Modeling credit risk and credit derivatives

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This thesis is about managing risk, which is an important task of banks and other financial institutions. These companies have large portfolios with different risk-bearing objects. We will contribute to the theory of the quantification of risk, and in particular credit risk. The first part of this thesis focuses on measuring the credit risk in large portfolios, whereas the second part deals with the valuation of credit derivatives.

1.1 Risk

We start by presenting different types of risk that a bank, or other financial institution, has to deal with. We first decompose the general notion of risk into three risk types, which are thereafter discussed more detail. We mainly focus on the credit risk, as this is the main topic of this thesis.

1.1.1 Risk Types

In its day-to-day business, a bank or other financial institution is always exposed to risk. This risk can be split up into a number of components, which we discuss in this section. Before we focus on the three main types of risk, we consider some examples of risks that a bank could encounter.

One of the core functions of a bank is lending out money to its customers, either private or corporate. The main risk that a bank faces when lending out money, is that the customer fails to make interest payments or that he fails to repay the loan. In such a case a the bank suffers financially.
Banks also face the risk that their investments drop in value. For example, banks usually own large numbers of stocks in several companies. In case these stocks drop in value, a bank incurs a loss. Also changes in foreign exchange rates could lead to losses for a bank. When, for example, a bank expects to receive a certain amount in a foreign currency one month from now, which is worth one million Euro today, could be worth less at the time the money is actually received, leading to loss for the bank.

Further, a bank also faces risks that are not directly related to loans or to their investments. These risks are related to the functioning of the bank. Banks for example suffer from fraud, either by employees or by clients, but also computer network problems could lead to serious problems and loss of clients.

In all these cases the bank suffers financially, and thus the occurrence of any of the events above has negative consequences for the bank. Therefore it is important that the bank measures and manages risk. In a first step to do so, we split up the risk into a number of components. In general we can identify three different types of risk.

- Credit risk, the risk that a debtor, or obligor, does not honor its payment obligations.
- Market risk, the risk that an investment decreases in value due to moves in market factors, such as the stock values or interest rates.
- Operational risk, the risk arising from the bank’s business function.

Looking at the examples above, we can classify the first example as credit risk, the second as market risk and the last example as operational risk. The three risk types can be decomposed into more detailed risk types. Following [Sch03a], credit risk consists of the components

- Arrival risk, the risk that a default occurs.
- Timing risk, when the default occurs.
- Recovery risk, in case that a default occurs, which fraction of the investment is lost.
- Default correlation risk, one default could indicate a bad state of the economy, which could yield more defaults.

The arrival risk and timing risk are very much related. Clearly, when one knows the timing of the default, it is directly clear that the default event has occurred. In Section 1.1.2 we consider credit risk in more detail.

Market risk can be decomposed into the, self-evident, market factors

- equity risk, the risk of changes in stock prices,
- interest rate risk, the risk that interest rates change,
- currency risk, the risk that foreign exchange rates change,
- commodity risk, the risk that prices of commodities change.

Clearly, in most transactions that a bank enters into, it faces at least one of these risks. Operational risk can be decomposed into many different types of risk, which cannot be seen as either market or credit risk. Event types that fall under operational risk are fraud, employment practice and workplace safety, clients product and business practice, damage to physical assets, business disruption and system failures, and execution, delivery and process management.

1.1.2 Credit Risk

In the previous section we have discussed a number of risk types to which banks, and other financial institutions are exposed. In this section we consider one specific type of risk, credit risk, which is commonly referred to as default risk as well. Above we have introduced the notion of credit risk, and we have identified four components of credit risk. All components rely in some way on the occurrence of the default event. In the previous section we have already seen some examples of default events. In this section we discuss two ways banks manage credit risk. A bank can reserve a certain amount of money to cover potential losses due to default events. Further, a bank can transfer the credit risk to other market parties, by entering a credit derivatives contract.

To determine the amount of money the bank has to reserve, it has to determine the total amount of money that could be at risk, and next to this it has to determine the probability that the default events occur. For this purpose the timing and size of the defaults have to modeled. When modeling these two quantities one has to take into account that

- default events are rare events,
- defaults occur unexpectedly,
- default events involve large losses,
- the size of the loss is not known until the default occurs,
- defaults are correlated.

The last point in this list in only relevant, and very important, in case one models more than one default time. In Section 1.2 we discuss the modeling of the size and timing of the losses, where the above points are discussed.
Next to reserving cash to cover losses, a bank can transfer default risk by entering into a credit derivatives contract. Such contract provide protection against losses due to defaults. If for example a bank has invested a large sum into a company, a credit default swap ensures that in case this company defaults, the incurred loss is repaid. Section 1.3 describes a number of credit derivatives and how they can be used to transfer default risk.

1.2 Credit Risk Modeling

Above we have seen that credit risk can be split up into several types of risk. When we model the credit risk, we have to focus on two main components, namely the timing of the default and the amount of the loss. The definition of the default event is not relevant in the modeling, as we are only interested in the timing of the default and the size of the resulting loss.

It is common to model the default time and the loss size separately. There are two main streams of modeling the default time, structural models and reduced form models, which are discussed in Sections 1.2.1 and 1.2.2, respectively. The size of the loss is modeled as the recovery rate, which is the fraction of the loan, or other type of debt, that is repaid, and hence not lost. In Section 1.2.3 some models for the recovery rate are considered. The section is concluded with an overview of techniques that can be used to measure portfolio credit risk.

1.2.1 Structural Models

In structural models, one models the ability of a company to repay its debt. A (continuous) process \( V_t \) is considered that represents the value of the company at time \( t > 0 \). Further a (continuous) threshold, or barrier, \( B_t \), is considered. This barrier can be interpreted as the level of the (total) debt of the company. We say that the company defaults the first time the value of the firm is below the barrier. This means that the default time \( \tau \) is defined as

\[
\tau = \inf\{ t > 0 \mid V_t < B_t \}.
\]

Clearly, the possible choices for the value process \( V_t \) and the barrier \( B_t \) are endless. Historically, models based on Brownian motion, such as geometric Brownian motion with drift, have become popular, since such models, under suitable conditions, allow one to derive explicit expressions for e.g. default probabilities.

In this section we consider two types of structural models in more detail. First we consider Merton’s model, which is one of the simplest structural models. Thereafter we discuss a simple first passage time model, which allows us to calculate the distribution of \( \tau \) explicitly.
Merton’s Model

The simplest form of structural models is due to Merton [Mer74]. Debt maturing at a fixed time horizon, $T$, is considered. The company only defaults in case its value is below the level of the debt at $T$. The evolution of the company’s value process and its debt before this time $T$ is not considered. The value process, $(V_t)_{t \geq 0}$, is modeled as geometric Brownian motion

$$dV_t = \mu V_t dt + \sigma V_t dW_t,$$

where $W_t$ is a standard Brownian motion, $\mu$ is the drift parameter and $\sigma$ is a volatility parameter. Clearly we have $V_0 > 0$ and hence $V_t > 0$ for all $t$. It is assumed that the level of the debt is constant over time. As we assume that defaults can only occur at $T$, we set $B_t = K 1_{\{t = T\}}$. The default time is thus defined as

$$\tau = \begin{cases} T & \text{if } V_T < K \\ \infty & \text{else.} \end{cases}$$

The probability, at time zero and under the physical measure $P$, that the company defaults at time $T$ can easily be calculated, since $V_T$ is lognormally distributed. This yields

$$P(\tau \leq T) = P(V_T < K) = \Phi\left(\frac{\log\left(\frac{K}{V_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma \sqrt{T}}\right),$$

where $V_0$ is the initial value of the company, and $\Phi$ is the cumulative distribution function of the standard normal distribution.

First Passage Time Model

A clear disadvantage of Merton’s model is that only a fixed point in time is considered, and hence the evolution of the company’s value and the company’s debt before this time is ignored. In the first passage time that we consider in this section, we deal with this disadvantage with the introduction of a constant barrier $D$. This corresponds to setting $B_t = D$. We allow for defaults in the whole interval $[0, T]$. The value of the company is again modeled as geometric Brownian motion, (1.1), and the default time can be defined as

$$\tau = \inf\{t \leq T : V_t < D\}.$$  

We consider a slightly more general model by combining both Merton’s model and the simple first passage model. We define the default time as

$$\tau = \min\{\tau_1, \tau_2\},$$

where $\tau_1$ is given by (1.2) and $\tau_2$ by (1.3).
Under these settings it is possible to calculate the probability, at time zero, that the default occurs before time $T$.

$$\mathbb{P}(\tau \leq T) = \Phi\left( \frac{\log(K/V_0) - mT}{\sigma \sqrt{T}} \right) + \left( \frac{D}{V_0} \right)^{2m/\sigma^2} \Phi\left( \frac{\log(D^2/(KV_0)) + mT}{\sigma \sqrt{T}} \right),$$

where $m = \mu - \frac{1}{2}\sigma^2$. The derivation of this expression relies on several properties of Brownian motion, such as the reflection principle. The distribution of $\tau$ for $t < T$, can easily be calculated from this expression.

The first passage model that we consider here is actually a special case of the first passage model considered by Black and Cox, [BC76], who consider the time dependent barrier

$$B_t = Ke^{-\gamma(T-t)}.$$  

It is clear that this agrees with the model we consider here in case $\gamma = 0$.

Many other authors have considered geometric Brownian motion as a basis for a structural default model. Among others, Longstaff and Schwartz, [LS95], and Kim et al., [KRS93], both consider stochastic interest rates instead, and the company defaults if the firm value is below some constant barrier. Other variations to the model by Black and Cox, involve jump-diffusion models for the firm value process, e.g. [Sch96] and [Zho96]. A partial information approach, where one assumes that the firm value is not, or only partially, observed, is considered by Giesecke [Gie04]. Bielecki and Rutkowski [BR02] and [Sch03a] provide a more elaborate discussion of structural models.

**Disadvantages of Structural Models**

Most of the structural models suffer from a number of disadvantages. First of all, default times are predictable when they are modeled using structural models. It is possible to construct a sequence of increasing stopping times that converge to the default time. This is not a realistic property, since defaults in general cannot be anticipated. By the introduction of random jumps in the firm value process, one can partially solve this problem, since the company can ‘jump to default’. On the other hand the company value can still pass the barrier as a continuous process.

In practice, credit spreads, which are the difference between the risk free rate and the rate at which a company can borrow, is relatively high for loans with a short maturity. When modeling the default time using structural models, and thus predictable default times, it is not possible to reproduce such spread, as the default probability is very small over a short period.

Another drawback of the structural models is that in general the value process $V_t$ and the level of the debt are not, or only partially, observable. Due to this lack of information, there is uncertainty about the occurrence of the default event. Without this information it is not possible to accurately estimate or calibrate the model parameters. The partial information approach deals with this drawback explicitly.
1.2.2 Reduced Form Models

The second type of default time models that we consider concerns reduced form models. In these models the distribution of the default time is exogenously given, in contrast to the structural models, where the (distribution of the) default time corresponds directly to the value of the company. As the default time is based on an exogenous specification of the distribution, there is not a formal definition of the default time. Instead the default time is modeled as the first jump of a Cox Process, or doubly stochastic Poisson Process. The Cox process is an extension of the inhomogeneous Poisson process. Recall that for an inhomogeneous Poisson process $N = (N_t)_{t \geq 0}$, one has for $0 < s < t$,

$$P(N_t - N_s = k) = \frac{\left( \int_s^t \lambda(u)du \right)^k}{k!} \exp\left(-\int_s^t \lambda(u)du\right),$$

(1.4)

where $\lambda(t) \geq 0$ is the deterministic (default) intensity, for $t \geq 0$. In case of Cox processes, stochastic intensities are considered instead, and (1.4) becomes a conditional probability. The application of Cox processes to the modeling of default times has been considered by e.g. [Lan98].

The default time of a company is modeled as the first jump time of $N$. This means that we can use (a variation to) (1.4) to calculate the distribution of the default time $\tau$.

In the remainder of this section we consider deterministic and stochastic intensities separately.

Deterministic Intensity

We first consider a deterministic intensity, $(\lambda(t))_{t \geq 0}$, and thus we model the default time as the first jump of a inhomogeneous Poisson process. Based on (1.4) we can easily determine the distribution of the default time. The probability that the company defaults before time $t > 0$ is given by

$$P(\tau \leq t) = P(N_t > 0) = 1 - P(N_t = 0) = 1 - e^{-\int_0^t \lambda(u)du}.$$  

In practical applications the deterministic default intensity is given a simple specification, typically (piecewise) constant or (piecewise) linear. Such specifications allow one the match the default probabilities to certain quantities observed in practice. The application of such intensities to the valuation of credit default swaps (introduced in Section 1.3.1) is considered in e.g. [Luo05].

Stochastic Intensity

When we use a stochastic, or random, intensity, the corresponding default process $(N_t)_{t \geq 0}$, is called a Cox process. Conditioned on the evolution of the intensity it is distributed as an inhomogeneous Poisson process. Thus, the distribution of the
default time can easily be obtained by conditioning on the realization of the intensity. For \( t > 0 \) we obtain,

\[
\mathbb{P}(\tau \leq t) = 1 - \mathbb{E}\left[ 1_{\{N_t > 0\}} \right] = 1 - \mathbb{E}\left[ e^{-\int_0^t \lambda(u)du} \right].
\]

Here we observe a close connection to the modeling of stochastic interest rates. Therefore many researchers have been inspired by well known interest rate models, such as the Vasicek model [Vas77], the Ho-Lee model [HL86] and the Cox-Ingersoll-Ross model [CIR85], for the modeling of the intensity. Further [BM06] gives an overview interest rate models, that could also be used to model the default intensity.

**Advantages of Reduced Form Modeling**

In Section 1.2.1 we have discussed a number of disadvantages of structural models. The two most important disadvantages are the predictability of the default time and the lack of information about a company’s value and debt, which are required to estimate model parameters. In this section we show, without going into too much detail, that reduced form models do not suffer from these disadvantages.

To show that default times in reduced form models are not predictable, we observe that \((N_t)_{t \geq 0}\) is a nondecreasing process, and hence it is a sub-martingale with respect to its natural filtration if \( \mathbb{E}[N_t] < \infty \) for all \( t \). The Doob-Meyer decomposition, see [LS78], (under suitable conditions) yields that there exists a increasing process \( A \) such that \( N - A \) is a martingale. This process \( A \) is called the compensator. Further, we known from p. 243 in [LS78] that the compensator is continuous if and only if the stopping time, or default time, \( \tau \) is totally inaccessible. This means that for all predictable stopping times \( \sigma \) we have \( \mathbb{P}(\tau = \sigma) = 0 \). In the reduced form models we actually directly model the compensator process, which defines the Poisson or Cox process. Clearly, the integrated intensity is a continuous process, which means that the default time \( \tau \) is not predictable.

As the reduced form models are based on an exogenous definition of the default time, we do not need information about the company’s value and debt to estimate model parameters. Instead, we can estimate, or calibrate, the parameters of reduced form models directly from market prices of relevant instruments, such as defaultable bonds or credit default swaps.

**1.2.3 Recovery Modeling**

In the previous two sections we have considered the modeling of the default time. As we have mentioned above, we also have to model the loss incurred after a default. After a company has defaulted, one could still receive a fraction of the loan or bond notional. In this section we discuss recovery rate models, which model the fraction that is repaid in case of default. We focus on the modeling of the value of a defaultable bond after a default has occurred. However, the results can be applied to loans and other credit derivatives as well.
Deterministic Recovery

When one models the recovery rate as a deterministic quantity there are two different approaches, the fractional recovery of par value and the fractional recovery of market value. The former approach models the loss given default as a fraction of the face value of the bond. The valuation of the bond is then straightforward, as one can easily distinguish the default and no default cases. The bond’s payoff at maturity $T$ can be written as

$$D_T = \left(1_{\{\tau > T\}} + 1_{\{\tau \leq T\}} R\right) N,$$

where $D_T$ denotes the payoff, $0 \leq R \leq 1$ the recovery rate and $N$ the face value of the bond. Here we have assumed that in case of default the recovered value of the bond is repaid at maturity. If we assume that the recovered value is paid at the time of default, we can accrue this amount at the risk free interest rate. This yields the payoff at maturity

$$D_T = \left(1_{\{\tau > T\}} + 1_{\{\tau \leq T\}} R/B(\tau, T)\right) N,$$

where $B(t, T)$ is a (stochastic) discount factor, i.e. the value at $t$ of receiving 1 at $T$.

In the fractional recovery of market value, it is assumed that at the time of default a fraction of the pre-default value of the bond is paid. The payoff of the bond at maturity can be written as

$$D_T = \left(1_{\{\tau > T\}} + 1_{\{\tau \leq T\}} RD_{\tau -} / B(\tau, T)\right) N,$$

where $D_{\tau -}$ is the pre-default value of the bond.

These two approaches can easily be extended, for example by using a combination of the two approaches, or by making the recovery rate time dependent. More detailed discussions on the modeling of the deterministic recovery rate can be found in [Sch03a] and [BR02].

Stochastic Recovery

We have seen that when we model the recovery rate deterministically, bond values can easily be evaluated, as one can decompose the payoff into a default and no default scenario. When we model the recovery rate as a random quantity this property does not hold in general. Clearly, we could assume that the recovery rate $R$ is a random variable independent of $\tau$, with distribution $F$. In such a case the bond can be valued similar to the deterministic models, for example by conditioning on the outcomes of the recovery rate.

Alternatively, one could model the recovery rate as a random quantity, with the first passage time models from Section 1.2.2. In case the value of the company drops below the value of the debt, the company defaults. The recovery value in such a case is $V_\tau < D_\tau$. 
For a more detailed discussion of stochastic recovery rates we refer to [Sch03a, Chapter 6]. There, a more general and formal framework is introduced, in which various recovery rate models are considered.

**1.2.4 Measuring Portfolio Credit Risk**

In general credit portfolios are very large, consisting of thousands of obligors. The measurement of the credit risk in such portfolios focusses on the tails of the credit loss distribution of these portfolios. Often banks are interested in the 95%, 99% or even more extreme percentiles of this distribution. In the previous section we have encountered two approaches to the modeling of the default time of a single company. In some cases it is straightforward to extend this to the modeling of the default times of several companies in a portfolio, which could in theory be used to measure portfolio credit risk. When the size of the portfolio is very large, which is usually the case for banks, such an approach could be very time consuming and therefore less interesting.

In this section we discuss a number of methods that can be used to measure the credit risk in large portfolios. First we briefly mention some popular risk measures, which can be evaluated using techniques that are discussed in the remainder of this section.

**Risk Measures**

One of the most popular risk measure is the value-at-risk (VaR) measure. The $\alpha$ percent VaR, for some portfolio, states the amount of money that can be lost with $\alpha$ percent certainty. In general VaR can be calculated for any kind of portfolio and any kind of risk type. In all cases one focusses on the loss on a single point in time, usually one year into the future. The VaR measure is discussed in more detail in e.g. [Jor06]. Further, other risk measures have been proposed to measure the risk in a portfolio. In any case one has to model the loss distribution of the portfolio in order to determine certain characteristics, such as the $\alpha$ percent quantile. Many different approaches can be considered for this purpose, and we list a number of these below.

**Asymptotic Approximations**

A common approach to modeling the loss in a very large credit portfolio is to consider a limiting portfolio, where the size of the portfolio tends to infinity. Under suitable assumptions, such as homogeneity and identical distributed underlyings, one can determine the distribution of the infinite portfolio, and use this as an approximation for the distribution of the finite portfolio.

A good example is the large pool model, where the portfolio is assumed to be homogeneous and of infinite size. The correlation between the different obligors is modeled using the one-factor Gaussian copula, which is discussed in more detail in Section 1.4. The advantage of this model is that one can calculate the loss distri-
bution explicitly in terms of the normal distribution, which in turn could be used to easily determine quantiles of the loss distribution.

Another approach to the modeling of extreme losses are large deviation approximations of the tails of the loss distributions. Large deviations theory, through the large deviation principle, provides asymptotic upper and lower bounds for the logarithm of a family of distributions, such as the probability that the average loss in a portfolio of size \( n \) is above a certain level, as \( n \to \infty \). Existing large deviation results, see e.g. Dembo and Zeitouni [DZ98], can easily be used to model to tails of the loss distribution at a single point in time. In Chapter 3 we give a brief introduction into the theory of large deviations and we present a large deviation principle for the whole path of the loss process.

**Historical Data**

A technique that is often used to construct the loss distribution is by using historical data. Here the loss distribution is constructed using a time series of historical data, for example as the empirical distribution function. When one considers the losses due to changes in asset prices, many data are available, and one can build a loss distribution. For the modeling of the credit loss distribution little data are available, as defaults in general are rare. Therefore such data might not suitable to construct the loss distribution. On the other hand, one can use daily market data on market quotes for credit derivatives, which are discussed in detail in Section 1.3, to approximate loss distribution. Such derivatives are used to transfer credit risk, and the associated premia give an indication of the credit risk involved.

**Filtering**

As an extension of the reduced form models, one can model the loss process as a Cox Process. Each jump of the Cox process represents a default of one of the companies in the portfolio. As is clear from the examples in Section 1.2.2 for the models for the (stochastic) intensities, it is common to model the intensity as

\[
d\lambda_t = \mu(\lambda_t, t)dt + \sigma(\lambda_t, t)dW_t,
\]

for some functions \( \mu, \sigma \), and Brownian motion \( W_t \) with respect to some filtration \( \mathcal{F}_t \). Following Bielecki and Rutkowski [BR02], one can write \( \mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{G}_t \), where \( \mathcal{F}_t^N \) is the filtration generated by the default counting process \( N_t \), and \( \mathcal{G}_t \) is another filtration.

In general, models assume that \( \lambda_t \) (or \( W_t \)) can be observed, as \( \lambda_t \) should be \( \mathcal{F}_t \) measurable. In practice, one can usually only observe very little information about the a company, and therefore one could question the assumption that \( \lambda_t \) is observable. When we would assume that one can only observe the number of defaults in the portfolio, where the intensity \( \lambda_t \) is given by (1.5), we can use the theory of filtering for point process observations, which is described in e.g. Brémaud [Bré81], to compute relevant quantities with respect to the loss distribution. In Chapter 2 we show that we can derive the conditional moment generating function when we model the
intensity as a CIR model. In that chapter, one also finds a brief introduction to filtering with point process observations.

1.3 Credit Derivatives

In this section we describe a number of credit derivatives, which can be used to transfer the default risk from one party to another. The payoff of these derivatives is contingent on the occurrence of a default event. Depending on the definitions of the contract of the credit derivative, the default event could be an unpaid coupon on a specific bond, the bankruptcy of the underlying company or a restructuring of the company.

In a credit derivative contract there are usually three parties involved, the protection buyer, the protection seller and the reference entity. The protection buyer has to pay a premium to the protection seller until maturity or until the default event with respect to the reference entity occurs. In case a default occurs before maturity, the protection seller has to compensate the loss incurred by the protection buyer. This can be done with physical settlement, i.e. the protection buyer delivers defaulted bonds to the protection seller, and receives the notional amount of the bond. Alternatively, the derivative can be settled in cash. In such a case, the loss is typically determined by asking several banks to provide prices for the defaulted bond. In both cases it is clear that the buyer of protection transfers the default risk to the protection seller. The only default risk that the protection buyer faces, is that the protection seller might not be able to cover the loss in case of default. This risk is known as counterparty risk, and it is present in almost any financial contract.

The reference entity can either be a (bond of a) single company, or it can be a basket of companies. In the former case we speak of single name credit derivatives, and in the latter case we speak of multi name credit derivatives. In this section we discuss the most popular credit derivatives traded.

1.3.1 Credit Default Swaps

The credit default swap (CDS) is the most liquid, and simplest credit derivative. It can be seen as an insurance against the default of the referenced company. An investor facing the default risk of a certain party, can enter into a CDS contract, by which he effectively eliminates the default risk with respect to this party. We illustrate the process behind a CDS in a simple example.

Example 1.1. Consider an investor that owns a bond with a notional value of 1000 Euro issued by ING bank that matures in two years, and say that the investor wants to eliminate the default risk with respect to ING bank. He can then enter into a CDS contract maturing after two years with, for example, ABN Amro bank, where the default event is defined with respect to this specific bond. The investor becomes the protection buyer and ABN Amro becomes the protection seller.
During the life of the CDS contract the investor has to make regular, e.g. quarterly, premium payments. These payments are defined as a fraction of the notional of the CDS, say 0.8% or 80 basis points per year. In case ING does not default with respect to this bond, the only payments made are those by the investor. Thus, in this case, eight payments of two Euro are made.

In case ING does default, ABN Amro has to pay the amount lost by the investor. Depending on the specifications of the contract there are two ways to make this payment, either physically or by cash settlement. Say that ING defaults after one year and two months. First assume that the CDS is physically settled. In the first year four payments of 2 Euro are paid by the investor. After the default the investor hands over the bond to ABN Amro and in return he receives 1000 Euro. Further, the investor has to pay a premium of 1.33 Euro, for the two month of protection during the fifth quarter of the contract.

Alternatively, the CDS contract could been cash settled. In such a case a number of dealers is requested to give prices for the defaulted bond. Say that the eventual price is 378 Euro, then ABN Amro has to pay the investor an amount of 1000 \( - 378 = 622 \) Euro.

The market for credit default swaps has become very liquid over the last two decades. This has led to a standardization of this market. Therefore market quotes are available for a wide range of maturities. For many companies, quotes with maturities, or tenors, of three and six months, and one, two, three, four, five, seven, ten, twelve and fifteen years can be found in the market. Further, due to standardization the market for credit default swaps, the time until the contract matures is usually not exactly equal to the periods above. The contracts last for at least this period, but they are extended until the first of four standard maturities, which are the twentieth of March, June, September or December. If, for example, two parties enter into a CDS contract with a tenor of 2 years on September 28, 2009, this contract will end on December 20, 2011, unless a default occurs before this date, which causes an earlier termination of the contract. Naturally, an investor can always try to negotiate different contract specifications, where the maturity can differ from the standard maturities.

1.3.2 Index Credit Default Swap

When an investor is exposed to the default risk of more than one company, he can choose to enter into CDS contracts for all companies to whose default risk it is exposed. If one of the companies defaults, the respective CDS covers the resulting loss, and the investor continues to make premium payments for the remaining CDS contracts. Alternatively the investor could enter into an index credit default swap, or index swap. This swap is similar to the CDS, but instead of a single underlying, a basket of companies is referenced. In case the CDS contracts are cash settlement, and the weights of the companies in the basket are the same as for the separate CDS contracts, the index swap makes the same default payments. The premium payments are
different however. The premium is paid with respect to the outstanding, or remaining, notional of the basket. At the start of the contract the premium is agreed which has to be paid. After a default, less premium is paid, since the notional of the basket is reduced. The contract ends at its maturity or when all underlying companies have defaulted. Similar to the regular credit default swaps, the more liquid index credit default swaps are standardized, such that the underlying index consists of the same underlyings and the contracts expire at the standard maturities introduced in the previous section.

In the following section we discuss another credit derivative that can be used to transfer the default risk of a basket of reference entities.

### 1.3.3 Collateralized Debt Obligations

In the previous section we have seen a credit derivative that can be used to deal with the default risk in a basket of reference entities. The advantage of this is that a protection buyer does not incur losses in case any of these companies defaults. On the other hand, one has to pay premium with respect to the whole basket. Further, an investor could reserve a certain amount of money to cover the first losses in this basket himself, and he wants to receive protection against a certain fraction of the losses. The remaining default risk should then be transferred. The collateralized debt obligation (CDO) can be used to do this.

First, we consider a basket $B$ of $N$ reference entities. With respect to this basket we define the loss process

$$L^B_t = \sum_{i=1}^{N} 1_{\{\tau_i \leq t\}} (1 - R_i) N_i,$$  \hspace{1cm} (1.6)

where $\tau_i$ is the default time of reference entity $i$, $R_i$ is the (fixed) recovery rate of reference entity $i$ and $N_i$ is the notional amount of the reference entity. A CDO tranche provides protection on a part of the losses in this basket. The level where the protection starts, the attachment point, is given as a percentage of the notional of the basket. When the losses in the basket exceed this level the protection buyer receives default payments. When the losses exceed the detachment point of the CDO tranche, and the contract has not matured, the protection buyer no longer receives default payments. One says that the tranche has been exhausted. The protection buyer has to make regular premium payments with respect to the notional amount of the tranche, which is equal to $(d - a)\%$ of the notional of the basket at the start of the contract, where $d$ denotes the detachment point and $a$ the attachment point. The tranche loss, $L^B_{a,d}(t)$, as a fraction of the tranche notional is given by,

$$L^B_{a,d}(t) = \frac{d \cdot L^B_d(t) - a \cdot L^B_a(t)}{d - a},$$

$$L^B_x(t) = \frac{1}{x} \min \left( L^B_t, x \right), \text{ for } x = a, d.$$
In general the basket is split up into a number of consecutive tranches \((a_i, d_i)\), such that \(a_i = d_{i-1}\), and \(a_1 = 0\). The tranche \((a_1, d_1) = (0\%, d_1)\), clearly is the most risky tranche, since the first default of one of the reference entities leads to losses on this tranche. This tranche is usually called the equity tranche. The subsequent tranches are referred to as mezzanine tranches. The last tranche is called the super senior tranche.

**Example 1.2.** Consider a basket of 100 reference entities, each with a notional of 1000 Euro. Then the total notional of the basket is 100,000 Euro. Suppose that an investor is exposed to the default risk of (a part of) these reference entities, and that he wants to cover losses between 5\% and 15\% in the basket, for a period of five years. This means that when the loss process exceeds 5,000 Euro the investor starts to receive default payments, until the loss process exceeds 15,000 Euro. Further assume that the premium for this protection is 2\% or 200 basis points, and say that the premium is paid quarterly.

When no defaults occur during the first three month, the investor has to pay 50 Euro premium, as the notional amount of the tranche is Euro 10,000. Suppose that in the following three months, four reference entities default, with recovery rates of 10\%, 25\%, 40\% and 60\%. The loss process then grows to 2,650 Euro. As this is still below 5,000 Euro, the investor has to pay 50 Euro premium after six months. Assume that until the end of the first year no further defaults occur. Then the investor has to make two more premium payments of 50 Euro.

When three more reference entities default in the first quarter of the next year, with an average recovery rate of 25\%, the loss process after the fifteenth month equals Euro 4,900, yielding another premium payment of 50 Euro. If another company defaults, say after one year and four months, with a recovery rate of 50\%, the loss process, now at 5,400 Euro, exceeds the attachment point of the CDO tranche, corresponding to 5,000 Euro. The investor thus receives 400 Euro. In return it has to make a premium payment of 48.33 Euro, as the notional of the tranche has been reduced after the default. (Note that over the first month of this quarter still full premium has to be paid).

The premium and default payments are made until the contract matures, or until the loss process exceeds the detachment point of 15\%, which corresponds to 15,000 Euro.

Originally, the basket underlying the CDO is constructed of several bonds, loans or other cash instruments. The main problem with this construction is that each bond can have a different maturity, and loans might be repaid early. This requires a manager that replaces bonds and loans if necessary. As the market for CDS contracts has become more liquid, it was realized that the basket of bonds and loans could be replaced by a basket of CDSs, resulting in synthetic CDO tranches. The advantages of using CDSs, is that these all have the same maturity, and that the basket becomes much easier to manage.
The introduction of synthetic CDOs has lead to a large increase in liquidity for CDO tranches, and to the introduction of CDO tranches with respect to standard baskets, or indices. Two of the most popular referenced indices are the iTraxx index, consisting of CDSs on 125 investment grade European companies, and the CDX index, consisting of 125 North-American companies. In the Chapters 4 and 5 these indices and CDO tranches in general are discussed in more detail.

1.3.4 Other Credit Derivatives

In the previous sections we have described some of the most popular credit derivatives. Next to these several other credit derivatives have emerged. Similar to the regular CDS, the constant maturity CDS has been introduced, where the premium payments depend on the market quote of a regular CDS at the beginning of a coupon period.

Another single name credit derivative, is the credit default swaption, which gives the holder the right to enter into a CDS contract at a certain future time against a certain premium. The details of the option specifies if one enters as protection buyer or seller.

Next to the CDO, the \( N \)-th to default swap is also a popular multi name credit derivative. In such a contract one is protected against the \( N \)-th default that occurs in the underlying basket. Usually this is the first or second default.

A popular variation of the CDO, is the forward starting CDO, where the protection starts at a future time \( T \). Companies that default before this time are excluded from the basket or one can assume that the loss process increases, without triggering default payments.

1.4 Modeling Credit Derivatives

In this section we discuss models for the valuation of credit derivatives. We mainly focus on the credit default swap (CDS), the index swap and the collateralized debt obligation. These are the most liquid instruments. We first focus on the modeling of a single default time, which allows us to value a CDS contract. In addition we can value an index swap contract as well, as this does not depend on the dependence between the different companies in the underlying basket. Thereafter we discuss models that can be used to value (synthetic) CDO tranches.

1.4.1 Valuation of Credit Default Swaps

In Section 1.2 we have encountered two different approaches to the modeling of default times, the structural and reduced form approaches. As both approaches model a single default time, they can easily be used to determine the value of a CDS contract. Assume that the contract has a maturity of \( T \) years, and that premium has to be paid at times \( T_1 < \cdots < T_N \). Further assume that the percentage of premium, or spread,
that has to be paid is denoted by \( s \). Then we can determine the value of the contract as the difference between the expected present value of future premium payment and the expected present value of a possible payment in case of default. If, for simplicity, we assume that the notional amount of the CDS contract equals 1, then the expected value of the premium payments, or premium leg (PL), is given by

\[
PL = E \left[ \sum_{i=1}^{N} \delta_i D(T_i) 1_{\{\tau > T_i\}} \right] \\
= \sum_{i=1}^{N} \delta_i D(T_i) \mathbb{P}(\tau > T_i),
\]

(1.8)

where \( D(t) \) is the discount factor, i.e. the present value of receiving 1 unit of currency at time \( t \), \( \tau \) is the default time of the company underlying the CDS, and \( \delta_i \) is the year fraction for the period from \( T_{i-1} \) to \( T_i \). For simplicity we have ignored the partial premium payment in case of default. The expected amount of premium that will be paid, can be obtained by multiplying by the spread \( s \).

The expected present value of the possible default payment, or default leg (DL), is given by

\[
DL = E \left[ D(\tau) 1_{\{\tau \leq T\}} (1 - R) \right] \\
\approx \sum_{i=1}^{M} D(t_i^*) (\mathbb{P}(\tau \leq t_i) - \mathbb{P}(\tau \leq t_{i-1})) (1 - R_{\text{fix}}).
\]

(1.9)

Here \( R \) is the recovery rate, the time points \( t_i \) form a discretization of the interval \([0, T]\), and \( t_i^* = (t_{i-1} + t_i)/2 \). It is a common approach to assume a fixed, deterministic recovery rate \( R_{\text{fix}} \). Further one usually approximates the expectation by assuming that the default can only occur on a certain time grid \( t_1 < \cdots < t_M \), i.e. one computes the integral numerically. In case the recovery rate is modeled as a stochastic quantity (independent of \( \tau \)) one can calculate the expectation by conditioning on the realization of \( R \).

The value of the CDS contract is obtained as the difference between the two legs. The value for the protection buyer is thus given as \( DL - s \cdot PL \), and the value for the protection seller as \( s \cdot PL - DL \). Usually the spread \( s \) is chosen such that the contract initially has zero value. Clearly this allows one to calibrate model parameters from market quotes for CDS contracts with different maturities. Especially the levels of the piecewise constant intensity that was discussed in Section 1.2.2, can easily be determined. One can choose the first level of the intensity such that the first CDS quote is matched. Next, one can match subsequent CDS quotes by adjusting the following levels of the intensity. In such a way the intensity is constant between the maturities of the CDS contracts used. This process is often referred to as bootstrapping of the intensity.
From Equations (1.8) and (1.9) it is clear that, in case a deterministic recovery rate is assumed, one only needs to know the distribution of the default time to value a CDS contract. This implies that we can (easily) use all models from Sections 1.2.1 and 1.2.2 to value CDSs.

For the valuation of an index CDS, it is sufficient to add the respective premium and default legs for CDSs on the companies underlying the index CDS, to obtain the premium and default leg of the index CDS.

### 1.4.2 Models for Collateralized Debt Obligations

Above we have seen that the valuation of CDS contracts is straightforward once the distribution of the default time can be calculated. In this section we look at the valuation of the more complex collateralized debt obligation (CDO). First we look at general pricing formulas. Thereafter we introduce the base correlation framework, which is the most popular model to value CDO tranches. The section is concluded by considering other models that have been proposed for the valuation of CDO tranches.

#### Pricing Formulas

To value a CDO tranche one has to determine the value of the protection and premium legs. These values depend directly on the cumulative loss process $L^B_t$, as in Equation (1.6). We write $a$ for the attachment point of the tranche and $d$ for the detachment point, and write $T$ for the maturity of the contract. Further we assume that premium payments are made at times $T_1 < \cdots < T_N$. The value of the premium leg, where the tranche loss is calculated as fraction of the tranche notional, is given by

$$PL_{a,d}^{B,T} = \mathbb{E} \left[ \sum_{i=1}^{N} \delta_i D(T_i) \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T_i} \left( 1 - L^B_{a,d}(t) \right) dt \right],$$

where the tranche loss is denoted by $L^B_{a,d}(t)$, as defined in (1.7). Instead of calculating the integral above, one usually uses the average of the loss process at the start and the end of the period as an approximation. This yields

$$PL_{a,d}^{B,T} \approx \sum_{i=1}^{N} \delta_i D(T_i) \left( 1 - \frac{1}{2} \left( \mathbb{E} \left[ L^B_{a,d}(T_i) \right] + \mathbb{E} \left[ L^B_{a,d}(T_i) \right] \right) \right).$$

The default leg, i.e. the expected present value of the payments in case defaults occur, is given by

$$DL_{a,d}^{B,T} = \mathbb{E} \left[ \int_0^T D(t) dL^B_{a,d}(t) \right] \approx \sum_{i=1}^{K} D(t_i^*) \left( \mathbb{E} \left[ L^B_{a,d}(t_i) \right] - \mathbb{E} \left[ L^B_{a,d}(t_{i-1}) \right] \right).$$
where the latter summation expression is an approximation of the integral.

From the formulas above it is clear that we need to calculate the expected tranche loss at certain points in time in order to determine the value of a CDO tranche. Further one can observe from Equation (1.7) that the tranche loss can be written as the difference between two equity tranches. Therefore we only need to compute expected equity tranche losses, i.e. $\mathbb{E} [L_d^E(t)]$, for various $d$ and $t$. In the remainder of this section we consider some popular models for the valuation of CDO tranches.

**Copulas**

The first type of models we consider are so-called copulas. Such models allow one to model the dependence between random variables, in such a way that the marginal distributions are not affected. If we write $F_1, \ldots, F_N$, for the marginal distributions of a set of random variables $X_1, \ldots, X_N$, then we can calculate the joint distribution of $X_1, \ldots, X_N$ as

$$
\mathbb{P} (X_1 \leq x_1, \ldots, X_N \leq x_N) = C (F_1(x_1), \ldots, F_N(x_N)),
$$

where $C : [0,1]^N \to [0,1]$ such that there exists a collection of uniform variables $U_1, \ldots, U_N$ for which the joint distribution function is given by the function $C$, and in addition $C(1, \ldots, 1, F_i(x_i), 1, \ldots) = F_i(x_i)$ for all $i$. The function $C$ is called a copula function. A good introduction to copulas is by [Nel99]. Some well known copulas are the product copula, which results when independence is assumed, and the Gaussian copula, which is based on the multivariate normal distribution.

We consider a special, and intuitively clear, way of creating copulas, through the factor copula approach. We define the set of random variables $X_1, \ldots, X_N$ as

$$
X_i = a_i \cdot Y + \sqrt{1 - \|a_i\|^2} \cdot \varepsilon_i, \quad i \leq N
$$

(1.10)

where $Y \in \mathbb{R}^M$, $Y$ and $\varepsilon_i$ are independent, and $\|a_i\| \leq 1$ for all $i \leq N$. The factor $Y$ is usually referred to as the common factor, and the $\varepsilon_i$ as the idiosyncratic factor. One can interpret the role of $Y$ as a general state of the economy, and the $\varepsilon_i$ as the state of company $i$.

We write $F_i$ for the marginal distribution of company $i$, which is obtained from CDS market data for this company. In the factor copula setup we assume that the company defaults when for the first time $X_i \leq \chi_i(t)$, for some nondecreasing function $\chi_i(t)$, which is chosen such that for all $t$

$$
F_i(t) = \mathbb{P} (\tau_i \leq t) = \mathbb{P} (X_i \leq \chi_i(t)).
$$

In this way the marginal distributions are matched.

To calculate the expected tranche losses we assume that recovery rates are deterministic, but possibly different for each underlying. This fixes the size of losses, and what remains is the calculation of the probability that the loss ends up at a certain level,
i.e. \( \mathbb{P}(L^B(t) = x) \). The advantage of the factor copula approach is that the variables \( X_i \) are independent when we condition on the realization of the common factor \( Y \). This allows us to easily build the conditional distribution \( \mathbb{P}(L^B(t) = x|Y = y) \), as in explained in Chapter 4. The unconditional distribution can then be obtained by integration.

**Base Correlation**

The most popular factor copula model is the one-factor Gaussian Copula, which was introduced to credit risk modeling by Li [Li00], where both \( Y \) and \( \varepsilon_i \) are standard normally distributed, and one sets \( a_i = \rho \) for all \( i \), where \( 0 \leq \rho \leq 1 \) denotes the correlation. Very accurate and fast approximations to the normal distribution allow for a fast valuation of CDO tranches.

The problem with the one-factor Gaussian copula is that one cannot calibrate to market prices with a single correlation value. In a first attempt to deal with this issue, the implied, or compound, correlation was introduced. Here a different correlation is used for each CDO tranche such that the tranche is repriced. This approach is analogous to the calculation of implied volatilities for call and put options in other markets, such as the equity and the foreign exchange markets. The disadvantage of the implied correlation approach is that the correlation depends on the attachment as well as the detachment point, which makes it difficult to interpolate the correlation in order to value a tranche with nonstandard attachment or detachment points, especially since market quotes are available for a small number of tranches.

Above we have observed that both legs of a CDO tranche contract can be valued as the difference of two equity tranches. This observation has led to the introduction of the base correlation framework. In general, market quotes are available for a number of consecutive tranches, and one can iteratively solve for a correlation for equity tranches.

After all base correlations have been calculated on can use interpolation to value any tranche \((a, d)\), using the two correlations \( \rho_a \) and \( \rho_d \). A big advantage over the implied correlation is that we now have to interpolate a function of a single variable. As we shall see in Chapter 4, the base correlation curve is increasing, whereas the implied correlation curve is usually not. In Chapter 4, we present a more detailed description of the base correlation framework, and we consider three different ways in which one can interpolate the base correlations, and we discuss the consequences of each interpolation method.

**Other Models for CDOs**

Due to the disadvantages of the base correlation framework, many different models for the valuation of CDO tranches have been introduced. In this section we briefly discuss a number of these models.
As an extension to the one-factor Gaussian copula, several authors have considered one-factor copulas. Different kinds of distributions are proposed for the common and idiosyncratic factors, or the correlation parameter is assumed to be a random variable. Each of these factor copulas attempts at matching a range of CDO quotes with a single set of parameters. Below, we briefly consider a number of these alternative one-factor copulas, which are discussed in more detail in Chapter 5.

Instead of assuming that the factors $Y$ and $\varepsilon_i$, as in (1.10), have a standard normal distribution, Hull [HW04] considers the Student $t$-distribution for both factors, to obtain the Double-$t$ factor copula. The Student $t$-distribution has much fatter tails compared to the normal distribution.

Similar to the double-$t$ factor copula, Kalemanova et al. [KSW05] have considered the normal inverse Gaussian (NIG) distribution for both factors, which resulted in the NIG factor copula. This distribution also shows fatter tails than the normal distribution. Further, in contrast to the normal and student $t$-distribution, this distribution is not symmetric, which allows for fat left (or right) tails, and thin right (or left) tails.

A different kind of extension has been consider by, amongst others, Burtschell et al. [BGL05]. Here the distribution of the factors $Y$ and $\varepsilon_i$ is still the standard normal distribution, but now the correlation parameter $\rho$ is assumed to be a random variable. In particular, it is assumed that the correlation can take on a finite number of values $\rho_i$, with probabilities $p_i$. The resulting factor copula model is referred to as the mixture factor copula.

The idea of a random correlation has also been adopted by Andersen and Sidenius [AS04], which let the correlation depend on the value of the factor $Y$, i.e. the state of the economy. The interpretation behind this assumption is that correlations tend to increase when the state of the economy deteriorates. In its simplest form, the correlation can take on two different values, which depends on $Y$ being above or below a certain level. In Chapter 5 we consider a somewhat more advanced version of the random factor loadings factor copula.

Next to these four alternative factor copula models, which are described in more detail in Chapter 5, several other versions have been proposed. Some of these are briefly described in this chapter as well.

Besides the factor copula framework, many authors have attempted to match CDO market data with different kinds of models. In the remainder of this section we provide a brief, and far from complete, overview of such models.

The factor copula models that we have considered above model the loss distribution by modeling the individual default processes, and the dependence is introduced through a copula. This approach, where the individual default processes are modeled, are often referred to as bottom-up.

Next to the bottom-up modeling, some authors have modeled the loss process directly, often ignoring the individual characteristics. Such models are usually referred
to as top-down models. Giesecke et al. have used this approach in [GGD05] and [EGG09], where the general setup of top-down modeling is introduced, together with more detailed examples. The top-down approach has also been followed by Brigo et al. [BPT06] and [BPT07], where a sum of Poisson models is used to model the loss process from the top down.

In an attempt to introduce a full dynamical model for the valuation of more advanced credit derivatives, such as options on CDO tranches, Schönbucher [Sch06] and Sidenius et al. [SPA06] have considered dynamical models for the evaluation of the loss process, which are inspired on the Heath-Jarrow-Morton modeling approach [HJM92] used in interest rates.

1.5 Credit Crisis

After the market for credit derivatives had grown explosively for almost two decades, this market collapsed when the credit crisis started halfway 2007. Many house owners in the United States could not pay their mortgages, which led to large losses on mortgage portfolios owned by banks or insurance companies. Banks had sold these mortgages in CDO-like, and more complex, structures to other banks or investors. Such structures were initially seen as very safe investments, as in general high ratings were given to these structures. Soon after house owners defaulted on their mortgages, it turned out that the structures were in fact bearing high risks. This led to large losses and a lot of uncertainty. Therefore banks had to write off huge amounts of money on these and other types of credit derivatives off their balance sheets, incurring losses of many billions. Following these events, stock markets world wide collapsed in 2008, leading to a financial crisis. Several banks, such as Lehmann Brothers and Bear Stearns defaulted, and the trust between banks almost completely disappeared, making it almost impossible for banks to borrow money from other banks. Many banks received enormous amounts of money from governments in order to survive.

Many people have blamed the mathematical models, and especially the Gaussian copula, for the credit crisis. The Gaussian copula had been widely used to value all sorts of credit derivatives, despite its limitations. When this model was first introduced by Li [Li00], he already stressed that one should be careful applying this model in practice, as the normal distribution is not always a suitable distribution. In practice such warnings were ignored on a large scale, and practitioners invented ways to deal with the flaws of the Gaussian copula, e.g. by the base correlation approach [MBAW04]. Base correlations, however, can only be obtained for a limited number of underlying baskets, which meant that many credit derivatives were valued based on ‘the wrong data’ obtained with ‘the wrong model’. Despite many people being aware of such limitations, the Gaussian copula still became the most popular model because of its simplicity and as it is easy to implement.
The credit crisis has shown that it is very important to correctly model loss process and dependence structures for (large) portfolios which bear credit risk. Furthermore, it became clear that many banks did not reserve enough money to account for credit losses. In this thesis we do not explicitly focus on the credit crisis, as many of the issues related to this crisis were outside the scope of the research underlying this thesis. However, in Chapters 2 and 3 we discuss two alternative ways of modeling the loss process, and analyzing some of its properties, first by modeling the loss process under the assumption of incomplete information, where the process driving the loss process is not observed. Secondly we focus on the asymptotic behavior of the loss process as the size of the portfolio increases. We derive characteristics of the path of the loss process, both as a large deviation principle and, for a special case, as the exact asymptotic behavior. The two modeling approaches address some of the shortcomings of current models, where full information is assumed or where the loss process is only considered at a fixed point in time. In Chapters 4 and 5 we present empirical analysis of pre-crisis CDO data, where the base correlation and other factor copula models are investigated. We show that the way base correlations are interpolated, can have a large influence, and in many cases substantial differences can be observed between the modeled and true values. Further we show that alternative factor copula models can do a reasonable task at fitting to pre-crisis data.

1.6 Outline

The remaining chapters in this thesis can be divided in two parts. The first part, Chapters 2 and 3, deals with two mainly mathematical topics, applied to credit risk. In the second part, Chapters 4 and 5, more practical results are discussed.

In Chapter 2 we model the number of defaults in a large credit portfolio as a Cox process where the intensity follows the square-root, or a Cox-Ingersoll-Ross process. We assume here that we cannot observe the Brownian motion that is driving the intensity process. Using filtering theory for point process observations we are able to derive explicit expressions for the conditional moment generating function of the loss counting process. We first derive partial differential equations for the evolution of the moment generating function between default times, which can be solved explicitly. Further we find the expression for this function at default times. By combining these results, and by assuming that the intensity initially has a gamma distribution, we are able to derive a recursive expression for the moment generating function.

In Chapter 3 we derive a number of asymptotic results for the loss process of a large credit portfolio. We consider a portfolio of $n$ independent and identically distributed obligors, and we model their default time and loss given default separately. Under mild restrictions we derive a large deviation principle for the path of the loss process on a finite time grid. This large deviation principle allows us to calculate exponential bounds on the probability that the path of the loss process is in certain sets. Thereafter we present an (easy-to-check) condition under which the result holds. In the second
part of the chapter, we derive the exact asymptotic behavior of the probability of the loss, or the increment of the loss, ever (on a countable time grid) exceeding a certain (time dependent) threshold. It turns out that the asymptotic behavior is completely determined by the ‘most likely’ time point. At the end of this chapter extensions of the presented results are discussed, where we consider a more diversified portfolio or dependence between the different obligors. Further we discuss how our results change if we consider a continuous time interval or a countable time grid, and the associated complications.

In Chapter 4 we analyze a large data set of market prices for CDO tranches, running from December 2004 up to November 2006, on iTraxx and CDX indices. Next to quotes for CDO tranches with five and ten year maturities, this data set contains CDS data for the underlying companies of the CDO tranches, and interest rates that are used for discounting payments. First a regression analysis is performed on the CDO quotes, to see up to which degree the levels of CDO tranche quotes can be explained by quotes for the index CDS on the same basket. Thereafter we calculate base correlations for all CDO tranche quotes, and we compute the correlation between the base correlations for all tranches, where it is observed that these correlations are in general very high. In the last part of the chapter we compare three different interpolation methodologies for the base correlation skew. We focus on the ability to price nonstandard CDO tranches, by pricing CDO tranches with five year maturities with the ten year skew, and vice versa. Further we price CDO tranches on the iTraxx index with the skew from the CDX index, and vice versa. The test results show that the most advanced interpolation method performs best, but the differences with the actual prices can still be significant.

In Chapter 5 we consider some of the alternative one-factor copula models from Section 1.4.2. We calibrate these models to market quotes from the same set of market data as in Chapter 4. Our main objective is to see how well the models are able to reproduce the market quotes. Further, we compare the performance of the models against each other. It turns out that the performance of the alternative factor copula models is of the same order. We conclude the chapter by investigating the models individually, where they are not only judged by there ability to match market quotes, but also by parameter stability, where unstable parameters can be an indication of over fitting to the data.