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Affine pure-jump processes on positive Hilbert–Schmidt operators

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Abstract

We show the existence of a broad class of affine Markov processes on the cone of positive self-adjoint Hilbert–Schmidt operators. Such processes are well-suited as infinite-dimensional stochastic covariance models. The class of processes we consider is an infinite-dimensional analogue of the affine processes on the cone of positive semi-definite and symmetric matrices studied in Cuchiero et al. (2011).

As in the finite-dimensional case, the processes we construct allow for a drift depending affine linearly on the state, as well as jumps governed by a jump measure that depends affine linearly on the state. The fact that the cone of positive self-adjoint Hilbert–Schmidt operators has empty interior calls for a new approach to proving existence: instead of using standard localization techniques, we employ the theory on generalized Feller semigroups introduced in Dörsek and Teichmann (2010) and further developed in Cuchiero and Teichmann (2020). Our approach requires a second moment condition on the jump measures involved, consequently, we obtain explicit formulas for the first and second moments of the affine process.

1. Introduction

In this article we show the existence of time-homogeneous *affine* Markov processes on the *cone of positive self-adjoint Hilbert–Schmidt* operators. The affine class is known for its tractability and flexibility.

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It is tractable because the Fourier–Laplace transform of such processes depends in an exponentially affine way on the initial state vector of the process. More specifically, denote by $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ the Hilbert space of self-adjoint Hilbert–Schmidt operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ and by $\mathcal{H}^+ \subseteq \mathcal{H}$ the cone of positive self-adjoint Hilbert–Schmidt operators. A $\mathcal{H}^+$-valued time-homogeneous Markov process $(X_t)_{t\geq 0}$ is affine, if there exist functions $\phi : \mathbb{R}^+ \times \mathcal{H}^+ \to \mathbb{R}^+$, $\psi : \mathbb{R}^+ \times \mathcal{H}^+ \to \mathcal{H}^+$ such that

$$\mathbb{E}\left[\mathrm{e}^{-\langle X_t, u\rangle} | X_0 = x\right] = \mathrm{e}^{-\phi(t,u) - \langle x, \psi(t,u)\rangle}, \qquad t \geq 0, \tag{1.1}$$

for all $u \in \mathcal{H}^+$. The functions ϕ and ψ are typically solutions of ordinary differential equations given in terms of the parameters of the model.

The affine class is flexible because the parameters of the model satisfy certain assumptions that allow for desired features such as constant and bounded linear drifts and constant and affine state-dependent jumps of infinite-variation.

Our motivation for studying affine processes in the state space $\mathcal{H}^+$ lies in the fact that such processes are well-qualified as models for infinite dimensional covariance processes, i.e., they can be used for the modeling of stochastic volatility in, for example, bond and commodity markets. See e.g. [17,6,2,3] for the modeling of forward price dynamics in bond and commodity markets as a process with values in a Hilbert space. In particular, in [4] a stochastic volatility model is constructed that involves a covariance process driven by Lévy noise and taking values in the positive Hilbert–Schmidt operators. Our model extends the covariance model in [4] from Lévy driven processes to processes allowing for state-dependent jumps (see also [7, Section 4.1]). More specifically, the affine processes we consider in this paper are of *pure-jump* type where the jumps can be *state-dependent* and of *infinite variation*.

Let us state our main result in an abbreviated form, see also Theorem 2.8 and its proof:

Theorem 1.1. *Let (b, B, m, μ) be a tuple consisting of a vector $b \in \mathcal{H}$, a bounded linear operator $B \in \mathcal{L}(\mathcal{H})$, a measure m on the Borel-σ-algebra $\mathcal{B}(\mathcal{H}^+ \setminus \{0\})$ and a $\mathcal{H}$-valued measure μ on $\mathcal{B}(\mathcal{H}^+ \setminus \{0\})$, satisfying the* admissibility assumptions *posed in Definition 2.3. Then there exists an affine process $(X_t)_{t\geq 0}$ in $\mathcal{H}^+$, such that the functions ϕ and ψ in Eq. (1.1) are the unique solution to the so called* generalized Riccati equations *associated to (b, B, m, μ):*

$$\frac{\partial}{\partial t}\phi(t,u) = \langle b, \psi(t,u)\rangle - \int_{\mathcal{H}^+\setminus\{0\}} \left(\mathrm{e}^{-\langle \xi, \psi(t,u)\rangle} - 1 + \langle \chi(\xi), \psi(t,u)\rangle\right) m(\mathrm{d}\xi), \tag{1.2a}$$

$$\frac{\partial}{\partial t}\psi(t,u) = B^*(\psi(t,u)) - \int_{\mathcal{H}^+\setminus\{0\}} \left(\mathrm{e}^{-\langle \xi, \psi(t,u)\rangle} - 1 + \langle \chi(\xi), \psi(t,u)\rangle\right) \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}, \tag{1.2b}$$

with initial values $\phi(0,u) = 0$ and $\psi(0,u) = u$ for $u \in \mathcal{H}^+$.

More specifically, the processes we consider have a constant drift vector b, a linear drift term B, a constant jump measure m, and a state-dependent jump measure μ. In addition to Theorem 1.1, and as a by-product of our method of proof, we establish explicit formulas for the first and second moments of the affine processes, see Proposition 4.17.

Note that Eq. (1.2b) is a non-linear differential equation on the cone of positive self-adjoint Hilbert–Schmidt operators which, in general, cannot be solved explicitly. Numerical methods for approximating solutions to infinite-dimensional Riccati equations are considered in e.g. [15,34]. A numerical approximation method tailored for this specific equation will be analyzed in forthcoming work [23].

There is a vast number of articles dealing with affine processes in several state spaces in *finite dimensions*, we mention, for example, [8,14,26,25,36,21,9]. In [14,9], the authors considered affine processes respectively on the canonical state space $\mathbb{R}_+^d \times \mathbb{R}^m$, $d, m \in \mathbb{N}$, and on the cone of positive semi-definite symmetric matrices. Both articles give sufficient and necessary admissible parameter conditions and characterize the class of stochastically continuous affine processes by means of their Markovian generator. The literature on affine processes in *infinite-dimensional* state spaces is more sparse. Existence of affine diffusion processes on Hilbert spaces was investigated in [35]. In [20], the author investigated affine processes in general locally convex vector spaces and in [10], existence of affine Markovian lifts of finite-dimensional Volterra processes was shown. The Markovian lift process takes values in a certain cone in a space of measures and shares many features of the affine processes which we consider.

The biggest challenge we face is that like many infinite-dimensional cones, the cone of positive self-adjoint Hilbert–Schmidt operators has empty interior. One consequence is that one cannot employ classical localization arguments to establish existence of the desired processes; we take a different approach outlined below. Another consequence is that it is difficult to incorporate a diffusion term. Indeed, it remains an open question whether and under what conditions infinite-dimensional affine processes on positive Hilbert–Schmidt operators allow for a diffusion term.

Our new approach involves approximating the transition semigroup associated with our Markov process by simpler transition semigroups corresponding to affine finite-activity jump processes. We then exploit the *generalized Feller theory* introduced in [13] and the approximation results [10, Proposition 3.3 and Theorem 3.2] as well as a version of the Kolmogorov extension theorem proven in [10, Theorem 2.11] to show that the limiting semigroup gives rise to a generalized Feller process. Note that the idea of showing the existence of affine processes with jumps of infinite variation through an approximation with simpler affine processes was already used on e.g. convex sets in finite dimensions, where it is known that affine processes are (classical) Feller processes (see [14,9]). However, our approach is somewhat different, and a considerable amount of effort goes into verifying that the approximating generalized Feller semigroups satisfy all necessary conditions to ensure convergence. In particular, a subtle analysis of the regularity of ϕ and ψ is conducted and we derive a uniform growth bound for the approximating semigroups.

1.1. Layout of the article

In Section 2 we provide the definition of admissible parameter sets and we state our main result (Theorem 2.8) on the existence of affine pure-jump processes on the cone of positive self-adjoint Hilbert–Schmidt operators. Moreover, we specify the exact form of the weak generator of these Markov affine processes on the linear span of the Fourier basis elements in terms of the introduced admissible parameter set. A brief outline of the proof of Theorem 2.8 is presented in Section 2 and the full proof is left to Section 4. In Section 3 we show the existence and uniqueness of the solution to the generalized Riccati equations (1.2) and we study the regularity of this solution with respect to its initial value. We briefly discuss the issue of solving the Riccati equations numerically in Section 3.2. We recall the generalized Feller setting in Section 4.1. Then in Sections 4.2 and 4.3 we make use of the results in Section 3 and some intricate approximation techniques for generalized Feller semigroups to complete the proof of Theorem 2.8. In Appendices A–C, we, respectively, add a comparison theorem that we need in our derivations, collect some ‘standard’ results on integration with respect to vector-valued measures, and provide a regularity result of the solution to our considered generalized Riccati equations.

1.2. Notation

We set $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$. For a vector space X and $U \subseteq X$ we denote the linear span of U by $\mathrm{lin}(U)$. For (X, τ) a topological space and $S \subseteq X$ we let $\mathcal{B}(S)$ denote the *Borel-σ-algebra* generated by the relative topology on S. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Then we denote by $C(S, H)$ the space of H-valued functions on S that are continuous with respect to the relative topology and we denote by $C_b(S, H)$ the space of bounded H-valued continuous functions on S. This is a Banach space when endowed with the supremum norm $\|\cdot\|_{C(S)}$. Notice that when $H = \mathbb{R}$, we typically omit H in the notation: $C(S) := C(S, \mathbb{R})$. Let $\mathcal{L}(X)$ denote the space of *bounded linear operators* from a Banach space X to X. This is a Banach space when equipped with the operator norm $\|\cdot\|_{\mathcal{L}(X)}$. If $\mathcal{G}$ is a linear operator on a Banach space X, we denote its *domain* by $\mathrm{dom}(\mathcal{G})$ and denote by I the identity in $\mathcal{L}(X)$. We denote unbounded operators by a calligraphic font and bounded ones by the standard font, e.g., $\mathcal{G}$ versus G. Let $\mathcal{L}^{(2)}(H \times H, H)$ denote the space of continuous bilinear forms from $H \times H$ to H. The *adjoint* of an operator $A: H \to H$ is denoted by A^*. An operator $A \in \mathcal{L}(H)$ is *positive* if $\langle Ax, x\rangle_H \geq 0$ for all $x \in H$. We let $\mathcal{L}_2(H)$ denote the space of *Hilbert–Schmidt operators* from H to H, this is a Hilbert space when endowed with the inner product

$$\langle A, B\rangle_{\mathcal{L}_2(H)} = \sum_{n=1}^{\infty} \langle Ae_n, Be_n\rangle_H,$$

where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis for H and $\langle\cdot,\cdot\rangle_{\mathcal{L}_2(H)}$ is independent of the choice of the orthonormal basis (see, e.g., [37, Section VI.6]). A nonempty subset K of a vector space is called a *wedge* if $K + K \subseteq K$ and $\alpha K \subseteq K$ for all $\alpha \geq 0$, if moreover $K \cap (-K) = \{0\}$ then we call K a *cone*. A cone K in a vector space X induces a partial ordering: we write $x \leq_K y$ if $y - x \in K$ (and $x \geq_K y$ if $x - y \in K$). If $K \subset H$ is a wedge, we define the *dual* of K by

$$K^* = \{x \in H : \langle x, y\rangle_H \geq 0 \text{ for all } y \in K\}, \tag{1.3}$$

and we say that K is *self-dual* if $K = K^*$. Note that if K is self-dual then $0 \leq_K x \leq_K y$ implies $\|x\|_H^2 \leq \langle x, y\rangle_H \leq \|x\|_H\|y\|_H$, i.e.,

$$0 \leq_K x \leq_K y \Rightarrow \|x\|_H \leq \|y\|_H \tag{1.4}$$

(in other words, K is *monotonic*).

We say that a cone K is *regular* if for all $y, x_1, x_2, \ldots \in K$ satisfying $x_1 \leq_K x_2 \leq_K \cdots \leq_K y$ there exists an $x \in H$ such that $\lim_{n\to\infty} \|x_n - x\|_H = 0$. A cone K is said to have *generating dual* if $B^* = K^* - K^*$. It is true that K has generating dual if and only if K is *normal*, i.e. $0 \leq_K x \leq_K y$ for $y \in K$, implies $\|x\| \leq \lambda\|y\|$ where $\lambda > 0$, see e.g. [22]. In finite dimensions, self-dual normal cones have non-empty interior. However, in infinite dimensions, the property $H = K - K$ does in general not imply that K has non-empty interior, see [27]. Let $(S, \mathcal{S})$ be a measurable space and $U \subseteq H$. A mapping $\mu: \mathcal{S} \to U$ is called a *U-valued measure* (on S) if it is weakly countably additive, i.e., if for every pairwise disjoint sequence $U_1, U_2 \ldots \in \mathcal{S}$ satisfying $\cup_{n\in\mathbb{N}} U_n = U$ it holds that

$$\langle \mu(U)x, y\rangle_H = \sum_{k\in\mathbb{N}} \langle \mu(U_k)x, y\rangle_H$$

for all $x, y \in H$. We know from the work of Pettis [33] that if $\mu: \mathcal{F} \to H$ is weakly σ-additive, then it is also strongly σ-additive. For a H-valued measure μ and $h \in H$ we define the signed measure $\langle \mu, h\rangle: \mathcal{F} \to \mathbb{R}$ by $\langle \mu, h\rangle(A) = \langle \mu(A), h\rangle_H$, $A \in \mathcal{F}$. Throughout this work we are required to integrate with respect to vector-valued measures, for a better readability we added a section on this matter to Appendix B.

1.3. Setting

Throughout this article we let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable infinite-dimensional real Hilbert space. For notational brevity we reserve $\langle\cdot,\cdot\rangle$ to denote the inner product on $\mathcal{L}_2(H)$, and $\|\cdot\|$ for the norm induced by $\langle\cdot,\cdot\rangle$. In addition, we define $\mathcal{H}$ to be the space of all self-adjoint Hilbert–Schmidt operators on H and $\mathcal{H}^+$ to be the cone of all positive operators in $\mathcal{H}$:

$$\mathcal{H} := \{A \in \mathcal{L}_2(H) \colon A = A^*\}, \text{ and } \mathcal{H}^+ := \{A \in \mathcal{H} \colon \langle Ah, h\rangle_H \geq 0 \text{ for all } h \in H\}.$$

Note that $\mathcal{H}$ is a closed subspace of $\mathcal{L}_2(H)$, and that $\mathcal{H}^+$ is a self-dual cone in $\mathcal{H}$ (indeed, $(\mathcal{H}^+)^* \subseteq \mathcal{H}^+$ by the spectral theorem for compact operators, and the reverse inclusion is trivial). Consequently, $\mathcal{H}$ is monotonic. Moreover, $\mathcal{H}^+$ is regular (see, e.g., [24, Theorem 1]), we have $\mathcal{H} = \mathcal{H}^+ - \mathcal{H}^+$ and $\mathcal{H}^+$ has empty interior.

We define the truncation function $\chi : \mathcal{H} \to \mathcal{H}$ by $\chi(\xi) = \xi \mathbf{1}_{\{\|\xi\|\leq 1\}}$ and fix it throughout this work.

2. Affine processes on $\mathcal{H}^+$ and statement of main result

In this section we give a detailed definition of affine processes on the state space $\mathcal{H}^+$ and introduce the notion of admissible parameter sets. We compare our admissible parameter conditions with the matrix valued case, this is done in Remark 2.4. Given an admissible parameter set we deduce first properties of the right-hand side functions of the differential equations in (1.2). At the end of this section we state our main result of this article in Theorem 2.8, which guarantees the existence of affine Markov processes on $\mathcal{H}^+$ associated with a given admissible parameter set and specifies the form of their weak generator on the Fourier-basis elements. However, we postpone the proof to Section 4.3 and only give a brief outline at the end of this section.

We consider a time-homogeneous Markov process X with state space $\mathcal{H}^+$ and transition semigroup $(P_t)_{t\geq 0}$ acting on functions $f \in C_b(\mathcal{H}^+)$,

$$P_t f(x) = \int_{\mathcal{H}^+} f(\xi) p_t(x, \mathrm{d}\xi), \qquad x \in \mathcal{H}^+,$$

where $p_t(x,\cdot)$, $t \geq 0$, $x \in \mathcal{H}^+$, is the transition kernel of X. Moreover for $x \in \mathcal{H}^+$, we denote the law of X given $X_0 = x$ by $\mathbb{P}_x$.

Definition 2.1. The Markov process $(X, (\mathbb{P}_x)_{x\in\mathcal{H}^+})$ is called *affine* if its Laplace transform has exponential-affine dependence on the initial state, i.e., if

$$P_t \,\mathrm{e}^{-\langle x,u\rangle} = \int_{\mathcal{H}^+} \mathrm{e}^{-\langle u,\xi\rangle}\, p_t(x, \mathrm{d}\xi) = \mathrm{e}^{-\phi(t,u)-\langle x,\psi(t,u)\rangle}, \tag{2.1}$$

for all $t \geq 0$, and $u, x \in \mathcal{H}^+$, for some functions $\phi\colon \mathbb{R}_+ \times \mathcal{H}^+ \to \mathbb{R}_+$ and $\psi\colon \mathbb{R}_+ \times \mathcal{H}^+ \to \mathcal{H}^+$.

We follow the approach in [9] and consider the Laplace transform instead of the characteristic function which is justified by the non-negativity of X.

Note, that we do not require stochastic continuity of the affine process here, as in this work we are not aiming to provide a characterization of affine processes. As discussed in the introduction, our existence result requires an analysis of the corresponding generalized Riccati equations. In particular, a direct consequence of our approach (see Theorem 2.8) is that the processes we consider are *regular* in the sense of [9, Def. 2.2]. We recall this concept for the reader's convenience:

Definition 2.2. We call the affine process *regular*, whenever the functions

$$\frac{\partial \phi(t,u)}{\partial t}\Big|_{t=0+} \qquad \text{and} \qquad \frac{\partial \psi(t,u)}{\partial t}\Big|_{t=0+},$$

exist and are continuous at $u = 0$.

As we will see, the established class of affine processes satisfies an even stronger regularity condition, see Section 3.3. In finite dimensions stochastically continuous affine processes are always regular (see [26]), however, there exist finite-dimensional affine processes that are not stochastically continuous. Arguably, such processes are of minor interest in applications. In infinite dimensions the regularity condition is somewhat more restrictive, as it implies e.g. that the operator B in Definition 2.3 must be bounded. We refer to [23, Section 3] for a construction of an infinite-dimensional affine process involving unbounded B.

In order to identify *pure-jump* affine processes, we introduce an *admissible parameter set* in the following definition. We think of b as the **constant drift vector**, B the **linear term in the drift**, m the **constant jump measure**, and μ the **state-dependent jump measure**.

Recall that Appendix B summarizes theory on integration with respect to a Hilbert space valued measure.

Definition 2.3. An *admissible parameter set* (b, B, m, μ) consists of

(i) a measure $m : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \to [0, \infty]$ such that

  (a) $\int_{\mathcal{H}^+\setminus\{0\}} \|\xi\|^2 \, m(\mathrm{d}\xi) < \infty$ and
  (b) $\int_{\mathcal{H}^+\setminus\{0\}} |\langle \chi(\xi), h\rangle| \, m(\mathrm{d}\xi) < \infty$ for all $h \in \mathcal{H}$ and there exists an element $I_m \in \mathcal{H}$ such that $\langle I_m, h\rangle = \int_{\mathcal{H}^+\setminus\{0\}} \langle \chi(\xi), h\rangle \, m(\mathrm{d}\xi)$ for every $h \in \mathcal{H}$;

(ii) a vector $b \in \mathcal{H}$ such that

$$\langle b, v\rangle - \int_{\mathcal{H}^+\setminus\{0\}} \langle \chi(\xi), v\rangle \, m(\mathrm{d}\xi) \geq 0 \quad \text{for all } v \in \mathcal{H}^+ ; \tag{2.2}$$

(iii) a $\mathcal{H}^+$-valued measure $\mu : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \to \mathcal{H}^+$ such that

$$\int_{\mathcal{H}^+\setminus\{0\}} \langle \chi(\xi), u\rangle \frac{\langle \mu(\mathrm{d}\xi), x\rangle}{\|\xi\|^2} < \infty,$$

for all $u, x \in \mathcal{H}^+$ satisfying $\langle u, x\rangle = 0$;

(iv) an operator $B \in \mathcal{L}(\mathcal{H})$ with adjoint B^* satisfying

$$\left\langle B^*(u), x\right\rangle - \int_{\mathcal{H}^+\setminus\{0\}} \langle \chi(\xi), u\rangle \frac{\langle \mu(\mathrm{d}\xi), x\rangle}{\|\xi\|^2} \geq 0,$$

for all $x, u \in \mathcal{H}^+$ satisfying $\langle u, x\rangle = 0$.

Remark 2.4 (*Comparison to the Finite-Dimensional Case*)**.** Definition 2.3 is analogous to the definition of an admissible parameter set for $\mathbb{R}^d_+$-valued processes see [14, Def. 2.6]) and the case of positive semi-definite and symmetric matrices, see [9, Def. 2.3]. However, as mentioned in the introduction, we do not consider any diffusion terms in this work. A more subtle difference is that we require second moment conditions on the measures $m(\mathrm{d}\xi)$ and $\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}$, whereas no moment conditions are needed in the finite-dimensional setting. These second moment conditions are a consequence of our *generalized Feller approach*, for which we take the weight function $\rho = \|\cdot\|^2 + 1$. See Remark 4.18 for a detailed discussion regarding the necessity of these moment conditions to our approach.

In what follows we will frequently use the following observation:

$$\begin{aligned}&\forall \xi, u \in \mathcal{H}^+:\\ -\min(\langle \xi, u\rangle, 1)\mathbf{1}_{\{\|\xi\|>1\}} \leq \mathrm{e}^{-\langle \xi, u\rangle} &-1 + \langle \chi(\xi), u\rangle \\ &\leq \tfrac{1}{2}|\langle \xi, u\rangle|^2\mathbf{1}_{\{\|\xi\|\leq 1\}} \leq \tfrac{1}{2}\|\xi\|^2\|u\|^2\mathbf{1}_{\{\|\xi\|\leq 1\}}.\end{aligned} \tag{2.3}$$

Given admissible parameters (b, B, m, μ), we define $F\colon \mathcal{H}^+ \to \mathbb{R}$ and $R\colon \mathcal{H}^+ \to \mathcal{H}$, respectively, by

$$F(u) = \langle b, u\rangle - \int_{\mathcal{H}^+\setminus\{0\}} \big(\mathrm{e}^{-\langle \xi, u\rangle} - 1 + \langle \chi(\xi), u\rangle\big) m(\mathrm{d}\xi), \tag{2.4a}$$

$$R(u) = B^*(u) - \int_{\mathcal{H}^+\setminus\{0\}} \big(\mathrm{e}^{-\langle \xi, u\rangle} - 1 + \langle \chi(\xi), u\rangle\big) \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}\,. \tag{2.4b}$$

Note that the admissibility conditions (see Definition 2.3), Corollary B.4, and (2.3) ensure that F and R are well-defined. We also have that F and R are continuous and grow at most quadratically:

Lemma 2.5. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3 and let F and R be given by (2.4). Then F and R are continuous on $\mathcal{H}^+$.*

Proof. This follows immediately from (2.3) and Theorem B.5. □

Lemma 2.6. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3 and let F and R be given by (2.4). Then for all $u \in \mathcal{H}^+$ we have*

$$|F(u)| \leq \left(\|b\| + \int_{\mathcal{H}^+\setminus\{0\}} \|\xi\|^2 m(\mathrm{d}\xi)\right)(1 + \|u\|^2)\,, \tag{2.5}$$

and

$$\|R(u)\| \leq \big(\|B^*\|_{\mathcal{L}(\mathcal{H})} + \|\mu(\mathcal{H}^+ \setminus \{0\})\|\big)(1 + \|u\|^2)\,. \tag{2.6}$$

Proof. This follows immediately from the admissibility conditions, (2.3), (B.7), and (B.4). □

Inspired by the finite-dimensional theory, we consider a system of ordinary differential equations associated with the admissible parameter set (b, B, m, μ) as introduced in the Eqs. (1.2). The equations are commonly known as the associated *generalized Riccati equations* which is due to the typically quadratic growth of F and R. By using the formulas for F and R in (2.4) Eqs. (1.2) can be rephrased as:

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, u) = F(\psi(t, u))\,, \quad t \geq 0; \qquad \phi(0, u) = 0\,,\\ \frac{\partial \psi}{\partial t}(t, u) = R(\psi(t, u))\,, \quad t \geq 0; \qquad \psi(0, u) = u\,. \end{cases} \tag{2.7}$$

Definition 2.7. Let $u \in \mathcal{H}^+$. We say that $(\phi(\cdot, u), \psi(\cdot, u))\colon [0, \infty) \to \mathbb{R}\times\mathcal{H}$ is a *solution to* (2.7) if $(\phi(\cdot, u), \psi(\cdot, u))$ is continuously differentiable, takes values in $\mathbb{R}^+ \times \mathcal{H}^+$, and satisfies (2.7).

For a transition semigroup $(P_t)_{t\geq 0}$ defined on bounded measurable functions on $\mathcal{H}^+$ we recall the notion of a *weak generator* $(\mathcal{A}, \mathrm{dom}(\mathcal{A}))$ of $(P_t)_{t\geq 0}$ (see [32, Definition 9.36]) i.e. $f \in C_b(\mathcal{H}^+)$ belongs to $\mathrm{dom}(\mathcal{A})$, whenever $\mathcal{A}f(x) := \lim_{t\to 0+} \frac{P_t f(x) - f(x)}{t}$ exists for every

$x \in \mathcal{H}^+$; $\mathcal{A}f \in C_b(\mathcal{H}^+)$ and

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{A} f(x) \mathrm{d}s, \quad x \in \mathcal{H}^+.$$

The following theorem is our main result, it asserts the existence of affine pure-jump processes on the cone of positive self-adjoint Hilbert–Schmidt operators admitting for state-dependent jumps of infinite variation and it specifies the form of the weak generator on a space of functions containing the Fourier basis elements. For the proof see Section 4.3, which relies on Sections 3 and 4.

Theorem 2.8. *Let (b, B, m, μ) be an admissible parameter set (cf. Definition 2.3). Then there exist constants $M, \omega \in [1, \infty)$ and a time-homogeneous $\mathcal{H}^+$-valued Markov process X with transition semigroup $(P_t)_{t\geq 0}$ such that*

$$\mathbb{E}[\|X_t\|^2 | X_0 = x] \leq M e^{\omega t}(\|x\|^2 + 1) \tag{2.8}$$

and

$$P_t \left(\mathrm{e}^{-\langle \cdot, u\rangle} \right)(x) = \mathrm{e}^{-\phi(t,u) - \langle x, \psi(t,u)\rangle},$$

for all $t \geq 0$ and $u, x \in \mathcal{H}^+$, where $(\phi(\cdot, u), \psi(\cdot, u))$ is the unique solution to the associated generalized Riccati equations in (2.7). Moreover let $(\mathcal{A}, \mathrm{dom}(\mathcal{A}))$ be the weak generator of $(P_t)_{t\geq 0}$, then $\mathrm{lin}\left\{\mathrm{e}^{-\langle \cdot, u\rangle} : u \in \mathcal{H}^+\right\} \subseteq \mathrm{dom}(\mathcal{A})$ and for every $f \in \mathrm{lin}\left\{\mathrm{e}^{-\langle \cdot, u\rangle} : u \in \mathcal{H}^+\right\}$ we have:

$$\mathcal{A} f(x) = \langle b + B(x), f'(x)\rangle + \int_{\mathcal{H}^+ \setminus \{0\}} \left(f(x+\xi) - f(x) - \langle \chi(\xi), f'(x)\rangle \right) \nu(x, \mathrm{d}\xi), \tag{2.9}$$

where $\nu(x, \mathrm{d}\xi) := m(\mathrm{d}\xi) + \frac{\langle \mu(\mathrm{d}\xi), x\rangle}{\|\xi\|^2}$.

Outline of the proof. The proof is based on the approximation procedure that we conduct in detail in Section 4.2, where we work in the realm of generalized Feller semigroups, see the preliminaries given in Section 4.1. Here we limit ourselves to give a brief outline of the proof that shall give a rough guidance for the upcoming sections and condensing the main ideas therein. The detailed proof is then given in Section 4.3. Inspired by [10], we approximate the Kolmogorov type operator $\mathcal{A}$ in (2.9) by operators $(\mathcal{A}^{(k)})_{k\in\mathbb{N}}$ corresponding to processes of pure-jump type with finite activity, i.e. for every $k \in \mathbb{N}$ we replace the constant jump measure $m(\mathrm{d}\xi)$ in formula (2.9) by $\mathbf{1}_{\{\xi \geq 1/k\}} m(\mathrm{d}\xi)$ and the linear jump measures $\mu(\mathrm{d}\xi)$ by $\mathbf{1}_{\{\xi \geq 1/k\}} \mu(\mathrm{d}\xi)$. The approximation operators $\mathcal{A}^{(k)}$ generate strongly continuous semigroups $(P_t^{(k)})_{t\geq 0}$ on a space of functions, being weakly continuous with sub-quadratic growth, see Proposition 4.13. Having established the existence of affine processes of pure-jump type associated with the strongly continuous semigroups $(P_t^{(k)})_{t\geq 0}$, we next apply a Trotter–Kato type result from [10] to obtain the limiting semigroup $(P_t)_{t\geq 0}$, see Proposition 4.16. To this end we first need to establish growth bounds on $(P_t^{(k)})_{t\geq 0}$, that are uniform in k, see Proposition 4.15. This requires understanding the associated generalized Riccati equations (1.2). We provide global existence and uniqueness results in Section 3. The crucial importance of the associated ODEs is that they substitute for the Kolmogorov equations, hence semigroup theoretic arguments involving the Kolmogorov type operators or the abstract Cauchy problem can be reduced to ODE theoretic arguments.

Lastly, we apply a version of Kolmogorov's extension theorem (see Theorem 4.5) to the limiting semigroup $(P_t)_{t\geq 0}$, which then yields the existence of an underlying Markovian process. This process associated via the semigroup to the operator $(\mathcal{A}, \mathrm{dom}(\mathcal{A}))$ is the desired affine process identified by the admissible parameter set (b, B, m, μ).

The second equation for $\psi(\cdot, u)$ in the generalized Riccati equations (2.7) is a non-linear differential equation on the cone of positive self-adjoint Hilbert–Schmidt operators. This type of infinite-dimensional differential equations has been of interest in the literature as they also show up e.g. in optimal control problems and stochastic filtering theory [11,19,29]. Hence several articles deal with the problem of numerical tractability of this type of equations. See, e.g. [34] where Galerkin approximation and convergence theory was developed for operator-valued Riccati differential equations formulated in the space of Hilbert–Schmidt operators and [15] where the author studied a backward Euler approximation scheme and convergence results for this type of equations. In a subsequent article [23], we investigate the Galerkin approximation further and draw a connection to matrix-valued affine processes. As the tractability of affine processes hinges on the ability to numerically approximate the Riccati equations (1.2), we have included a short discussion of the results in [23] in Section 3.2.

An example of a stochastic volatility model where the covariance process is an affine Markov process on $\mathcal{H}^+$ is the infinite-dimensional lift of the BNS model constructed in [4] to model forward rates in commodity markets. In [7, Section 4] we constructed several other examples to model stochastic volatility in this context of forward rates in commodity markets and we showed that our model class allow multiple modeling options for the instantaneous covariance process, including state-dependent jump intensity. Moreover, in [18] we studied infinite-dimensional stochastic volatility models with a stationary affine covariance process on $\mathcal{H}^+$.

3. Analysis of the generalized Riccati equations

In this section we investigate the generalized Riccati equations given by (2.7). In Section 3.1 we introduce Lipschitz continuous approximations of the mappings R and F in (2.4) and use these approximations to show existence and uniqueness of a solution to (2.7). In Section 3.3 we establish regularity properties of R and F and use this to show that the solution map depends in a differentiable way on its initial value.

3.1. Solving the generalized Riccati equations (2.7)

The goal of this subsection is to prove the existence of a unique solution to the generalized Riccati equations given an admissible parameter set (b, B, m, μ). A common approach in the finite-dimensional case, e.g. in the case of the cone of positive semi-definite and symmetric matrices, is to use a localization argument exploiting the fact that the function R is analytic on the interior of the cone. Note, however, that in general R fails to be Lipschitz continuous on the boundary of the cone. The cone of positive self-adjoint Hilbert–Schmidt operators has an empty interior, a property that is shared by many cones in infinite dimensions. This has the consequence that localization arguments for solving Eqs. (2.7) on the interior of $\mathbb{R}^+ \times \mathcal{H}^+$ are not valid anymore. Instead, for every $k \in \mathbb{N}$ we introduce approximations $F^{(k)}$ of F in Eq. (3.2) and $R^{(k)}$ of R in Eq. (3.3), which involve only finite-activity jump-measures, see (3.1). These approximations are Lipschitz continuous on $\mathcal{H}^+$, and in Proposition 3.7 we show that the solution to the generalized Riccati equations associated with $(b, B, m^{(k)}, \mu^{(k)})$ converges to the (unique) solution to Eq. (2.7).

We begin by introducing the approximating functions for F and R: for $k \in \mathbb{N}$ we set

$$m^{(k)}(\mathrm{d}\xi) := \mathbf{1}_{\{\|\xi\|>1/k\}} m(\mathrm{d}\xi) \quad \text{and} \quad \mu^{(k)}(\mathrm{d}\xi) := \mathbf{1}_{\{\|\xi\|>1/k\}} \mu(\mathrm{d}\xi), \tag{3.1}$$

and we introduce the functions $F^{(k)}: \mathcal{H}^+ \to \mathbb{R}$ and $R^{(k)}: \mathcal{H}^+ \to \mathcal{H}$ defined respectively as follows

$$F^{(k)}(u) = \langle b, u\rangle - \int_{\mathcal{H}^+\setminus\{0\}} \left(\mathrm{e}^{-\langle \xi, u\rangle} - 1 + \langle \chi(\xi), u\rangle\right) m^{(k)}(\mathrm{d}\xi), \tag{3.2}$$

$$R^{(k)}(u) = B^*(u) - \int_{\mathcal{H}^+\setminus\{0\}} \left(\mathrm{e}^{-\langle \xi, u\rangle} - 1 + \langle \chi(\xi), u\rangle\right) \frac{\mu^{(k)}(\mathrm{d}\xi)}{\|\xi\|^2}. \tag{3.3}$$

We denote the generalized Riccati equations associated to $(b, B, m^{(k)}, \mu^{(k)})$ by:

$$\begin{cases} \frac{\partial \phi^{(k)}}{\partial t}(t,u) = F^{(k)}(\psi^{(k)}(t,u)), \quad t \geq 0; \qquad \phi^{(k)}(0,u) = 0, \\ \frac{\partial \psi^{(k)}}{\partial t}(t,u) = R^{(k)}(\psi^{(k)}(t,u)), \quad t \geq 0; \qquad \psi^{(k)}(0,u) = u. \end{cases} \tag{3.4}$$

The notion of quasi-monotonicity will be needed to guarantee that the solution to (3.4) stays in $\mathbb{R}^+ \times \mathcal{H}^+$.

Definition 3.1. Let $(V, \|\cdot\|_V)$ be a Hilbert space and let $K \subset V$ be a self-dual cone. In addition, let $D \subseteq V$ and let $f: D \to V$, then f is called *quasi-monotone with respect to* K if for all $v_1, v_2 \in D$ satisfying $v_1 \leq_K v_2$ and for all $u \in K$ satisfying $\langle v_2 - v_1, u\rangle = 0$ we have

$$\langle f(v_2) - f(v_1), u\rangle \geq 0.$$

Intuitively, quasi-monotone functions are pointing 'inwards' at the boundary points, which ensures that solutions stay in a cone (see Theorem A.1). For details on quasi-monotone functions on Banach spaces and their connection to differential equations see [12, Section 5.3].

The following lemma states that the admissibility of parameters implies that $R^{(k)}$, $k \in \mathbb{N}$, is quasi-monotone with respect to $\mathcal{H}^+$. The proof is analogous to the proof of [9, Lemma 5.1], we present an abridged version.

Lemma 3.2. *Let B and μ satisfy the admissibility conditions (iii) and (iv) in Definition 2.3. Then for all $k \in \mathbb{N}$ the function $R^{(k)}$ given by (3.3) is quasi-monotone with respect to $\mathcal{H}^+$.*

Proof. The admissibility condition (iv) in Definition 2.3 (which makes sense thanks to condition (iii) in Definition 2.3) and the monotonicity of the exponential function imply the quasi-monotonicity of $R^{(k)}$. $\square$

By removing the small jumps and since m and μ have finite first moment, we obtain Lipschitz continuous mappings on $\mathcal{H}^+$:

Lemma 3.3. *Let B and μ satisfy the admissibility conditions (iii) and (iv) in Definition 2.3. Let $k \in \mathbb{N}$ and $R^{(k)}$ given by (3.3). Then for all $u, v \in \mathcal{H}^+$ we have*

$$\|R^{(k)}(u) - R^{(k)}(v)\| \leq \left(\|B\|_{\mathcal{L}(\mathcal{H})} + 2k\|\mu(\mathcal{H}^+ \setminus \{0\})\|\right) \|u - v\| \tag{3.5}$$

Proof. Observe that for all $u, v, \xi \in \mathcal{H}^+$ we have

$$\left|\mathrm{e}^{-\langle \xi, u\rangle} - \mathrm{e}^{-\langle \xi, v\rangle}\right| \leq \|\xi\| \|u - v\|.$$

Thus, (B.4) and (B.7) imply that

$$\begin{aligned}\|R^{(k)}(u)-R^{(k)}(v)\| &\le \left\|B^*(u-v)\right\| + \left\| \int_{\mathcal{H}^+\setminus\{0\}\cap\{\frac{1}{k}<\|\xi\|<1\}} \langle \xi, u-v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}\right\| \\ &\quad + \left\| \int_{\mathcal{H}^+\setminus\{0\}\cap\{\|\xi\|>\frac{1}{k}\}} (\mathrm{e}^{-\langle\xi,u\rangle}-\mathrm{e}^{-\langle\xi,v\rangle})\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}\right\| \\ &\le \left(\|B\|_{\mathcal{L}(\mathcal{H})}+2k\|\mu(\mathcal{H}^+\setminus\{0\})\|\right)\|u-v\|. \quad \square\end{aligned}$$

Note that R is typically not Lipschitz continuous on the whole $\mathcal{H}^+$:

Remark 3.4. Note that

$$\left|e^{-\langle\xi,u\rangle}-e^{-\langle\xi,v\rangle}+\langle\xi,u-v\rangle\right| \le \left|\int_{\langle\xi,u\rangle}^{\langle\xi,v\rangle} s\,ds\right| \le \|\xi\|^2(\|u\|\vee\|v\|)\|u-v\| \tag{3.6}$$

for all $\xi, u, v \in \mathcal{H}^+$. This implies that R is in general Lipschitz continuous only on bounded sets in $\mathcal{H}^+$.

By Lemmas 3.2 and 3.3 we have that $R^{(k)}$ is Lipschitz continuous on $\mathcal{H}^+$ and quasi-monotone with respect to $\mathcal{H}^+$. Hence classical infinite dimensional ODE theory guarantees the existence of a global solution to Eqs. (3.4):

Proposition 3.5. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3 and let $R^{(k)}$, $k \in \mathbb{N}$, be given by Eq. (3.3). Then for every $k \in \mathbb{N}$ and $u \in \mathcal{H}^+$ there exists a solution $(\phi^{(k)}(\cdot,u), \psi^{(k)}(\cdot,u))$ to (3.4). Moreover,*

$$\psi^{(k)}(t,u) \le_{\mathcal{H}^+} \psi^{(k)}(t,v), \quad \forall u, v \in \mathcal{H}^+ \text{ satisfying } u \le_{\mathcal{H}^+} v, \tag{3.7}$$

for all $t \ge 0$ and

$$\|\psi^{(k)}(t,u)-\psi^{(k)}(t,v)\| \le \exp\left(\left(\|B\|_{\mathcal{L}(\mathcal{H})}+2k\|\mu(\mathcal{H}^+\setminus\{0\})\|\right)t\right)\|u-v\| \tag{3.8}$$

for all $t \ge 0$ and $u, v \in \mathcal{H}^+$.

Proof. Let $k \in \mathbb{N}$. By Lemma 3.3 the function $R^{(k)}$ is Lipschitz continuous on $\mathcal{H}^+$, by (3.5) with $v = 0$ the function $R^{(k)}$ satisfies the linear growth condition $\|R^{(k)}(u)\| \le \left(\|B\|_{\mathcal{L}(\mathcal{H})}+2k\|\mu(\mathcal{H}^+\setminus\{0\})\|\right)\|u\|$ and by Lemma 3.2 $R^{(k)}$ is quasi-monotone with respect to $\mathcal{H}^+$, thus by [30, VI.3. Theorem 3.1 and Proposition 3.2] there exists a unique global solution $\psi^{(k)}(\cdot,u)\colon[0,\infty)\to\mathcal{H}^+$ to the second equation of (3.4). Now, setting $\phi^{(k)}(t,u) = \int_0^t F^{(k)}(\psi^{(k)}(s,u))\,\mathrm{d}s$, for all $t \ge 0$, we obtain by continuity of $F^{(k)}$ and $\psi^{(k)}(\cdot,u)$ a solution $(\phi^{(k)}(\cdot,u), \psi^{(k)}(\cdot,u))$ to (3.4) satisfying the inequality (3.7). Finally, observe that Lemma 3.3 implies that

$$\begin{aligned}&\frac{\partial}{\partial t}\|\psi^{(k)}(t,u)-\psi^{(k)}(t,v)\|^2 \\ &= 2\left\langle \psi^{(k)}(t,u)-\psi^{(k)}(t,v), R^{(k)}(\psi^{(k)}(t,u))-R^{(k)}(\psi^{(k)}(t,v))\right\rangle \\ &\le 2\left(\|B\|_{\mathcal{L}(\mathcal{H})}+2k\|\mu(\mathcal{H}^+\setminus\{0\})\|\right)\|\psi^{(k)}(t,u)-\psi^{(k)}(t,v)\|^2.\end{aligned}$$

This and Gronwall's lemma implies the second inequality (3.8). $\square$

The next proposition guarantees the existence of a unique solution to the original generalized Riccati equations (2.7) on $[0,\infty)$. First, we prove the following lemma:

Lemma 3.6. *Let B and μ satisfy the admissibility conditions (iii) and (iv) in Definition 2.3, let $R^{(k)}$ and R be respectively given by Eqs. (3.3) and (2.7). Then for every $M > 0$ we have*

$$\lim_{k\to\infty} \sup_{u\in\mathcal{H}^+:\|u\|\leq M} \|R^{(k)}(u) - R(u)\| = 0\,.$$

Proof. It follows immediately from (B.7) and (2.3) that

$$\|R^{(k)}(u) - R(u)\| \leq \|\mu(\{\xi \in \mathcal{H}^+ : \|\xi\| \leq \tfrac{1}{k}\})\| \|u\|^2. \tag{3.9}$$

The assertion follows from the above and the continuity of μ, see (B.2). $\square$

Proposition 3.7. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. Then for every $u \in \mathcal{H}^+$ there exists a unique solution $(\phi(\cdot, u), \psi(\cdot, u))$ to (2.7). Moreover,*

$$\psi(t, u) \leq_{\mathcal{H}^+} \psi^{(k)}(t, u) \quad \forall\, k \in \mathbb{N},\ t \geq 0 \text{ and } u \in \mathcal{H}^+,$$

and $\psi(t, u) = \lim_{k\to\infty} \psi^{(k)}(t, u)$ for all $t \geq 0$ and $u \in \mathcal{H}^+$, as well as

$$\psi(t, u) \leq_{\mathcal{H}^+} \psi(t, v), \quad \forall\, t \geq 0 \text{ and } u, v \in \mathcal{H}^+ \text{ with } u \leq_{\mathcal{H}^+} v, \tag{3.10}$$

and

$$\|\psi(t, u)\| \leq \exp\left(\left(\|B\|_{\mathcal{L}(\mathcal{H})} + 2\|\mu(\mathcal{H}^+ \setminus \{0\})\|\right) t\right) \|u\|, \quad \forall\, t \geq 0,\ u \in \mathcal{H}^+. \tag{3.11}$$

Finally, for all $M, T \geq 0$ there exists a $K(M, T) \geq 0$ such that for all $u, v \in \mathcal{H}^+$ satisfying $\|u\|, \|v\| \leq M$ and all $t \in [0, T]$ it holds that

$$\|\psi(t, u) - \psi(t, v)\| \leq K(M, T)\|u - v\|. \tag{3.12}$$

Proof. First of all note that uniqueness of a solution follows from the fact that R is Lipschitz continuous on bounded sets, see Remark 3.4. Observe that by (B.5), (2.3), and (3.3) we have, for all $u \in \mathcal{H}^+$ and $k \in \mathbb{N}$,

$$\begin{aligned} R^{(k)}(u) - R^{(k+1)}(u) &= \int_{\mathcal{H}^+\cap\{\frac{1}{k+1}<\|\xi\|\leq\frac{1}{k}\}} \left(\mathrm{e}^{-\langle u,\xi\rangle} - 1 + \langle \xi, u\rangle\right) \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2} \\ &\geq_{\mathcal{H}^+} 0. \end{aligned} \tag{3.13}$$

Now fix $u \in \mathcal{H}^+$. By Proposition 3.5 we know that there exists a unique global solution $\psi^{(k)}(\cdot, u)$ to Eq. (3.4) for every $k \in \mathbb{N}$. This combined with (3.13) implies that for all $k \in \mathbb{N}$ and $t \geq 0$ we have

$$\begin{aligned} \frac{\partial \psi^{(k+1)}}{\partial t}(t, u) - R^{(k+1)}(\psi^{(k+1)}(t, u)) &= \frac{\partial \psi^{(k)}}{\partial t}(t, u) - R^{(k)}(\psi^{(k)}(t, u)) \\ &\leq_{\mathcal{H}^+} \frac{\partial \psi^{(k)}}{\partial t}(t, u) - R^{(k+1)}(\psi^{(k)}(t, u)). \end{aligned}$$

It follows from Lemma 3.3 and Theorem A.1 with $K = \mathcal{H}^+$, $F = R^{(k+1)}$, $f = \psi^{(k+1)}(\cdot, u)$ and $g = \psi^{(k)}(\cdot, u)$ that

$$\psi^{(k+1)}(t, u) \leq_{\mathcal{H}^+} \psi^{(k)}(t, u)\,, \quad t \geq 0. \tag{3.14}$$

As moreover $\psi^{(k)}(t, u) \geq_{\mathcal{H}^+} 0$ for all $t \geq 0$ and $k \in \mathbb{N}$, the regularity of the cone $\mathcal{H}^+$ implies that for all $t \geq 0$ there exists a $\psi(t, u) \in \mathcal{H}^+$ such that

$$\psi(t, u) = \lim_{k\to\infty} \psi^{(k)}(t, u). \tag{3.15}$$

Note that by (3.14), the monotonicity of $\mathcal{H}^+$, and the continuity of $\psi^{(1)}(\cdot, u)$ we have, for all $T > 0$,

$$\sup_{k\in\mathbb{N}, s\in[0,T]} \|\psi^{(k)}(s,u)\| \leq \sup_{s\in[0,T]} \|\psi^{(1)}(s,u)\| < \infty\,. \tag{3.16}$$

It follows from this, (3.15), the dominated convergence theorem, and Lemmas 3.6 and 2.6 that for all $t \geq 0$ we have

$$\begin{aligned}
\psi(t,u) &= \lim_{k\to\infty} \psi^{(k)}(t,u) \\
&= u + \lim_{k\to\infty} \int_0^t R^{(k)}(\psi^{(k)}(s,u))\mathrm{d}s \\
&= u + \lim_{k\to\infty} \int_0^t \left(R^{(k)}(\psi^{(k)}(s,u)) - R(\psi^{(k)}(s,u))\right)\mathrm{d}s \\
&\quad + \lim_{k\to\infty} \int_0^t R(\psi^{(k)}(s,u))\mathrm{d}s \\
&= u + \int_0^t R(\psi(s,u))\mathrm{d}s.
\end{aligned}$$

The equation above combined with Lemma 2.6 implies that the map $\psi(\cdot, u)$ is continuous, whence Lemma 2.5 and the fundamental theorem of calculus imply that $\psi(\cdot,u) \in C^1([0,\infty),\mathcal{H})$ and

$$\frac{\partial\psi}{\partial t}(t,u) = R(\psi(t,u)), \quad t \geq 0; \qquad \psi(0,u) = u. \tag{3.17}$$

Moreover, the continuity of F and of $\psi(\cdot,u)$ ensures that by setting

$$\phi(t,u) = \int_0^t F(\psi(s,u))\mathrm{d}s, \quad t \geq 0, \tag{3.18}$$

we obtain that $(\phi(\cdot,u), \psi(\cdot,u))$ is a solution to (2.7).

Next, note that (3.10) follows from (3.7) and (3.15). Moreover, (3.11) follows from (3.8) with $k = 1$, (3.14), (3.15), and the fact that $\psi^{(1)}(t,0) \equiv 0$. Finally, (3.12) follows from the Lipschitz continuity of R on bounded sets (see Remark 3.4), (3.11), and the same reasoning as we used to obtain (3.8). □

3.2. *Numerical approximation of the generalized Riccati equations (2.7)*

The appeal of affine processes lies in their tractability, more specifically, in the fact that the characteristic function is given in terms of the solution to a Riccati equation (see (1.1) and (1.2)/(2.7)). This is particularly relevant when numerical approximations of the SDE describing the associated process X converge slowly and/or are difficult to implement (e.g. because the SDE contains a square-root term or state-dependent jumps). However, the tractability of affine processes relies on the assumption that the Riccati equation can be (approximatively) solved, which is not immediately clear, especially in the infinite-dimensional setting.

In a forthcoming article [23], we reduce the infinite-dimensional generalized Riccati equations (2.7) to finite-dimensional (essentially matrix valued) equations using Galerkin-type approximations, and provide error bounds for these approximations. As solvers are available for the finite-dimensional setting, this paves the way for explicit numerical examples.

More specifically, let $u \in \mathcal{H}^+$, $d \in \mathbb{N}$, and let $\mathbf{P}_d : \mathcal{H} \to \mathcal{H}_d$ be an appropriately chosen projection onto a finite-dimensional subspace $\mathcal{H}_d$ (in particular, such that $\mathbf{P}_d(\mathcal{H}^+)|_{H_d} \simeq \mathcal{S}^d_+$, the cone of symmetric and positive semi-definite matrices, and where H_d is a d-dimensional subspace of H). Denoting the associated $d \times d$-dimensional Galerkin approximation of $\psi(\cdot, u)$ (the solution to (1.2b)) by $\psi_d(\cdot, \mathbf{P}_d(u))$, [23] provides an upper bound for the error

$$\sup_{t\in[0,T]} \|\psi_d(t, \mathbf{P}_d(u)) - \psi(t, u)\|$$

in terms of $\|(\mathsf{I} - \mathbf{P}_d)\mu(\mathcal{H}^+ \setminus \{0\})\|$, $\|(\mathsf{I} - \mathbf{P}_d)\mathrm{e}^{t\mathbf{B}^*} u\|$, and the Lipschitz constant of the function $R|_{B_u(M)\cap\mathcal{H}}$, where $B_u(M)$ is the ball with origin u and radius $M = \sup_{t\in[0,T]} \|\psi(t, u)\|$, see also Remark 3.4. Moreover, in [23] we show that the Galerkin approximation $\psi_d(\cdot, \mathbf{P}_d(u))$ is essentially the solution to a matrix-valued generalized Riccati equation as in [9].

3.3. Regularity with respect to the initial value of the solution

Having established the existence of a unique solution to (2.7), we now turn to the regularity of the solution with respect to the initial value. To this end we first must introduce a fitting concept of differentiability:

Definition 3.8. Let X and Y be Banach spaces and $D \subseteq X$ a convex subset. We say that a function $f : D \subseteq X \to Y$ has a *one-sided derivative at $x \in D$ in the direction $v \in X$*, whenever $x + \lambda v \in D$ for all λ sufficiently small and the limit

$$\lim_{\lambda\to 0+} \frac{f(x + \lambda v) - f(x)}{\lambda},$$

exists in Y. We denote this limit by $\mathrm{d}_+ f(x)(v)$. We define the second one-sided derivative in $x \in D$ in direction $(v, w) \in X \times X$ as

$$\lim_{\lambda\to 0+} \frac{\mathrm{d}_+ f(x + \lambda w)(v) - \mathrm{d}_+ f(x)(v)}{\lambda},$$

whenever $x + \lambda w \in D$ and $\mathrm{d}_+ f(x + \lambda w)(v)$ exists for all λ sufficiently small and moreover the limit exists in Y. We denote the second one-sided derivative of f at x in directions (v, w) by $\mathrm{d}^2_+ f(x)(v, w)$.

Lemma 3.9. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3 and let F and R be given by (2.4). For $u \in \mathcal{H}^+$ define $\mathrm{d}R(u) \in \mathcal{L}(\mathcal{H})$ by*

$$\mathrm{d}R(u)v = B^*(v) + \int_{\mathcal{H}^+\setminus\{0\}} \langle \xi, v\rangle \mathrm{e}^{-\langle \xi,u\rangle} - \langle \chi(\xi), v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}, \quad v \in \mathcal{H}, \tag{3.19}$$

and $\mathrm{d}F(u) \in \mathcal{L}(H, \mathbb{R})$ by

$$\mathrm{d}F(u)v = \langle b, v\rangle + \int_{\mathcal{H}^+\setminus\{0\}} \langle \xi, v\rangle \mathrm{e}^{-\langle \xi,u\rangle} - \langle \chi(\xi), v\rangle m(\mathrm{d}\xi), \quad v \in \mathcal{H}. \tag{3.20}$$

Moreover define $\mathrm{d}^2 R(u) \in \mathcal{L}^{(2)}(\mathcal{H} \times \mathcal{H}, \mathcal{H})$ by

$$\mathrm{d}^2 R(u)(v, w) = -\int_{\mathcal{H}^+\setminus\{0\}} \langle \xi, v\rangle\langle \xi, w\rangle \mathrm{e}^{-\langle \xi,u\rangle} \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}, \quad v, w \in \mathcal{H}. \tag{3.21}$$

and $\mathrm{d}^2 F(u) \in \mathcal{L}^{(2)}(\mathcal{H} \times \mathcal{H}, \mathbb{R})$ by

$$\mathrm{d}^2 F(u)(v, w) = -\int_{\mathcal{H}^+\setminus\{0\}} \langle \xi, v\rangle\langle \xi, w\rangle \mathrm{e}^{-\langle \xi,u\rangle} m(\mathrm{d}\xi), \quad v, w \in \mathcal{H}. \tag{3.22}$$

Then the operator $\mathrm{d}R(u)$ *is quasi-monotone for all* $u \in \mathcal{H}^+$*, and for all* $u_0, u_1 \in \mathcal{H}^+$ *and* $v, w \in \mathcal{H}$ *we have*

$$\|\mathrm{d}R(u_0)(v)\| \le \|B^*\|_{\mathcal{L}(\mathcal{H})}\|v\| + \left\|\mu(\mathcal{H}^+ \setminus \{0\})\right\| (1 + \|u_0\|)\|v\| \tag{3.23}$$

$$\|\mathrm{d}R(u_0)(v) - \mathrm{d}R(u_1)(v)\| \le \left\|\mu(\mathcal{H}^+ \setminus \{0\})\right\| \|u_0 - u_1\| \|v\|, \tag{3.24}$$

$$\|\mathrm{d}^2 R(u_0)(v, w)\| \le \left\|\mu(\mathcal{H}^+ \setminus \{0\})\right\| \|v\| \|w\|, \tag{3.25}$$

and $u \mapsto \mathrm{d}^2 R(u)(v, w)$ *is continuous. Moreover,* F *and* R *are two-times one-sided differentiable in* u *in the direction* (v, w) *for all* $u, v, w \in \mathcal{H}^+$*, and for all* $u, v, w \in \mathcal{H}^+$ *we have:*

$$\mathrm{d}_+ R(u)(v) = \mathrm{d}R(u)v, \tag{3.26}$$

$$\mathrm{d}^2_+ R(u)(v, w) = \mathrm{d}^2 R(u)(v, w), \tag{3.27}$$

$$\mathrm{d}_+ F(u)(v) = \mathrm{d}F(u)v, \tag{3.28}$$

$$\mathrm{d}^2_+ F(u)(v, w) = \mathrm{d}F(u)(v, w). \tag{3.29}$$

Proof. The quasi-monotonicity of $\mathrm{d}R$ follows directly from the admissibility assumption. As

$$\left|\langle \xi, v\rangle \mathrm{e}^{-\langle \xi, u\rangle} - \langle \chi(\xi), v\rangle\right| \le \|\xi\| \|v\| (\mathbf{1}_{\{\|\xi\|>1\}} + \|\xi\| \|u\| \mathbf{1}_{\{\|\xi\|\le 1\}})$$

for all $u, \xi \in \mathcal{H}^+$ and all $v \in \mathcal{H}$, we obtain (3.23). Estimate (3.24) is obtained similarly, estimate (3.25) is immediate from the definition, and the continuity of $u \mapsto \mathrm{d}^2 R(u)(v, w)$ follows from the dominated convergence theorem (Theorem B.5).

We next confirm the asserted differentiability of the map $u \mapsto R(u)$. Let $u, v \in \mathcal{H}^+$ then

$$\begin{aligned}
\mathrm{d}_+ R(u)(v) &= \lim_{\lambda \to 0+} \frac{R(u + \lambda v) - R(u)}{\lambda} \\
&= B^*(v) - \lim_{\lambda \to 0+} \int_{\mathcal{H}^+ \setminus \{0\}} \frac{\mathrm{e}^{-\langle \xi, u+\lambda v\rangle} - \mathrm{e}^{-\langle \xi, u\rangle}}{\lambda} + \langle \chi(\xi), v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2} \\
&= B^*(v) - \int_{\mathcal{H}^+ \setminus \{0\}} \lim_{\lambda \to 0+} \frac{\mathrm{e}^{-\langle \xi, u+\lambda v\rangle} - \mathrm{e}^{-\langle \xi, u\rangle}}{\lambda} + \langle \chi(\xi), v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2} \\
&= B^*(v) + \int_{\mathcal{H}^+ \setminus \{0\}} \langle \xi, v\rangle \mathrm{e}^{-\langle \xi, u\rangle} - \langle \chi(\xi), v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}.
\end{aligned} \tag{3.30}$$

where the interchange of the integral and the limit in Eq. (3.30) is justified, since $\lambda \mapsto \mathrm{e}^{-\langle u+\lambda v, \xi\rangle}$ is a convex mapping, hence its differential quotient is non-decreasing in λ and non-negative and thus we can apply the monotone convergence theorem to obtain that the one-sided derivative of R exists in u in the direction v and (3.26) holds. An analogous derivation for F leads to Eq. (3.28).

The proof that the *second* one-sided directional derivative of both F and R exist and that (3.27)–(3.29) hold is again analogous. Note in particular that for the existence of the second derivatives we use that the measures $m(\mathrm{d}\xi)$ and $\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}$ have finite second moments. □

Proposition 3.11 states that the solution $(\phi(\cdot, u), \psi(\cdot, u))$ to (2.7) is such that the mappings $u \mapsto \psi(t, u)$ and $u \mapsto \phi(t, u)$ are twice one-sided differentiable in 0 in all directions. The techniques to prove this are well-known, however, as we are dealing with a non-standard concept of differentiability we provide the details of the proof in Appendix C.

Remark 3.10. In fact, one can prove that $u \mapsto \psi(t,u)$ and $u \mapsto \phi(t,u)$ are twice one-sided differentiable in u for every $u \in \mathcal{H}^+$, in every direction $(v,w) \in \mathcal{H}^+ \times \mathcal{H}^+$. We do not need this, but we *do* need the existence of the first derivative in $u \in \mathcal{H}^+$ for u sufficiently small in order to obtain the second derivative. See also Appendix C.

Proposition 3.11. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3, for every $u \in \mathcal{H}^+$ let $(\phi(\cdot,u), \psi(\cdot,u))$ be the solution to (2.7), and let $\mathrm{d}R$, $\mathrm{d}F$, d^2R, and d^2F be defined by (3.19)–(3.22). Then the maps $u \mapsto \psi(t,u)$ and $u \mapsto \phi(t,u)$ are twice one-sided differentiable in 0 in all directions $(v,w) \in \mathcal{H}^+ \times \mathcal{H}^+$. Moreover, $\mathrm{d}_+\psi(t,0)(v), \mathrm{d}_+^2\psi(t,0)(v,w) \in \mathcal{H}^+$ for all $v,w \in \mathcal{H}^+$ and the mappings $t \mapsto \mathrm{d}_+\phi(t,0)(v)$ and $t \mapsto \mathrm{d}_+\psi(t,0)(v)$ solves the following pair of differential equations:*

$$\frac{\partial}{\partial t}\mathrm{d}_+\phi(t,0)(v) = \mathrm{d}F(0)\big(\mathrm{d}_+\psi(t,0)(v)\big), \quad t \geq 0; \quad \mathrm{d}_+\phi(0,0)(v) = 0, \tag{3.31}$$

$$\frac{\partial}{\partial t}\mathrm{d}_+\psi(t,0)(v) = \mathrm{d}R(0)\big(\mathrm{d}_+\psi(t,0)(v)\big), \quad t \geq 0; \quad \mathrm{d}_+\psi(0,0)(v) = v, \tag{3.32}$$

Moreover, the mappings $t \mapsto \mathrm{d}_+^2\psi(t,0)(v,w)$ and $t \mapsto \mathrm{d}_+^2\phi(t,0)(v,w)$ solve the following pair of differential equations:

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{d}_+^2\phi(t,0)(v,w) &= \mathrm{d}^2F(0)(\mathrm{d}_+\psi(t,0)(v), \mathrm{d}_+\psi(t,0)(w)) \\ &\quad + \mathrm{d}F(0)\big(\mathrm{d}_+^2\psi(t,0)(v,w)\big), \quad t \geq 0; \quad \mathrm{d}_+^2\phi(0,0)(v,w) = 0,\end{aligned} \tag{3.33}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{d}_+^2\psi(t,0)(v,w) &= \mathrm{d}^2R(0)\big(\mathrm{d}_+\psi(t,0)(v), \mathrm{d}_+\psi(t,0)(w)\big) \\ &\quad + \mathrm{d}R(0)(\mathrm{d}_+^2\psi(t,0)(v,w)), \quad t \geq 0; \quad \mathrm{d}_+^2\psi(0,0)(v,w) = 0.\end{aligned} \tag{3.34}$$

Proof. See Appendix C. □

For $u = 0$ we derive explicit formulas for the solutions to the pairs of differential equations in (3.32) and (3.34) of Proposition 3.11, as those will be needed for proving Lemma 4.14 in the approximating case and for Proposition 4.17. First, note that

$$\mathrm{d}_+R(0)(v) = B^*(v) + \int_{\mathcal{H}^+\cap\{\|\xi\|\geq 1\}} \langle \xi, v\rangle \frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}.$$

Recall the definition of $\mathrm{d}R(0)$ from (3.19). The solution of Eq. (3.32) is then given by

$$\mathrm{d}_+\psi(t,0)(v) = \mathrm{e}^{t\mathrm{d}R(0)}v. \tag{3.35}$$

By inserting formula (3.35) into Eq. (3.34) (note that $e^{t\mathrm{d}R(0)}v \in \mathcal{H}^+$) and solving this inhomogeneous linear equation we obtain

$$\mathrm{d}_+^2\psi(t,0)(v,w) = \int_0^t \mathrm{e}^{(t-s)\mathrm{d}R(0)}\mathrm{d}^2R(0)(\mathrm{e}^{s\mathrm{d}R(0)}v, \mathrm{e}^{s\mathrm{d}R(0)}w)\mathrm{d}s. \tag{3.36}$$

4. Existence of affine pure-jump processes in $\mathcal{H}^+$

In this section we use the well-posedness and regularity results of the generalized Riccati equations (2.7) from Section 3 to show the existence of an affine process in $\mathcal{H}^+$ associated to a given admissible parameter set (b, B, m, μ) conform Definition 2.3. Due to the lack of local compactness of the underlying state space, standard Feller theory cannot be employed in our context and we use the theory of generalized Feller processes as introduced in [13]. The existence proof is based on the approximation procedure roughly sketched at the end of Section 2. In this section we rigorously build up this approximation procedure in the *generalized Feller* setting. Essentially, we approximate the transition semigroup $(P_t)_{t\geq 0}$, that can be associated to an affine process in $\mathcal{H}^+$ with infinite-activity jump behavior, by simpler transition semigroups corresponding to affine finite-activity jump processes. The considered semigroups are strongly continuous semigroups on a certain Banach space of real functions being weakly-continuous on compact sets and having at most quadratic growth in the tails. We briefly introduce the *generalized Feller* setting, that is we define *generalized Feller* semigroups and processes in Section 4.1 and consequently in Section 4.2 we apply approximation results from the theory of strongly continuous semigroups adapted to the generalized Feller setting by [10].

4.1. Preliminaries: generalized Feller semigroups

We recall the concept of generalized Feller semigroups introduced in [13] and further developed in [10].

Throughout this section let (Y, τ) be a complete regular Hausdorff space.

Definition 4.1. A function $\rho\colon Y \to (0,\infty)$ such that for every $R > 0$ the set $K_R := \{x \in Y :\ \rho(x) \leq R\}$ is compact is called an *admissible weight function*. The pair (Y, ρ) is called *weighted space*.

Let $\rho\colon Y \to (0,\infty)$ be an admissible weight function. For $f\colon Y \to \mathbb{R}$ we define $\|f\|_\rho \in [0,\infty]$ by

$$\|f\|_\rho := \sup_{x\in Y} \tfrac{|f(x)|}{\rho(x)}. \tag{4.1}$$

Note that $\|\cdot\|_\rho$ defines a norm on the vector space $B_\rho(Y) := \left\{f\colon Y \to \mathbb{R}\colon \|f\|_\rho < \infty\right\}$ which renders $(B_\rho(Y), \|\cdot\|_\rho)$ a Banach space. Recall that $C_b(Y)$ denotes the space of bounded $\mathbb{R}$-valued τ-continuous functions on Y. As any admissible weight function satisfies $\inf_{x\in Y} \rho(x) > 0$, we have that $C_b(Y) \subseteq B_\rho(Y)$.

Definition 4.2. We define $\mathcal{B}_\rho(Y)$ to be the closure of $C_b(Y)$ in $B_\rho(Y)$.

The following useful characterization of $\mathcal{B}_\rho(Y)$ is proven in [13, Theorem 2.7]:

Theorem 4.3. *Let (Y, ρ) be a weighted space. Then $f \in \mathcal{B}_\rho(Y)$ if and only if $f|_{K_R} \in C(K_R)$ for all $R > 0$ and*

$$\lim_{R\to\infty} \sup_{x\in Y\setminus K_R} \tfrac{|f(x)|}{\rho(x)} = 0\,. \tag{4.2}$$

We can now present the definition of a generalized Feller semigroup, as introduced in [13, Section 3].

Definition 4.4. A family of bounded linear operators $(P_t)_{t\geq 0}$ in $\mathcal{L}(\mathcal{B}_\rho(Y))$ is called a *generalized Feller semigroup (on $\mathcal{B}_\rho(Y)$)*, if

(i) $P_0 = I$, the identity on $\mathcal{B}_\rho(Y)$,

(ii) $P_{t+s} = P_t P_s$ for all $t, s \geq 0$,
(iii) $\lim_{t\to 0+} P_t f(x) = f(x)$ for all $f \in \mathcal{B}_\rho(Y)$ and $x \in Y$,
(iv) there exist constants $C \in \mathbb{R}$ and $\varepsilon > 0$ such that $\|P_t\|_{\mathcal{L}(\mathcal{B}_\rho(Y))} \leq C$ for all $t \in [0, \varepsilon]$,
(v) $(P_t)_{t\geq 0}$ is a positive semigroup, i.e., $P_t f \geq 0$ for all $t \geq 0$ and for all $f \in \mathcal{B}_\rho(Y)$ satisfying $f \geq 0$.

By [13, Theorem 3.2] any generalized Feller semigroup is strongly continuous. Moreover, generalized Feller semigroups allow for a Kolmogorov extension theorem, see [10, Theorem 2.11] for a proof:

Theorem 4.5. *Let $(P_t)_{t\geq 0}$ be a generalized Feller semigroup on $\mathcal{B}_\rho(Y)$ satisfying $P_t 1 = 1$ for all $t \geq 0$. Then there exist a filtered measurable space $(\Omega, (\mathcal{F}_t)_{t\geq 0})$ with a right-continuous filtration and a family of functions $X_t : \Omega \to Y$, $t \geq 0$, such that X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$ and for any initial value $x \in Y$ there exists a probability measure $\mathbb{P}_x$ such that*

$$\mathbb{E}_{\mathbb{P}_x}[f(X_t)] = P_t f(x) \tag{4.3}$$

for every $t \geq 0$ and every $f \in \mathcal{B}_\rho(Y)$. Moreover, for all $x \in Y$ the process $(X_t)_{t\geq 0}$ is a time-homogeneous $\mathbb{P}_x$-Markov process, i.e., for all $x \in Y$, $0 \leq s < t$, $f \in \mathcal{B}_\rho(Y)$ we have

$$\mathbb{E}_{\mathbb{P}_x}[f(X_t) \mid \mathcal{F}_s] = P_{t-s} f(X_s), \tag{4.4}$$

almost surely with respect to $\mathbb{P}_x$.

Let $(P_t)_{t\geq 0}$ be a generalized Feller semigroup satisfying $P_t 1 = 1$ for all $t \geq 0$. The process $(X_t)_{t\geq 0}$, the existence of which is guaranteed by Theorem 4.5, is called a *generalized Feller process* with initial value x with respect to the measure $\mathbb{P}_x$.

From now on we write $\mathbb{E}_x$ for expectations with respect to the probability measure $\mathbb{P}_x$.

Remark 4.6. Let $(P_t)_{t\geq 0}$ be a generalized Feller semigroup and let $x \in Y$, then by a Riesz representation-type result (see [10, Theorem 2.4 and Remark 2.8]) $P_t\rho(x) \in \mathbb{R}$ can be defined by the integral of ρ with respect to the measure representing the linear functional $f \mapsto P_t f(x)$, $f \in \mathcal{B}_\rho(Y)$. Moreover, as there exist $M > 1$, $\omega \in \mathbb{R}$ such that $|P_t f(x)| \leq M \exp(\omega t)\rho(x)\|f\|_\rho$ for all $f \in \mathcal{B}_\rho(Y)$, we obtain

$$P_t \rho \leq M \exp(\omega t)\rho \tag{4.5}$$

for $t \geq 0$. If moreover $(P_t)_{t\geq 0}$ is associated to a Markov process $(X_t)_{t\geq 0}$ such that Eq. (4.3) holds, we obtain:

$$\mathbb{E}_x[\rho(X_t)] = P_t\rho(x) \leq M \exp(\omega t)\rho.$$

This can be seen by Eq. (4.5) and a monotone convergence argument by choosing for every $n \in \mathbb{N}$ the approximations $\rho_n = \sum_{i=1}^n \langle \cdot, e_i \rangle^2 \wedge n \in \mathcal{B}_\rho(Y)$, where $(e_i)_{i\in\mathbb{N}}$ is an ONB of $\mathcal{H}$, then $\rho_n \to \rho$ in pointwise as $n \to \infty$ and $\rho_n \leq \rho_{n+1}$ for all $n \in \mathbb{N}$.

4.2. Approximation of semigroups associated to affine processes in $\mathcal{H}^+$

We equip the Hilbert space $\mathcal{H}$ with its weak topology $\sigma(\mathcal{H}, \mathcal{H}')$ (which, by the Riesz representation theorem, is the weak-$*$-topology). Note that as $\mathcal{H}^+$ is self-dual, it is closed in $(\mathcal{H}, \sigma(\mathcal{H}, \mathcal{H}'))$. For brevity of notation we let $\mathcal{H}^+_{\mathrm{w}}$ denote the complete regular Hausdorff space $(\mathcal{H}^+, \sigma(\mathcal{H}, \mathcal{H}')_{\mathcal{H}^+})$, where $\sigma(\mathcal{H}, \mathcal{H}')_{\mathcal{H}^+}$ denotes the relative topology $\sigma(\mathcal{H}, \mathcal{H}')$ on $\mathcal{H}^+$. In addition, we define $\rho : \mathcal{H}^+ \to \mathbb{R}$ by

$$\rho(x) := 1 + \|x\|^2 , \quad x \in \mathcal{H}^+ , \tag{4.6}$$

and observe that ρ is an admissible weight function on $\mathcal{H}^+_{\mathrm{w}}$ by the Banach–Alaoglu theorem, i.e., $(\mathcal{H}^+_{\mathrm{w}}, \rho)$ is a weighted space. Note that for every $R > 0$, the pre-image $\{x \in \mathcal{H}^+ : \rho(x) \le R\}$ is compact in $\mathcal{H}^+$ equipped with the norm topology, if and only if $\mathcal{H}$ is finite-dimensional. As we assume throughout the article that $\mathcal{H}$ is infinite-dimensional, we see that ρ is not an admissible weight function in the norm topology.

The linear span of the set of Fourier basis elements $\{\mathrm{e}^{-\langle \cdot, u \rangle} : u \in \mathcal{H}^+\}$ is denoted by

$$\mathcal{D} := \operatorname{lin}\left(\left\{\mathrm{e}^{-\langle \cdot, u \rangle} : u \in \mathcal{H}^+\right\}\right) . \tag{4.7}$$

The relevance of this set lies in the following lemma.

Lemma 4.7. *The set $\mathcal{D}$ is dense in $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$.*

Proof. It suffices to prove that for every $\varepsilon > 0$ and every $f \in C_b(\mathcal{H}^+_{\mathrm{w}})$ there exists an $f_\varepsilon \in \mathcal{D}$ such that $\|f - f_\varepsilon\|_\rho < \varepsilon$. To this end, observe that for every $\varepsilon > 0$ and every $f \in C_b(\mathcal{H}^+_{\mathrm{w}})$ there exists an $R > 0$ such that $\sup_{x \in \mathcal{H}^+, \|x\| > R} \frac{f(x)}{\rho(x)} < \frac{\varepsilon}{2}$, and apply Stone–Weierstrass to $C(\mathcal{H}^+_{\mathrm{w}} \cap \{x \in \mathcal{H}^+ : \|x\| \le R\})$. $\square$

Corollary 4.8. *The space $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ is separable.*

Proof. Let U be a countable dense set in $(\mathcal{H}^+, \|\cdot\|)$ (recall from Section 1.3 that $\mathcal{H}^+$ is separable). Then by Lemma 4.7 the set $\left\{\sum_{j=1}^n q_j \mathrm{e}^{-\langle \cdot, u_j \rangle} : n \in \mathbb{N}, q_j \in \mathbb{Q}, u_j \in U\right\}$ is dense in $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$. $\square$

Throughout the remainder of this section let (b, B, m, μ) be an admissible parameter set, see Definition 2.3. First, we define for $k \in \mathbb{N}$, $\tilde{B}^{(k)} \in \mathcal{L}(\mathcal{H})$ and $\tilde{b}^{(k)} \in \mathcal{H}^+$ by

$$\tilde{B}^{(k)}(x) := B(x) - \int_{\mathcal{H}^+ \cap \{0 < \|\xi\| \le 1\}} \xi \, \frac{\langle \mu^{(k)}(\mathrm{d}\xi), x \rangle}{\|\xi\|^2} , \quad x \in \mathcal{H}^+ ,$$

$$\tilde{b}^{(k)} := b - \int_{\mathcal{H}^+ \cap \{0 < \|\xi\| \le 1\}} \xi \, m^{(k)}(\mathrm{d}\xi) ,$$

where $m^{(k)}$ and $\mu^{(k)}$ are as defined in (3.1). Note that the fact that $B \in \mathcal{L}(\mathcal{H})$ and that μ is an $\mathcal{H}^+$-valued measure, as well as (B.7) and (3.1) ensure that $\tilde{B}^{(k)} \in \mathcal{L}(\mathcal{H})$ is well-defined. Moreover, (i) in Definition 2.3 and (2.2) ensure that $\tilde{b}^{(k)} \in \mathcal{H}^+$ is well-defined. For $x \in \mathcal{H}^+$ and $k \in \mathbb{N}$ we consider the following deterministic equation in differential form:

$$\begin{cases} \mathrm{d}\mathbf{x}_t^{(x,k)} &= \big(\tilde{b}^{(k)} + \tilde{B}^{(k)}(\mathbf{x}_t^{(x,k)})\big)\mathrm{d}t, \qquad t \ge 0, \\ \mathbf{x}_0^{(x,k)} &= x . \end{cases} \tag{4.8}$$

Standard infinite-dimensional ODE theory ensures that for all $x \in \mathcal{H}^+$ and $k \in \mathbb{N}$ the unique classical solution to (4.8) is given by

$$\mathbf{x}_t^{(x,k)} := \mathrm{e}^{t\tilde{B}^{(k)}} x + \int_0^t \mathrm{e}^{(t-s)\tilde{B}^{(k)}} \, \tilde{b}^{(k)} \mathrm{d}s , \quad t \ge 0 . \tag{4.9}$$

The following lemma provides some properties of $\mathbf{x}^{(x,k)}$, $x \in \mathcal{H}^+$, $k \in \mathbb{N}$.

Lemma 4.9. *Let (b, B, m, μ) be an admissible parameter set cf. Definition 2.3. For $x \in \mathcal{H}^+$ and $k \in \mathbb{N}$ let $\mathbf{x}^{(x,k)}$ be given by (4.9). Then*

$$0 \leq_{\mathcal{H}^+} \mathbf{x}_t^{(x,k+1)} \leq_{\mathcal{H}^+} \mathbf{x}_t^{(x,k)} \tag{4.10}$$

for all $k \in \mathbb{N}$, $x \in \mathcal{H}^+$, and $t \geq 0$.

Proof. It follows immediately from Definition 2.3 (iv) that $\mathcal{H} \ni x \mapsto \tilde{b}^{(k)} + \tilde{B}^{(k)}(x) \in \mathcal{H}$ is quasi-monotone with respect to $\mathcal{H}^+$. As $\tilde{b}^{(k)} \in \mathcal{H}^+$, Theorem A.1 with $K = \mathcal{H}^+$, $F(\cdot) = \tilde{b}^{(k)} + \tilde{B}^{(k)}(\cdot)$, $f \equiv 0$, and $g(\cdot) = \mathbf{x}_{\cdot}^{(x,k)}$ ensures that $\mathbf{x}_t^{(x,k)} \in \mathcal{H}^+$ for all $t \geq 0$, $x \in \mathcal{H}^+$, $k \in \mathbb{N}$.

Moreover, for all $k \in \mathbb{N}$ and $x \in \mathcal{H}^+$ we have

$$\tilde{b}^{(k)} + \tilde{B}^{(k)}(x) - \left(\tilde{b}^{(k+1)} + \tilde{B}^{(k+1)}(x)\right) \geq_{\mathcal{H}^+} 0.$$

This implies that for every $x \in \mathcal{H}^+$, $k \in \mathbb{N}$, and $t \geq 0$ we have

$$\begin{aligned}\frac{\partial \mathbf{x}_t^{(x,k+1)}}{\partial t} - \left(\tilde{b}^{(k+1)} + \tilde{B}^{(k+1)}(\mathbf{x}_t^{(x,k+1)})\right) &= \frac{\partial \mathbf{x}_t^{(x,k)}}{\partial t} - \left(\tilde{b}^{(k)} + \tilde{B}^{(k)}(\mathbf{x}_t^{(x,k)})\right) \\ &\leq_{\mathcal{H}^+} \frac{\partial \mathbf{x}_t^{(x,k)}}{\partial t} - \left(\tilde{b}^{(k+1)} + \tilde{B}^{(k+1)}(\mathbf{x}_t^{(x,k)})\right).\end{aligned}$$

Again applying Theorem A.1 with $K = \mathcal{H}^+$, $F(\cdot) = \tilde{b}^{(k)} + \tilde{B}^{(k)}(\cdot)$, $f(t) = \mathbf{x}_t^{(x,k+1)}$ and $g(t) = \mathbf{x}_t^{(x,k)}$, $t \geq 0$, implies that $\mathbf{x}_t^{(x,k+1)} \leq_{\mathcal{H}^+} \mathbf{x}_t^{(x,k)}$ for all $t \geq 0$. $\square$

For $k \in \mathbb{N}$, $t \geq 0$ and $f \in \mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ define $P_t^{(\det,k)} f : \mathcal{H}^+ \to \mathbb{R}$ by

$$(P_t^{(\det,k)} f)(x) := f(\mathbf{x}_t^{(x,k)}), \quad x \in \mathcal{H}^+. \tag{4.11}$$

Lemma 4.10. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. Let $k \in \mathbb{N}$, $t \geq 0$, $f \in C_b(\mathcal{H}_{\mathrm{w}}^+)$ and let $P_t^{(\det,k)} f : \mathcal{H}^+ \to \mathbb{R}$ be defined by (4.11). In addition, let*

$$M := \max\{1 + 2\|\tilde{B}^{(1)}\|_{\mathcal{L}(\mathcal{H})}^{-2} \|\tilde{b}^{(1)}\|^2, 2\}, \tag{4.12}$$

$$\omega := 2\|\tilde{B}^{(1)}\|_{\mathcal{L}(\mathcal{H})}. \tag{4.13}$$

Then $P_t^{(\det,k)} f \in C_b(\mathcal{H}_{\mathrm{w}}^+)$,

$$\|P_t^{(\det,k)} f\|_\rho \leq M\, \mathrm{e}^{\omega t}\, \|f\|_\rho, \tag{4.14}$$

and

$$\|P_t^{(\det,k)} f\|_{\sqrt{\rho}} \leq \sqrt{M}\, \mathrm{e}^{\omega t/2}\, \|f\|_{\sqrt{\rho}}. \tag{4.15}$$

Proof. For every $t \geq 0$ the operator $\mathrm{e}^{t\tilde{B}^{(k)}}$ is strong-to-strong continuous, hence it is also weak-to-weak continuous, and thus $P_t^{(\det,k)} f \in C_b(\mathcal{H}_{\mathrm{w}}^+)$. Next note that Lemma 4.9 implies that

$$\begin{aligned}\frac{1+\|\mathbf{x}_t^{(x,k)}\|^2}{1+\|x\|^2} &\leq \frac{1+\|\mathbf{x}_t^{(x,1)}\|^2}{1+\|x\|^2} \leq \frac{1+2\,\mathrm{e}^{2t\|\tilde{B}^{(1)}\|_{\mathcal{L}(\mathcal{H})}}(\|\tilde{B}^{(1)}\|_{\mathcal{L}(\mathcal{H})}^{-2}\|\tilde{b}^{(1)}\|^2+\|x\|^2)}{1+\|x\|^2} \\ &\leq M\,\mathrm{e}^{\omega t}\end{aligned} \tag{4.16}$$

for all $x \in \mathcal{H}^+$. Using the above estimate and (4.11) we obtain

$$\begin{aligned}\|P_t^{(\det,k)} f\|_\rho &= \sup_{x\in\mathcal{H}^+} \frac{(P_t^{(\det,k)} f)(x)}{1+\|x\|^2} = \sup_{x\in\mathcal{H}^+} \frac{f(\mathbf{x}_t^{(x,k)})}{1+\|x\|^2} \leq \|f\|_\rho \sup_{x\in\mathcal{H}^+} \frac{1+\|\mathbf{x}_t^{(x,k)}\|^2}{1+\|x\|^2} \\ &\leq M\,\mathrm{e}^{\omega t}\,\|f\|_\rho.\end{aligned}$$

Similarly,

$$\| P_t^{(\mathrm{det},k)} f \|_{\sqrt{\rho}} = \sup_{x\in\mathcal{H}^+} \frac{f(\mathbf{x}_t^{(x,k)})}{\sqrt{1+\|x\|^2}} \leq \| f \|_{\sqrt{\rho}} \sup_{x\in\mathcal{H}^+} \frac{\sqrt{1+\|\mathbf{x}_t^{(x,k)}\|^2}}{\sqrt{1+\|x\|^2}} \leq \sqrt{M}\, \mathrm{e}^{\omega t/2} \, \| f \|_{\sqrt{\rho}} . \quad \square$$

Recall that if $(A, \mathrm{dom}(A))$ is the generator of a strongly continuous semigroup $S = (S_t)_{t\geq 0}$ on a Banach space X, then a subspace $D \subseteq \mathrm{dom}(A)$ is a *core* for A if D is dense in $\mathrm{dom}(A)$ for the graph norm $\| \cdot \|_{\mathrm{dom}(A)} = \| \cdot \|_X + \| A \cdot \|_X$ (see [16, Chapter II, Def. 1.6]). By [16, Chapter II, Prop. 1.7] any subspace $D \subseteq \mathrm{dom}(A)$ that is dense in X and invariant under S is a core.

Lemma 4.11. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. For all $k \in \mathbb{N}$, $t \geq 0$, $f \in \mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ let $P_t^{(\mathrm{det},k)} f : \mathcal{H}^+ \to \mathbb{R}$ be defined by (4.11). Then $(P_t^{(\mathrm{det},k)})_{t\geq 0}$ is a generalized Feller semigroup on both $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ and $\mathcal{B}_{\sqrt{\rho}}(\mathcal{H}_{\mathrm{w}}^+)$ for all $k \in \mathbb{N}$. Moreover $\mathcal{D}$ is a core for the generator $\mathcal{G}_{\mathrm{det}}^{(k)}$ of $(P_t^{(\mathrm{det},k)})_{t\geq 0}$ on $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ and for all $f \in \mathcal{D}$ we have*

$$(\mathcal{G}_{\mathrm{det}}^{(k)} f)(x) = \langle \tilde{b}^{(k)} + \tilde{B}^{(k)}(x), f'(x) \rangle, \quad x \in \mathcal{H}^+. \tag{4.17}$$

Proof. Let $k \in \mathbb{N}$. It follows from Lemma 4.10 that $(P_t^{(\mathrm{det},k)})_{t\geq 0}$ is a family of bounded linear operators on both $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ and $\mathcal{B}_{\sqrt{\rho}}(\mathcal{H}_{\mathrm{w}}^+)$. Moreover, properties (i), (ii), and (v) in Definition 4.4 are trivially satisfied. Property (iv) follows from Lemma 4.10. Finally, property (iii) follows from Theorem 4.3 and the fact that $\lim_{t\to 0^+} \|\mathbf{x}_t^{(x,k)} - x\| = 0$.

It is easily verified that $\mathcal{D}$ is a subspace of $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ that is invariant for $(P_t^{(\mathrm{det},k)})_{t\geq 0}$. We know from Lemma 4.7 that $\mathcal{D}$ is dense in $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$, thus by [16, Chapter II, Prop. 1.7] it remains to prove that $\mathcal{D} \subseteq \mathrm{dom}(\mathcal{G}_{\mathrm{det}}^{(k)})$ and that (4.17) holds. To this end, let $u \in \mathcal{H}^+$ and consider $f(\cdot) = \mathrm{e}^{-\langle u, \cdot\rangle} \in \mathcal{D}$. For f of this latter form, we define $f'(x) := -\mathrm{e}^{-\langle u,x\rangle} u$, for $u, x \in \mathcal{H}^+$ and $f''(x)$ to be the bounded linear map on $\mathcal{H}^+$ defined for $u, x \in \mathcal{H}^+$ by $f''(x)(v) := \mathrm{e}^{-\langle u,x\rangle} u \langle u, v\rangle$, $v \in \mathcal{H}^+$. Now, observe that for $\tilde{B}(x) := \tilde{B}^{(k)}(x) + \tilde{b}^{(k)}$, we have

$$\begin{aligned}
&\frac{(P_t^{(\mathrm{det},k)} f)(x) - f(x)}{t} - \langle f'(x), \tilde{B}(x)\rangle \\
&= \int_0^1 \left\langle f'(s(\mathbf{x}_t^{(x,k)} - x) + x), \frac{\mathbf{x}_t^{(x,k)} - x}{t} - \tilde{B}(x) \right\rangle \mathrm{d}s \\
&\quad + \int_0^1 \int_0^1 \left\langle f''\left(us\left(\mathbf{x}_t^{(x,k)} - x\right) + x\right) \left(s\left(\mathbf{x}_t^{(x,k)} - x\right)\right), \tilde{B}(x) \right\rangle \mathrm{d}u\,\mathrm{d}s,
\end{aligned} \tag{4.18}$$

where we used Lemma C.1 twice, which is applicable as the one-sided derivatives of f, considered as a function on $\mathcal{H}^+$, exist. Observe that

$$\begin{aligned}
&\lim_{t\to 0+} \sup_{x\in\mathcal{H}^+} \frac{\left| \frac{1}{t}\left(\mathbf{x}_t^{(x,k)} - x\right) - \left(\tilde{B}^{(k)} x + \tilde{b}^{(k)}\right)\right|}{\sqrt{\rho(x)}} \\
&\leq \lim_{t\to 0+} \sup_{x\in\mathcal{H}^+} \frac{\|\tilde{B}^{(k)}\|_{\mathcal{L}(\mathcal{H})} \|e^{t\tilde{B}^{(k)}} - I\|_{\mathcal{L}(\mathcal{H})} \|x\| + \frac{1}{t}\|\tilde{b}^{(k)}\| \int_0^t \|e^{(t-s)\tilde{B}^{(k)}} - I\|_{\mathcal{L}(\mathcal{H})} \mathrm{d}s}{\sqrt{1+\|x\|^2}} = 0.
\end{aligned} \tag{4.19}$$

Moreover we have

$$\lim_{t\to 0+} \sup_{x\in\mathcal{H}^+} \frac{|\mathbf{x}_t^{(x,k)} - x|}{\sqrt{\rho(x)}} \leq \lim_{t\to 0+} \sup_{x\in\mathcal{H}^+} \frac{\|e^{t\tilde{B}^{(k)}} - I\|_{\mathcal{L}(\mathcal{H})} \|x\| + \int_0^t \|e^{(t-s)\tilde{B}^{(k)}} \tilde{b}^{(k)}\| \mathrm{d}s}{\sqrt{\rho(x)}} = 0. \tag{4.20}$$

Since $\sup_{x\in\mathcal{H}^+}|\rho(x)|^{-\frac{1}{2}}\|f'(x)\| < \infty$ and $\sup_{x\in\mathcal{H}^+}\|f''(x)\|_{\mathcal{L}(\mathcal{H})} < \infty$, it follows from Eqs. (4.18), (4.19), and (4.20) that

$$\lim_{t\to 0+}\left\| \tfrac{(P_t^{(\mathrm{det},k)}f)(x)-f(x)}{t} - \langle f'(x), \tilde{B}^{(k)}(x)+\tilde{b}^{(k)}\rangle \right\|_{\rho} = 0. \tag{4.21}$$

This, the linearity of $\mathcal{G}^{(k)}_{\mathrm{det}}$ and the fact that $\mathcal{D}$ is invariant for $P_t^{(\mathrm{det},k)}$ (and thus $P_t^{(\mathrm{det},k)}f \in \mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ whenever $f\in\mathcal{D}$) implies that $\mathcal{D}\subseteq \mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{det}})$ and that (4.17) holds. □

We now introduce the family of measures $\nu^{(k)}\colon \mathcal{H}^+\times\mathcal{B}(\mathcal{H}^+\setminus\{0\})\to[0,\infty)$ for every $x\in\mathcal{H}^+$ given by

$$\nu^{(k)}(x,\mathrm{d}\xi) = m^{(k)}(\mathrm{d}\xi) + \frac{\langle \mu^{(k)}(\mathrm{d}\xi), x\rangle}{\|\xi\|^2} \tag{4.22}$$

and define the operator $\mathcal{G}^{(k)}_{\mathrm{jump}}\colon \mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})\subseteq \mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})\to\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ by

$$\begin{aligned}&\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})\\ &= \left\{ f\in\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})\colon \left(x\mapsto \int_{\mathcal{H}^+\setminus\{0\}} (f(\xi+x)-f(x))\,\nu^{(k)}(x,\mathrm{d}\xi)\right)\in\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})\right\}\end{aligned} \tag{4.23}$$

and for $f\in\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})$:

$$\mathcal{G}^{(k)}_{\mathrm{jump}}f(x) := \int_{\mathcal{H}^+\setminus\{0\}} (f(\xi+x)-f(x))\,\nu^{(k)}(x,\mathrm{d}\xi), \quad x\in\mathcal{H}^+. \tag{4.24}$$

Note that for all $k\in\mathbb{N}$ the measure $\nu^{(k)}(x,\mathrm{d}\xi)$ is finite, i.e. $\nu^{(k)}(x,\mathcal{H}^+\setminus\{0\})<\infty$ for all $x\in\mathcal{H}^+$, but it is an affine function in x and hence unbounded in the first component. For that reason $\mathcal{G}^{(k)}_{\mathrm{jump}}f$ may not be in $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ for all $f\in\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$. However, the following lemma ensures that $C_b(\mathcal{H}^+_{\mathrm{w}})\subseteq\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})$:

Lemma 4.12. *Let (b,B,m,μ) be an admissible parameter set conform Definition 2.3. Let $k\in\mathbb{N}$, and let $\mathcal{G}^{(k)}_{\mathrm{jump}}$ be as defined in (4.23) and (4.24). Then $C_b(\mathcal{H}^+_{\mathrm{w}})\subseteq\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})$.*

Proof. Let $f\in C_b(\mathcal{H}^+_{\mathrm{w}})$ and let $g_f\colon\mathcal{H}^+\to\mathbb{R}$ be defined by

$$g_f(x) = \int_{\mathcal{H}^+\setminus\{0\}} f(x+\xi)\tfrac{\langle\mu^{(k)}(\mathrm{d}\xi),x\rangle}{\|\xi\|^2} \tag{4.25}$$

We will prove that $g_f\in\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ using Theorem 4.3. All other terms in the definition of $\mathcal{G}^{(k)}_{\mathrm{jump}}f$ can be dealt with in a similar (simpler) way.

To see that g_f is continuous on $K_R:=\{\rho\le R\}$ for all $R>0$ it suffices to show that g_f is sequentially continuous on K_R for every $R>0$ as the weak topology restricted to K_R is metrizable. Fix $R>0$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K_R converging (weakly) to an $x\in K_R$. By the dominated convergence theorem (Theorem B.5) and the fact that $\sup_{n\in\mathbb{N}}\|x_n\|\le\sqrt{R}$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\left|g_f(x_n)-g_f(x)\right| &\le \lim_{n\to\infty}\left\|\int_{\mathcal{H}^+\setminus\{0\}}(f(x_n+\xi)-f(x+\xi))\tfrac{\mu^{(k)}(\mathrm{d}\xi)}{\|\xi\|^2}\right\|\|x_n\|\\ &\quad+\lim_{n\to\infty}\left|\int_{\mathcal{H}^+\setminus\{0\}} f(x+\xi)\tfrac{\langle\mu^{(k)},x_n-x\rangle(\mathrm{d}\xi)}{\|\xi\|^2}\right| = 0.\end{aligned}$$

Finally, observe that $\lim_{R\to\infty}\sup_{x\in\mathcal{H}^+:\rho(x)\geq R}|\rho(x)|^{-1}|g_f(x)| = 0$ as f is bounded and $\int_{\mathcal{H}^+\setminus\{0\}}\frac{\mu^{(k)}(\mathrm{d}\xi)}{\|\xi\|^2}\in\mathcal{H}$ (recall (B.7)). By Theorem 4.3 this ensures that $g_f\in\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$, which completes the proof of the lemma. □

In the next proposition we achieve an important intermediate stage, that allows us to conclude the existence of generalized Feller processes in $\mathcal{H}^+$ admitting for bounded drifts and finite-activity jump behavior, as well as satisfying the exponential affine formula (1.1):

Proposition 4.13. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. Let $k\in\mathbb{N}$, and let $(\phi^{(k)}(\cdot,u),\psi^{(k)}(\cdot,u))$ be the unique solution to (3.4) (cf. Proposition 3.5). Let $\mathcal{D}\subseteq\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ be given by (4.7) and $\mathcal{G}^{(k)}_{\mathrm{det}}$ and $\mathcal{G}^{(k)}_{\mathrm{jump}}$ be as defined in (4.17), respectively (4.24). Consider the operator $\mathcal{G}^{(k)}_{\mathrm{det}}+\mathcal{G}^{(k)}_{\mathrm{jump}}\colon \mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{det}})\cap\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})\subseteq\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})\to\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$. Then $\mathcal{D}\subseteq\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{det}})\cap\mathrm{dom}(\mathcal{G}^{(k)}_{\mathrm{jump}})$. Moreover, there exists a generalized Feller semigroup $(P^{(k)}_t)_{t\geq 0}$ with generator $(\mathcal{G}^{(k)},\mathrm{dom}(\mathcal{G}^{(k)}))$ such that*

(i) $\mathcal{D}\subseteq\mathrm{dom}(\mathcal{G}^{(k)})$,
(ii) $\mathcal{G}^{(k)}f=(\mathcal{G}^{(k)}_{\mathrm{det}}+\mathcal{G}^{(k)}_{\mathrm{jump}})f$ for all $f\in\mathcal{D}$,
(iii) $P^{(k)}_t 1=1$ for all $t\geq 0$, and
(iv) for all $u,x\in\mathcal{H}^+$, $t\geq 0$ we have

$$\left(P^{(k)}_t \mathrm{e}^{-\langle\cdot,u\rangle}\right)(x)=\mathrm{e}^{-\phi^{(k)}(t,u)-\langle x,\psi^{(k)}(t,u)\rangle}. \tag{4.26}$$

Proof of Proposition 4.13. Roughly speaking, we can ensure the existence of a generalized Feller semigroup $P^{(k)}_t$ satisfying (ii) in Proposition 4.13 by verifying that all conditions of [10, Proposition 3.3] are satisfied. However, the assertions of [10, Proposition 3.3] do not immediately give us (i), (iii), and (iv). In order to obtain these statements we need to dig into the proof of [10, Proposition 3.3], which makes this proof somewhat technical and tricky. To enhance the readability, we split the proof in to several parts.

Step 1: Verifying the assumptions of [10, Proposition 3.3]. We consider, in the notation of that Proposition, $(X,\rho)=(\mathcal{H}^+_{\mathrm{w}},\rho)$, $A=\mathcal{G}^{(k)}_{\mathrm{det}}$, ω as in (4.13), $M_1=M$ where M is as in (4.12), $\mu(x,E)=\nu^{(k)}(x,E-x\cap\mathcal{H}^+)$ (recall the definition of $\nu^{(k)}$ from (4.22); here $E-x:=\{y\in\mathcal{H}\colon y+x\in E\}$), and $B=\mathcal{G}^{(k)}_{\mathrm{jump}}$. By Lemma 4.11, $\mathcal{G}^{(k)}_{\mathrm{det}}$ is the generator of a generalized Feller semigroup $(P^{(\mathrm{det},k)}_t)_{t\geq0}$ of transport type on both $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ and $\mathcal{B}_{\sqrt{\rho}}(\mathcal{H}^+_{\mathrm{w}})$. In particular, by [13, Theorem 3.2], $(P^{(\mathrm{det},k)}_t)_{t\geq0}$ defines a strongly continuous semigroup on both $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ and $\mathcal{B}_{\sqrt{\rho}}(\mathcal{H}^+_{\mathrm{w}})$, i.e., it automatically holds that the domain of $\mathcal{G}^{(k)}_{\mathrm{det}}$ is dense and that $(P^{(\mathrm{det},k)}_t)_{t\geq0}$ allows for exponential bounds (see Lemma 4.10 for explicit bounds). Lemma 4.12 implies that $\mathcal{G}^{(k)}_{\mathrm{jump}}f$ is weakly continuous on compact sets $\{\rho\leq R\}$ for all $R\geq 0$ and all $f\in C_b(\mathcal{H}^+_{\mathrm{w}})$.

Moreover, one easily verifies that there exists a constant K (possibly depending on k) such that for all $x\in\mathcal{H}^+$ we have

$$\int_{\mathcal{H}^+\setminus\{0\}}\rho(y+x)\,\nu^{(k)}(x,\mathrm{d}y)\leq\int_{\mathcal{H}^+\setminus\{0\}}(1+2\|x\|^2+2\|y\|^2)\,\nu^{(k)}(x,\mathrm{d}y)\leq K|\rho(x)|^2\,, \tag{4.27}$$

$$\int_{\mathcal{H}^+\setminus\{0\}}\sqrt{\rho(y+x)}\,\nu^{(k)}(x,\mathrm{d}y)\leq\int_{\mathcal{H}^+\setminus\{0\}}(1+\|x\|+\|y\|)\,\nu^{(k)}(x,\mathrm{d}y)\leq K\rho(x)\,, \tag{4.28}$$

and

$$\int_{\mathcal{H}^+\setminus\{0\}} \nu^{(k)}(x, \mathrm{d}y) \leq K\sqrt{\rho(x)}\,. \tag{4.29}$$

Next, observe that by Lemma 4.9 and the fact that $(0, B, 0, \mu)$ is also an admissible parameter set, we have $e^{t\tilde{B}^{(k)}}\xi \in \mathcal{H}^+$ whenever $\xi \in \mathcal{H}^+$. Thus

$$\begin{aligned} P_t^{(\det,k)}\rho(\xi + x) &= 1 + \|\mathbf{x}_t^{(\xi+x,k)}\|^2 = 1 + \|e^{t\tilde{B}^{(k)}}\xi + \mathbf{x}_t^{(\xi,k)}\|^2 \\ &= P_t^{(\det,k)}\rho(x) + 2\langle e^{t\tilde{B}^{(k)}}\xi, \mathbf{x}_t^{(\xi,k)}\rangle + \|\,\mathrm{e}^{t\tilde{B}^{(k)}}\,\xi\,\|^2 \geq P_t^{(\det,k)}\rho(x) \end{aligned} \tag{4.30}$$

for all $x, \xi \in \mathcal{H}^+$. This together with estimates similar to (4.16) yields (note that $\|\,\mathrm{e}^{t\tilde{B}^{(k)}}\,\xi\,\| \leq \|\,\mathrm{e}^{t\tilde{B}^{(1)}}\,\xi\,\|$, and recall ω from (4.13))

$$\begin{aligned} &\left| \frac{\sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(\xi + x) - \sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(x)}{\sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(x)} \right| \\ &= \frac{\sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(\xi + x) - \sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(x)}{\sup_{t\geq 0} \mathrm{e}^{-\omega t}\, P_t^{(\det,k)}\rho(x)} \\ &\leq \frac{\sup_{t\geq 0} \mathrm{e}^{-\omega t} \left| \|\,\mathrm{e}^{t\tilde{B}^{(k)}}\,\xi\,\|^2 + 2\|\,\mathrm{e}^{t\tilde{B}^{(k)}}\,\xi\,\|\|\mathbf{x}_t^{(x,k)}\| \right|}{1 + \|x\|^2} \leq \frac{\|\xi\|^2 + 2\|\xi\|(\|x\| + \sqrt{M})}{1 + \|x\|^2} \\ &\leq \frac{(M + 2\|\xi\|^2)(1 + \|x\|)}{1 + \|x\|^2} \leq \tfrac{2M + 4\|\xi\|^2}{1 + \|x\|} \end{aligned} \tag{4.31}$$

for all $x, \xi \in \mathcal{H}^+$. It follows that for all $x \in \mathcal{H}^+$ we have

$$\begin{aligned} &\int_{\mathcal{H}^+\setminus\{0\}} \left| \frac{\sup_{t\geq 0} \mathrm{e}^{-\omega t} \left(P_t^{(\det,k)}\rho\right)(\xi + x) - \sup_{t\geq 0} \mathrm{e}^{-\omega t} \left(P_t^{(\det,k)}\rho\right)(x)}{\sup_{t\geq 0} \mathrm{e}^{-\omega t} \left(P_t^{(\det,k)}\rho\right)(x)} \right| \nu^{(k)}(x, \mathrm{d}\xi) \\ &\leq \sup_{y\in\mathcal{H}^+} \left| \int_{\mathcal{H}^+\setminus\{0\}} \left(\tfrac{2M + 4\|\xi\|^2}{1 + \|y\|}\right) \nu^{(k)}(y, \mathrm{d}\xi) \right| =: \tilde{\omega}_k < \infty. \end{aligned} \tag{4.32}$$

This ensures that all conditions of [10, Proposition 3.3] are satisfied.

Step 2: Presenting the assertions of [10, Proposition 3.3]. As in the proof of [10, Proposition 3.3], we introduce the operator $\mathcal{G}_{\text{jump}}^{(k,n)} \in \mathcal{L}(\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+))$ which satisfies

$$(\mathcal{G}_{\text{jump}}^{(k,n)} f)(x) = \int_{\mathcal{H}^+\setminus\{0\}} (f(\xi + x) - f(x))\tfrac{n}{\rho(\xi+x)\wedge n}\, \nu^{(k)}(x, \mathrm{d}\xi)$$

for all $x \in \mathcal{H}^+$, $f \in \mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$. Note that $\mathcal{D} \subseteq \operatorname{dom}(\mathcal{G}_{\text{jump}}^{(k)})$ by Lemma 4.12. For future reference (see Proposition 4.19) we also introduce $\tilde{\rho}_k \colon \mathcal{H}^+ \to \mathbb{R}$, $\tilde{\rho}_k(x) = \sup_{t\geq 0} e^{-\omega t} P_t^{(\det,k)}\rho(x)$. It follows from [10, Remark 2.9] that $\tilde{\rho}_k$ is an admissible weight function and that $\|\cdot\|_\rho \leq \|\cdot\|_{\tilde{\rho}_k} \leq M\,\|\cdot\|_\rho$. Moreover, it follows from the proof of [10, Proposition 3.3] (with $A = \mathcal{G}_{\det}^{(k)}$ and $B_n = \mathcal{G}_{\text{jump}}^{(k,n)}$) that $\mathcal{G}_{\det}^{(k)} + \mathcal{G}_{\text{jump}}^{(k,n)}$ is the generator of a generalized Feller semigroup $(P_t^{(k,n)})_{t\geq 0}$ on $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ for all $n \in \mathbb{N}$, such that

(a) $\|P_t^{(k,n)}\|_{\mathcal{L}(\mathcal{B}_{\tilde{\rho}_k}(\mathcal{H}_{\mathrm{w}}^+))} \leq e^{(\omega+\tilde{\omega}_k)t}$ for all $t \geq 0$, $n \in \mathbb{N}$,

(b) $\|P_t^{(k,n)}\|_{\mathcal{L}(\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+))} \leq M\,\mathrm{e}^{(\omega+\tilde{\omega}_k)t}$ for all $t \geq 0$, $n \in \mathbb{N}$,

(c) $\lim_{n\to\infty} \|(\mathcal{G}_{\text{jump}}^{(k,n)} - \mathcal{G}_{\text{jump}}^{(k)})f\|_\rho = 0$ for all $f \in \mathcal{D}$.

It moreover follows from the proof of [10, Proposition 3.3] that there exists a generalized Feller semigroup $(P_t^{(k)})_{t\geq 0}$ on $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ with generator $\mathcal{G}^{(k)}$ satisfying

$$\lim_{n\to\infty}\sup_{s\in[0,t]}\|(P_s^{(k,n)}-P_s^{(k)})f\|_\rho=0,\ \text{ for all } f\in\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+),\ t\geq 0. \tag{4.33}$$

Step 3: Proof of (i) and (ii). Fix $f\in\mathcal{D}$. Let $u_{k,n}(t)=P_t^{(k,n)}f$, $t\geq 0$ and $n\in\mathbb{N}$, let $u_k(t)=P_t^{(k)}f$, $t\geq 0$, and let $v_k(t)=P_t^{(k)}(\mathcal{G}_{\mathrm{det}}^{(k)}+\mathcal{G}_{\mathrm{jump}}^{(k)})f$. Observe that $u'_{k,n}(t)=P_t^{(k,n)}(\mathcal{G}_{\mathrm{det}}^{(k)}+\mathcal{G}_{\mathrm{jump}}^{(k,n)})f$. By (a), (b), and (4.33) we have, for all $T\geq 0$, that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left(\|u_{k,n}(t)-u_k(t)\|_\rho+\|u'_{k,n}(t)-v_k(t)\|_\rho\right)=0. \tag{4.34}$$

This implies that u_k is differentiable and $u_k'(t)=v_k(t)$, which implies that $f\in\operatorname{dom}(\mathcal{G}^{(k)})$ and $\mathcal{G}^{(k)}f=u_k'(0)=(\mathcal{G}_{\mathrm{det}}^{(k)}+\mathcal{G}_{\mathrm{jump}}^{(k)})f$.

Step 4: Proof of (iii). In order to verify that $P_t^{(k)}1=1$ for all $t\geq 0$, observe that $\mathcal{G}_{\mathrm{jump}}^{(k,n)}1=0$ (whence $e^{t\mathcal{G}_{\mathrm{jump}}^{(k,n)}}1=1$ for all $t\geq 0$), whence the Trotter product formula (see, e.g., [16, Chapter III, Corollary 5.8]) implies that $P_t^{(k,n)}1=1$ for all $t\geq 0$. It follows that $P_t^{(k)}1=1$ for all $t\geq 0$.

Step 5: Proof of (iv). Recall the definition of $R^{(k)}$ and $F^{(k)}$ from (3.2) and (3.3). Recall from Lemmas 4.11 and 4.12 that $\mathrm{e}^{-\langle\cdot,u\rangle}\in\mathcal{D}\subseteq\operatorname{dom}(\mathcal{G}_{\mathrm{det}}^{(k)})\cap\operatorname{dom}(\mathcal{G}_{\mathrm{jump}}^{(k)})$ for all $u\in\mathcal{H}^+$, and that

$$\begin{aligned}\mathcal{G}^{(k)}(\mathrm{e}^{-\langle\cdot,u\rangle})(x)&=(\mathcal{G}_{\mathrm{det}}^{(k)}+\mathcal{G}_{\mathrm{jump}}^{(k)})(\mathrm{e}^{-\langle\cdot,u\rangle})(x)\\&=\Big(-\langle\tilde{b}^{(k)}+\tilde{B}^{(k)}(x),u\rangle+\int_{\mathcal{H}^+\setminus\{0\}}\big(\mathrm{e}^{-\langle\xi,u\rangle}-1\big)\nu^{(k)}(x,\mathrm{d}\xi)\Big)\mathrm{e}^{-\langle x,u\rangle}\\&=\big(-F^{(k)}(u)-\langle x,R^{(k)}(u)\rangle\big)\mathrm{e}^{-\langle x,u\rangle}\end{aligned} \tag{4.35}$$

for all $u,x\in\mathcal{H}^+$. On the other hand, Proposition 3.5 implies that

$$\begin{aligned}&\frac{\partial}{\partial t}\mathrm{e}^{-\phi^{(k)}(t,u)-\langle x,\psi^{(k)}(t,u)\rangle}\\&=\big(-F^{(k)}(\psi^{(k)}(t,u))-\langle x,R^{(k)}(\psi^{(k)}(t,u))\rangle\big)\mathrm{e}^{-\phi^{(k)}(t,u)-\langle x,\psi^{(k)}(t,u)\rangle}\end{aligned}$$

for all $u,x\in\mathcal{H}^+$. Therefore for all $u\in\mathcal{H}^+$ it holds that the function $[0,\infty)\ni t\mapsto\mathrm{e}^{-\phi^{(k)}(t,u)-\langle\cdot,\psi^{(k)}(t,u)\rangle}\in\mathcal{D}\subseteq\operatorname{dom}(\mathcal{G}^{(k)})$ is a classical solution to the following abstract Cauchy problem:

$$\begin{cases}\dfrac{\partial}{\partial t}v(t)=\mathcal{G}^{(k)}v(t),\\ \quad v(0)=\mathrm{e}^{-\langle\cdot,u\rangle}.\end{cases}$$

By the uniqueness of the classical solution we conclude (4.26). □

From Proposition 4.13 on the existence of the generalized Feller semigroup $(P_t^{(k)})$ with $P_t^{(k)}1=1$, together with the version of Kolmogorov's extension Theorem 4.5, we conclude that there exists a generalized Feller process associated to $(P_t^{(k)})_{t\geq 0}$, denoted by $(X_t^{(k)})_{t\geq 0}$, such that $\mathbb{E}_x\left[f(X_t^{(k)})\right]=P_t^{(k)}f(x)$ for every $f\in\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$. Item (a) and Eq. (4.33) in the proof of Proposition 4.13 result in exponential bounds on $\|P_t^{(k)}\|_{\mathcal{L}(\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+))}$ *that depend on* $k\in\mathbb{N}$. In order to proceed, we need to establish bounds that are *uniform* in k. We begin with a lemma that builds on top of the results in Proposition 3.11:

Lemma 4.14. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. Moreover for every $k \in \mathbb{N}$, let $(\phi^{(k)}(\cdot, u), \psi^{(k)}(\cdot, u))$ be the solution of (3.4), the existence of which is established in Proposition 3.5, and the mappings $\mathrm{d}_+\phi(\cdot, 0)$, $\mathrm{d}_+\psi(\cdot, 0)$, $\mathrm{d}_+^2\phi^{(k)}(\cdot, 0)$ and $\mathrm{d}_+^2\psi^{(k)}(\cdot, 0)$ be as in Proposition 3.11 for the admissible parameter set $(b, B, m^{(k)}, \mu^{(k)})$. Moreover, let $(X_t^{(k)})_{t\geq 0}$ be the generalized Feller process associated to $(P_t^{(k)})_{t\geq 0}$. Then for every $v, w \in \mathcal{H}$ and $t \geq 0$ the following formulas hold true:*

$$\mathbb{E}_x\left[\langle X_t^{(k)}, v\rangle\right] = \mathrm{d}_+\phi(t, 0)(v) + \langle x, \mathrm{d}_+\psi(t, 0)(v)\rangle, \tag{4.36}$$

and

$$\begin{aligned}\mathbb{E}_x\left[\langle X_t^{(k)}, v\rangle\langle X_t^{(k)}, w\rangle\right] = &-\mathrm{d}_+^2\phi^{(k)}(t, 0)(v, w) - \langle x, \mathrm{d}_+^2\psi^{(k)}(t, 0)(v, w)\rangle\\ &+ \big(\mathrm{d}_+\phi(t, 0)(v) + \langle x, \mathrm{d}_+\psi(t, 0)(v)\rangle\big)\\ &\times \big(\mathrm{d}_+\phi(t, 0)(w) + \langle x, \mathrm{d}_+\psi(t, 0)(w)\rangle\big).\end{aligned} \tag{4.37}$$

Proof. Let $k \in \mathbb{N}$ arbitrary, but fixed. Recall from Remark 4.6 that for all $t \geq 0$:

$$\mathbb{E}_x\left[\|X_t^{(k)}\|^2\right] < \infty, \quad \forall x \in \mathcal{H}^+. \tag{4.38}$$

We first show that the formulas (4.36) and (4.37) hold for $v, w \in \mathcal{H}^+$ and subsequently extend these to $v, w \in \mathcal{H}$. Let $u \in \mathcal{H}^+$, $x \in \mathcal{H}^+$ and $t \geq 0$, then we set

$$\Phi^{(k)}(t, u, x) := \mathrm{e}^{-\phi^{(k)}(t,u)-\langle x,\psi^{(k)}(t,u)\rangle},$$

and by the affine property of $(X_t^{(k)})_{t\geq 0}$ from Eq. (4.26) we have

$$\mathbb{E}_x\left[\mathrm{e}^{-\langle X_t^{(k)},u\rangle}\right] = \Phi^{(k)}(t, u, x). \tag{4.39}$$

By Proposition 3.11 the right-hand side of Eq. (4.39) is one-sided differentiable in $u \in \mathcal{H}^+$ in the direction v for every $v \in \mathcal{H}^+$. In particular, by applying the chain-rule at $u = 0$ we have:

$$\begin{aligned}\mathrm{d}_+\Phi^{(k)}(t, 0, x)(v) &= \big(-\mathrm{d}_+\phi^{(k)}(t, 0)(v) - \langle x, \mathrm{d}_+\psi^{(k)}(t, 0)(v)\rangle\big)\Phi^{(k)}(t, 0, x)\\ &= -\mathrm{d}_+\phi^{(k)}(t, 0)(v) - \langle x, \mathrm{d}_+\psi^{(k)}(t, 0)(v)\rangle,\end{aligned} \tag{4.40}$$

where $\mathrm{d}_+\phi^{(k)}(t, 0) = \mathrm{d}_+\phi(t, 0)$ and $\mathrm{d}_+\psi^{(k)}(t, 0) = \mathrm{d}_+\psi(t, 0)$ for all $t \geq 0$ and $k \in \mathbb{N}$, see Lemma 3.9. Moreover, note that for $\theta \in \mathbb{R}^+$ the random variable $\mathrm{e}^{-\langle X_t^{(k)},\theta v\rangle}$ is integrable and for $\mathbb{P}_x$-almost all $\omega \in \Omega$ the mapping $\theta \mapsto \mathrm{e}^{-\langle X_t^{(k)}(\omega),\theta v\rangle}$ is differentiable. Due to Eq. (4.38) the term

$$\sup_{\theta\in[0,1]} \Big|\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{-\langle X_t^{(k)},\theta v\rangle}\Big| = \sup_{\theta\in[0,1]} \big|-\langle X_t^{(k)}, v\rangle\mathrm{e}^{-\langle X_t^{(k)},\theta v\rangle}\big|$$

is integrable. Hence, all the requirements for switching the derivative with respect to θ and the expectation with respect to $\mathbb{P}_x$ are fulfilled, thus the left-hand side of Eq. (4.39) together with Eq. (4.40) yields:

$$\mathbb{E}_x\left[\langle X_t^{(k)}, v\rangle\right] = \mathrm{d}_+\phi(t, 0)(v) + \langle x, \mathrm{d}_+\psi(t, 0)(v)\rangle. \tag{4.41}$$

Again due to Eq. (4.38) we obtain by differentiating both sides of Eq. (4.39) at $u = 0$ twice in the direction v and w the formula in (4.37). Note that for every $v \in \mathcal{H}$ there exist $v^+, v^- \in \mathcal{H}^+$

such that $v = v^+ - v^-$, by linearity of the formula (4.36) in v, we have:

$$\begin{aligned}\mathbb{E}_x\left[\langle X_t^{(k)}, v\rangle\right] &= \mathbb{E}_x\left[\langle X_t^{(k)}, v^+\rangle\right] - \mathbb{E}_x\left[\langle X_t^{(k)}, v^-\rangle\right]\\ &= \mathrm{d}_+\phi(t,0)(v^+) - \mathrm{d}_+\phi(t,0)(v^-)\\ &\quad + \langle x, \mathrm{d}_+\psi(t,0)(v^+) - \mathrm{d}_+\psi(t,0)(v^-)\rangle\\ &= \mathrm{d}_+\phi(t,0)(v) + \langle x, \mathrm{d}_+\psi(t,0)(v)\rangle.\end{aligned}$$

By introducing the linear functional

$$\langle\!\langle \cdot, v\otimes w\rangle\!\rangle : \mathcal{H}\otimes\mathcal{H} \to \mathbb{R} \text{ defined by } \langle\!\langle x\otimes x, v\otimes w\rangle\!\rangle := \langle x, v\rangle\langle x, w\rangle, \tag{4.42}$$

we can write $\mathbb{E}_x\left[\langle X_t^{(k)}, v\rangle\langle X_t^{(k)}, w\rangle\right] = \mathbb{E}_x\left[\langle\!\langle X_t^{(k)}\otimes X_t^{(k)}, v\otimes w\rangle\!\rangle\right]$ for every $v, w \in \mathcal{H}^+$ and we have

$$\begin{aligned}\mathbb{E}_x\left[\langle\!\langle X_t^{(k)}\otimes X_t^{(k)}, v\otimes w\rangle\!\rangle\right] = &-\langle\!\langle \mathrm{d}_+^2\phi^{(k)}(t,0) + \mathrm{d}_+^2\psi^{(k)}(t,0)^*(x), v\otimes w\rangle\!\rangle\\ &+ \langle\!\langle \mathrm{d}_+\phi(t,0)\otimes \mathrm{d}_+\phi(t,0), v\otimes w\rangle\!\rangle\\ &+ \langle\!\langle \mathrm{d}_+\phi(t,0)\otimes \mathrm{d}_+\psi(t,0)^*(x), v\otimes w\rangle\!\rangle\\ &+ \langle\!\langle \mathrm{d}_+\psi(t,0)(x)\otimes \mathrm{d}_+\phi(t,0), v\otimes w\rangle\!\rangle\\ &+ \langle\!\langle \mathrm{d}_+\psi(t,0)^*(x)\otimes \mathrm{d}_+\psi(t,0)^*(x), v\otimes w\rangle\!\rangle,\end{aligned} \tag{4.43}$$

where we conveniently identified functionals on $\mathcal{H}$ with elements of $\mathcal{H}$. Written in this form the right-hand side in formula (4.37) reveals its linearity in $v\otimes w$ and for $v\otimes w \in \mathcal{L}_2(\mathcal{H})$, we have

$$v\otimes w = v^+\otimes w^+ - v^+\otimes w^- - v^-\otimes w^+ + v^-\otimes w^-$$

and thus expanding both sides by linearity in Eq. (4.37), shows the validity of the formula for all $v, w \in \mathcal{H}$. □

Note that by inserting the formulas from (3.35)–(3.36) and (3.19)–(3.22) into the corresponding terms in (4.36) and (4.37), the latter become explicit up to the parameters (b, B, m, μ). To save some space, we give those explicit formulas only for the limit case in Proposition 4.17.

Using the formulas from Lemma 4.14, we establish uniform growth bounds for the semigroups $(P_t^{(k)})_{t\geq 0}$ in the next proposition. Let us note here that in general we do not obtain an uniform growth bound $w \in \mathbb{R}^+$ with $M = 1$:

Proposition 4.15. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3 and for every $k \in \mathbb{N}$ let $(P_t^{(k)})_{t\geq 0}$ be the generalized Feller semigroup on $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$ associated with $(b, B, m^{(k)}, \mu^{(k)})$, the existence of which is guaranteed by Proposition 4.13. Then there exist a constant $w \in \mathbb{R}^+$ and $M \geq 1$, both independent of $k \in \mathbb{N}$, such that*

$$\|P_t^{(k)}\|_{\mathcal{L}(\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+))} \leq M\,\mathrm{e}^{wt} \quad \textit{for all } k \in \mathbb{N}, t \geq 0. \tag{4.44}$$

Proof. Recall from Remark 4.6, that in order to show the existence of a $M \geq 1$ and $w \in \mathbb{R}^+$ such that Eq. (4.44) holds, it suffices to show the existence of a $\epsilon > 0$ and $C \geq 0$, independent of $k \in \mathbb{N}$, such that

$$\mathbb{E}_x\left[\rho(X_t^{(k)})\right] \leq C\rho(x)\,, \quad \forall t \in [0,\epsilon] \text{ and } x \in \mathcal{H}^+. \tag{4.45}$$

Let $k \in \mathbb{N}$ be arbitrary, but fixed and denote by $(e_n)_{n\in\mathbb{N}}$ an ONB of $\mathcal{H}$, then by Parseval's identity and monotone convergence we have:

$$\mathbb{E}_x\left[\rho(X_t^{(k)})\right] = \mathbb{E}_x\left[1+\|X_t^{(k)}\|^2\right] = 1+\sum_{n=1}^{\infty}\mathbb{E}_x\left[\langle X_t^{(k)}, e_n\rangle^2\right],$$

for every $t \geq 0$ and $x \in \mathcal{H}^+$. By Eq. (4.37), in particular using the notation in Eq. (4.43), we have for all $n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}_x\left[\langle X_t^{(k)}, e_n\rangle^2\right] &= \langle\!\langle -\mathrm{d}_+^2\phi^{(k)}(t,0) - \mathrm{d}_+^2\psi^{(k)}(t,0)^*(x), e_n\otimes e_n\rangle\!\rangle \\ &\quad + \langle\!\langle \big(\mathrm{d}_+\phi(t,0)+\mathrm{d}_+\psi(t,0)^*(x)\big)^{\otimes 2}, e_n\otimes e_n\rangle\!\rangle. \end{aligned} \tag{4.46}$$

We show separately for the first and second terms on the right-hand side of Eq. (4.46) that, when summing over all $n \in \mathbb{N}$, we find a $\epsilon > 0$ and $C \geq 0$ such that Eq. (4.45) holds. Since

$$\sum_{n=1}^{\infty}\langle \mathrm{d}_+\phi(t,0)+\mathrm{d}_+\psi(t,0)^*(x), e_n\rangle^2 = \|\mathrm{d}_+\phi(t,0)+\mathrm{d}_+\psi(t,0)^*(x)\|^2,$$

we deduce for the second term on the right hand side of (4.46):

$$\sum_{n=1}^{\infty}\langle\!\langle \big(\mathrm{d}_+\phi(t,0)+\mathrm{d}_+\psi(t,0)^*(x)\big)^{\otimes 2}, e_n\otimes e_n\rangle\!\rangle \leq C(t)(1+\|x\|^2),$$

for

$$C(t) = \big(\|\mathrm{d}_+\phi(t,0)\| + \|\mathrm{d}_+\psi(t,0)^*\|_{\mathcal{L}(\mathcal{H})}\big)^2.$$

The terms $\|\mathrm{d}_+\phi(t,0)\|$ and $\|\mathrm{d}_+\psi(t,0)^*\|_{\mathcal{L}(\mathcal{H})}$ are bounded for all $t \geq 0$. Therefore, we deduce the existence of $\epsilon > 0$ and $C \geq 0$, independent of $k \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty}\langle\!\langle \big(\mathrm{d}_+\phi(t,0)+\mathrm{d}_+\psi(t,0)^*(x)\big)^{\otimes 2}, e_n\otimes e_n\rangle\!\rangle \leq C(1+\|x\|^2), \tag{4.47}$$

for all $t \in [0,\epsilon]$ and $x \in \mathcal{H}^+$. We continue with the first term on the right hand side of (4.46). Recall formulas (3.20), (3.22), (3.33), (3.35) and (3.36), from which we obtain:

$$\begin{aligned}&\langle\!\langle \mathrm{d}_+^2\psi^{(k)}(t,0)^*(x), e_n\otimes e_n\rangle\!\rangle \\ &\quad = -\int_0^t\int_{\mathcal{H}^+\setminus\{0\}}\langle \mathrm{e}^{s\mathrm{d}R(0)^*}\xi, e_n\rangle^2\big\langle x, \mathrm{e}^{(t-s)\mathrm{d}R(0)}\big\rangle\frac{\mu^{(k)}(\mathrm{d}\xi)}{\|\xi\|^2}\mathrm{d}s,\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}\langle\!\langle \mathrm{d}_+^2\phi^{(k)}(t,0), e_n\otimes e_n\rangle\!\rangle = -\int_0^t\Big(&\int_{\mathcal{H}^+\setminus\{0\}}\langle \mathrm{e}^{s\mathrm{d}R(0)^*}\xi, e_n\rangle^2\, m^{(k)}(\mathrm{d}\xi) \\ &+ \langle\!\langle \mathrm{d}_+^2\psi^{(k)}(s,0)^*(b), e_n\otimes e_n\rangle\!\rangle\Big)\mathrm{d}s \\ &+\int_0^t\int_{\mathcal{H}^+\cap\{\|\xi\|\geq 1\}}\langle\langle \mathrm{d}_+^2\psi^{(k)}(s,0)^*(\xi), e_n\otimes e_n\rangle\rangle\, m(\mathrm{d}\xi)\mathrm{d}s\,.\end{aligned} \tag{4.49}$$

Hence the two terms on the right hand side of Eq. (4.46) can be estimated by

$$\sum_{n=1}^{\infty}\int_0^t\int_{\mathcal{H}^+\setminus\{0\}}\langle \mathrm{e}^{s\mathrm{d}R(0)^*}\xi, e_n\rangle^2\langle x, \mathrm{e}^{(t-s)\mathrm{d}R(0)}\rangle\frac{\mu^{(k)}(\mathrm{d}\xi)}{\|\xi\|^2}\mathrm{d}s$$
$$\leq \Big(\int_0^t \|\mathrm{e}^{s\mathrm{d}R(0)^*}\|^2_{\mathcal{L}(\mathcal{H})}\|\mathrm{e}^{(t-s)\mathrm{d}R(0)}\|_{\mathcal{L}(\mathcal{H})}\|\mu(\mathcal{H}^+\setminus\{0\})\|\mathrm{d}s\Big)\|x\|,$$

and

$$\sum_{n=1}^{\infty}\langle\!\langle \mathrm{d}^2_+\phi^{(k)}(t,0), e_n\otimes e_n\rangle\!\rangle$$
$$\leq 2(\|b\|+\|\mu(\mathcal{H}^+\setminus\{0\})\|+\int_{\mathcal{H}^+\setminus\{0\}}\|\xi\|^2+\|\xi-\chi(\xi)\|m(\mathrm{d}\xi))$$
$$\times\int_0^t\int_0^s\|\mathrm{e}^{\tau\mathrm{d}R(0)^*}\|^2_{\mathcal{L}(\mathcal{H})}\|\mathrm{e}^{(s-\tau)\mathrm{d}R(0)}\|_{\mathcal{L}(\mathcal{H})}\mathrm{d}\tau\,\mathrm{d}s\,,$$

where we used that for all $k\in\mathbb{N}$:

$$\|\mu^{(k)}(\mathcal{H}^+\setminus\{0\})\|\leq\|\mu(\mathcal{H}^+\setminus\{0\})\|<\infty$$

and

$$\int_{\mathcal{H}^+\setminus\{0\}}\|\xi\|^2+\|\xi-\chi(\xi)\|m^{(k)}(\mathrm{d}\xi)\leq\int_{\mathcal{H}^+\setminus\{0\}}\|\xi\|^2+\|\xi-\chi(\xi)\|m(\mathrm{d}\xi)<\infty.$$

Therefore there exist $\epsilon>0$ and $\tilde{C}\geq 0$ such that

$$\sum_{n=1}^{\infty}\langle\!\langle -\mathrm{d}^2_+\phi^{(k)}(t,0)-\mathrm{d}^2_+\psi^{(k)}(t,0)^*(x), e_n\otimes e_n\rangle\!\rangle$$
$$\leq\tilde{C}(1+\|x\|^2),$$

for all $t\in[0,\epsilon]$ and $x\in\mathcal{H}^+$. Taking the sum of the latter constant $\tilde{C}$ and the constant C found in Eq. (4.47) yields (4.45). □

In the next step we show that the family $(P_t)_{t\geq 0}$, defined by $P_t:=\lim_{k\to\infty}P_t^{(k)}$ for $t\geq 0$, gives rise to a generalized Feller semigroup and deduce the existence of a generalized Feller process $(X_t)_{t\geq 0}$ with generator $\mathcal{G}$ as in formula (2.9).

Proposition 4.16. *Let (b,B,m,μ) be an admissible parameter set conform Definition 2.3. Then there exists a generalized Feller semigroup $(P_t)_{t\geq 0}$ on $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ such that*

$$\left(P_t\mathrm{e}^{-\langle\cdot,u\rangle}\right)(x)=\mathrm{e}^{-\phi(t,u)-\langle x,\psi(t,u)\rangle},\tag{4.50}$$

for all $t\geq 0$ and $x,u\in\mathcal{H}^+$, where $(\phi(\cdot,u),\psi(\cdot,u))$ is the unique solution to the generalized Riccati equation (2.7). The semigroup $(P_t)_{t\geq 0}$ gives rise to a generalized Feller process $(X_t)_{t\geq 0}$ in $(\mathcal{H}^+_{\mathrm{w}},\|\cdot\|^2+1)$ such that

$$\mathbb{E}_x\left[f(X_t)\right]=P_tf(x),\quad t\geq 0,\quad x\in\mathcal{H}^+,$$

and the generator $\mathcal{G}$ of $(P_t)_{t\geq 0}$ is of the form in Eq. (2.9) on $\mathcal{D}$.

Proof. Hereto we check that the conditions of Theorem 3.2 in [10] hold. From Proposition 4.15, we know that the sequence of semigroups $(P_t^{(k)})_{t\geq 0,k\in\mathbb{N}}$ with generators $(\mathcal{G}^{(k)})_{k\in\mathbb{N}}$ satisfy the following growth bound

$$\|P_t^{(k)}\|_{\mathcal{L}(\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+))} \leq M\mathrm{e}^{wt}, \qquad \forall\, n \in \mathbb{N} \quad \text{and } t \geq 0\,, \tag{4.51}$$

where $w \in \mathbb{R}$.

Recall the definition of $\mathcal{D}$ from Eq. (4.7) and recall from Lemma 4.7 that $\mathcal{D}$ is a dense subspace of $\mathcal{B}_\rho(\mathcal{H}_{\mathrm{w}}^+)$. Thus (i) in Theorem 3.2 in [10] is satisfied.

Note that the operator $\mathcal{G}^{(n)}$, $n \in \mathbb{N}$, applied to the function $\mathrm{e}^{-\phi^{(k)}(s,u)-\langle\cdot,\psi^{(k)}(s,u)\rangle}$, with $(\phi^{(k)}(\cdot,u), \psi^{(k)}(\cdot,u))$ being a solution to (3.4), gives (see also Eq. (4.35))

$$\begin{aligned}
&\left(\mathcal{G}^{(n)}\mathrm{e}^{-\phi^{(k)}(s,u)-\langle\cdot,\psi^{(k)}(s,u)\rangle}\right)(x)\\
&\quad= \mathrm{e}^{-\phi^{(k)}(s,u)}\mathcal{G}^{(n)}\mathrm{e}^{-\langle\cdot,\psi^{(k)}(s,u)\rangle}(x)\\
&\quad= \left(-F^{(n)}(\psi^{(k)}(s,u)) - \langle R^{(n)}(\psi^{(k)}(s,u)), x\rangle\right)\mathrm{e}^{-\phi^{(k)}(s,u)-\langle x,\psi^{(k)}(s,u)\rangle}\,,
\end{aligned}$$

for $x, u \in \mathcal{H}^+$, $s \geq 0$ From the latter and Eq. (4.26), we infer

$$\begin{aligned}
&\frac{1}{\|x\|^2+1}\left|\mathcal{G}^{(n)}P_s^{(k)}\mathrm{e}^{-\langle\cdot,u\rangle}(x) - \mathcal{G}^{(k)}P_s^{(k)}\mathrm{e}^{-\langle\cdot,u\rangle}(x)\right|\\
&\leq \frac{\mathrm{e}^{-\phi^{(k)}(s,u)-\langle x,\psi^{(k)}(s,u)\rangle}}{\|x\|^2+1}\left[b_{s,u}^{(n,k)} + \|x\|a_{s,u}^{(n,k)}\right]\,,
\end{aligned} \tag{4.52}$$

where

$$a_{s,u}^{(n,k)} := \left\|R^{(n)}(\psi^{(k)}(s,u)) - R^{(k)}(\psi^{(k)}(s,u))\right\|$$

and

$$b_{s,u}^{(n,k)} := \left|F^{(n)}(\psi^{(k)}(s,u)) - F^{(k)}(\psi^{(k)}(s,u))\right|$$

From Eqs. (2.3) and (3.16) we have, for all $0 \leq s \leq T < \infty$:

$$\begin{aligned}
&\left|\left(\mathrm{e}^{-\langle\xi,\psi^{(k)}(s,u)\rangle} - 1 - \langle\chi(\xi), \psi^{(k)}(s,u)\rangle\right)\left(\mathbf{1}_{\{\|\xi\|>1/n\}} - \mathbf{1}_{\{\|\xi\|>1/k\}}\right)\right|\\
&\quad\leq \|\psi^{(k)}(s,u)\|^2\|\xi\|^2\mathbf{1}_{\{\|\xi\|\leq 1\}}\\
&\quad\leq \sup_{s\in[0,T]}\|\psi^{(1)}(s,u)\|^2\|\xi\|^2\mathbf{1}_{\{\|\xi\|\leq 1\}} =: g(\xi)\,.
\end{aligned}$$

Observe that for $h \in \mathcal{H}^+$, we have $\int_{\mathcal{H}^+\setminus\{0\}} g(\xi)\,\frac{\langle\mu(\mathrm{d}\xi),h\rangle}{\|\xi\|^2} < \infty$. Hence Lemma B.3 implies that $g(\cdot)/\|\cdot\|^2 \in \mathcal{L}^1(\mathcal{H}^+,\mu)$ and from Theorem B.5, we deduce that $\sup_{s\in[0,T]} a_{s,u}^{(n,k)}$ converges to 0 as $n, k \to \infty$. By the admissibility condition (i) in Definition 2.3, we infer $\int_{\mathcal{H}^+\setminus\{0\}} g(\xi)\,m(\mathrm{d}\xi) < \infty$ and applying the dominated convergence theorem we also deduce that $\sup_{s\in[0,T]} b_{s,u}^{(n,k)}$ converge to 0 as $n, k \to \infty$. Observing that $\phi^{(k)}(s,u) \in \mathbb{R}^+$ and $\psi^{(k)}(s,u) \in \mathcal{H}^+$ for all $s \geq 0$, we can bound $\mathrm{e}^{-\phi^{(k)}(s,u)-\langle x,\psi^{(k)}(s,u)\rangle}$ by 1 for all $x \in \mathcal{H}^+$ and get from Eq. (4.52), that for all $s > 0$:

$$\begin{aligned}
&\left\|\mathcal{G}^{(n)}P_s^{(k)}\mathrm{e}^{-\langle\cdot,u\rangle} - \mathcal{G}^{(k)}P_s^{(k)}\mathrm{e}^{-\langle\cdot,u\rangle}\right\|_\rho\\
&\quad\leq \sup_{x\in\mathcal{H}^+}\frac{\|x\|+1}{\|x\|^2+1}\left(a_{s,u}^{(n,k)} + b_{s,u}^{(n,k)}\right)\\
&\quad\leq \left(\sup_{s\in[0,T]} a_{s,u}^{(n,k)} + \sup_{s\in[0,T]} b_{s,u}^{(n,k)}\right)C_u\|\mathrm{e}^{-\langle\cdot,u\rangle}\|_\infty\,,
\end{aligned}$$

where $C_u = \sup_{x\in\mathcal{H}^+}(\|x\|+1)/(\|x\|^2+1)$. Thus condition (ii) in Theorem 3.2 in [10] is satisfied with $\|\cdot\|_{\mathcal{D}} = \|\cdot\|_\infty$ and we deduce the existence of a generalized Feller semigroup $(P_t)_{t\geq 0}$ with the same growth bound as the semigroup $(P_t^{(k)})_{t\geq 0}$ and such that $P_t f = \lim_{k\to\infty} P_t^{(k)} f$, for all $f \in \mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$, uniformly on compacts in time. Since $P_t 1 = 1$, for all $t \geq 0$, we deduce from Theorem 4.5 that there exists a generalized Feller process $(X_t)_{t\geq 0}$ such that $P_t f(x) = \mathbb{E}_x\left[f(X_t)\right]$ for all $t \geq 0$ and $x \in \mathcal{H}^+$. The exponential affine formula (4.50) follows from formula (4.26) and the fact that $\lim_{k\to\infty}\phi^{(k)}(t,u) = \phi(t,u)$ and $\lim_{k\to\infty}\psi^{(k)}(t,u) = \psi(t,u)$ for all $t \geq 0$ and $u \in \mathcal{H}^+$. From this we further derive the particular form of the generator $\mathcal{G}$ on the space $\mathcal{D}$ by noting that $t \mapsto P_t \mathrm{e}^{-\langle\cdot,u\rangle}(x)$ uniquely solves the abstract Cauchy problem associated to $(\mathcal{G}, \mathrm{dom}(\mathcal{G}))$ and hence by mimicking the proof of the approximation case in Proposition 4.13, we conclude formula (2.9). $\square$

Analogous to the approximating processes $(X_t^{(k)})_{t\geq 0}$, for $k \in \mathbb{N}$ in Lemma 4.14, we now deduce explicit formulas for the expressions $\mathbb{E}_x\left[\langle X_t, v\rangle\right]$ as well as for $\mathbb{E}_x\left[\langle X_t, v\rangle^2\right]$, where $x \in \mathcal{H}^+$, $t \geq 0$ and $v \in \mathcal{H}^+$.

Proposition 4.17. *Let (b, B, m, μ) be an admissible parameter set conform Definition 2.3. Recall the definition of $\mathrm{d}R(0)$, $\mathrm{d}^2R(0)$, $\mathrm{d}F(0)$, and $\mathrm{d}^2F(0)$ from (3.19)–(3.22). Then for all $v, w \in \mathcal{H}^+$ the following formulas hold true:*

$$\mathbb{E}_x\left[\langle X_t, v\rangle\right] = \int_0^t \langle b, \mathrm{e}^{s\mathrm{d}R(0)}v\rangle + \int_{\mathcal{H}^+\cap\{\|\xi\|>1\}} \langle \xi, \mathrm{e}^{s\mathrm{d}R(0)}v\rangle\, m(\mathrm{d}\xi)\mathrm{d}s + \langle x, \mathrm{e}^{t\mathrm{d}R(0)}v\rangle \tag{4.53}$$

and

$$\begin{aligned}
&\mathbb{E}_x\left[\langle X_t, v\rangle\langle X_t, w\rangle\right] \\
&\quad= -\int_0^t \mathrm{d}^2F(0)(\mathrm{e}^{s\mathrm{d}R(0)}v, \mathrm{e}^{s\mathrm{d}R(0)}w)\,\mathrm{d}s \\
&\qquad - \int_0^t\int_0^s \mathrm{d}F(0)\left(\mathrm{e}^{(s-u)\mathrm{d}R(0)}\mathrm{d}^2R(0)(\mathrm{e}^{u\mathrm{d}R(0)}v, \mathrm{e}^{u\mathrm{d}R(0)}w)\right)\,\mathrm{d}u\,\mathrm{d}s \\
&\qquad - \int_0^t \left\langle x, \mathrm{e}^{(t-s)\mathrm{d}R(0)}\mathrm{d}^2R(0)(\mathrm{e}^{s\mathrm{d}R(0)}v, \mathrm{e}^{s\mathrm{d}R(0)}w)\right\rangle\,\mathrm{d}s \\
&\qquad + \left(\int_0^t \mathrm{d}F(0)(\mathrm{e}^{s\mathrm{d}R(0)}v)\,\mathrm{d}s + \left\langle x, \mathrm{e}^{t\mathrm{d}R(0)}v\right\rangle\right) \\
&\qquad\quad \times \left(\int_0^t \mathrm{d}F(0)(\mathrm{e}^{s\mathrm{d}R(0)}w)\,\mathrm{d}s + \left\langle x, \mathrm{e}^{t\mathrm{d}R(0)}w\right\rangle\right).
\end{aligned} \tag{4.54}$$

Moreover, for $v \in \mathcal{H}^+$, $\langle\cdot, v\rangle \in \mathrm{dom}(\mathcal{G})$ and

$$\mathcal{G}\langle\cdot, v\rangle(x) = \langle b + B(x), v\rangle + \int_{\mathcal{H}^+\cap\{\|\xi\|>1\}} \langle \xi, v\rangle\,\nu(x, \mathrm{d}\xi)\,, \qquad x \in \mathcal{H}^+\,. \tag{4.55}$$

Proof. Formulas (4.53) and (4.54) can be obtained analogous to the computation of the formulas (4.36) and (4.37) derived for the approximating case, combined with the explicit formulas (3.35)–(3.36). As in the proof of Lemma 4.14 we use Proposition 3.11 and the finite second moments of the process $(X_t)_{t\geq 0}$ to interchange the operations of the expectation and the one-sided derivatives. To obtain more explicit formulas, we consider the analogous of the formulas (4.36) and (4.37) and recall that $\mathrm{d}_+\phi(t,0)(v)$, $\mathrm{d}^2_+\phi(t,0)(v,v)$ can be expressed in terms of $\mathrm{d}F(0)$, $\mathrm{d}^2F(0)$, $\mathrm{d}_+\psi(t,0)(v)$, and $\mathrm{d}^2_+\psi(t,0)(v,w)$, see (3.31) and (3.33). Then, we recall the expressions (3.35) and (3.36) for $\mathrm{d}_+\psi(t,0)(v)$, and $\mathrm{d}^2_+\psi(t,0)(v,w)$.

To prove (4.55), observe that using the analogue of (4.36), we get

$$\begin{aligned}&\frac{1}{t}\left|P_t\langle\cdot,v\rangle(x)-\langle x,v\rangle-\langle b+B(x),v\rangle-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,\nu(x,\mathrm{d}\xi)\right|\\&\quad\le\frac{1}{t}\left|\mathrm{d}_+\phi(t,0)(v)-\langle b,v\rangle-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,m(\mathrm{d}\xi)\right|\\&\qquad+\frac{1}{t}\|x\|\left\|\mathrm{d}_+\psi(t,0)(v)-v-B^*(v)-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}\right\|.\end{aligned}$$

The latter together with formulas (3.31) and (3.32), yield

$$\begin{aligned}&\lim_{t\to0^+}\sup_{x\in\mathcal{H}^+}\frac{\frac{1}{t}\left|P_t\langle\cdot,v\rangle(x)-\langle x,v\rangle-\langle b+B(x),v\rangle-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,\nu(x,\mathrm{d}\xi)\right|}{1+\|x\|^2}\\&\quad\le\left|\mathrm{d}F(0)(v)-\langle b,v\rangle-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,m(\mathrm{d}\xi)\right|\\&\qquad+\left\|\mathrm{d}R(0)(v)-B^*(v)-\int_{\mathcal{H}^+\cap\{\|\xi\|>1\}}\langle\xi,v\rangle\,\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}\right\|\end{aligned}$$

and recalling the formulas for $\mathrm{d}R(0)$ and $\mathrm{d}F(0)$ respectively in (3.19) and (3.20), we conclude that $\langle\cdot,v\rangle\in\mathrm{dom}(\mathcal{G})$, for $v\in\mathcal{H}^+$ and that (4.55) holds. □

Remark 4.18. As observed in Remark 2.4, the second moment conditions are a consequence of our generalized Feller approach with weight function $\rho=\|\cdot\|^2+1$. More specifically, the uniform bounds established in Proposition 4.15 rely on the existence of second moments as established in Lemma 4.14. A natural question to ask is whether one could perform the analysis with a different (weaker) weight function. However, in the proof of Lemma 4.11 we consider the square root of the weight function $\sqrt{\rho}$, more specifically, we need that $\sqrt{\rho(x)}\ge c\|x\|$, $x\in\mathcal{H}^+$, for some constant $c\in(0,\infty)$.

Naturally, the second moments of m and μ are also used to derive the explicit formulas for the first and second moments of the affine process in Proposition 4.17. Finally, we note that the existence of a *first* moment of $\frac{\mu(\mathrm{d}\xi)}{\|\xi\|^2}$ is already used in Lemma 3.3 to ensure that the approximating mappings $R^{(k)}$ are Lipschitz continuous.

In general we do not obtain a version of the process X in Proposition 4.16 with càdlàg paths. By Theorem 2.13 in [10] a càdlàg version exists when the associated semigroup $(P_t)_{t\ge0}$ is quasi-contractive on $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$, i.e., if one can take $M=1$ in Proposition 4.15. We do not know whether this holds in general. However, we can show that X admits a càdlàg version in the finite activity setting:

Proposition 4.19. *Assume the setting of Proposition 4.16 and assume moreover that $m(\mathcal{H}^+\setminus\{0\})<\infty$ and that $\mathcal{H}^+\setminus\{0\}\ni\xi\mapsto\|\xi\|^{-2}$ is μ-integrable. Then there exists a version of X with càdlàg paths.*

Proof. By [10, Theorem 2.13] it in fact suffices to prove that the generalized Feller semigroup $(P_t)_{t\ge0}$ associated to X is quasi-contractive on $\mathcal{B}_{\tilde\rho}(\mathcal{H}^+_{\mathrm{w}})$, where $\tilde\rho\colon\mathcal{H}^+\to[0,\infty)$ is an admissible weight function such that its associated norm $\|\cdot\|_{\tilde\rho}$ is equivalent to $\|\cdot\|_\rho$. Note that in the finite activity setting we can apply Proposition 4.13 with $k=\infty$ (with the understanding that $m^{(\infty)}:=m$ and $\mu^{(\infty)}:=\mu$) to directly obtain $(P_t)_{t\ge0}$ (i.e., no approximation over k

is necessary). In particular $\tilde{\omega}_\infty < \infty$, where $\tilde{\omega}_\infty$ is defined by taking $k = \infty$ in (4.32). It then follows from statement (a) on "Step 2: Presenting the assertions of [10, Proposition 3.3]" that $(P_t)_{t\geq 0}$ is quasi-contractive on $\mathcal{B}_{\tilde{\rho}_\infty}(\mathcal{H}^+_{\mathrm{w}})$ where $\tilde{\rho}_\infty$ is an admissible weight function with associated norm equivalent to $\|\cdot\|_\rho$. $\square$

In the next section we give the proof of Theorem 2.8. The proof is based on collecting the results from this section and transferring from a generalized Feller setting to the classical setting that we used for presenting the results in Section 2.

4.3. *Proof of Theorem 2.8*

Let (b, B, m, μ) be an admissible parameter set. Then by Proposition 4.16 there exists a generalized Feller semigroup $(P_t)_{t\geq 0}$ and the associated generalized Feller process $(X_t)_{t\geq 0}$ in $\mathcal{H}^+$ such that

$$\mathbb{E}_x\left[f(X_t)\right] = P_t f(x) \quad \text{for } t \geq 0,$$

and the Markov property (4.4) holds. The existence of constants $M, \omega \in [1, \infty)$ such that (2.8) is satisfied follows from Remark 4.6. The space $\mathcal{H}$ is a separable Hilbert space and hence the Borel-σ-algebras $\mathcal{B}(\mathcal{H}^+)$ and $\mathcal{B}(\mathcal{H}^+_{\mathrm{w}})$ coincide. This means that the transition kernels $(p_t(x, \mathrm{d}y))_{t\geq 0}$ defining the semigroup $(P_t)_{t\geq 0}$ stay unaffected under the change of topology and hence the process $(X_t)_{t\geq 0}$ is also a Markov process in $\mathcal{H}^+$ with the strong topology.

The asserted exponential-affine formula in (2.1) is precisely formula (4.50) from Proposition 4.16. By this and Proposition 3.7 we have for all $x \in \mathcal{H}^+$:

$$\begin{aligned}\lim_{t\to 0+} \frac{P_t \mathrm{e}^{-\langle\cdot,u\rangle}(x) - \mathrm{e}^{-\langle\cdot,u\rangle}(x)}{t} &= \lim_{t\to 0+} \frac{\mathrm{e}^{-\phi(t,u)-\langle x,\psi(t,u)\rangle} - \mathrm{e}^{-\langle x,u\rangle}}{t} \\ &= (-F(u) - \langle x, R(u)\rangle)\mathrm{e}^{-\langle x,u\rangle}. \end{aligned} \tag{4.56}$$

In particular, we see that $\mathcal{A}(\mathcal{D}) \subseteq C_b(\mathcal{H}^+)$ and since $(P_t)_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{B}_\rho(\mathcal{H}^+_{\mathrm{w}})$ we have $\left(P_t \mathrm{e}^{-\langle\cdot,u\rangle}\right)(x) = \mathrm{e}^{-\langle\cdot,x\rangle}(x) + \int_0^t \left(P_s \mathcal{A}\mathrm{e}^{-\langle\cdot,u\rangle}\right)(x)\mathrm{d}s$. Consequently, we have shown that $\mathcal{D} \subseteq \mathrm{dom}(\mathcal{A})$ and from formula (4.56) we see that formula (2.9) holds true on $\mathcal{D}$.

5. Conclusions and outlook

With Theorem 2.8 we have proven the existence of affine Markov processes in the cone of positive self-adjoint Hilbert–Schmidt operators by a novel approach inspired by [10]. In particular, our approach relies on the theory of generalized Feller processes, taking the weight function $\rho = \|\cdot\|^2 + 1$. This approach requires the existence of first and second moments of the jump measures m and μ. A beneficial by-product is that we obtain explicit formulas for the first and second moments of the affine Markov process, see Proposition 4.17. See Remark 4.18 for a discussion regarding the necessity of the second-moment condition.

Below, we discuss and motivate three further directions of research.

On relaxing the condition on existence of moments.

A possible direction of further research is to investigate whether one can adapt the proof in such a way to allow for the weight function $\rho = \|\cdot\| + 1$. In this case a first moment conditions on m and μ should suffice. On a more abstract level, the question arises whether it is possible to establish existence without any moment conditions, as can be done in the finite dimensional setting where the cone of interest does not have empty interior. Another tantalizing question is to what degree an infinite dimensional affine process on the cone of positive self-adjoint Hilbert–Schmidt operators allows for diffusion. It is clear from [5] that certain constructions are possible.

On the construction of stochastic volatility models.

Our main motivation for considering affine processes on the space of positive self-adjoint Hilbert–Schmidt operators is that such processes qualify as infinite dimensional stochastic covariance processes. Hence we consider in [7] stochastic volatility models in Hilbert spaces, where the introduced class of affine pure-jump processes will be used for modeling the operator-valued instantaneous variance process. Specifically, we will consider a process $(Y_t)_{t\geq 0}$ in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ given by

$$\mathrm{d}Y_t = \mathcal{A}Y_t\,\mathrm{d}t + \sigma_t Q^{1/2}\,\mathrm{d}W_t, \qquad t \geq 0\,, \tag{5.1}$$

where $\mathcal{A}\colon \mathrm{dom}(\mathcal{A}) \subseteq H \to H$ is a possibly unbounded operator with dense domain $\mathrm{dom}(\mathcal{A})$, $(W_t)_{t\geq 0}$ is a cylindrical Brownian motion in H, $Q \in \mathcal{H}^+$, and $(\sigma_t)_{t\geq 0}$ is an operator valued stochastic process given by the square-root of an affine pure-jump process, the existence of which is guaranteed by our main result Theorem 2.8.

On considering a different state space for the covariance process.

Note that we take σ in (5.1) to be the *square root* of an affine process in order to obtain that Y is again affine. However, this means that the 'natural' state space for σ is not the cone of positive self-adjoint Hilbert–Schmidt operators, but the cone of positive self-adjoint *trace class* operators. Unfortunately, this is no longer a cone in a Hilbert space. As self-duality of the cone was used at various instances in the proof of Theorem 2.8, it is not clear how much can be salvaged if we consider trace class operators. This would be a further interesting direction of research.

Appendix A. A comparison theorem

A more general version of the following comparison theorem can be found, e.g., as [12, Theorem 5.4].

Theorem A.1. *Let $(H,(\cdot,\cdot))$ be a Hilbert space, $K \subset H$ a cone, let $T > 0$, and let $F\colon[0,T]\times H \to H$. Assume that $F(t,\cdot)$ is quasi-monotone with respect to K for all $t \in [0,T]$, and that there exists a constant $L \in [0,\infty)$ such that*

$$\|F(t,x) - F(t,y)\|_H \leq L\|x-y\|_H, \quad t \in [0,T], x, y \in H. \tag{A.1}$$

Let $f, g \in C^1([0,T],H)$ satisfy $f(0) \leq_K g(0)$ and $f'(t) - F(t,f(t)) \leq_K g'(t) - F(t,g(t))$ for all $t \in [0,T]$. Then $f(t) \leq_K g(t) \in K$ for all $t \in [0,T]$.

Appendix B. Integration with respect to a vector-valued measure

We summarize some results on vector-valued measures and integration. The theory goes back to the work of Bartle, Dunford, and Schwartz (see, e.g., [1]) and Lewis [28]. A good overview can be found in [31, Chapter 2]. As we work in the Hilbert-space setting (in particular, as Hilbert spaces are reflexive), the theory simplifies considerably.

Throughout this section let $(S,\mathcal{F})$ be a measurable space, let $(H,\langle\cdot,\cdot\rangle_H)$ be a real Hilbert space, and let $\mu\colon\mathcal{F} \to H$ be an H-valued measure.

Definition B.1. We say that $f : S \to \mathbb{R}$ is *μ-integrable* if the following two conditions are satisfied:

(i) f is $\langle \mu, h\rangle$-integrable for all $h \in H$ (i.e., $f : S \to \mathbb{R}$ is measurable and $\int_S |f| \mathrm{d}|\langle \mu, h\rangle| < \infty$ for all $h \in H$), and
(ii) for all $A \in \mathcal{F}$ there exists an $h_A \in H$ such that for all $h \in H$ we have $\langle h_A, h\rangle_H = \int_A f \mathrm{d}\langle \mu, h\rangle$.

In this case we denote h_A by $\int_A f \,\mathrm{d}\mu$. In addition, we define

$$\mathcal{L}^1(S, \mu) := \{f : S \to \mathbb{R} : f \text{ is } \mu\text{-integrable}\} \tag{B.1}$$

Example B.2. If f is a $\mathcal{F}$-simple function, then $f \in \mathcal{L}^1(S, \mu)$.

The following characterization is useful (see also [28, p. 163]):

Lemma B.3. *We have that $f \in \mathcal{L}^1(S, \mu)$ if and only if f is $\langle \mu, h\rangle$-integrable for all $h \in H$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions such that $f_n \to f$ μ-a.s. and $|f_n| \le |f|$ for all $n \in \mathbb{N}$. Let $A \in \mathcal{F}$. Note that the mapping $T : H \to \mathbb{R}$, $T(h) = \int_A f \mathrm{d}\langle \mu, h\rangle$ is linear and that

$$T(h) = \lim_{n\to\infty} \int_A f_n \mathrm{d}\langle \mu, h\rangle = \lim_{n\to\infty} \langle \int_A f_n \,\mathrm{d}\mu, h\rangle_H,$$

for all $h \in H$ by the dominated convergence theorem. It follows from this and the uniform boundedness principle that $\sup_{n\in\mathbb{N}} \| \int_A f_n \,\mathrm{d}\mu\|_H < \infty$, whence $T \in H^*$. The Riesz representation theorem thus ensures that there exists an $h_A \in H$ such that $\langle h_A, h\rangle_H = T(h)$ for all $h \in H$. □

Corollary B.4. *If $f \in \mathcal{L}^1(S, \mu)$ and $g : S \to \mathbb{R}$ is measurable and satisfies $|g| \le f$ μ-a.s., then $g \in \mathcal{L}^1(S, \mu)$. In particular, $\mathcal{L}^1(S, \mu)$ contains all bounded measurable $\mathbb{R}$-valued functions on S.*

By [28, Corollary 1.4] we have, for any $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ converging to $E \in \mathcal{F}$, that

$$\lim_{n\to\infty} \mu(E_n) = \mu(E). \tag{B.2}$$

Moreover, the dominated convergence theorem remains valid for H-valued measures:

Theorem B.5 (*Theorem 2.1.7 in [31]*)**.** *Let $g \in L^1(S, \mu)$, let $f : S \to \mathbb{R}$ be μ-measurable and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of μ-measurable functions on S satisfying $|f_n(s)| \le g(s)$ for all $s \in S$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(s) = f(s)$ for all $s \in S$. Then $f, f_n \in L^1(S, \mu)$, $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} \left\| \int_S f_n \mathrm{d}\mu - \int_S f \mathrm{d}\mu \right\|_H = 0\,. \tag{B.3}$$

Finally, let $K \subset H$ be a self-dual cone and assume that $\mu : \mathcal{F} \to K$ is a K-valued measure. In this case we have $0 \le_K \mu(E) \le_K \mu(F)$ for all $E, F \in \mathcal{F}$ satisfying $E \subseteq F$, and thus also (by monotonicity of K)

$$\|\mu(E)\|_H \le \|\mu(F)\|_H. \tag{B.4}$$

Moreover, as K is self-dual, $\langle \mu, h\rangle$ is a positive measure for all $h \in K$, whence (again by self-duality) we have

$$f \in L^1(S,\mu),\, f \geq 0 \Rightarrow \int_S f\,\mathrm{d}\mu \in K. \tag{B.5}$$

In particular, if $f \in L^1(S,\mu)$ is positive, and $E \in \mathcal{F}$, then

$$\int_E f\,\mathrm{d}\mu \leq_K \operatorname*{ess\,sup}_{s\in E} f(s)\mu(E). \tag{B.6}$$

This combined with the monotonicity of K implies that for every every $f \in L^1(S,\mu)$ and every $E \in \mathcal{F}$ we have (by considering f^+ and f^- separately) that

$$\left\| \int_E f\,\mathrm{d}\mu \right\|_H \leq \operatorname*{ess\,sup}_{s\in E} |f(s)| \|\mu(E)\|_H. \tag{B.7}$$

Appendix C. Proof of Proposition 3.11

To prove Proposition 3.11, we need the following consequence of the fundamental theorem of calculus:

Lemma C.1. *Let X, Y be Banach spaces, let $F : D \subset X \to Y$, let $x, y \in D$ and assume that the one-sided derivative of F in z exists in the direction $y - x$ for all $z \in \{x + s(y - x) : s \in [0, 1]\}$ and that the mapping*

$$[0,1] \ni s \mapsto \mathrm{d}_+F(x + s(y - x))(y - x) \in Y \tag{C.1}$$

is continuous. Then $F(y) - F(x) = \int_0^1 \mathrm{d}_+F(x + s(y - x))(y - x)\,\mathrm{d}s$.

Proof. The continuity of $[0,1] \ni s \mapsto \mathrm{d}_+F(x + s(y - x))(y - x) \in Y$ and the fundamental theorem of calculus imply that the right derivative of the mapping $[0,1] \ni t \mapsto \left(F(x + t(y - x)) - F(x) - \int_0^t \mathrm{d}_+F(x + s(y - x))(y - x)\,\mathrm{d}s\right) \in Y$ equals zero. As any function with right derivative equal to zero is constant, this leads to the desired assertion. □

Proof of Proposition 3.11. Note that in order to prove that the *second* directional derivative in 0 of a mapping exists, we need that its *first* directional derivative exists in $u \in \mathcal{H}^+$ for all $u \in \mathcal{H}^+$ sufficiently small. Hence, we begin by proving that the first derivative of $u \mapsto \psi(t, u)$ exists in u in the direction v for all $u, v \in \mathcal{H}^+$ and all $t \in [0, \infty)$. To this end we fix $u, v \in \mathcal{H}^+$.

Recall the definition of the operators $\mathrm{d}R(u) \in \mathcal{L}(\mathcal{H})$ and $\mathrm{d}^2R(u) \in \mathcal{L}^{(2)}(\mathcal{H} \times \mathcal{H}, \mathcal{H})$ from (3.19) and (3.21). Define the operator $C_\theta(t) \in \mathcal{L}(\mathcal{H})$, $\theta, t \in [0, \infty)$, by

$$C_\theta(t)w = \int_0^1 \mathrm{d}R\left(\psi(t,u) + s(\psi(t,u+\theta v) - \psi(t,u))\right) w\,\mathrm{d}s \tag{C.2}$$

(note that the integral is well-defined as the integrand is continuous in s by (3.24) and bounded by (3.11) and (3.23)). Lemma C.1, (3.26), the fact that $(1 - s)\psi(t, u) + s\psi(t, u + \theta v) \in \mathcal{H}^+$ for all $s \in [0, 1]$, $t \in [0, \infty)$, and the fact that $\psi(t, u + \theta v) \geq_{\mathcal{H}^+} \psi(t, u)$ for all $t \in [0, \infty)$ by (3.10) imply that

$$C_\theta(t)(\psi(t,u+\theta v) - \psi(t,u)) = R(\psi(t,u+\theta v)) - R(\psi(t,u)), \quad \theta, t \in [0,\infty).$$

This and (2.7) imply

$$\frac{\partial}{\partial t}(\psi(t,u+\theta v) - \psi(t,u)) = C_\theta(t)(\psi(t,u+\theta v) - \psi(t,u)), \quad \theta, t \in [0,\infty).$$

It follows that

$$\psi(t,u+\theta v)-\psi(t,u)=\theta\exp\left(\int_0^t C_\theta(s)\,\mathrm{d}s\right)v,\quad \theta,t\in[0,\infty)$$

(note that $\int_0^t C_\theta(s)\,\mathrm{d}s$ is well-defined in $\mathcal{L}(\mathcal{H})$ as the $\mathcal{L}(\mathcal{H})$-valued integrand is continuous in s by (3.24) and bounded due to (3.23)). This implies that for all $\theta\in(0,\infty)$ we have

$$\begin{aligned}&\left\|\frac{\psi(t,u+\theta v)-\psi(t,u)}{\theta}-\exp\left(\int_0^t C_0(s)\,\mathrm{d}s\right)v\right\|\\&=\left\|\left(\exp\left(\int_0^t C_\theta(s)\,\mathrm{d}s\right)-\exp\left(\int_0^t C_0(s)\,\mathrm{d}s\right)\right)v\right\|.\end{aligned}\tag{C.3}$$

Using the identity $\|\mathrm{e}^A-\mathrm{e}^B\|_{\mathcal{L}(\mathcal{H})}\le\|A-B\|_{\mathcal{L}(\mathcal{H})}\mathrm{e}^{\|A\|_{\mathcal{L}(\mathcal{H})}\vee\|B\|_{\mathcal{L}(\mathcal{H})}}$, $A,B\in\mathcal{L}(\mathcal{H})$, we obtain from (C.2), (C.3), (3.11), (3.12), and (3.24) that the one-sided derivative $\mathrm{d}_+\psi(t,u)(v)$ exists. Moreover, the fact that $C_0(t)v=\mathrm{d}R(\psi(t,u))v$ implies that $t\mapsto\mathrm{d}_+\psi(t,u)(v)$ is the solution to the following ODE

$$\frac{\partial}{\partial t}\mathrm{d}_+\psi(t,u)(v)=\mathrm{d}R(\psi(t,u))\big(\mathrm{d}_+\psi(t,u)(v)\big),\quad t\ge 0;\quad \mathrm{d}_+\psi(0,u)(v)=v.\tag{C.4}$$

This together with the quasi-monotonicity of $\mathrm{d}R(\psi(t,u))$ (see Lemma 3.9) and Theorem A.1 implies that $\mathrm{d}_+\psi(t,u)(v)\in\mathcal{H}^+$. Regarding the derivative of ϕ, note that estimates analogous to (3.23) and (3.24) hold for $\mathrm{d}F$, which, in combination with the fact that $\mathrm{d}_+\psi(t,u)(v)\in\mathcal{H}^+$, (2.7), (3.28), and Lemma C.1 implies that

$$\begin{aligned}&\frac{\phi(t,u+\theta v)-\phi(t,u)}{\theta}\\&=\int_0^t\int_0^1\mathrm{d}F(\psi(s,u)+r(\psi(s,u+\theta v)-\psi(s,u)))\,\mathrm{d}r\,\frac{\psi(s,u+\theta v)-\psi(s,u)}{\theta}\,\mathrm{d}s\end{aligned}$$

for all $\theta\in(0,\infty)$, $t\in[0,\infty)$. This in combination with (3.12) and (3.10) implies that the dominated convergence theorem can be applied to obtain that $\mathrm{d}_+\phi(t,u)$ exists for all t and satisfies

$$\frac{\partial}{\partial t}\mathrm{d}_+\phi(t,u)(v)=\mathrm{d}F(\psi(t,u))\big(\mathrm{d}_+\psi(t,u)(v)\big),\quad t\ge 0;\quad \mathrm{d}_+\phi(0,u)(v)=0.\tag{C.5}$$

This proves in particular that $u\mapsto(\phi(t,u),\psi(t,u))$ is differentiable in 0 in the direction $v\in\mathcal{H}^+$ for all $v\in\mathcal{H}^+$ and that the corresponding derivatives solve the ODEs (3.31) and (3.32).

We now turn to the second derivative in 0. To this end, fix $v,w\in\mathcal{H}^+$ and observe that Lemma C.1, the boundedness and continuity of d^2R (see Lemma 3.9), (3.27) and the fact that $\psi(t,\theta v),\mathrm{d}_+\psi(t,\theta v)\in\mathcal{H}^+$ for all $\theta\in[0,\infty)$ imply that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\mathrm{d}_+\psi(t,\theta v)(w)-\mathrm{d}_+\psi(t,0)(w)\right)&=\int_0^1\mathrm{d}^2R(s\psi(t,\theta v))(\mathrm{d}_+\psi(t,\theta v)(w),\psi(t,\theta v))\,\mathrm{d}s\\&\quad+\mathrm{d}R(0)\left(\mathrm{d}_+\psi(t,\theta v)(w)-\mathrm{d}_+\psi(t,0)(w)\right)\end{aligned}$$

for all $\theta\in[0,\infty)$, $t\in[0,\infty)$. As $\mathrm{d}_+\psi(0,\theta v)(w)-\mathrm{d}_+\psi(0,0)(w)=0$ this implies

$$\begin{aligned}&\frac{\mathrm{d}_+\psi(t,\theta v)(w)-\mathrm{d}_+\psi(t,0)(w)}{\theta}\\&=\int_0^t\mathrm{e}^{(t-r)\mathrm{d}R(0)}\int_0^1\mathrm{d}^2R(s\psi(r,\theta v))\left(\mathrm{d}_+\psi(r,\theta v)(w),\frac{\psi(r,\theta v)}{\theta}\right)\mathrm{d}s\,\mathrm{d}r\end{aligned}\tag{C.6}$$

for all $\theta \in (0, \infty)$, $t \in [0, \infty)$. Note that (3.12), (3.24), and (C.4) imply that $\lim_{\theta \to 0+} \mathrm{d}_+\psi(t, \theta v)(w) = \mathrm{d}_+\psi(t, 0)(w)$. Moreover, we have already established that $\lim_{\theta \to 0+} \frac{\psi(t,\theta v)}{\theta} = \mathrm{d}_+\psi(t, 0)(v)$. Combining these observations with (3.11), (3.25), and (C.6) implies that $\mathrm{d}^2_+\psi(t, 0)(v, w)$ exists and that $\mathrm{d}^2_+\psi(t, 0)(v, w)$ satisfies (3.33). We leave it to the reader to now verify that also $\mathrm{d}^2_+\phi(t, u)(v, w)$ exists and that $\mathrm{d}^2_+\phi(t, u)(v, w)$ satisfies (3.34). □

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