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Abstract. This paper is concerned with the status of $1 / 0$ and ways to deal with it. These matters are treated in the setting of Komori fields, also known as non-trivial cancellation meadows. Different viewpoints on the status of $1 / 0$ exist in mathematics and theoretical computer science. We give a simple account of how mathematicians deal with $1 / 0$ in which a customary convention among mathematicians plays a prominent part, and we make plausible that a convincing account, starting from the popular computer science viewpoint that $1 / 0$ is undefined, by means of some logic of partial functions is not attainable.

Keywords: partial Komori field, imperative Komori field, relevant division convention, logic of partial functions.

MSC2000 codes: 12E99, 12L99.

1 Introduction

In [5], meadows are proposed as alternatives for fields with a purely equational specification. A meadow is a commutative ring with identity and a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero. Thus, meadows are total algebras. Recently, we found in [12] that meadows were already introduced by Komori [8] in a report from 1975, where they go by the name of *desirable pseudo-fields*. This finding induced us to propose the name *Komori field* for a meadow satisfying $0 \neq 1$ and $x \neq 0 \to x \cdot x^{-1} = 1$ (see [2]). The prime example of Komori fields is the field of rational numbers with the multiplicative inverse operation made total by imposing that the multiplicative inverse of zero is zero.

In [3], we renamed meadows to inversive meadows and introduced divisive meadows as inversive meadows with the multiplicative inverse operation replaced by a division operation. We also introduced simple constructions of variants of inversive and divisive meadows with a partial multiplicative or division operation. Moreover, we took a survey of logics of partial functions, but did not succeed in determining their adequacy for reasoning about the partial variants of meadows. This made us reflect on the way in which mathematicians deal with $1 / 0$ and how that way relates to the viewpoint, reflected in the partial variants of meadows, that $1 / 0$ is undefined.

In the current paper, we give a simple account of how mathematicians deal with $1 / 0$ in mathematical works. Dominating in this account is the concept of an imperative Komori field, a concept in which a customary convention among mathematicians plays a prominent part. We also make plausible that a convincing account, starting from the usual viewpoint of theoretical computer scientists that $1 / 0$ is undefined, by means of some logic of partial functions is not attainable.

It is quite usual that neither the division operator nor the multiplicative inverse operator is included in the signature of number systems such as the field of rational numbers and the field of real numbers. However, the abundant use of the division operator in mathematical practice makes it very reasonable to include the division operator, or alternatively the multiplicative inverse operator, in the signature. It appears that excluding both of them creates more difficulties than that it solves. At the least, the problem of division by zero cannot be avoided by excluding $1 / 0$ from being written.

This paper is organized as follows. First, we discuss the main prevailing viewpoints on the status of $1 / 0$ in mathematics and theoretical computer science (Section 2). Next, we give a brief summary of meadows and Komori fields (Section 3). After that, we introduce partial Komori fields and imperative Komori fields (Section 4) and discuss the convention that is involved in imperative Komori fields (Section 5). Then, we make plausible the inadequacy of logics of partial functions for a convincing account of how mathematicians deal with $1 / 0$ (Section 6). Finally, we make some concluding remarks (Section 7).

2 Viewpoints on the Status of 1 / 0

In this section, we shortly discuss two prevailing viewpoints on the status of $1 / 0$ in mathematics and one prevailing viewpoint on the status of $1 / 0$ in theoretical computer science. To our knowledge, the viewpoints in question are the main prevailing viewpoints. We take the case of the rational numbers, the case of the real numbers being essentially the same.

One prevailing viewpoint in mathematics is that $1 / 0$ has no meaning because 1 cannot be divided by 0. The argumentation for this viewpoint rests on the fact that there is no rational number z such that $0 \cdot z = 1$. Moreover, in mathematics, syntax is not prior to semantics and posing the question "what is $1 / 0$" is not justified by the mere existence of $1 / 0$ as a syntactic object. Given the fact that there is no rational number that mathematicians intend to denote by $1 / 0$, this means that there is no need to assign a meaning to $1 / 0$.

Another prevailing viewpoint in mathematics is that the use of $1 / 0$ is simply disallowed because the intention to divide 1 by 0 is non-existent in mathematical practice. This viewpoint can be regarded as a liberal form of the previous one: the rejection of the possibility that $1 / 0$ has a meaning is circumvented by disallowing the use of $1 / 0$. Admitting that $1 / 0$ has a meaning, such as 0 or "undefined", is consistent with this viewpoint.

The prevailing viewpoint in theoretical computer science is that the meaning of $1 / 0$ is "undefined" because division is a partial function. Division is identified as a partial function because there is no rational number z such that $0 \cdot z = 1$. This viewpoint presupposes that the use of $1 / 0$ should be allowed, for otherwise assigning a meaning to $1 \,/\, 0$ does not make sense. Although this viewpoint is more liberal than the previous one, it is remote from ordinary mathematical practice. This will be illustrated in Section 6.

The first of the two prevailing viewpoints in mathematics discussed above only leaves room for very informal concepts of expression, calculation, proof, substitution, etc. For that reason, we refrain from considering that viewpoint any further.

The prevailing viewpoint in theoretical computer science corresponds to two of the partial meadows of rational numbers obtained from the inversive and divisive meadows of rational numbers by a simple construction in [3]. Because $0 \neq 1$ and $x \neq 0 \to x \cdot x^{-1} = 1$ are satisfied, the latter are also called Komori fields of rational numbers and the former are also called partial Komori fields of rational numbers.

The prevailing viewpoint in mathematics considered further in this paper corresponds to the inversive and divisive meadows of rational numbers together with an imperative about the use of the multiplicative inverse operator and division operator, respectively. These combinations are called imperative Komori fields.

3 Meadows and Komori Fields

In this section, we give a brief summary of meadows and Komori fields.

The signature of inversive meadows consists of the following constants and operators:

- the constants 0 and 1;
- the binary *addition* operator $+$;
- the binary *multiplication* operator $\cdot$;
- the unary *additive inverse* operator $-$;
- the unary *multiplicative inverse* operator ${}^{-1}$.

We use infix notation for the binary operators, prefix notation for the unary operator $-$, and postfix notation for the unary operator ${}^{-1}$. Moreover, we use the usual precedence convention to reduce the need for parentheses.

The set of all terms over the signature of inversive meadows constitutes the *inversive meadow notation*.

An inversive meadow is an algebra over the signature of inversive meadows that satisfies the equations given in Tables 1 and 2. The equations given in Table 1 are the axioms of a commutative ring with identity. From the equations given in Tables 1 and 2, the equation $0^{-1} = 0$ is derivable.

Henceforth, we will write $\Sigma^{\mathrm{i}}_{\mathrm{Md}}$ for the signature of inversive meadows and $E^{\mathrm{i}}_{\mathrm{Md}}$ for the set of axioms for inversive meadows.

Table 1. Axioms of a commutative ring with identity

$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x + y = y + x$	$x \cdot y = y \cdot x$
$x + 0 = x$	$x \cdot 1 = x$
$x + (-x) = 0$	$x \cdot (y + z) = x \cdot y + x \cdot z$

Table 2. Additional axioms for an inversive meadow

$(x^{-1})^{-1} = x$
$x \cdot (x \cdot x^{-1}) = x$

Table 3. Additional axioms for a divisive meadow

$1 \,/\, (1 \,/\, x) = x$
$(x \cdot x) \,/\, x = x$
$x \,/\, y = x \cdot (1 \,/\, y)$

A *inversive Komori field* is an inversive meadow that satisfies the *separation axiom* $0 \neq 1$ and the *general inverse law* $x \neq 0 \to x \cdot x^{-1} = 1$.

The inversive Komori field that we are most interested in is $\mathcal{Q}_0^{\mathrm{i}}$, the inversive Komori field of rational numbers:[1]

$$\mathcal{Q}_0^{\mathrm{i}} = I(\Sigma_{\mathrm{Md}}^{\mathrm{i}}, E_{\mathrm{Md}}^{\mathrm{i}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\}) \;.$$

$\mathcal{Q}_0^{\mathrm{i}}$ differs from the field of rational numbers only in that the multiplicative inverse of zero is zero.

The signature of divisive meadows is the signature of inversive meadows with the unary multiplicative inverse operator ${}^{-1}$ replaced by:

- the binary *division* operator $/$.

The set of all terms over the signature of divisive meadows constitutes the *divisive meadow notation*.

A divisive meadow is an algebra over the signature of divisive meadows that satisfies the equations given in Tables 1 and 3.

Henceforth, we will write $\Sigma_{\mathrm{Md}}^{\mathrm{d}}$ for the signature of divisive meadows and $E_{\mathrm{Md}}^{\mathrm{d}}$ for the set of axioms for divisive meadows.

A *divisive Komori field* is an divisive meadow that satisfies the *separation axiom* $0 \neq 1$ and the *general division law* $x \neq 0 \to x \,/\, x = 1$.

The divisive Komori field that we are most interested in is $\mathcal{Q}_0^{\mathrm{d}}$, the divisive Komori field of rational numbers:

$$\mathcal{Q}_0^{\mathrm{d}} = I(\Sigma_{\mathrm{Md}}^{\mathrm{d}}, E_{\mathrm{Md}}^{\mathrm{d}} \cup \{(1 + x^2 + y^2) \,/\, (1 + x^2 + y^2) = 1\}) \;.$$

$\mathcal{Q}_0^{\mathrm{d}}$ differs from $\mathcal{Q}_0^{\mathrm{i}}$ only in that the multiplicative inverse operation is replaced by a division operation such that $x \,/\, y = x \cdot y^{-1}$.

[1] We write $I(\Sigma, E)$ for the initial algebra among the algebras over the signature Σ that satisfy the equations E (see e.g. [4]).

4 Partial Komori Fields and Imperative Komori Fields

In this section, we introduce partial inversive and divisive Komori fields and imperative inversive and divisive Komori fields.

Let $\mathcal{K}_i$ be an inversive Komori field. Then it make sense to construct one partial inversive Komori field from $\mathcal{K}_i$:

- $0^{-1} \uparrow \mathcal{K}_i$ is the partial algebra that is obtained from $\mathcal{K}_i$ by making 0^{-1} undefined.

Let $\mathcal{K}_d$ be a divisive Komori field. Then it make sense to construct two partial divisive Komori fields from $\mathcal{K}_d$:

- $Q \mathbin{/} 0 \uparrow \mathcal{K}_d$ is the partial algebra that is obtained from $\mathcal{K}_d$ by making $q \mathbin{/} 0$ undefined for all q in the domain of $\mathcal{K}_d$;
- $(Q \setminus \{0\}) \mathbin{/} 0 \uparrow \mathcal{K}_d$ is the partial algebra that is obtained from $\mathcal{K}_d$ by making $q \mathbin{/} 0$ undefined for all q in the domain of $\mathcal{K}_d$ different from 0.

$(Q \setminus \{0\}) \mathbin{/} 0 \uparrow \mathcal{K}_d$ expresses a view on the partiality of division by zero that cannot be expressed if only multiplicative inverse is available.

The partial Komori field constructions are special cases of a more general partial algebra construction for which we have coined the term *punching* in [3]. Presenting the details of the general construction is outside the scope of the current paper.

The partial Komori fields that we are most interested in are the ones that can be obtained from $\mathcal{Q}_0^{\mathrm{i}}$ and $\mathcal{Q}_0^{\mathrm{d}}$ by means of the partial Komori field constructions introduced above. It yields three partial Komori fields of rational numbers:

$$0^{-1} \uparrow \mathcal{Q}_0^{\mathrm{i}}\,, \qquad Q \mathbin{/} 0 \uparrow \mathcal{Q}_0^{\mathrm{d}}\,, \qquad (Q \setminus \{0\}) \mathbin{/} 0 \uparrow \mathcal{Q}_0^{\mathrm{d}}\,.$$

Notice that these partial algebras have been obtained by means of the well-known initial algebra construction and a straightforward partial algebra construction. This implies that only equational logic for total algebras has been used as a logical tool for their construction.

The first two of the partial Komori fields of rational numbers introduced above correspond most closely to the prevailing viewpoint on the status of $1 \mathbin{/} 0$ in theoretical computer science that is mentioned in Section 2. In the sequel, we will focus on $Q \mathbin{/} 0 \uparrow \mathcal{Q}_0^{\mathrm{d}}$ because the divisive notation is used more often than the inversive notation.

An imperative Komori field of rational numbers is a Komori field of rational numbers together with an imperative to comply with a very strong convention with regard to the use of the multiplicative inverse or division operator.

Like with the partial Komori field of rational numbers, we introduce three imperative Komori fields of rational numbers:

- $0^{-1} \Uparrow \mathcal{Q}_0^{\mathrm{i}}$ is $\mathcal{Q}_0^{\mathrm{i}}$ together with the imperative to comply with the convention that q^{-1} is not used with $q = 0$;
- $Q \mathbin{/} 0 \Uparrow \mathcal{Q}_0^{\mathrm{d}}$ is $\mathcal{Q}_0^{\mathrm{d}}$ together with the imperative to comply with the convention that $p \mathbin{/} q$ is not used with $q = 0$;

– $(Q \setminus \{0\}) \,/\, 0 \Uparrow \mathcal{Q}_0^{\mathrm{d}}$ is $\mathcal{Q}_0^{\mathrm{d}}$ together with the imperative to comply with the convention that $p \,/\, q$ is not used with $q = 0$ if $p \neq 0$.

The conventions are called the *relevant inversive convention*, the *relevant division convention* and the *liberal relevant division convention*, respectively.

The conventions are very strong in the settings in which they must be complied with. For example, the relevant division convention is not complied with if the question "what is $1 \,/\, 0$" is posed. Using $1 \,/\, 0$ is disallowed, although we know that $1 \,/\, 0 = 0$ in $\mathcal{Q}_0^{\mathrm{d}}$.

The first two of the imperative Komori fields of rational numbers introduced above correspond most closely to the second of the two prevailing viewpoints on the status of $1 / 0$ in mathematics that are mentioned in Section 2. In the sequel, we will focus on $Q / 0 \Uparrow \mathcal{Q}_0^{\mathrm{d}}$ because the divisive notation is used more often than the inversive notation.

Komori fields go by the name of non-trivial cancellation meadows in previous work (see e.g. [3]). Therefore, we take the names partial non-trivial cancellation meadow and imperative non-trivial cancellation meadow as alternatives for partial Komori field and imperative Komori field, respectively.

5 Discussion on the Relevant Division Convention

In this section, we discuss the relevant division convention, i.e. the convention that plays a prominent part in imperative Komori fields.

The existence of the relevant division convention can be explained by assuming a context in which two phases are distinguished: a definition phase and a working phase. A mathematician experiences these phases in this order. In the definition phase, the status of $1 \,/\, 0$ is dealt with thoroughly so as to do away with the necessity of reflection upon it later on. As a result, $\mathcal{Q}_0^{\mathrm{d}}$ and the relevant division convention come up. In the working phase, $\mathcal{Q}_0^{\mathrm{d}}$ is simply used in compliance with the relevant division convention when producing mathematical texts. Questions relating to $1 \,/\, 0$ are understood as being part of the definition phase, and thus taken out of mathematical practice. This corresponds to a large extent with how mathematicians work.

In the two phase context outlined above, the definition phase can be made formal and logical whereas the results of this can be kept out of the working phase. Indeed, in mathematical practice, we find a world where logic does not apply and where validity of work is not determined by the intricate details of a very specific formal definition but rather by the consensus obtained by a group of readers and writers.

Whether a mathematical text, including definitions, questions, answers, conjectures and proofs, complies with the relevant division convention is a judgement that depends on the mathematical knowledge of the reader and writer. For example, $\forall x \bullet (x^2 + 1) / (x^2 + 1) = 1$ complies with the relevant division convention because the reader and writer of it both know that $\forall x \bullet x^2 + 1 \neq 0$.

Whether a mathematical text complies with the relevant division convention may be judged differently even with sufficient mathematical knowledge. This is illustrated by the following mathematical text, where $>$ is the usual ordering on the set of rational numbers:

> **Theorem.** If $p \,/\, q = 7$ then $\dfrac{q^2 + p \,/\, q - 7}{q^4 + 1} > 0$.
>
> *Proof.* Because $q^4 + 1 > 0$, it is sufficient to show that $q^2 + p \,/\, q - 7 > 0$. It follows from $p \,/\, q = 7$ that $q^2 + p \,/\, q - 7 = q^2$, and $q^2 > 0$ because $q \neq 0$ (as $p \,/\, q = 7$). $\square$

Reading from left to right, it cannot be that first $p \,/\, q$ is used while knowing that $q \neq 0$ and that later on $q \neq 0$ is inferred from the earlier use of $p \,/\, q$. However, it might be said that the first occurrence of the text fragment $p \,/\, q = 7$ introduces the knowledge that $q \neq 0$ at the right time, i.e. only after it has been entirely read.

The possibility of different judgements with sufficient mathematical knowledge looks to be attributable to the lack of a structure theory of mathematical text. However, with a formal structure theory of mathematical text, we still have to deal with the fact that compliance with the relevant division convention is undecidable.

The imperative to comply with the relevant division conventions boils down to the disallowance of the use of $1 \,/\, 0$, $1 \,/\, (1 + (-1))$, etcetera in mathematical text. The usual explanation for this is the non-existence of a z such that $0 \cdot z = 1$. This makes the legality of $1 \,/\, 0$ comparable to the legality of $\sum_{m=1}^{\infty} 1 \,/\, m$, because of the non-existence of the limit of $(\sum_{m=1}^{n+1} 1 \,/\, m)_{n \in \mathbb{N}}$. However, a mathematical text may contain the statement "$\sum_{m=1}^{\infty} 1 \,/\, m$ is divergent". That is, the use of $\sum_{m=1}^{\infty} 1 \,/\, m$ is not disallowed. So the fact that there is no rational number that mathematicians intend to denote by an expression does not always lead to the disallowance of its use.

In the case of $1 \,/\, 0$, there is no rational number that mathematicians intend to denote by $1 \,/\, 0$, there is no real number that mathematicians intend to denote by $1 \,/\, 0$, there is no complex number that mathematicians intend to denote by $1 \,/\, 0$, etcetera. A slightly different situation arises with $\sqrt{2}$: there is no rational number that mathematicians intend to denote by $\sqrt{2}$, but there is a real number that mathematicians intend to denote by $\sqrt{2}$. It is plausible that the relevant division convention has emerged because there is no well-known extension of the field of rational numbers with a number that mathematicians intend to denote by $1 \,/\, 0$.

6 Partial Komori Fields and Logics of Partial Functions

In this section, we adduce arguments in support of the statement that partial Komori fields together with logics of partial functions do not quite explain how mathematicians deal with $1 \,/\, 0$ in mathematical works. It needs no explaining that a real proof of this statement is out of the question. However, we do not preclude the possibility that more solid arguments exist. Moreover, as it stands, it is possible that our argumentation leaves room for controversy.

In the setting of a logic of partial functions, there may be terms whose value is undefined. Such terms are called non-denoting terms. Moreover, often three truth values, corresponding to true, false and neither-true-nor-false, are considered. These truth values are denoted by T, F, and $*$, respectively.

In logics of partial functions, three different kinds of equality are found (see e.g. [11, 3]). They only differ in their treatment of non-denoting terms:

- *weak equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is $*$;
- *strong equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is T whenever both t and t' are non-denoting and F otherwise;
- *existential equality*: if either t or t' is non-denoting, then the truth value of $t = t'$ is F.

With strong equality, the truth value of $1 \,/\, 0 = 1 \,/\, 0 + 1$ is T. This does not at all fit in with mathematical practice. With existential equality, the truth value of $1 \,/\, 0 = 1 \,/\, 0$ is F. This does not at all fit in with mathematical practice as well. Weak equality is close to mathematical practice: the truth value of an equation is neither T nor F if a term of the form $p \,/\, q$ with $q = 0$ occurs in it.

This means that the classical logical connectives and quantifiers must be extended to the three-valued case. Many ways of extending them must be considered uninteresting for a logic of partial functions because they lack an interpretation of the third truth value that fits in with its origin: dealing with non-denoting terms. If those ways are excluded, only four ways to extend the classical logical connectives to the three-valued case remain (see e.g. [1]). Three of them are well-known: they lead to Bochvar's strict connectives [6], McCarthy's sequential connectives [10], and Kleene's monotonic connectives [7]. The fourth way leads to McCarthy's sequential connectives with the role of the operands of the binary connectives reversed.

In mathematical practice, the truth value of $\forall x \bullet x \neq 0 \rightarrow x \,/\, x = 1$ is considered T. Therefore, the truth value of $0 \neq 0 \rightarrow 0 \,/\, 0 = 1$ is T as well. With Bochvar's connectives, the truth value of this formula is $*$. With McCarthy's or Kleene's connectives the truth value of this formula is T. However, unlike with Kleene's connectives, the truth value of the seemingly equivalent $0 \,/\, 0 = 1 \lor 0 = 0$ is $*$ with McCarthy's connectives. Because this agrees with mathematical practice, McCarthy's connectives are closest to mathematical practice.

The conjunction and disjunction connectives of Bochvar and the conjunction and disjunction connectives of Kleene have natural generalizations to quantifiers, which are called Bochvar's quantifiers and Kleene's quantifiers, respectively. Both Bochvar's quantifiers and Kleene's quantifiers can be considered generalizations of the conjunction and disjunction connectives of McCarthy.[2]

With Kleene's quantifiers, the truth value of $\forall x \bullet x \,/\, x = 1$ is $*$ and the truth value of $\exists x \bullet x \,/\, x = 1$ is T. The latter does not at all fit in with mathematical practice. Bochvar's quantifiers are close to mathematical practice: the truth value

[2] In [9], Bochvar's quantifiers are called McCarthy's quantifiers, but McCarthy combines his connectives with Kleene's quantifiers (see e.g. [7]).

of a quantified formula is neither T nor F if a term of the form $p \,/\, q$ with $q = 0$ occurs in it.

What precedes suggest that mathematical practice is best approximated by a logic of partial functions with weak equality, McCarthy's connectives and Bochvar's quantifiers. We call this logic the *logic of partial meadows*, abbreviated $\mathcal{L}_{\mathrm{PMd}}$.

In order to explain how mathematicians deal with $1 \,/\, 0$ in mathematical works, we still need the convention that a sentence is not used if its truth value is neither T nor F. We call this convention the *two-valued logic convention*.

$\mathcal{L}_{\mathrm{PMd}}$ together with the imperative to comply with the two-valued logic convention gets us quite far in explaining how mathematicians deal with $1 \,/\, 0$ in mathematical works. However, in this setting, not only the truth value of $0 \neq 0 \rightarrow 0 \,/\, 0 = 1$ is T, but also the truth value of $0 = 0 \vee 0 \,/\, 0 = 1$ is T. In our view, the latter does not fit in with how mathematicians deal with $1 \,/\, 0$ in mathematical works. Hence, we conclude that $\mathcal{L}_{\mathrm{PMd}}$, even together with the imperative to comply with the two-valued logic convention, fails to provide a convincing account of how mathematicians deal with $1 \,/\, 0$ in mathematical works.

7 Conclusions

We have given a simple account of how mathematicians deal with $1 \,/\, 0$ in mathematical works. Dominating in this account is the concept of an imperative Komori field. The concept of an imperative Komori field is a special case of the more general concept of an *imperative algebra*, i.e. an algebra together with the imperative to comply with some convention about its use. An example of an imperative algebra is imperative stack algebra: stack algebra, whose signature consists of *empty*, *push*, *pop* and *top*, together with the imperative to comply with the convention that $\mathit{top}(s)$ is not used with $s = \mathit{empty}$.

Moreover, we have argued that a logic of partial functions with weak equality, McCarthy's connectives and Bochvar's quantifiers, together with the imperative to comply with the convention that sentences whose truth value is neither T nor F are not used, approximates mathematical practice best, but after all fails to provide a convincing account of how mathematicians deal with $1 \,/\, 0$ in mathematical works.

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