obtaining $\Phi_{w_0}(t, q^\lambda k^{-\delta}) \in \text{SOL}_{q^\lambda k^{-\delta}}$. By (2.5.3) we have

$$W_\kappa(t, w_0(q^\lambda k^{-\delta})) = k^{(\delta, \lambda)}\theta(\kappa)^{-N} t^\lambda,$$

hence by Proposition 2.5.9,

$$\Phi_{w_0}(t, q^\lambda k^{-\delta}) = k^{(\delta, \lambda)}\theta(\kappa)^{-N} \sum_{\alpha \in Q_+} \Gamma^{w_0}_\alpha t^\lambda - \alpha$$

with $\Gamma^{w_0}_\alpha = C_{(e, w_0)}(q^\lambda k^{-\delta})\Gamma_\alpha(q^{w_0(\lambda)k^\delta})$. From the definitions of $C_{(e, w_0)}$, $d_w$, $\eta$ and $\xi_{w_0}$ (see Subsections 2.2.3, 2.3.1 and 2.4.1) we have

$$C_{(e, w_0)}(\gamma)T_{w_0} = d_{w_0}(\gamma^{-1})^{-1}\eta(T_{w_0})\xi_{w_0}(\gamma) = \left( \prod_{\alpha \in R^+} \frac{1}{k - k^{-1}\gamma^\alpha} \right)\eta(T_{w_0})\xi_{w_0}(\gamma)$$

as $H_0$-valued regular functions in $\gamma \in T$. By Theorem 2.5.10, the leading coefficient $\Gamma^{w_0}_0$ thus simplifies to

$$\Gamma_0^{w_0} = K(q^{w_0(\lambda)k^\delta})C_{(e, w_0)}(q^\lambda k^{-\delta})T_{w_0} = K(q^{w_0(\lambda)k^\delta})d_{w_0}(q^{-\lambda k^{-\delta}})^{-1}\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta}),$$

where $K$ is given by (2.5.17). Combined with the previous lemma, we conclude that

$$\Phi_{w_0}(t, q^\lambda k^{-\delta}) = k^{(\delta, \lambda)}\theta(\kappa)^{-N} K(q^{w_0(\lambda)k^\delta})d_{w_0}(q^{-\lambda k^{-\delta}})^{-1}\Xi_\lambda(t).$$

In view of (4.1.1), $\Phi_{w_0}(t, q^\lambda k^{-\delta})$ thus is a nonzero constant multiple of $\Xi_\lambda(t)$.

Next, we consider $0 \neq Q_\lambda \in \text{SOL}_{q^\lambda k^{-\delta}}$. By Lemma 4.4.1 and (4.2.1), it suffices to note that $K_0(\lambda)$ is a constant multiple of $\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta})$, which follows directly from the fact that $K_0(\lambda) \in H_0$ satisfies

$$\tilde{A}^{i(0)}_k K_0(\lambda) = K_0(\lambda), \quad \forall 1 \leq i \leq N,$$

where $\tilde{A}_i$ is given by (4.4.2); see the proof Lemma of 4.4.1. Thus, $Q_\lambda(t)$ is a nonzero constant multiple of $\Xi_\lambda(t)$, and we conclude that

$$Q_\lambda(t) = r_\kappa(\lambda)\Phi_{w_0}(t, q^\lambda k^{-\delta}),$$
for some \( r_\kappa(\lambda) \in \mathbb{C}^\times \). We first show that \( r_\kappa(\lambda) \) is independent of \( \lambda \in \Lambda \).

For \( w \in S_N \), we write \( C_{(w,e)}(t) \) for the \( \gamma \)-independent value \( C_{(w,e)}(t, \gamma) \) of the cocycle. Let \( \lambda, \mu \in \Lambda \). By the \( S_N \)-invariance of \( Q_\lambda \), we then have, on the one hand,

\[
Q_\lambda(q^{-\mu}k^\delta) = C_{(w_0,e)}(q^{-\mu}k^\delta)Q_\lambda(q^{-w_0(\mu)}k^{-\delta}) = r_\kappa(\lambda)C_{(w_0,e)}(q^{-\mu}k^\delta)C_{(e,w_0)}(q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta) = r_\kappa(\lambda)C_{(w_0,w_0)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta).
\]

On the other hand, using the self-duality of \( Q_\lambda \) (see Proposition 4.3.2) and of \( \Phi_\kappa \) (see Theorem 2.5.6),

\[
Q_\lambda(q^{-\mu}k^\delta) = C_{(e,w_0)}(q^{-\lambda}k^\delta) = C_{(w_0,e)}(q^{-\lambda}k^\delta)Q_\mu(q^{-w_0(\lambda)}k^{-\delta}) = r_\kappa(\mu)C_{(w_0,e)}(q^{-\lambda}k^\delta)C_{(e,w_0)}(q^\mu k^{-\delta})\Phi_\kappa(q^{-w_0(\lambda)}k^{-\delta}, q^{w_0(\mu)}k^\delta) = r_\kappa(\mu)C_{(w_0,w_0)}(q^{-\lambda}k^\delta, q^\mu k^{-\delta})\Phi_\kappa(q^{-w_0(\lambda)}k^{-\delta}, q^{w_0(\mu)}k^\delta) = r_\kappa(\mu)C_{(w_0,w_0)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta).
\]

We conclude that \( r_\kappa(\lambda) = r_\kappa(\mu) \) if \( Q_\lambda(q^{-\mu}k^\delta) \neq 0 \). In particular, since \( Q_\lambda(k^\delta) = v_+ \neq 0 \), we have \( r_\kappa(\lambda) = r_\kappa(0) \) for all \( \lambda \in \Lambda \).

It remains to compute \( r_\kappa := r_\kappa(0) \). Using the fact that \( C_{(e,s_1)}(\gamma) = C_{(s_1,\gamma)}(\gamma^{-\alpha_1})C_{s_1} \) with \( R_i(z) = c_k(z)^{-1}(\eta(T_i^{-1}) - k^{-1}) + 1 \) for \( 1 \leq i < N \), as well as that \( C_{(s_1,\gamma)}(T_{w_0}^{-1}T_{w_0}) = T_{w_0}^{-1} \) for all \( w \in S_N \), we get \( C_{(e,w_0)}(\gamma)T_{w_0} = \sum_{w \leq w_0} c_w(\gamma)T_{w_0}^{-1} \) as \( H_0 \)-valued regular function in \( \gamma \in T \) with \( c_w \in \mathbb{C}[T] \) and with

\[
e_{w_0}(\gamma) = \prod_{\beta \in R^+} c_k(\gamma^{-\beta})^{-1}.
\]

Taking the \( T_\gamma \)-coefficient in the expansion of the formula

\[
v_+ = Q_0 = r_\kappa \Phi_{w_0}(\gamma, k^{-\delta}) = r_\kappa \theta(\kappa)^{-N}K(k^\delta)C_{(e,w_0)}(k^{-\delta})T_{w_0}
\]

with respect to the \( \mathbb{C} \)-basis \( \{T_w\}_{w \in S_N} \) of \( H_0 \), we conclude that

\[
r_\kappa = \theta(\kappa)^{-N}K(k^\delta)^{-1} \prod_{\beta \in R^+} c_k(k^{\delta,\beta}).
\]

Substituting the explicit expressions (2.2.5) and (2.5.17) of \( c_k \) and \( K \), respectively, we get the desired formula (4.4.4) for \( r_\kappa \).

The following formula is an analog for the \( Q_\lambda(\lambda \in \Lambda) \) of the evaluation formula for the self-dual symmetric Macdonald polynomials (see Section 4.5).
Corollary 4.4.3. Let $\lambda \in \Lambda$ and write $Q_\lambda(t) = \sum_{\alpha \in Q_+} K_\alpha(\lambda) t^{\lambda - \alpha}$ with $K_\alpha(\lambda) \in H_0$ (see Proposition 4.2.4). The leading coefficient $K_0(\lambda)$ is given by

$$K_0(\lambda) = k^{(\delta, \lambda)} \left( \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j - 1}{1 - q^{-m} k^{2(j-i)}} \right) \left( \prod_{m=0}^{1} \frac{1 - q^{-m} k^{2(j-i+1)}}{1 - k^{2(j-i)}} \right).$$

with

$$P(k^2) = \prod_{1 \leq i < j \leq N} \frac{1}{1 - k^{2(j-i)}}.$$ (4.4.8)

Proof. By (4.4.3) and Theorem 2.5.10, we have for $\lambda \in \Lambda$,

$$K_0(\lambda) = r_\lambda k^{(\delta, \lambda)} \theta(k)^{-N} K(q^{w_0(\lambda)} k^\delta) C_{(e, w_0)}(q^{\lambda} k^{-\delta}) T_{w_0},$$

with $K$ given by (2.5.17) and $r_\lambda$ given by (4.4.4). Substituting the explicit expressions for $K$ and $r_\lambda$ we get the desired expression.

The following consequence should be compared with the general expansion formula of $v_+ = \sum_{w \in S_N} k^{l(w)} T_w \in H_0$ in terms of the $\xi_w(\gamma)$ ($w \in S_N$); see [47, Lemma 2.27 (2)].

Corollary 4.4.4. The element $v_+ = \sum_{w \in S_N} k^{l(w)} T_w \in H_0$ can be written as

$$v_+ = k^{\binom{N}{2}} P(k^2) C_{(e, w_0)}(q^{\lambda} k^{-\delta}) T_{w_0}$$

$$= \left( \prod_{1 \leq i < j \leq N} \frac{1}{1 - k^{2(j-i)}} \right) \eta_{k^{-\delta}}(T_{w_0}) \xi_{w_0}(k^{-\delta}).$$ (4.4.9)

Proof. We have $v_+ = Q_0 = K_0(0)$, hence the previous corollary gives the first equality of (4.4.9). The second equality then follows from (4.4.6).

Applying the map $\chi_+$ to the first line of (4.4.9) gives

$$\sum_{w \in S_N} k^{2l(w)} = P(k^2)$$

with $P(k^2)$ given by (4.4.8), which is a well-known product formula for the Poincaré series of $S_N$; see [39, Corollary (2.5)].

4.5 Relation to symmetric self-dual Macdonald polynomials

In this section we collect various consequences of the previous sections for the symmetric Laurent polynomials $\chi_+(Q_\lambda) \in C[T]^{S_N}$ ($\lambda \in \Lambda$). We keep the generic conditions (4.1.1) on $k \in C^\times$. We define

$$E_\lambda := P(k^2)^{-1} \chi_+(Q_\lambda) \in C[T]^{S_N}, \quad \lambda \in \Lambda.$$
By Proposition 4.2.4, we have

\[ E_\lambda(t) = \sum_{\alpha \in Q_+} K_\alpha^+ \lambda \mu^{-\alpha}, \]

with \( K_\alpha^+ \lambda (\mu) \in \mathbb{C} \) all but finitely many zero, and with leading coefficient \( K_0^+ \lambda (\mu) \neq 0 \) by Corollary 4.4.3 and (4.1.1).

**Theorem 4.5.1.** The \( E_\lambda \in \mathbb{C}[T]^{S_N} (\lambda \in \Lambda) \) are the symmetric self-dual Macdonald polynomials. In other words, the \( E_\lambda \) are the unique symmetric regular functions on \( T \) satisfying

\[ L_p(E_\lambda) = p(q^{-\lambda}k^\delta) E_\lambda \quad \forall p \in \mathbb{C}[T]^{S_N} \quad (4.5.1) \]

and \( E_\lambda(k^\delta) = 1 \) for all \( \lambda \in \Lambda \).

**Proof.** By Proposition 3.5.1, \( E_\lambda \in \mathbb{C}[T]^{S_N} \) satisfies (4.5.1). Because the \( S_N \)-orbits \( S_N(q^{-\lambda}k^\delta) (\lambda \in \Lambda) \) in \( T \) are pairwise different by (4.1.1), the eigenvalue equations (4.5.1) uniquely characterize \( E_\lambda \in \mathbb{C}[T]^{S_N} \) up to a nonzero constant multiple. Now

\[ E_\lambda(k^\delta) = P(k^2)^{-1} \chi_+ (Q_\lambda(k^\delta)) = 1 \]

by Lemma 4.3.1, which fixes the normalization of the solution \( E_\lambda \in \mathbb{C}[T]^{S_N} \) of (4.5.1) uniquely.

The duality property of \( Q_\lambda \) (see Proposition 4.3.2) immediately gives the well-known duality property of the Macdonald polynomials.

**Corollary 4.5.2.** The Macdonald polynomials \( E_\lambda (\lambda \in \Lambda) \) are self-dual, in the sense that

\[ E_\lambda(q^{-\mu}k^\delta) = E_\mu(q^{-\lambda}k^\delta) \]

for all \( \lambda, \mu \in \Lambda \).

**Remark 4.5.3.** The self-duality of (the suitably normalized) Macdonald polynomials was initially proved by Koornwinder using Pieri formulas in an unpublished manuscript (the argument is reproduced in [41, VI (6.6)]). Cherednik ([10, Theorem 1.4.6] and [9, Theorem 3.2]) reproduced the self-duality of the Macdonald polynomials using the anti-involution \( * \) (see Theorem 2.2.8) on the double affine Hecke algebra.

We also immediately reobtain the well-known evaluation formula for the symmetric Macdonald polynomials; see [41, VI (6.11)] (the parameters \((n, q, t)\) in [41, Chapter VI] correspond to \((N, q^{-1}, k^2)\) in our notations).

**Corollary 4.5.4.** For \( \lambda \in \Lambda \) let \( P_\lambda := K_\lambda^+ (\lambda)^{-1} E_\lambda \in \mathbb{C}[T]^{S_N} \) be the monic symmetric Macdonald polynomial of degree \( \lambda \). Then

\[ P_\lambda(k^\delta) = k^{-\langle \delta, \lambda \rangle} \prod_{1 \leq i < j \leq N} \prod_{m=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^{-m} k^{2(j-i+1)}}{1 - q^{-m} k^{2(j-i)}}. \quad (4.5.2) \]
Proof. By the previous theorem we have $P_\lambda(k^\delta) = K_0^+(\lambda)^{-1}$. Corollary 4.4.3 gives

$$K_0^+(\lambda) = k^{(\delta, \lambda)} \prod_{1 \leq i < j \leq N}^{\lambda_i - \lambda_j - 1} \prod_{m=0}^{1 - q^{-m} k^{2(j-i)}} \frac{1 - q^{-m} k^{2(j-i+1)}}{1 - q^{-m} k^{2(j-i+1)}},$$

which implies the desired result. \qed