Bispectral quantum Knizhnik-Zamolodchikov equations
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Appendix: Holonomic systems of $q$-difference equations

In the appendix we detail the construction of power series solutions of holonomic systems of $q$-difference equations. Special cases have been investigated in, e.g., [2], [21] and [15, §12]. Many arguments go back to classical works [1], [3], [4], [61] on ordinary linear $q$-difference equations.

We begin with the construction of formal asymptotic solutions to holonomic systems of $q$-difference equations. Let $C[[z]] = C[[z_1, \ldots, z_M]]$ denote the ring of formal power series in $M$ indeterminates $z_1, \ldots, z_M$ over the complex numbers. Let $V$ be a finite-dimensional complex vector space and let $A_i \in C[[z]] \otimes \text{End}(V)$ for $i = 1, \ldots, M$. Since $C[[z]] \otimes \text{End}(V)$ is isomorphic to $\text{End}_C(C[[z]] \otimes V)$ as $C[[z]]$-module, we can view the $A_i$ as $C[[z]]$-linear endomorphisms of $C[[z]] \otimes V$. Fix $0 < q_i < 1$ for $1 \leq i \leq M$. Define the $q_i$-dilation operators $T_i: C[[z]] \to C[[z]]$

for $i = 1, \ldots, M$ as the complex linear maps

$$T_i \left( \sum_m d_m z^m \right) := \sum_m q_i^m d_m z^m \quad (d_m \in C),$$

where we use multi-index notation $z^m = z_1^{m_1} \cdots z_M^{m_M}$ for $m = (m_1, \ldots, m_M)$ with $m_j \in \mathbb{Z}_{\geq 0}$. We also view $T_i$ as operators on $C[[z]] \otimes V$ and on $C[[z]] \otimes \text{End}(V)$. Consider the system of first-order linear $q$-difference equations

$$A_i T_i f = f, \quad (i = 1, \ldots, M) \quad (A.7.7)$$

for $f \in C[[z]] \otimes V$.

For $f \in C[[z]] \otimes V$ and $A \in C[[z]] \otimes \text{End}(V)$, we introduce the notations

$$f^{(m)} := f|_{z_{m+1} = \ldots = z_M = 0} \in C[[z_1, \ldots, z_m]] \otimes V$$

$$A^{(m)} := A|_{z_{m+1} = \ldots = z_M = 0} \in C[[z_1, \ldots, z_m]] \otimes \text{End}(V)$$
for $0 \leq m \leq M$, with the convention that $f^{(M)} = f$ and $A^{(M)} = A$. We make the following assumptions on the system of $q$-difference equations (A.7.7):

(a) The system (A.7.7) is holonomic, that is

$$A_i T_i(A_j) = A_j T_j(A_i)$$

for all $1 \leq i, j \leq M$. Note that the holonomy implies that the leading coefficients $A_i^{(0)} \in \text{End}(V)$ of $A_i$ mutually commute, i.e.,

$$[A_i^{(0)}, A_j^{(0)}] = 0$$

for all $1 \leq i, j \leq M$.

(b) The complex linear endomorphisms $A_1^{(0)}, \ldots, A_M^{(0)}$ of $V$ are semisimple. Combined with (a) we thus have

$$V = \bigoplus_{\gamma \in S} V[\gamma]$$

with $V[\gamma] := \{v \in V \mid A_i^{(0)} v = \gamma_i v \ \forall i \}$ ($\gamma \in \mathbb{C}^M$) and $S := \{\gamma \in \mathbb{C}^M \mid V[\gamma] \neq \{0\}\}$. $$(1^M) := (1, \ldots, 1) \in \mathbb{C}^M$ belongs to $S$.

(c) $1^M_n \notin qk^{-N}$ for all $\gamma \in S$ and $1 \leq k \leq M$.

(d) $\gamma_k \in \mathbb{C}^M$ for all $\gamma \in S$ and $1 \leq k \leq M$.

**Proposition A.1.** Fix $v \in V[(1^M)]$. Consider the system (A.7.7) of $q$-difference equations and suppose that (a)-(d) are satisfied. Then there exists a unique solution $\Phi_v \in \mathbb{C}[[z]] \otimes V$ of (A.7.7) such that

$$\Phi_v^{(0)} = v.$$  

**Proof.** The proposition is a consequence of the following lemma.

**Lemma A.2.** Let $0 \leq m < M$. Suppose one has a solution

$$f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$$

of the system of equations

$$A_r^{(m)} T_r f_m = f_m, \quad 1 \leq r \leq m,$$

$$A_s^{(m)} f_m = f_m, \quad m < s \leq M.$$  

(A.7.9)

Then there exists a unique

$$f_{m+1} = \sum_{n \geq 0} f_{m,n} z_{m+1}^n \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V$$

with $f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m,0} = f_m$ satisfying (A.7.9) with the role of $m$ replaced by $m + 1$:

$$A_r^{(m+1)} T_r f_{m+1} = f_{m+1}, \quad 1 \leq r \leq m + 1,$$

$$A_s^{(m+1)} f_{m+1} = f_{m+1}, \quad m + 1 < s \leq M.$$  

(A.7.10)
The proposition follows directly from the lemma as follows. Note that \( f_0 := v \in V[(1^M)] \) is a solution of (A.7.9) for \( m = 0 \). The formal \( V \)-valued series \( f_M \in \mathbb{C}[[z]] \otimes V \), obtained by repeated application of the lemma starting from \( f_0 = v \), gives a formal \( V \)-valued series solution of (A.7.7) satisfying \( f_M^{(0)} = v \). For uniqueness, assume that \( f \in \mathbb{C}[[z]] \otimes V \) is another formal \( V \)-valued series satisfying \( f^{(0)} = v \) and solving (A.7.7). We have \( f^{(0)} = v = f_0 \) and \( f^{(m)} \) solves (A.7.9) for all \( 0 \leq m < M \). Hence, by the uniqueness part of the lemma, \( f = f(M) = f_M \).

We now proceed to prove the lemma. We assume that we have a formal power series solution \( f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) of (A.7.9) for some \( 0 \leq m < M \). We write

\[
A_r^{(m+1)} = \sum_{n \geq 0} A_r^{(m)} z_n^{m+1},
\]

where \( A_r^{(m)} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes \text{End}(V) \) and \( A_r^{(m)} = A_r^{(m)} \). By a direct computation one verifies that

\[
f_{m+1} = \sum_{n \geq 0} f_{m,n} z_n^{m+1} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V
\]

with \( f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and \( f_{m,0} = f_m \) satisfies the \( q \)-difference equation

\[
A_r^{(m+1)} f_{m+1} = f_{m+1}
\]

if and only if

\[
(1 - q_m^{n+1} A_r^{(m+1)}) f_{m,n} = \sum_{l=1}^{n} q_{m+1}^{n-l} A_r^{(m)} f_{m,n-l}
\]

for all \( n \in \mathbb{Z}_{\geq 0} \). The recurrence relations (A.7.12) admit a unique solution \( (f_{m,n})_{n \in \mathbb{Z}_{\geq 0}} \) with \( f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and with initial condition \( f_{m,0} = f_m \). Indeed, note that (A.7.12) is valid for \( n = 0 \) since \( f_{m,0} = f_m \) satisfies (A.7.9). For \( n \geq 1 \), we have

\[
\det (1 - q_m^n A_r^{(m+1)}) \in \mathbb{C}[[z_1, \ldots, z_m]]^x,
\]

since

\[
det (1 - q_m^n A_r^{(m+1)}) \mid_{z_1 = \cdots = z_m = 0} = det (1 - q_m^n A_r^{(0)}) \prod_{\gamma \in S} (1 - q_m^n \dim(V[\gamma])) \neq 0
\]

by assumption (d). Cramer’s rule then implies that (A.7.12) admits a unique solution \( (f_{m,n})_{n \in \mathbb{Z}_{\geq 0}} \) with \( f_{m,0} = f_m \).

We conclude that there exists a unique

\[
f_{m+1} = \sum_{n \geq 0} f_{m,n} z_n^{m+1}
\]

with \( f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and \( f_{m,0} = f_m \) satisfying the \( q \)-difference equation

\[
A_r^{(m+1)} f_{m+1} = f_{m+1}.
\]
It remains to show that \( f_{m+1} \) also satisfies (A.7.10) for \( r = 1, \ldots, m \) and for \( s = m + 2, \ldots, M \).

Fix \( 1 \leq r \leq m \) and write \( g_r := A_r^{(m+1)} \mathcal{T}_r f_{m+1} \). Its expansion in powers of \( z_{m+1} \) is written as

\[
g_r = \sum_{n \geq 0} g_{r; n} z_{m+1}^n
\]

with \( g_{r; n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and \( g_{r; 0} = A_r^{(m)} \mathcal{T}_r f_m = f_m \), where the last equality follows from the fact that \( f_m \) is assumed to satisfy (A.7.9). Furthermore, using the holonomy (A.7.8) and the \( q \)-difference equation (A.7.13) in \( z_{m+1} \) satisfied by \( f_{m+1} \), we have

\[
A_r^{(m+1)} \mathcal{T}_r f_{m+1} = A_r^{(m+1)} \mathcal{T}_m f_{m+1} = A_r^{(m+1)} \mathcal{T}_m (A_r^{(m+1)} f_m) = A_r^{(m+1)} (A_r^{(m+1)} f_m) = f_{m+1}.
\]

We conclude that \( g_r \) satisfies the characterizing properties of \( f_{m+1} \). Hence \( g_r = f_{m+1} \), i.e.,

\[
A_r^{(m+1)} \mathcal{T}_r f_m = f_{m+1}.
\]

Fix \( m + 1 < s \leq M \) and write \( g_s := A_s^{(m+1)} f_{m+1} \). By a similar argument as used in the previous paragraph, we now show that \( g_s = f_{m+1} \). We write

\[
g_s = \sum_{n \geq 0} g_{s; n} z_{m+1}^n
\]

with \( g_{s; n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and \( g_{s; 0} = A_s^{(m)} f_m = f_m \), where the last equality follows by the assumption that \( f_m \) satisfies (A.7.9). Using the holonomy (A.7.8), the \( q \)-difference equation (A.7.13), and the obvious fact that \( \mathcal{T}_s (A_s^{(m+1)}) = A_{s+1}^{(m+1)} \) since \( s > m + 1 \), we have

\[
A_s^{(m+1)} \mathcal{T}_{m+1} f_s = A_s^{(m+1)} \mathcal{T}_s (A_s^{(m+1)}) \mathcal{T}_m f_{m+1} = A_s^{(m+1)} \mathcal{T}_s (A_s^{(m+1)}) \mathcal{T}_m f_{m+1} = A_s^{(m+1)} \mathcal{T}_s (A_s^{(m+1)}) \mathcal{T}_m f_{m+1} = A_s^{(m+1)} f_{m+1} = g_s.
\]

We conclude that \( g_s \) satisfies the characterizing properties of \( f_{m+1} \). Hence \( g_s = f_{m+1} \), i.e.,

\[
A_s^{(m+1)} f_{m+1} = f_{m+1}.
\]

This completes the proof of Lemma A.2, and hence the proof of Proposition A.1. □
Appendix: Holonomic sytems of $q$-difference equations

We investigate the analytical properties of the solution $\Phi_1$ when the $q$-connection matrices $A_i$ ($1 \leq i \leq M$) satisfy, besides the conditions (a)-(d), the following analyticity condition:

(e) For some $\epsilon > 0$ the formal $\text{End}(V)$-valued series $A_i \in \mathbb{C}[[z]] \otimes \text{End}(V)$ ($1 \leq i \leq M$) converges normally on compacta of the open polydisc $D^M_{\epsilon} := \{ z \in \mathbb{C}^M \mid |z_i| < \epsilon \ \forall i \}$.

In other words, if we expand $A_i$ along a basis of $\text{End}(V)$, condition (e) requires its coefficients in $\mathbb{C}[[z]]$ to converge normally on compacta of $D^M_{\epsilon}$.

**Proposition A.3.** Suppose that the $q$-connection matrices $A_i \in \mathbb{C}[[z]] \otimes \text{End}(V)$ ($1 \leq i \leq M$) satisfy (a)-(e). Let $v \in V[(1^M)]$. There exists an $\epsilon > 0$ such that the formal $V$-valued series $\Phi_1 \in \mathbb{C}[[z]] \otimes V$ converges normally on compacta of $D^M_{\epsilon}$.

**Proof.** For ease of notation, we will write $\Phi$ instead of $\Phi_v$. By induction on $m = 0, \ldots, M$ we prove that there exists $\epsilon > 0$ such that $\Phi^{(m)} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ converges normally on compacta of $D^m_{\epsilon}$.

For $m = 0$, there is nothing to prove. Fix $0 \leq m < M$ and suppose $\Phi^{(m)}$ converges normally on compacta of $D^m_{\epsilon}$. Write

$$\Phi^{(m+1)} = \sum_{n \geq 0} \Phi_{m,n} z_{m+1}^n$$

with $\Phi_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $\Phi_{m,0} = \Phi^{(m)}$. Recall from the proof of Lemma A.2 that the formal $V$-valued power series $\Phi_{m,n} (n \geq 1)$ are unique characterized by the recurrence relations

$$\Phi_{m,n} = \sum_{l=1}^{n} a_{m+1}^{n-l} (1 - q^n_{m+1} A^{(m)}_{m+1})^{-1} A^{(m)}_{m+1; l} \Phi_{m,n-l}$$ (A.7.14)

for all $n \geq 1$. We use this recurrence formula to find bounds for $\Phi_{m,n}$ in a neighborhood of $0 \in \mathbb{C}^m$.

Turn the finite-dimensional complex vector space $V$ into an inner product space, with corresponding norm denoted by $\| \cdot \|$. We also write $\| \cdot \|$ for the operator norm of the associated finite-dimensional normed space $\text{End}(V)$. We continue the proof of the proposition with two technical sublemmas. First we find a proper uniform bound for $A^{(m)}_{m+1;l}$ for all $l$ (see A.7.14).

**Lemma A.4.** There exists $\epsilon > 0$ and $M > 0$ such that $\| A^{(m)}_{m+1;l} \| \leq M \epsilon^{-l}$ on $D^m_{\epsilon}$ for all $l \geq 0$.

**Proof.** By (e) there exists an $\epsilon > 0$ such that $A^{(m+1)}_{m+1} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes \text{End}(V)$ converges normally on compacta of the polydisc $D^m_{\epsilon}$. Consequently, for $\epsilon < \epsilon' < 2\epsilon$ we have that $\| A^{(m+1)}_{m+1} \|$ is uniformly bounded on the polydisc $D^m_{\epsilon}$, say by $M > 0$. In particular, we get

$$\| (\partial_{z_{m+1}} A^{(m+1)}_{m+1})(z_1, \ldots, z_{m+1}) |_{z_{m+1}=0} \| \leq M \epsilon^{-l}!$$
for all \((z_1, \ldots, z_m) \in \overline{D}_m\) and for all \(l \geq 0\) (see, e.g., [25] Theorem 2.2.7). This proves the lemma in view of the definition (A.7.11) of \(A_{m+1;l}'\).

**Lemma A.5.** There exists an \(\epsilon > 0\) such that \(\Phi_{m;n} \in C[[z_1, \ldots, z_m]] \otimes V\) converges normally on compacta of \(D^m\) for all \(n \geq 0\). Furthermore, there exists a constant \(C > 0\) (independent of \(n\)) such that

\[
\|\Phi_{m;n}\| \leq \frac{C}{1+C} \left( \frac{1+C}{q_{m+1}^\epsilon} \right)^n \|\Phi^{(m)}\|
\]

on \(D^m\) for all \(n \geq 1\).

**Proof.** In the proof of this lemma, we write \(q\) instead of \(q_{m+1}\). By assumption, \(\Phi_{m;0} = \Phi^{(m)}\) converges normally on compacta of \(D^m\) if \(0 < \epsilon < \delta\). We now use the recurrence relation (A.7.14) to obtain the desired results for \(\Phi_{m;n}\) with \(n \geq 1\).

By the proof of Lemma A.2 and since \(0 < q < 1\), there exists some \(\epsilon > 0\) (independent of \(n \geq 1\)) such that \(\det(1-q^nA_{m+1}^{(m)}\) is analytic on \(D^m\) for all \(n \geq 1\) and such that \(|\det(1-q^nA_{m+1}^{(m)}\) is bounded on the closure \(\overline{D}_m\) of \(D^m\), with bound independent of \(n \geq 1\). For such \(\epsilon\), it follows from (A.7.14) that \(\Phi_{m;n}\) converges normally on compacta of \(D^m\) for all \(n \geq 1\). Furthermore, by (e), \(0 < q < 1\), and Cramer’s rule, it implies that for \(\epsilon > 0\) small enough,

\[
\|(1-q^nA_{m+1}^{(m)}\) \leq C'\]

on \(\overline{D}_m\) for all \(n \geq 1\), with \(C' > 0\) also independent of \(n\). By (A.7.14), \(0 < q < 1\) and the previous lemma, we thus obtain for \(\epsilon > 0\) small enough,

\[
\|\Phi_{m;n}\| \leq C' \sum_{l=1}^n q^{n-l}\|A_{m+1;l}\| \|\Phi_{m;n-l}\| \leq C \sum_{l=1}^n \left( \frac{1}{q^\epsilon} \right)^l \|\Phi_{m;n-l}\|
\]

(A.7.15)

on \(\overline{D}_m\) for all \(n \geq 1\) with the constant \(C = C'M > 0\) (independent of \(n\)).

Now, we have the following claim (cf. [15] §10.6): the recurrence relation

\[
g_n = C \sum_{l=1}^n \left( \frac{1}{q^\epsilon} \right)^l g_{n-l}, \quad (n > 0)
\]

with \(g_0 \in \mathbb{R}\) fixed is uniquely solved by

\[
g_n = \frac{C}{C+1} \left( \frac{1+C}{q^\epsilon} \right)^n g_0
\]
for $n \geq 1$. Being obvious for $n = 1$, the claim follows using induction for $n > 1$ by

$$g_n = C \left(\frac{1}{q^\varepsilon}\right)^n g_0 + C \sum_{l=1}^{n-1} \left(\frac{1}{q^\varepsilon}\right)^l g_{n-l}$$

$$= C \left(\frac{1}{q^\varepsilon}\right)^n g_0 + C^2 \sum_{l=1}^{n-1} \left(\frac{1}{q^\varepsilon}\right)^l \left(\frac{1+C}{q^\varepsilon}\right)^{n-l-1} g_0$$

$$= C \left(\frac{1}{q^\varepsilon}\right)^n g_0 \left(1 + C \sum_{l=0}^{n-2} (1+C)^l\right)$$

$$= C \left(\frac{1}{q^\varepsilon}\right)^n g_0 \left(1 + C \left(\frac{(1+C)^{n-1} - 1}{1+C - 1}\right)\right)$$

$$= \frac{C}{C+1} \left(1+C\right)^n g_0.$$  

Combined with (A.7.15), the lemma now follows immediately. □

To conclude the proof of the proposition, note that the previous lemma shows that

$$\Phi^{(m+1)} = \sum_{n \geq 0} \Phi_{m,n} e^{n+1} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes \text{End}(V)$$

converges normally on compacta of $D^m_{\varepsilon}$ if we take $\varepsilon' > 0$ sufficiently small. This concludes the proof of the induction step. □

We interpret the $q$-dilation operators $T_i$ as automorphisms of $\mathcal{M}(\mathbb{C}^M)$ by

$$(T_i f)(z) = f(z_1, \ldots, z_{i-1}, q_i z_i, z_{i+1}, \ldots, z_M).$$

**Theorem A.6.** Suppose $A_i \in \mathcal{M}(\mathbb{C}^M) \otimes \text{End}(V)$ ($1 \leq i \leq M$) satisfy the holonomy conditions (A.7.8) as meromorphic $\text{End}(V)$-valued functions on $\mathbb{C}^M$. Suppose that the $A_i$ are analytic at $0 \in \mathbb{C}^M$ and that their power series expansions at $0 \in \mathbb{C}^M$ satisfy the conditions (b)-(d).

Let $v \in V[[1^M]]$. There exists a unique $\Phi_v \in \mathcal{M}(\mathbb{C}^M) \otimes V$ solving the holonomic system (A.7.7) of $q$-difference equations and coinciding, in a small neighborhood of $0 \in \mathbb{C}^M$, with the converging $V$-valued power series solution $\Phi_v$ from Proposition A.3.

**Proof.** Since the $A_i$ are assumed to be analytic at $0 \in \mathbb{C}^M$, their power series expansions at $0 \in \mathbb{C}^M$ are converging normally on compacta of some open polydisc $D^M_{\varepsilon}$ ($\varepsilon > 0$). Hence, condition (e) is automatically satisfied.

Let $\Phi_v \in \mathbb{C}[[z]] \otimes V$ be the power series solution from Proposition A.3 and let $\varepsilon > 0$ such that $\Phi_v$ converges normally on compacta of $D^M_{\varepsilon}$. Let $z' \in \mathbb{C}^M$ and $U \subset \mathbb{C}^M$ some open locally compact neighborhood of $z'$. Since $0 < q_i < 1$ ($1 \leq i \leq M$), there exists a $\lambda \in \mathbb{Z}_{\geq 0}$ such that $q^\lambda U \subset D^M_{\varepsilon}$, where $q^\lambda z = (q_1^\lambda z_1, \ldots, q_M^\lambda z_M)$. Define $\Phi_v$ as $V$-valued meromorphic function on $z \in U$ by

$$\Phi_v(z) = A_\lambda(z) \Phi_v(q^\lambda z),$$

(A.7.16)
where $A_\lambda \in \mathcal{M}(\mathbb{C}^M) \otimes \text{End}(V)$ is defined inductively by
\[
A_{\lambda+\mu}(z) = A_\lambda(z)A_\mu(q^\lambda z), \quad \forall \lambda, \mu \in \mathbb{Z}^M_{\geq 0},
\]
and $A_{\epsilon_i} = A_i (1 \leq i \leq M)$, where the $\epsilon_i (1 \leq i \leq M)$ are the standard generators of the additive monoid $\mathbb{Z}^M_{\geq 0}$. Of course, the definition of $A_\lambda(z)$ makes sense by the holonomy conditions for the $A_i$. Furthermore, (A.7.16) together with the holonomy conditions for the $A_i$ show that the power series solution $\Phi_v$ of (A.7.8) has a unique extension to a meromorphic $V$-valued solution on $\mathbb{C}^M$ of (A.7.8). \qed