Feshbach resonances in ultracold mixtures of the fermionic quantum gases 6Li and 40K

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2.1 Introduction

This chapter provides a background of the theoretical concepts used in this thesis. This chapter is by no means a review of the discussed subjects, it is rather a description of the different approaches as compared to common treatments, a description of the approximations which have been made and an introduction of notations. The field of ultracold gases is tremendously active and many up-to-date reviews are available, covering a very large part of the field. Here I will summarize some of the review articles which have shown to be very useful. An extensive description of optical trapping is given in Ref. [51] and for magnetic trapping in Ref. [52, 53, 54, 55]. Evaporative cooling, in particular in magnetic traps, is described in Ref. [56, 57] and the description of trapped ultracold Fermi gases is well covered in Ref. [58, 59]. Basic experimental techniques for experiments on ultracold gases can be found in [55, 60, 61, 58]. An extensive description of Feshbach resonances is given in Ref. [62], and in particular the square well problem, which offers a very good insight in the basic properties of Feshbach resonances, is discussed in various places [63, 58, 64]. Finally, many PhD theses offering very detailed descriptions of ultracold gases experiments have been written, see e.g. [65, 66, 67].

Three aspects of the theoretical background for this thesis require a bit of extra attention. First, the quantum statistics of particles are of great importance in experiments with ultracold gases, this will be addressed in Sect. 2.2. Second, the experimental and theoretical results presented in this thesis are in many aspects based on ultracold scattering theory. In Sect. 2.3 a brief description of ultracold scattering and in particular resonant scattering is given. Third, both magnetic and optical traps are used in the experiments presented in this thesis. A clear introduction on what assumptions are made to analyze and model the trapping is given in Sect. 2.4.

2.2 Cooling of fermions

Experiments with ultracold fermionic gases are in many aspects similar to experiments with ultracold bosonic gases, however there are a few distinct differences. For two indistinguishable particles the wavefunction describing both particles $\psi(r_1, r_2)$ can be constructed from the two single-particle wavefunctions $\psi_1(r_1)$ and $\psi_2(r_2)$ in two manners

$$\psi_\pm(r_1, r_2) = C [\psi_1(r_1)\psi_2(r_2) \pm \psi_1(r_2)\psi_2(r_1)]$$

where $C$ is a normalization factor. The + sign yields a two-particle wavefunction which is symmetric under exchange of particle 1 and 2 and the − sign yield an anti-symmetric wavefunction under exchange of the two particles. The former type of particles are called bosons and the latter fermions. The anti-symmetry of the fermionic wavefunction is called the Pauli exclusion principle and implies that two fermions cannot occupy the same internal state $\psi_1 = \psi_2$, because $\psi_-(r_1, r_2) = C [\psi_1(r_1)\psi_1(r_2) - \psi_1(r_2)\psi_1(r_1)] = 0$. 

Additionally, two fermions cannot be at the same spatial position \( r_1 = r_2 \), since
\[
\psi_- (r_1, r_2) = C [\psi_1 (r_1) \psi_2 (r_2) - \psi_1 (r_2) \psi_2 (r_1)] = 0.
\]
These two properties of fermions have dramatic consequences for experiments with ultracold atoms. The first aspect concerning the internal states is depicted in Fig. 2.1 for both types of particles in a harmonic trap. For \( T = 0 \) the bosons pile up in the ground state resulting in a macroscopic occupation of the ground state, a so-called Bose-Einstein condensate. The fermions cannot occupy the same state, therefore they will fill up each state up to a certain energy referred to as the Fermi-Energy \( E_F \).

The second property of fermions, that they cannot exist at the same spatial position \( r_1 = r_2 \) has different consequences. The wavefunction of the two bosons is symmetric under exchange of the two particles, therefore, the total probability density \( |\psi_+ (r_1, r_2)|^2 \) has a finite value for \( r_1 = r_2 \). The fermionic two particle wavefunction has to be antisymmetric under exchange of the particles, resulting in the fact that the total probability density \( |\psi_- (r_1, r_2)|^2 \) will vanish for \( r_1 = r_2 \). This has the important consequence that two indistinguishable fermions will not collide for \( T \to 0 \).

To achieve quantum degeneracy in ultracold gases only one method has been demonstrated to be successful, namely, evaporative cooling [56, 57]. To achieve an increase in phase-space density by means of evaporative cooling, rethermalizing collisions are essential down to \( T \to 0 \). Three fundamentally different pre-cooling steps have been successfully employed to achieve ultracold samples, laser-cooling [4], cooling by contact with a cryogenic surface [68] and very recently buffer gas cooling [69]. However, the final step to achieve quantum degeneracy has always been evaporative cooling, therefore collisions are essential for ultracold gases experiments. This has an important consequence for experiments with fermionic gases, namely, that at least two distinguishable species are required to obtain rethermalizing collisions and allow to perform evaporative cooling.

The second species next to the fermion can be another internal (spin)state of the same atomic species, a different bosonic species, or a different fermionic species. The first quantum degenerate Fermi gas was obtained by a spin mixture of two internal hyperfine states of \(^{40}\text{K}\) [7]. Surprisingly, this method was proven to be very efficient, however, this approach has never been repeated until this thesis project. In the past decade many other combinations of species have been used to cool fermions to quantum degeneracy. Cooling by means of rethermalization with a boson has been used to achieve quantum degeneracy in \(^6\text{Li}\) by cooling with \(^7\text{Li}\) [30, 31], \(^{23}\text{Na}\) [70] or \(^{87}\text{Rb}\) [71], for degeneracy in \(^{40}\text{K}\) by cooling with \(^{87}\text{Rb}\) [72], and for degeneracy in \(^3\text{He}^*\) by cooling with \(^4\text{He}^*\) [39] and for \(^{173}\text{Yb}\) by cooling with \(^{174}\text{Yb}\) [73]. The method using a homonuclear spin mixture to assure rether-
2.3 Ultracold scattering

Rethermalizing collisions has been used for $^{40}$K in two spin states [7], for $^6$Li in two spin states [4] and for $^{173}$Yb in six spin states [12]. Cooling fermionic atoms to quantum degeneracy by rethermalizing collisions with other fermions had not been performed prior to this thesis. Very recently in experiments performed in Innsbruck, simultaneous to the experiments presented in this thesis, quantum degeneracy has been achieved in $^{40}$K by cooling with two spin states of $^6$Li [73]. In this thesis the first results on cooling $^6$Li by rethermalizing collisions with a $^{40}$K spin-mixture are presented. The Fermi-Fermi mixture is cooled close to quantum degeneracy limited by technical aspects of the optical dipole trap. Cooling of two different fermionic species both in a single spin state by means of forced evaporation on both species has not been achieved at the moment this thesis is being written.

The cooling of $^6$Li with $^{40}$K or vice-versa has an additional problem of the mass-imbalance between the two species. To achieve rethermalization of a sample, not only collisions are required, but also momentum transfer has to occur. If the mass imbalance is very large, many collisions are required to achieve rethermalization. It can be easily shown [64] that for two particles with mass $m_1$ and $m_2$ the thermalization time $\tau_{th}$ is related to the collision rate $\tau_c^{-1}$ by

$$\tau_{th}^{-1} = \frac{\xi}{2(\gamma + 3/2)} \tau_c^{-1}$$

where $\xi = 4m_1m_2/(m_1 + m_2)^2$ is the mass-imbalance factor and $\gamma$ is the trap parameter, $\gamma = 3/2$ for a harmonic trap and $\gamma = 3$ for a linear trap. For the $^6$Li-$^{40}$K system $\xi \approx 0.455$ therefore rethermalization of the $^6$Li-$^{40}$K mixture takes about twice as long compared to a homonuclear system only due to the effect of the mass difference.

2.3 Ultracold scattering

Atomic scattering theory has been treated extensively in literature, see e.g. [76, 64]. In this section we will summarize the background required for Chapters 5 and 6. We will start in Sect. 2.3.1 with the Hamiltonian describing two body scattering. Subsequently, we will treat resonance scattering at $T = 0$ (Sect. 2.3.2) and at finite temperature (Sect 2.3.3). We will conclude the chapter with a brief description of the accumulated phase method in Sect. 2.3.4.

2.3.1 Two body hamiltonian

The two-body Hamiltonian of two interacting atoms can be written as:

$$H = H^{\text{rel}} + H^{\text{int}},$$

where $H^{\text{rel}}$ is the Hamiltonian of the relative motion and $H^{\text{int}}$ is the internal energy of the two atoms, given by

$$H^{\text{rel}} = \frac{\mathbf{p}^2}{2\mu} + V,$$

$$H^{\text{int}} = H_a^{\text{hf}} + H_b^{\text{hf}}$$

(2.1)
The single atom hyperfine energies as a function of the magnetic field $B$ for the electronic groundstates of $^6\text{Li}$ and $^{40}\text{K}$. The states are labeled with their zero-field quantum numbers $|fm_f\rangle$. Note the inverted hyperfine structure for $^{40}\text{K}$ and the relatively small hyperfine field $B_{hf}$ of $^6\text{Li}$.

\[ H_{hf}^\alpha = \frac{a_{hf}}{\hbar^2} \hat{i} \cdot \mathbf{s} + (\gamma_e \mathbf{s} - \gamma_i \hat{i}) \cdot \mathbf{B}, \tag{2.3} \]

where $a_{hf}$ is the hyperfine constant, $\gamma_e \equiv g_s \mu_B / \hbar$ and $\gamma_i \equiv g_i \mu_B / \hbar$ are the electronic and nuclear gyromagnetic factors, $\mathbf{B}$ is the magnetic field and $\hat{i}$ and $\mathbf{s}$ are the nuclear and electronic spin operators with the respective quantum numbers $m_i$ and $m_s$. The hyperfine structure of the ground state of $^6\text{Li}$ and $^{40}\text{K}$ play a central role throughout this thesis. In Fig. 2.2 the eigenvalues $E_{hf}$ of Eq. 2.3 are plotted as a function of the magnetic field dependence of the various hyperfine states of $^6\text{Li}$ and $^{40}\text{K}$. Both atoms have an electron spin of $s = 1/2$, $^6\text{Li}$ has a nuclear spin of $i = 1$ and $^{40}\text{K}$ has a nuclear spin of $i = 4$ (see Appendix A for more details on potassium). The hyperfine constants for $^6\text{Li}$ and $^{40}\text{K}$ are $a_{Li}^{hf}/\hbar = 152.1368407(20)$ MHz and $a_{K}^{hf}/\hbar = -285.7308(24)$ MHz respectively [77]. The states are labeled with the low-field quantum numbers $|fm_f\rangle$ where $f = i + s$ is the total spin operator with the corresponding quantum numbers $f$ and $m_f$. Note that for finite magnetic field $H_{hf}$ is not diagonal in the $|fm_f\rangle$ basis.

Relative Hamiltonian

The Hamiltonian of the relative motion, given by Eq. 2.1 describes the interaction of two particles with an effective interaction $V$ and a relative kinetic energy $p^2/2m$. The effective interaction $V$ can be written as the central Coulomb interaction $V_{Coul}(r)$ of the two atoms, where $r$ is the internuclear distance. The central interaction is a function of

\footnote{Note that the common reference for $g_i$, Ref. [77], uses a different sign convention. We use the convention [78] where the magnetic moment is parallel to the nuclear spin: $\mu = g_i \mu_B$, i.e. $g_i$ of the proton and $^6\text{Li}$ is positive and $g_i$ of $^{40}\text{K}$ is negative.}
the internuclear distance and depends on the total spin $S = s_\alpha + s_\beta$. It can be decomposed as

$$V^{cen}(r) = \sum_S |S\rangle V_S(r) \langle S|$$  \hspace{1cm} (2.4)

where $V_S(r)$ is the interaction potential for total electron spin quantum number $S$. For spin 1/2 atoms the total electron spin has a singlet ($S = 0$) or triplet ($S = 1$) symmetry. The singlet and triplet potentials are then given by $V_s(r)$ and $V_t(r)$ respectively and Eq. 2.4 can be rewritten as:

$$V^{cen}(r) = V_D(r) + J(r)s_\alpha \cdot s_\beta$$  \hspace{1cm} (2.5)

where $V_D(r) = \frac{1}{4}(V_s(r) + 3V_t(r))$ is called the direct interaction and $J(r) = V_t(r) - V_s(r)$ the exchange interaction. Asymptotically $V_D(r)$ corresponds to the van der Waals tail

$$V_D(r) = -\frac{C_6}{r^6}$$  \hspace{1cm} (2.6)

The characteristic length of this potential can be expressed in the so-called van der Waals range

$$r_0 = \frac{1}{2} \left( \frac{2\mu C_6}{\hbar^2} \right)^{1/4}$$  \hspace{1cm} (2.7)

The exchange term in Eq. 2.5 can be well described by

$$J(r) = J(\gamma_\alpha, \gamma_\beta, r)r^{\frac{-\gamma_\alpha}{2} + \frac{\gamma_\beta}{2} - \frac{1}{\gamma_\alpha + \gamma_\beta} - 1} e^{-\left(\frac{\gamma_\alpha + \gamma_\beta}{2}\right)r}$$  \hspace{1cm} (2.8)

where $-\gamma_\alpha^2/2$ and $-\gamma_\beta^2/2$ are the ionization energies of both atoms and $J(\gamma_\alpha, \gamma_\beta, r)$ is a normalization function, which reduces to a constant for homonuclear systems, $(\gamma_\alpha = \gamma_\beta)$. For heteronuclear systems, $\gamma_\alpha \neq \gamma_\beta$, a parameterization of $J(\gamma_\alpha, \gamma_\beta, r)$ can be found in Ref. [79], the values for the LiK system are given in Appendix B. Knowledge about the energy scales of the direct and exchange interactions are required to perform accumulated-phase calculations as described below in Sect. 2.3.4.

Scattering in a central potential  The elastic scattering properties can be obtained by solving the one-dimensional radial Schrödinger equation \[80\]

$$\left[ \frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \right] R(r) = ER(r)$$  \hspace{1cm} (2.9)

where $V'_S(r) = V_S(r) + \hbar^2(l + 1)/2\mu r^2$ and correspondingly, $V'_0(r) = V_0(r)$. For $E < 0$ the solutions correspond to the vibrational levels of the $V'_S(r)$ potential and for $E \downarrow 0$ and $l = 0$ the influence of the scattering potential $V_S(r)$ on the scattered wavefunction at $r \rightarrow \infty$ can be expressed in the so-called s-wave phase-shift $\eta_0$ [76]. This phase shift is related to the s-wave scattering length $a$ by

$$a = \lim_{k \rightarrow 0} \frac{\tan \eta_0}{k}$$  \hspace{1cm} (2.10)

\[^2\text{For a more accurate description of the direct term Eq. 2.6 can be easily extended with the }-C_8/r^8 \text{ and } -C_{10}/r^{10} \text{ dispersive terms.}\]
We define $a_s$ and $a_t$ as the $s$-wave scattering lengths for the $V_0(r)$ and $V_1(r)$ potentials. In the case of resonant scattering ($\eta_0 \rightarrow \pi/2$) the scattering length diverges, this is referred to as a potential resonance due to the resonance being induced by the scattering potential $V_{cen}(r)$.

**Internal Hamiltonian**

The internal Hamiltonian, given by Eq. 2.2, can be rewritten in a more convenient form

$$H_{int} = H_{hf}^\alpha + H_{hf}^\beta \equiv H_{+}^\int + H_{-}^\int$$

where

$$H_{+}^\int = \frac{1}{2\hbar^2} (a_f^\alpha i_\alpha + a_f^\beta i_\beta) \cdot S + (\gamma_{e,\alpha} i_\alpha - \gamma_{e,\beta} i_\beta) \cdot B$$

$$H_{-}^\int = \frac{1}{2\hbar^2} (a_f^\alpha i_\alpha - a_f^\beta i_\beta) \cdot (s_\alpha - s_\beta)$$

where $H_{+}^\int$ (for $B = 0$) contains all terms which conserve the total electron spin and $H_{-}^\int$ gives rise to the coupling between different $S$, additionally we have set $\gamma_{e,\alpha} = \gamma_{e,\beta} \equiv \gamma_e$. We define the total spin of the two-body system as $F = S + i_\alpha + i_\beta$. For distinguishable atomic species the nuclear spin does not need to be symmetrized and we take the basis $|SM_S i_\alpha S_i_\beta \rangle = |SM_S i_\alpha i_\beta \rangle$, where we omit the total nuclear spin $i_\alpha$ and $i_\beta$ of both atoms to simplify the notation. For indistinguishable atoms one has to define the total nuclear spin $I = i_\alpha + i_\beta$ with its associated quantum number $I$ and projection $M_I$, allowing to use the symmetrized basis $|SM_S IM_I \rangle$. We have neglected dipole-dipole coupling and second-order spin-orbit interaction, therefore, the relative Hamiltonian $H_{rel}$ conserves the total spin and only acts on the spatial part of the wavefunction. In contrast, $H_{int}$ only acts on the spin states and not on the spatial wavefunction.

**Inelastic Scattering**

The relative Hamiltonian (Eq. 2.1) conserves the total spin $F$ and its projection $M_F$. However, the exchange term of the central interaction (Eq. 2.5), $s_\alpha \cdot s_\beta$, couples the electron spins of both atoms and can couple different hyperfine states while conserving the total electron spin $S$. These collisions, which change the hyperfine state of a colliding atom pair, are called spin-exchange collisions. Only certain combinations of hyperfine channels are allowed due to conservation of total angular momentum and energy. For spin 1/2 atoms we can rewrite the exchange term of Eq. 2.5 as

$$s_\alpha \cdot s_\beta = \frac{1}{2} (S^2 - s_\alpha^2 - s_\beta^2) = \frac{1}{2} S^2 - \frac{3}{4}$$

(2.11)

The spin-exchange rate $K_2$ of a certain inelastic collision channel is given by [60]

$$K_2 = 4\pi (a_t - a_s)^2 v_f |\langle f'_a m'_{f,a} f'_b m'_{f,b} | s_\alpha \cdot s_\beta | f_a m_{f,a} f_b m_{f,b} \rangle|^2$$

(2.12)

where $|f_a m_{f,a} f_b m_{f,b} \rangle$ and $|f'_a m'_{f,a} f'_b m'_{f,b} \rangle$ are the initial and final hyperfine states respectively and $v_f$ is the relative velocity of the two atoms in the final state, given by
\[ \nu_f = \sqrt{\frac{2}{\mu} \left( E_{\text{kin}} + E_{h\alpha}^f + E_{h\beta}^f - E_{h\alpha}^{f'} - E_{h\beta}^{f'} \right)} \] 

Here \( E_{h\alpha}^f \) and \( E_{h\beta}^f \) are the hyperfine energies of the initial states with kinetic energy \( E_{\text{kin}} \) and \( E_{h\alpha}^{f'} \) and \( E_{h\beta}^{f'} \) are the hyperfine energies of the final states. Note that the hyperfine energies depend on the magnetic field value. The value of \( \nu_f \) reflects the amount of phase-space available for the final scattering states: a large energy difference between the initial and final state will result in a large inelastic collision rate. To obtain a numerical value for the spin-exchange rate for a certain magnetic field \( B \) and temperature \( T \) (see Sect. 3.6) we perform a basis transformation from the \( |f_\alpha m_\alpha f_\beta m_\beta \rangle \) basis to the total spin basis \( |S m_\mu \alpha \mu_\beta \rangle \), this allows us to calculate the coupling term given in Eq. 2.11.

### 2.3.2 Feshbach resonances

For a collision of two atoms the scattering properties are determined by the scattering potential \( V^{\text{cen}}(r) \) (see Sect. 2.3.1). The properties of the scattering potential \( V^{\text{cen}}(r) \) determine the s-wave scattering length \( a \) for \( k \to 0 \). However, the value of \( a \) can be drastically influenced due to resonant coupling to a two-body bound state, referred to as a closed channel. Since this closed channel can have a different hyperfine structure than the open channel the presence of an external magnetic field can modify the position of the closed-channel bound state with respect to the open channel. This phenomenon is depicted in Fig. 2.3. The hyperfine energy of the two colliding atoms at large separation \( (r \to 0) \) defines the total energy of the initial atom pair. Since the s-wave scattering is energetically allowed in this channel, even at \( T \to 0 \), this channel is called the open channel. Any scattering potential with a higher asymptotic energy is referred to as a closed channel (see Fig. 2.3). Since the hyperfine energy of different channels vary different due to the influence of an external magnetic field, the closed channel shifts with respect to the open channel (see Fig. 2.3). If the closed channel has a bound state with binding energy \( \epsilon_S \), it can become resonant with the asymptotic energy of the open channel for a certain magnetic field \( B_0 \). If a coupling between the open and closed channel exists the scattering length can be dramatically altered due to resonant coupling to the bound state. On this resonance the scattering length diverges (see Fig. 2.3b). This phenomenon is known as a Feshbach resonance. Occasionally these resonances are referred to as Fano-Feshbach resonances referring to the work of Fano in 1961 [81]. However, the resonances in the context of ultracold atoms differ significantly from the original Feshbach and Fano resonances. In the field of ultracold atoms the resonance occurs due to tuning of the internal structure of the collision partners even down to zero kinetic energy. Fano and Feshbach resonances occur due to the variation of energy of the collision partners, and not their internal structure.

In the presence of only one closed channel the scattering length can be written as:

\[ a(B) = a_{bg} + a_{\text{res}}(B) \]
where \( a_{bg} \) is the scattering length in the open channel, the so-called background scattering length and \( a_{res}(B) \) is the magnetically induced scattering length due to the presence of the closed channel. This resonance term is commonly denoted as

\[
a_{res}(B) = -a_{bg} \frac{\Delta B}{B - B_0} \tag{2.13}
\]

where \( B_0 \) is the resonance position and \( \Delta B \) is the resonance width defined as the difference between the magnetic field resonance \( B_0 \) and the magnetic field where \( a(B) = 0 \), hence, \( \Delta B \) is a function of \( a_{bg} \) itself. By introducing \( \Delta \mu = \mu_o - \mu_c \) as the difference in magnetic moment between the open and closed channel we can write

\[
a(B) = a_{bg} - \frac{a_{bg} \Delta B \Delta \mu}{\Delta \mu (B - B_0)}
\]

where the numerator of the second term \( a_{bg} \Delta B \Delta \mu \equiv C \) corresponds to the coupling strength between the open and closed channel and is always positive \[82\]. Figure 2.4 shows the possible configurations of the scattering length divergence. A Feshbach resonance width is fully characterized by the three parameters \( a_{bg} \), \( \Delta B \) and \( \Delta \mu \), and the position solely by \( B_0 \). In the literature various expressions are used to describe the resonance width for \( k \to 0 \). The length scale \( R^* \) \[83\] and the width \( \Gamma \) are commonly used and are related by

\[
R^* \equiv \frac{\hbar^2}{2\mu a_{bg} \Delta B \Delta \mu}, \quad \Gamma \equiv \frac{\hbar^2 k}{\mu R^*} = 2Ck
\]

Additionally the dimensionless parameter \( s_{res} = R^*/r_0 \) \[62\] is a convenient measure to describe the resonance strength.

\[^{3}\text{Note that } \mu = -\partial E/\partial B\]
2.3 Ultracold scattering

2.3.3 Ultracold scattering at finite energy

Up till now we have considered scattering at $T = 0$. However, even at ultracold temperatures the finite temperature plays a significant role. In this section we discuss the $s$-wave scattering properties in the presence of a Feshbach resonance. We do this in the context of the scattering amplitude $f_0$ given by

$$f_0 = \frac{1}{k \cot \eta_0 - ik}$$  \hfill (2.14)

where $\eta_0$ is the $s$-wave phase-shift as defined in 2.3.1. The cross-section of $s$-wave collisions for distinguishable particles is given by

$$\sigma(k) = 4\pi|f_0|^2 = \frac{4\pi}{k^2 \cot^2 \eta_0 + k^2}$$  \hfill (2.15)

where the $\sigma(k)$ has an energy dependence reflected in the unitarity-limit term $k^2$ in the denominator and in $\eta_0$. The $s$-wave phase shift is given by

$$\eta_0(E) = \eta_{bg}(E) + \eta_{res}(E)$$  \hfill (2.16)

where the resonant term is expressed as

$$\tan \eta_{res}(E) = -k a_{bg} \Delta B \Delta \mu \frac{\Delta \mu}{\hbar^2 k^2/2\mu - \Delta \mu(B - B_0)}$$  \hfill (2.17)

where $E = \hbar^2 k^2/2\mu$, this equation reduces to equation 2.13 for $k \rightarrow 0$. Generally, the finite energy behavior is described in terms of the effective range expansion. This method expands the $\cot \eta_0$ in terms of $-ka$. However, due to the resonant structure of $a$ we choose to not perform this expansion, instead we calculate the energy dependent cross section by assuming a non-resonance background scattering length ($\tan k a_{bg} \approx k a_{bg}$). We readily obtain

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Figure 2.4: The scattering length (solid red) is plotted as a function of magnetic field for the four possible symmetries of the divergence around a Feshbach resonance. The red dotted line indicates the $a_{bg}$ and the black dashed line indicates $\Delta \mu$. 
Theoretical Background

\[ \sigma(k) = \frac{4\pi}{k^2} \frac{(\Gamma/2 - ka_{bg} \delta)^2}{(1 + k^2a_{bg}^2)(\delta^2 + (\Gamma/2)^2)} \]  

(2.18)

where \( \delta = \Delta \mu (B - B_0) - E \) represents the detuning from the resonance. In the absence of a resonance (\( \Gamma = 0 \)) this expression reduces to the familiar form

\[ \sigma(k) = 4\pi \frac{a_{bg}^2}{1 + k^2a_{bg}^2} \]  

(2.19)

The cross-section has an asymmetric shape around the Feshbach resonance, this shape is called the Fano-profile. In the absence of a background scattering length Eq. 2.18 reduces to

\[ \sigma(k) = \frac{4\pi}{k^2} \frac{(\Gamma/2)^2}{\delta^2 + (\Gamma/2)^2} \]

which has a symmetric Lorentz-shape and is the familiar Breit-Wigner resonance [84]. To obtain a thermally averaged cross-section we need to average over all possible \( k \)-vectors. In particular we want to study the collision rate \( \Gamma = n \langle \sigma v \rangle \), where \( n \) is the density and \( v = \hbar k/\mu \) is the relative velocity of a collision pair. The distribution in position and momentum space is given by \( f(r,p) = \exp(-H(r,p)/k_B T) \), where the single-atom Hamiltonian \( H(r,p) \) describes the system at position \( r \) with momentum \( p \), and is given by

\[ H(r,p) = \frac{p^2}{2m} + U(r) \]  

(2.20)

where \( U(r) \) is the trapping potential. Using \( p = \hbar k \) and the fact that \( \langle \sigma v \rangle \) is independent of \( r \) we obtain

\[ \langle \sigma v \rangle = \frac{\int \frac{\hbar k}{\mu} \sigma(k) e^{-H(r,k)/k_B T} dk dr}{\int e^{-H(r,k)/k_B T} dk dr} \]

(2.21)

\[ = \frac{2^{3/2}}{\sqrt{\pi \mu k_B T}^{3/2}} \int_0^\infty \sigma(E) E e^{-E/k_B T} dE \]  

(2.22)

In particular a finite energy collision \( k > 0 \) results in a shift of the resonance position and of the zero-crossing. Additionally, the on-resonance scattering cross section is reduced due to the unitarity limited term \( 1/k^2 \). For a thermal sample with a distribution of \( k \)-values this results in a washing out of the zero crossing of the scattering length. This is depicted in figure 2.5; the finite temperature of the sample results in a shift of the maximum cross section.

2.3.4 Accumulated phase method

The Asymptotic Bound-state Model as described in Chapter 5 utilizes the Accumulated phase method [85]. Recently, an extensively discussion of this method with additional extension is presented in Ref. [86]. The accumulated phase method enables to neglect details of the inner part of the potential, i.e. for an internuclear separation of \( r < r_{in} \).
2.3 Ultracold scattering

The scattering cross section $\sigma(k)$ and the thermally averaged rate constant $\langle \sigma v \rangle$, both normalized to a background value of 1. The solid red and dashed green curves show Eq. 2.18 for $k = 0$ and $k > 0$. The dash-dotted blue curve shows the thermally averaged rate constant $\langle \sigma v \rangle$ given by Eq. 2.22. Note the shift of the maxima for finite temperatures.

At the distance $r_{in}$ a boundary condition is imposed on the wavefunction in the form of a phase. This phase represents the accumulated phase behavior of the wavefunction for $r < r_{in}$. A proper choice of $r_{in}$ allows the inner potential to be neglected and a very good approximation of the wavefunctions near the scattering threshold can be made. In particular to describe Feshbach resonances, originating from near-threshold bound states, this tool has proven to be very powerful as demonstrated on the $^{40}\text{K}^{87}\text{Rb}$ system in Chapter 5. The idea is to expand the radial wavefunctions in a singlet and triplet basis, if this is possible, knowledge about the singlet and triplet radial wavefunctions suffice to describe any superposition occurring due to hyperfine mixing by $H_{hf}$ of the singlet and triplet potentials.

Three conditions on $r_{in}$ have to be fulfilled to use the accumulated phase method. First, the accumulated phase radius $r_{in}$ has to be small enough that the exchange energy is larger than the hyperfine energy. If this is the case the hyperfine coupling term $H_{hf}$ can be neglected and the singlet-triplet basis is a good basis for the radial wavefunctions. Second, $r_{in}$ should be large enough that the singlet-and triplet potentials are known. This condition can be fulfilled if the exchange energy is smaller than the van der Waals energy and the potentials are well described by the van der Waals tail. Finally, the energy and angular momentum scales should be small enough that varying the energy or angular momentum $l$ does not significantly alter the accumulated phase at $r_{in}$.

The conditions given above are depicted in Fig. 2.6 for the $^{6}\text{Li}^{40}\text{K}$ system. The first condition is fulfilled for $r \lesssim 22 a_0$, the second for $r \gtrsim 14 a_0$, therefore taking $r_{in} = 18 a_0$ yields a good choice for the accumulated phase method. The third condition is always satisfied in the calculations presented in this thesis, for more detail on this condition we refer the reader to Ref. [86].

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4If this condition cannot be fulfilled a first order correction can be made to allow the phase to vary proportional to $E$ or $l(l + 1)$. [86].
Theoretical Background

Figure 2.6: Determination of the accumulated phase point \( r_{in} \) for the \(^{6}\text{Li}^{40}\text{K} \) system. The solid line is the van der Waals energy, (Eq. 2.6), the red dashed line is the hyperfine energy \( (E^{\alpha}_{hf} + E^{\beta}_{hf}) \) and the blue dash-dotted line is the exchange energy (Eq. 2.8).

Figure 2.7 illustrates an application of accumulated phase method where the first bound state \( E_1 \) of the \(^{6}\text{Li}^{40}\text{K} \) triplet potential is calculated from the triplet scattering length \( a_t = 63.5 \text{ a}_0 \) \[17\]. First, the accumulated phase at \( r = r_{in} \) is obtained by solving the radial Schrödinger equation (Eq. 2.9) for \( E = k_B \times 1 \text{ nK} \) using as a boundary condition at \( r \rightarrow \infty \) the s-wave phase shift obtained with equation 2.10. The obtained wavefunction is depicted as the blue solid curve in Fig. 2.7. Subsequently, the triplet binding energy is obtained by varying \( E \) using as boundary conditions the phase at \( r_{in} \) and \( \psi', \psi \rightarrow 0 \) for \( r \rightarrow \infty \). The obtained binding energy is \( E_1 = h \times 434 \text{ MHz} \), well in agreement with full coupled channel calculations yielding \( E_1 = h \times 427.44 \text{ MHz} \) \[17\]. The procedure can also be applied in a reversed manner to relate the binding energy to the scattering length. Using \( E_1 = h \times 427.44 \text{ MHz} \) as input parameter we obtain a scattering length of \( a_t = 63.9 \text{ a}_0 \), well in agreement with the coupled-channel calculation value \( (a_t = 63.5 \text{ a}_0) \).

The accumulated phase method is very useful for Feshbach resonance studies due to its accurate description of near-threshold states. For example calculating the Franck-Condon factors of singlet and triplet wavefunctions can be performed very accurately with the accumulated phase method. These Franck-Condon factors are essential for the generalized version of the ABM presented in Chapter 5.

In Fig. 2.8 another illustration of the accumulated phase method is given. The grey line is the triplet potential of \(^{6}\text{Li}^{40}\text{K} \), and the solid blue line depicts the radial wavefunction of the least bound state with binding energy \( E_1 = h \times 427.44 \text{ MHz} \). Two other wavefunctions are plotted, first the red dashed line represents the wavefunction with the accumulated phase method up to \( r_{in} = 18 \text{ a}_0 \). This wavefunction is obtained by solving Eq. 2.9 using \( V_{vdW}(r) = -C_6/r^6 \) and as a boundary condition \( \chi(r) \rightarrow 0 \) for \( r \rightarrow \infty \), where \( \chi(r) = rR(r) \). We will refer to such a state as an asymptotic-bound state since the probability density of the wavefunction extends far outside the range of the potential. The inner part of the wavefunction, i.e. for \( r < r_{in} \), contains less than 1% of the probability density of the wavefunction. Therefore, summarizing the short-range behavior in the phase of the wavefunction yields a good approximation. As a comparison the wavefunction where the least bound state is treated as a halo state is plotted (green dash-dotted curve). For a halo state the potential is approximated by a contact potential with a bound state at an...
2.4 Trapping of ultracold atoms

In this section the principles of trapping will be briefly resumed, in particular a background is given for the aspects special to our experimental setup which is presented in Chapter 3. In Sect. 2.4.1 optical trapping is discussed, followed by Sect. 2.4.2 on magnetic trapping. Finally, density distributions of thermal and degenerate gases are discussed in Sect. 2.4.3.

2.4.1 Optical dipole potentials

Optical dipole potentials for neutral atoms are based on the AC Stark effect. An intense laser beam is far-detuned from an optical transition, inducing an electric transition dipole-moment on the atom. The induced dipole moment in the external driving field results in a potential energy proportional to the driving intensity. An extensive discussion of optical dipole potentials for neutral atoms can be found in Ref. [51]. Here we consider the approximations required to describe the optical potentials used in our setup.

An optical potential formed by a far detuned laser beam with an intensity $I(r)$ interacting with a two-level system is given by [51]

$$U(r) = -\frac{3\pi e^2}{2\omega_0^3} \left( \frac{\Gamma}{\omega_0 - \omega_L} + \frac{\Gamma}{\omega_0 + \omega_L} \right) I(r),$$

(2.23)

where $\omega_0 = 2\pi c/\lambda_0$ and $\omega_L = 2\pi c/\lambda_L$ are the angular frequency of the atomic transition and of the laser respectively and $\Gamma$ is the spontaneous decay rate of the excited state. For
laser beams tuned relatively close to resonance $\omega_L/\omega_0 \approx 1$ the rotating wave approximation (RWA) is generally made. This approximation assumes $\omega_0 + \omega_L \gg \omega_0 - \omega_L$, however for lithium in a trap with $\lambda_L \simeq 1\mu m$ the counter-rotating term $\Gamma/(\omega_0 + \omega_L)$ has a contribution of 19% to the trapping potential, therefore we do not perform the RWA. For alkali atoms the excited state of the D-lines has an orbital angular momentum of $l = 1$ which results in a fine-structure splitting in the D1 and D2 lines. The optical transition can still be approximated by a two-level (ground state/exited state) model if the detuning of the laser from the atomic resonance, $\Delta \equiv \omega_0 - \omega_L$, is much larger than the fine structure splitting, $\Delta E_{FS}$, of the excited state. We take the atomic transition frequency $\omega_0$ at the average of the D1 and D2 lines. For all dipole potentials considered in this thesis $\Delta \gg \Delta E_{FS}$ and we can use the two-level model.

The intensity profile of a focused Gaussian beam is described by the cylindrically symmetric intensity distribution [88]:

$$I(\rho, z) = I_0 e^{-2(\rho/w(z))^2} \frac{1}{1 + (z/z_R)^2} \quad (2.24)$$

where $w(z) = w_0 \sqrt{1 + z/z_R}$ is the beam waist as a function of $z$, $w_0$ is the focal waist, $z_R = w_0^2 \pi / \lambda_L$ is the Rayleigh range, $\lambda_L$ is the wavelength of the far off resonant laser beam and $I_0 = 2P/(\pi w_0^2)$ is the peak intensity for a total laser power $P$. Substituting Eq. 2.24 into 2.23 we obtain the total trapping potential, with a peak value $U_0$ given by

$$U_0 = \frac{6Pc^2}{w_0^2\omega_0^2} \left( \frac{\Gamma}{\omega_L^2 - \omega_0^2} \right).$$

This results in an attractive potential for a red-detuned laser ($\omega_L < \omega_0$) or a repulsive potential for a blue detuned laser ($\omega_L > \omega_0$).
2.4 Trapping of ultracold atoms

Repulsive potentials

For a laser blue detuned with respect to the \(ns^2S \rightarrow np^2P\) transition, it is red detuned with respect to all \(ns^2S \rightarrow (n+i)p^2P\) \((i > 0)\) transitions. Summing over all these attractive terms potentially attenuates the repulsive potential originating from the lowest transition. The total trapping potential \(U_{total}\) is given by the sum over the contributions of all possible \(2S \rightarrow 2P\) transitions up to the ionization threshold [51]:

\[
U_{total} = \sum_i U_i
\]

where \(\omega_i\) and \(\Gamma_i\) are the angular frequency and decay rate of the \(ns \rightarrow (n+i)p\) optical transition. Since the linewidths of the higher excited states \(\Gamma_i\) become increasingly smaller for larger \(i\), the effective detuning becomes much larger. As an example we consider a blue detuned laser beam with \(\lambda_L = 532\) nm. For potassium the \(i = 1\) transition \((4s^2S_{1/2} \rightarrow 5p^2P_{3/2})\) has a wavelength of \(\lambda_1 = 404.4\) nm and a decay rate of \(\Gamma_1 = 2\pi \times 0.2\) MHz [39], therefore \(U_1 = 7 \times 10^{-3}U_0\). Summing over all transitions results in an attenuation of the potential by 0.7% for potassium and 0.4% for lithium, therefore we can safely consider it as a two-level system, also for far-off-resonance blue-detuned laser beams.

Attractive Potentials

For a red detuned laser \(\omega_L < \omega_0\), the optical dipole potential is attractive \((U_0 < 0)\) and can be used as a trap for an ultracold atomic sample, with the shape given by

\[
U(\rho, z) = U_0 \frac{e^{-2(\rho/w(z))^2}}{1 + (z/z_R)^2}
\]

(2.25)

The bottom can be well approximated by a harmonic potential with trapping frequencies in radial \(\omega_r\) and axial \(\omega_z\) directions of:

\[
\omega_r = \sqrt{\frac{4U_0}{mw_0^2}}, \quad \omega_z = \sqrt{\frac{2U_0}{m \lambda^2 w_0^2\pi}}
\]

(2.26)

where \(m\) is the mass of the trapped atom.

2.4.2 Magnetic potentials

Magnetic trapping of neutral atoms is based on the Zeeman effect: an atom in an externally applied magnetic field experiences an energy shift proportional to the magnetic field value. For an extensive discussion of magnetic trapping of neutral atoms see e.g. [60, 55, 90], here we will briefly introduce the concepts required to describe the magnetic trap used in our setup. Atoms in different hyperfine ground-states experience a different Zeeman effect and therefore a different trapping potential. The internal energy of a ground state \((l = 0)\) atom in an externally applied magnetic field is given by Eq. 2.3. The hyperfine interaction couples the nuclear and electronic spin to a total angular momentum \(f = i + s\), therefore, we take the appropriate \(|fm_f\rangle\) basis.

In the case of \(s = 1/2\) Eq. 2.3 can be diagonalized resulting in an analytic expression for the hyperfine energies, called the Breit-Rabi formula [91].
where $\mu_B = 9.27400915 \times 10^{-24} \text{ JT}^{-1}$ is the Bohr-magneton, $x = (g_s - g_i)\mu_B B/(a^{hf}(i + 1/2))$ and the sign corresponds to the manifolds with $f = i \pm s$. Figure 2.2 shows the hyperfine diagrams for $^6\text{Li}$ and $^{40}\text{K}$ respectively. For low magnetic fields the atomic energy shift is well described by the linear Zeeman effect. For typical magnetic fields achieved in our magnetic trap this assumption can be made for $^{40}\text{K}$, however for $^6\text{Li}$ at relatively high temperatures the quadratic Zeeman effect cannot be neglected as will be discussed in Sect. 3.4.3. In this section we will consider clouds cold enough ($T_{\text{Li}} \ll 0.6 \text{ mK}, T_K \ll 4 \text{ mK}$) that the linear Zeeman effect yields a good description.

For an atom in the hyperfine state $m_f$ of the $f = i + s$ manifold the energy shift due to the magnetic field is given by:

$$U(B) = -\mu \cdot B = m_f g_f \mu_B |B|$$

where $g_f$ is the Landé g-factor of the hyperfine manifold (see e.g. [55]). The magnetic field generating our trap is created by a pair of coils mounted in approximately an anti-Helmholtz configuration. This results in a three-dimensional quadrupole field, where the field-gradient along the symmetry axis of the coil is twice the gradient in the two orthogonal directions. The absolute value of the magnetic field is given by $B(x,y,z) = \alpha_B \sqrt{x^2 + y^2 + 4z^2}$, where $\alpha_B$ is the magnetic field gradient and $z$ is the symmetry axis of the coils (see Fig. 3.10). This magnetic field results in a potential given by:

$$U(x,y,z) = m_f g_f \mu_B \frac{\alpha_B}{2} \sqrt{x^2 + y^2 + 4z^2}$$

which results in a magnetic trap for low-field seeking states ($m_f g_f > 0$) simultaneously expelling high-field seeking states ($m_f g_f < 0$). In the origin of the coordinate system the magnetic field vanishes and different $m_f$ states within a $f$-manifold are degenerate, therefore, transitions from trapped to untrapped states can occur, resulting in trap loss. This process is known as Majorana loss and limits the densities which can be achieved in quadrupole traps.

**Optically plugged magnetic trap**

The linear trap offers very tight confinement and is therefore attractive to experiments with ultracold gases. Evaporative cooling in such a trap can be performed more efficient than in e.g. harmonic confinement. Therefore, it is favorable to use a linear trap which is stable against Majorana spin flips, rather than a harmonic confinement\footnote{For an experimental demonstration see Sect. 3.6.1 or e.g. Ref. [92].}. Trap losses due to Majorana spin flips can be suppressed, without sacrificing the linear confinement, by applying a strong repulsive potential at the trap origin. This method has been first demonstrated in 1995 at MIT allowing to reach Bose-Einstein condensation in sodium \footnote{For an extensive discussion on evaporative cooling in power-law traps see e.g. [57, 56].}. The method was abandoned mainly because of the complex shape of the combined trap, making analytic descriptions of the trapped BEC's complicated. In 2005 D. Naik et al \cite{15} reintroduced the method demonstrating it as an efficient pre-cooling stage before
loading an optical dipole trap. In both references \cite{3,15} the magnetic field gradient had to be decreased during the evaporative cooling to minimize three-body losses. This requires low stray fields to avoid the magnetic field zero to move with respect to the plug position while reducing the field gradient. In the presented experiments the gradient is constant during the evaporative cooling process, and therefore the setup is insensitive to even large (though constant) stray fields generated by the environment.

2.4.3 Properties of trapped gases

Fermi Energy

We start by discussing the Fermi energy $E_F$ which is defined as the energy of the highest occupied single particle state in a Fermi gas at $T = 0$ (see Fig. 2.1). This is related to the Fermi-temperature $T_F = E_F/k_B$. To obtain the Fermi energy we need to consider the density of states $\rho(\epsilon)$. The density of states describes the number of classical states with energy $\epsilon$ per unit phase-space. The density of states for an ideal gas, confined in a trapping potential $U(r)$, is given by

$$\rho(\epsilon) = \frac{1}{(2\pi\hbar)^3} \int \delta \left( \epsilon - \frac{\mathbf{p}^2}{2m} - U(r) \right) d\mathbf{r} d\mathbf{p}$$

For a harmonic trap, given by

$$U(x, y, z) = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

we can obtain the density of states by evaluating the integral in Eq. (2.29). This results in $\rho(\epsilon) = (1/\hbar\bar{\omega})^3 \epsilon^2/2$, where $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. To obtain the Fermi energy we integrate the density of states, $\rho(\epsilon)$, up to the Fermi energy $E_F$. By definition of $E_F$ we know that this equals the total number of atoms

$$\int_0^{E_F} d\epsilon \rho(\epsilon) \equiv N$$

For the harmonic confinement evaluating the integral (2.31) yields

$$k_B T_F = \hbar \bar{\omega} (6N)^{1/3}$$

For the pancake shaped linear trap, described by Eq. (2.28) we obtain the density of states by evaluating the integral

$$\rho(\epsilon) = \frac{128\sqrt{2}}{105\pi} \left( \frac{\sqrt{m}}{\mu_B \alpha_B \hbar} \right)^3 \epsilon^{7/2}$$

And for the Fermi energy in a linear trap we substitute Eq. (2.33) into Eq. (2.31) and obtain:

$$k_B T_F \simeq 1.5961 N^{2/9} \left( \mu_B \alpha_B \hbar / \sqrt{m} \right)^{2/3}.$$
Density distributions

In the classical limit, i.e. the temperature \( T \) is much larger than \( T_F \), the statistical nature of the particles is irrelevant and the probability distribution for an atom with energy \( \epsilon \) is given by the Maxwell-Boltzmann distribution

\[
f(\epsilon) = e^{-\epsilon/k_B T}
\]

For a non-interacting ideal gas the energy is given by the one-body Hamiltonian \( \epsilon = H(\mathbf{r}, \mathbf{p}) \), given by Eq. 2.20. The density distribution is obtained by integrating the Boltzmann distribution over all possible momenta

\[
n_{th}(\mathbf{r}) = \int d\mathbf{p} e^{-H(\mathbf{r}, \mathbf{p})/k_B T}
\]

For a harmonic confinement solving the integral yields a Gaussian density distribution given by

\[
n_{th}(\mathbf{r}) = n_0 e^{-m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2k_B T}
\]

where \( n_0 \) is the peak density and \( \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3} \).

Degenerate density distributions For the quantum-degenerate gas we take a different approach. We assume the thermal energy \( k_B T \) is much larger than the quantum mechanical level spacing \( \hbar \omega \), this allows us to calculate the density distribution \( n_{FD}(\mathbf{r}) \) of a trapped fermionic gas by using the semi-classical approximation

\[
n_{FD}(\mathbf{r}) = \frac{1}{(2\pi \hbar)^3} \int d\mathbf{p} \frac{1}{e^{(H(\mathbf{r}, \mathbf{p})-\mu)/k_B T} + 1}
\]

where the integrand is the Fermi-Dirac distribution function. To calculate the density distribution of a Fermi-degenerate cloud at a finite temperature, \( T < T_F \), we evaluate the integral \[2.35\]. This results in the expression

\[
n(\mathbf{r}) = -\left( \frac{m k_B T}{2\pi \hbar^2} \right)^{3/2} \text{Li}_{3/2} \left( -\zeta e^{-U(\mathbf{r})/k_B T} \right)
\]

where \( \text{Li}_n(z) \equiv \sum_{k=1}^{\infty} z^k/k^n \) is the polylogarithm function and \( \zeta \equiv \exp(\mu/k_B T) \) is the fugacity.

In typical experiments the density distribution is analyzed by absorption imaging. This method projects the three-dimensional density distributions onto a two-dimensional optical density distribution (see Sect. 3.5.3). We can calculate this two-dimensional density profile for a harmonic potential (Eq. 2.30) by integrating over the \( y \)-dimension. Additionally, we integrate over the \( z \)-dimension to obtain the one-dimensional density profile and once more over the \( x \)-dimension to obtain the total atom number. We readily obtain
2.4 Trapping of ultracold atoms

\[ n_{2D}(x, z) = -\frac{1}{\hbar^3} \frac{m (k_B T)^2}{2 \pi \omega_y} \text{Li}_2 \left( -\zeta e^{-m(\omega_x^2 x^2 + \omega_z^2 z^2)/(2 k_B T)} \right) \tag{2.36} \]

\[ n_{1D}(x) = -\frac{1}{\hbar^3} \sqrt{\frac{m (k_B T)^{5/2}}{2 \pi \omega_y \omega_z}} \text{Li}_{5/2} \left( -\zeta e^{-m(\omega_x^2 x^2)/(2 k_B T)} \right) \tag{2.37} \]

\[ N = -\left( \frac{k_B T}{\hbar \omega} \right)^3 \text{Li}_3(-\zeta) \tag{2.38} \]

The fugacity \( \zeta \) can be related to the degeneracy parameter \( T/T_F \) by combining Eq. 2.38 and 2.32

\[ \frac{T}{T_F} = (-6 \text{Li}_3(-\zeta))^{-1/3} \tag{2.39} \]

In the case of a time-of-flight experiment the gas is released from the trap and starts ballistically expanding. For a harmonic confinement the single particle Hamiltonian 2.20 is quadratic in momentum and in position space. Therefore, in the semiclassical approximation the time-of-flight distribution can be related to the spatial distribution by simply rescaling the coordinates

\[ \omega_i \rightarrow \frac{\omega_i}{\sqrt{1 + \omega_i^2 t^2}} \]

where \( \omega_i \) is \( \omega_x \), \( \omega_y \) or \( \omega_z \). A quantum mechanical treatment of this problem yields the same result \[93\].