Bispectral quantum Knizhnik-Zamolodchikov equations

van Meer, M.

Citation for published version (APA):
Appendix: Holonomic systems of $q$-difference equations

In the appendix we detail the construction of power series solutions of holonomic systems of $q$-difference equations. Special cases have been investigated in, e.g., [2], [21] and [15, §12]. Many arguments go back to classical works [1], [3], [4], [61] on ordinary linear $q$-difference equations.

We begin with the construction of formal asymptotic solutions to holonomic systems of $q$-difference equations. Let $C[[z]] = C[[z_1, \ldots, z_M]]$ denote the ring of formal power series in $M$ indeterminates $z_1, \ldots, z_M$ over the complex numbers. Let $V$ be a finite-dimensional complex vector space and let $A_i \in C[[z]] \otimes \text{End}(V)$ for $i = 1, \ldots, M$. Since $C[[z]] \otimes \text{End}(V)$ is isomorphic to $\text{End}_{C[[z]]}(C[[z]] \otimes V)$ as $C[[z]]$-module, we can view the $A_i$ as $C[[z]]$-linear endomorphisms of $C[[z]] \otimes V$. Fix $0 < q_i < 1$ for $1 \leq i \leq M$. Define the $q_i$-dilation operators $T_i : C[[z]] \to C[[z]]$ for $i = 1, \ldots, M$ as the complex linear maps $T_i(\sum_m d_m z^m) := \sum_m q_i^{m_i} d_{m+i} z^m \quad (d_m \in C)$, where we use multi-index notation $z^m = z_{m_1}^{m_1} \cdots z_{m_M}^{m_M}$ for $m = (m_1, \ldots, m_M)$ with $m_j \in \mathbb{Z}_{\geq 0}$. We also view $T_i$ as operators on $C[[z]] \otimes V$ and on $C[[z]] \otimes \text{End}(V)$.

Consider the system of first-order linear $q$-difference equations

$$A_i T_i f = f, \quad (i = 1, \ldots, M) \quad (A.7.7)$$

for $f \in C[[z]] \otimes V$.

For $f \in C[[z]] \otimes V$ and $A \in C[[z]] \otimes \text{End}(V)$, we introduce the notations

$$f^{(m)} := f|_{z_{m+1} = \ldots = z_M = 0} \in C[[z_1, \ldots, z_M]] \otimes V$$

$$A^{(m)} := A|_{z_{m+1} = \ldots = z_M = 0} \in C[[z_1, \ldots, z_M]] \otimes \text{End}(V)$$
for $0 \leq m \leq M$, with the convention that $f^{(M)} = f$ and $A^{(M)} = A$. We make the following assumptions on the system of $q$-difference equations (A.7.7):

(a) The system (A.7.7) is holonomic, that is

$$A_i T_i(A_j) = A_j T_j(A_i)$$

(A.7.8)

for all $1 \leq i, j \leq M$. Note that the holonomy implies that the leading coefficients $A_i^{(0)} \in \text{End}(V)$ of $A_i$ mutually commute, i.e.,

$$[A_i^{(0)}, A_j^{(0)}] = 0$$

for all $1 \leq i, j \leq M$.

(b) The complex linear endomorphisms $A_1^{(0)}, \ldots, A_M^{(0)}$ of $V$ are semisimple. Combined with (a) we thus have

$$V = \bigoplus_{\gamma \in S} V[\gamma]$$

with $V[\gamma] := \{v \in V \mid A_i^{(0)} v = \gamma_i v \ \forall i \} (\gamma \in \mathbb{C}^M)$ and $S := \{ \gamma \in \mathbb{C}^M \mid V[\gamma] \neq \{0\} \}$.

(c) $(1^M) := (1, \ldots, 1) \in \mathbb{C}^M$ belongs to $S$.

(d) $\gamma_k \notin q_k^{-N}$ for all $\gamma \in S$ and $1 \leq k \leq M$.

**Proposition A.1.** Fix $v \in V[(1^M)]$. Consider the system (A.7.7) of $q$-difference equations and suppose that (a)-(d) are satisfied. Then there exists a unique solution $\Phi_v \in \mathbb{C}[[z]] \otimes V$ of (A.7.7) such that $\Phi_v^{(0)} = v$.

**Proof.** The proposition is a consequence of the following lemma.

**Lemma A.2.** Let $0 \leq m < M$. Suppose one has a solution

$$f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$$

of the system of equations

$$A_r^{(m)} T_r f_m = f_m, \quad 1 \leq r \leq m,$$

$$A_s^{(m)} f_m = f_m, \quad m < s \leq M.$$  

(A.7.9)

Then there exists a unique

$$f_{m+1} = \sum_{n \geq 0} f_{m,n} z_{m+1}^n \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V$$

with $f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m,0} = f_m$ satisfying (A.7.9) with the role of $m$ replaced by $m + 1$:

$$A_r^{(m+1)} T_r f_{m+1} = f_{m+1}, \quad 1 \leq r \leq m + 1,$$

$$A_s^{(m+1)} f_{m+1} = f_{m+1}, \quad m + 1 < s \leq M.$$  

(A.7.10)
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The proposition follows directly from the lemma as follows. Note that $f_0 := v \in V[(1^M)]$ is a solution of (A.7.9) for $m = 0$. The formal $V$-valued series $f_M \in \mathbb{C}[[z]] \otimes V$, obtained by repeated application of the lemma starting from $f_0 = v$, gives a formal $V$-valued series solution of (A.7.7) satisfying $f^{(0)}_M = v$. For uniqueness, assume that $f \in \mathbb{C}[[z]] \otimes V$ is another formal $V$-valued series satisfying $f^{(0)} = v$ and solving (A.7.7). We have $f^{(0)} = v = f_0$ and $f^{(m)}$ solves (A.7.9) for all $0 \leq m < M$. Hence, by the uniqueness part of the lemma, $f = f^{(M)} = f_M$.

We now proceed to prove the lemma. We assume that we have a formal power series solution $f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ of (A.7.9) for some $0 \leq m < M$. We write

$$A^{(m+1)}_r = \sum_{n \geq 0} A^{(m)}_{r,n} z_n^{-m+1},$$

(A.7.11)

where $A^{(m)}_{r,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes \text{End}(V)$ and $A^{(m)}_{r,0} = A^{(m)}_r$. By a direct computation one verifies that

$$f_{m+1} = \sum_{n \geq 0} f_{m,n} z_{m+1} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V$$

with $f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m,0} = f_m$ satisfies the $q$-difference equation

$$A^{(m+1)}_{m+1} f_{m+1} = f_{m+1}$$

if and only if

$$(1 - q^{n+1}_{m+1} A^{(m)}_{m+1}) f_{m,n} = \sum_{l=1}^{n} q^{n-l}_{m+1} A^{(m)}_{m+1} f_{m,n-l}$$

(A.7.12)

for all $n \in \mathbb{Z}_{\geq 0}$. The recurrence relations (A.7.12) admit a unique solution $(f_{m,n})_{n \in \mathbb{Z}_{\geq 0}}$ with $f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and with initial condition $f_{m,0} = f_m$. Indeed, note that (A.7.12) is valid for $n = 0$ since $f_{m,0} = f_m$ satisfies (A.7.9). For $n \geq 1$, we have

$$\det(1 - q^{n}_{m+1} A^{(m)}_{m+1}) \in \mathbb{C}[[z_1, \ldots, z_m]] \times,$$

since

$$\det(1 - q^{n}_{m+1} A^{(m)}_{m+1})|_{z_1=\ldots=z_m=0} = \det(1 - q^{n}_{m+1} A^{(0)}_{m+1}) = \prod_{\gamma \in S}(1 - q^{n}_{m+1} \dim(V[\gamma])) \neq 0$$

by assumption (d). Cramer’s rule then implies that (A.7.12) admits a unique solution $(f_{m,n})_{n \in \mathbb{Z}_{\geq 0}}$ with $f_{m,0} = f_m$.

We conclude that there exists a unique

$$f_{m+1} = \sum_{n \geq 0} f_{m,n} z_{m+1}$$

with $f_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m,0} = f_m$ satisfying the $q$-difference equation

$$A^{(m+1)}_{m+1} f_{m+1} = f_{m+1}.$$

(A.7.13)
It remains to show that $f_{m+1}$ also satisfies (A.7.10) for $r = 1, \ldots, m$ and for $s = m + 2, \ldots, M$.

Fix $1 \leq r \leq m$ and write $g_r := A_r^{(m+1)} T_r f_{m+1}$. Its expansion in powers of $z_{m+1}$ is written as

$$g_r = \sum_{n \geq 0} g_{r,n} z_{m+1}^n$$

with $g_{r,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $g_{r,0} = A_r^{(m)} T_r f_m = f_m$, where the last equality follows from the fact that $f_m$ is assumed to satisfy (A.7.9). Furthermore, using the holonomy (A.7.8) and the $q$-difference equation (A.7.13) in $z_{m+1}$ satisfied by $f_{m+1}$, we have

$$A_r^{(m+1)} T_{m+1} f_{m+1} = A_r^{(m+1)} T_{m+1} (A_r^{(m+1)} T_r f_{m+1})$$

$$= A_r^{(m+1)} T_r (A_r^{(m+1)} T_{m+1} f_{m+1})$$

$$= A_r^{(m+1)} T_r T_{m+1} f_{m+1} = g_r.$$  

We conclude that $g_r$ satisfies the characterizing properties of $f_{m+1}$. Hence $g_r = f_{m+1}$, i.e.,

$$A_r^{(m+1)} T_r f_{m+1} = f_{m+1}. $$

Fix $m + 1 < s \leq M$ and write $g_s := A_s^{(m+1)} f_{m+1}$. By a similar argument as used in the previous paragraph, we now show that $g_s = f_{m+1}$. We write

$$g_s = \sum_{n \geq 0} g_{s,n} z_{m+1}^n$$

with $g_{s,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $g_{s,0} = A_s^{(m)} f_m = f_m$, where the last equality follows by the assumption that $f_m$ satisfies (A.7.9). Using the holonomy (A.7.8), the $q$-difference equation (A.7.13), and the obvious fact that $T_s(A_s^{(m+1)}) = A_s^{(m+1)}$ since $s > m + 1$, we have

$$A_s^{(m+1)} T_{m+1} f_{m+1} = A_s^{(m+1)} T_{m+1} (A_s^{(m+1)} T_m f_{m+1})$$

$$= A_s^{(m+1)} T_m (A_s^{(m+1)} T_{m+1} f_{m+1})$$

$$= A_s^{(m+1)} T_{m+1} f_{m+1} = g_s.$$  

We conclude that $g_s$ satisfies the characterizing properties of $f_{m+1}$. Hence $g_s = f_{m+1}$, i.e.,

$$A_s^{(m+1)} f_{m+1} = f_{m+1}. $$

This completes the proof of Lemma A.2, and hence the proof of Proposition A.1. □
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We investigate the analytical properties of the solution \( \Phi \), when the \( q \)-connection matrices \( A_i \) (\( 1 \leq i \leq M \)) satisfy, besides the conditions (a)-(d), the following analyticity condition:

(e) For some \( \epsilon > 0 \) the formal \( \text{End}(V) \)-valued series \( A_i \in \mathbb{C}[[z]] \otimes \text{End}(V) \) \( (1 \leq i \leq M) \) converges normally on compacta of the open polydisc \( D^M_\epsilon := \{ z \in \mathbb{C}^M \mid |z| < \epsilon \forall i \} \).

In other words, if we expand \( A_i \) along a basis of \( \text{End}(V) \), condition (e) requires its coefficients in \( \mathbb{C}[[z]] \) to converge normally on compacta of \( D^M_\epsilon \).

**Proposition A.3.** Suppose that the \( q \)-connection matrices \( A_i \in \mathbb{C}[[z]] \otimes \text{End}(V) \) \( (1 \leq i \leq M) \) satisfy (a)-(e). Let \( v \in V[(1^M)] \). There exists an \( \epsilon > 0 \) such that the formal \( V \)-valued series \( \Phi \in \mathbb{C}[[z]] \otimes V \) converges normally on compacta of \( D^M_\epsilon \).

**Proof.** For ease of notation, we will write \( \Phi \) instead of \( \Phi_v \). By induction on \( m = 0, \ldots, M \) we prove that there exists \( \epsilon > 0 \) such that \( \Phi^{(m)} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) converges normally on compacta of \( D^m_\epsilon \).

For \( m = 0 \), there is nothing to prove. Fix \( 0 \leq m < M \) and suppose \( \Phi^{(m)} \) converges normally on compacta of \( D^m_\delta \) for some \( \delta > 0 \). Write

\[
\Phi^{(m+1)} = \sum_{n \geq 0} \Phi_{m,n} z_{m+1}^n
\]

with \( \Phi_{m,n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V \) and \( \Phi_{m,0} = \Phi^{(m)}_v \). Recall from the proof of Lemma A.2 that the formal \( V \)-valued power series \( \Phi_{m,n} \) \( (n \geq 1) \) are unique characterized by the recurrence relations

\[
\Phi_{m,n} = \sum_{l=1}^{n} d_{m+1}^{n-l} (1 - q_{m+1}^{n})_{A_{m+1},l}^{(m+1)} A_{m+1,l}^{(m)} \Phi_{m-1,n-l}
\]

(A.7.14)

for all \( n \geq 1 \). We use this recurrence formula to find bounds for \( \Phi_{m,n} \) in a neighborhood of \( 0 \in \mathbb{C}^m \).

Turn the finite-dimensional complex vector space \( V \) into an inner product space, with corresponding norm denoted by \( \| \cdot \| \). We also write \( \| \cdot \| \) for the operator norm of the associated finite-dimensional normed space \( \text{End}(V) \). We continue the proof of the proposition with two technical sublemmas. First we find a proper uniform bound for \( A_{m+1,l}^{(m)} \) for all \( l \geq 0 \) (see (A.7.14)).

**Lemma A.4.** There exists \( \epsilon > 0 \) and \( M > 0 \) such that \( \| A_{m+1,l}^{(m)} \| \leq M \epsilon^{-l} \) on \( \overline{D}^m_\epsilon \) for all \( l \geq 0 \).

**Proof.** By (e) there exists an \( \epsilon > 0 \) such that \( A_{m+1}^{(m+1)} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes \text{End}(V) \) converges normally on compacta of the polydisc \( D^m_{\epsilon} \). Consequently, for \( \epsilon < \epsilon' < 2 \epsilon \) we have that \( \| A_{m+1}^{(m+1)} \| \) is uniformly bounded on the polydisc \( D^m_{\epsilon'} \), say by \( M > 0 \). In particular, we get

\[
\| (\partial^l_{z_{m+1}} A_{m+1}^{(m+1)})(z_1, \ldots, z_{m+1}) |_{z_{m+1}=0} \| \leq M \epsilon^{-l} !
\]
for all \((z_1, \ldots, z_m) \in \overline{D}_m\) and for all \(l \geq 0\) (see, e.g., [25] Theorem 2.2.7). This proves the lemma in view of the definition (A.7.11) of \(A_{m+1,l}^{(m)}\).

Lemma A.5. There exists an \(\epsilon > 0\) such that \(\Phi_{m,n} \in \mathbb{C}[z_1, \ldots, z_m] \otimes V\) converges normally on compacta of \(D_m^n\) for all \(n \geq 0\). Furthermore, there exists a constant \(C > 0\) (independent of \(n\)) such that

\[
\|\Phi_{m,n}\| \leq \frac{C}{1 + C} \left(1 + \frac{C}{q^{m+1}\epsilon}\right)^n \|\Phi^{(m)}\|
\]

on \(\overline{D}_m^n\) for all \(n \geq 1\).

Proof. In the proof of this lemma, we write \(q\) instead of \(q_{m+1}\). By assumption, \(\Phi_{m,0} = \Phi^{(m)}\) converges normally on compacta of \(D_m^n\) if \(0 < \epsilon < \delta\). We now use the recurrence relation (A.7.14) to obtain the desired results for \(\Phi_{m,n}\) with \(n \geq 1\).

By the proof of Lemma A.2 and since \(0 < q < 1\), there exists some \(\epsilon > 0\) (independent of \(n \geq 1\)) such that \(\det(1 - q^nA_{m+1}^{(m)})^{-1}\) is analytic on \(D_m^n\) for all \(n \geq 1\) and such that \(|\det(1 - q^nA_{m+1}^{(m)})^{-1}|\) is bounded on the closure \(\overline{D}_m^n\) of \(D_m^n\), with bound independent of \(n \geq 1\). For such \(\epsilon\), it follows from (A.7.14) that \(\Phi_{m,n}\) converges normally on compacta of \(D_m^n\) for all \(n \geq 1\). Furthermore, by (e), \(0 < q < 1\), and Cramer’s rule, it implies that for \(\epsilon > 0\) small enough,

\[
\|(1 - q^nA_{m+1}^{(m)})^{-1}\| \leq C'
\]

on \(\overline{D}_m^n\) for all \(n \geq 1\), with \(C' > 0\) also independent of \(n\). By (A.7.14), \(0 < q < 1\) and the previous lemma, we thus obtain for \(\epsilon > 0\) small enough,

\[
\|\Phi_{m,n}\| \leq C' \sum_{l=1}^{n} q^{n-l} \|A_{m+1,l}^{(m)}\| \|\Phi_{m,n-l}\| \leq C' \sum_{l=1}^{n} \left(\frac{1}{q\epsilon}\right)^l \|\Phi_{m,n-l}\|\quad (A.7.15)
\]

on \(\overline{D}_m^n\) for all \(n \geq 1\) with the constant \(C = C'M > 0\) (independent of \(n\)).

Now, we have the following claim (cf. [15] §10.6): the recurrence relation

\[
g_n = C \sum_{l=1}^{n} \left(\frac{1}{q\epsilon}\right)^l g_{n-l}, \quad (n > 0)
\]

with \(g_0 \in \mathbb{R}\) fixed is uniquely solved by

\[
g_n = \frac{C}{C' + 1} \left(\frac{1 + C}{q\epsilon}\right)^n g_0
\]
for $n \geq 1$. Being obvious for $n = 1$, the claim follows using induction for $n > 1$ by

\[
g_n = C \left( \frac{1}{q \epsilon} \right)^n g_0 + C \sum_{l=1}^{n-1} \left( \frac{1}{q \epsilon} \right)^l g_{n-l}
\]

\[
= C \left( \frac{1}{q \epsilon} \right)^n g_0 + \frac{C^2}{C+1} \sum_{l=1}^{n-2} \left( \frac{1}{q \epsilon} \right)^l \left( \frac{1+C}{q \epsilon} \right)^{n-l} g_0
\]

\[
= C \left( \frac{1}{q \epsilon} \right)^n g_0 \left( 1 + C \sum_{l=0}^{n-2} (1+C)^l \right)
\]

\[
= C \left( \frac{1}{q \epsilon} \right)^n g_0 \left( 1 + C \left( \frac{(1+C)^n - 1}{1+C-1} \right) \right)
\]

\[
= C \left( \frac{1}{C+1} \right) \left( \frac{1+C}{q \epsilon} \right)^n g_0.
\]

Combined with (A.7.15), the lemma now follows immediately. \hfill \Box

To conclude the proof of the proposition, note that the previous lemma shows that

\[
\Phi^{(m+1)} = \sum_{n \geq 0} \Phi_{m,n} z_{m+1}^n \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes \text{End}(V)
\]

converges normally on compacta of $D_{\epsilon^{'}, q}^{n+1}$ if we take $\epsilon' > 0$ sufficiently small. This concludes the proof of the induction step. \hfill \Box

We interpret the $q$-dilation operators $T_i$ as automorphisms of $\mathcal{M}(\mathbb{C}^M)$ by

\[(T_i f)(z) = f(z_1, \ldots, z_{i-1}, q_i z_i, z_{i+1}, \ldots, z_M).\]

**Theorem A.6.** Suppose $A_i \in \mathcal{M}(\mathbb{C}^M) \otimes \text{End}(V)$ ($1 \leq i \leq M$) satisfy the holonomy conditions (A.7.8) as meromorphic $\text{End}(V)$-valued functions on $\mathbb{C}^M$. Suppose that the $A_i$ are analytic at $0 \in \mathbb{C}^M$ and that their power series expansions at $0 \in \mathbb{C}^M$ satisfy the conditions (b)-(d).

Let $v \in V[(1^M)]$. There exists a unique $\Phi_v \in \mathcal{M}(\mathbb{C}^M) \otimes V$ solving the holonomic system (A.7.7) of $q$-difference equations and coinciding, in a small neighborhood of $0 \in \mathbb{C}^M$, with the converging $V$-valued power series solution $\Phi_v$ from Proposition A.3.

**Proof.** Since the $A_i$ are assumed to be analytic at $0 \in \mathbb{C}^M$, their power series expansions at $0 \in \mathbb{C}^M$ are converging normally on compacta of some open polydisc $D_{\epsilon}^{M}$ ($\epsilon > 0$). Hence, condition (e) is automatically satisfied.

Let $\Phi_v \in \mathbb{C}[[z]] \otimes V$ be the power series solution from Proposition A.3 and let $\epsilon > 0$ such that $\Phi_v$ converges normally on compacta of $D_{\epsilon}^{M}$. Let $z' \in \mathbb{C}^M$ and $U \subset \mathbb{C}^M$ some open locally compact neighborhood of $z'$. Since $0 < q_i < 1$ ($1 \leq i \leq M$), there exists a $\lambda \in \mathbb{Z}_{\geq 0}$ such that $q^\lambda U \subset D_{\epsilon}^{M}$, where $q^\lambda z = (q_1^\lambda z_1, \ldots, q_M^\lambda z_M)$. Define $\Phi_v$ as $V$-valued meromorphic function on $z \in U$ by

\[
\Phi_v(z) = A_{\lambda}(z) \Phi_v(q^\lambda z), \quad (A.7.16)
\]
where $A_\lambda \in \mathcal{M}(\mathbb{C}^M) \otimes \text{End}(V)$ is defined inductively by

$$A_{\lambda+\mu}(z) = A_\lambda(z)A_\mu(q^\lambda z), \quad \forall \lambda, \mu \in \mathbb{Z}^M_{\geq 0},$$

and $A_{\epsilon_i} = A_i$ ($1 \leq i \leq M$), where the $\epsilon_i$ ($1 \leq i \leq M$) are the standard generators of the additive monoid $\mathbb{Z}^M_{\geq 0}$. Of course, the definition of $A_\lambda(z)$ makes sense by the holonomy conditions for the $A_i$. Furthermore, (A.7.16) together with the holonomy conditions for the $A_i$ show that the power series solution $\Phi_v$ of (A.7.8) has a unique extension to a meromorphic $V$-valued solution on $\mathbb{C}^M$ of (A.7.8).