Anisotropic structural predictor in glassy materials

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There is growing evidence that relaxation in glassy materials, both spontaneous and externally driven, is mediated by localized soft spots. Recent progress made it possible to identify the soft spots inside glassy structures and to quantify their degree of softness. These softness measures, however, are typically scalars, not taking into account the tensorial, anisotropic nature of soft spots, which implies orientation-dependent coupling to external deformation. Here, we derive from first principles the linear response coupling between the local heat capacity of glasses, previously shown to provide a measure of glassy softness, and external deformation in different directions. We first show that this linear response quantity follows an anomalous, fat-tailed distribution related to the universal $\omega^\alpha$ density of states of quasilocalized, nonphononic excitations in glasses. We then construct a structural predictor as the product of the local heat capacity and its linear response to external deformation, and show that it offers an enhanced predictability of plastic rearrangements under deformation in different directions, compared to the purely scalar predictor.

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Introduction. At the heart of resolving the glass mystery resides the need to quantify the disordered structures inherently associated with glasses and to relate them to glass properties and dynamics, most notably spontaneous and driven structural relaxation [1,2]. Numerous attempts to address and meet this grand challenge have been made [3–20], aiming at defining most relevant here is the normalized local thermal energy [6], recently [21–24]. Among the structural predictors proposed, (with the exception of Refs. [15–18]), the LHC misses importantly test a theory that allows us to identify particularly soft spots have been related to quasilocalized, nonphononic excitations in glasses. These localized soft spots are the loci of glassy relaxation, and hence macroscopic theories of deformation and flow of glasses.

Recently accumulated evidence suggests that spatially localized soft spots are the loci of glassy relaxation, and hence are highly relevant for glass dynamics. These localized soft spots have been related to quasilocalized, nonphononic excitations in glasses [4–6], whose universal $\omega^\alpha$ density of states ($\omega$ is the vibrational frequency) has been also established recently [21–24]. Among the structural predictors proposed, most relevant here is the normalized local thermal energy [6], which quantifies the interparticle interaction contribution to the zero-temperature heat capacity, termed hereafter the local heat capacity (LHC) $c_\alpha$ ($\alpha$ is the interaction index).

The LHC $c_\alpha$ is a general (system- and model-independent), first-principles statistical mechanical quantity that reveals soft spots in glassy materials [6]. Yet, the LHC is a scalar that quantifies the resistance to motion in some unknown direction. That is, as previously proposed structural predictors in glasses (with the exception of Refs. [15–18]), the LHC misses important tensorial, anisotropic information about the coupling to deformation in a certain direction. For example, an extremely soft spot can be completely decoupled from external forces applied in a certain direction and hence irrelevant for the glass response in this direction.

In this Rapid Communication, we develop and quantitatively test a theory that allows us to identify particularly soft glassy structures, explicitly revealing their anisotropic nature and their intrinsic coupling to the direction of externally applied forces. The theory is developed in two steps; First, the linear response coupling of $c_\alpha$ to external deformation tensors $\mathcal{H}(\gamma)$, parametrized by a strain amplitude $\gamma$, is derived. The resulting quantity $dc_\alpha/d\gamma$ is shown to follow an anomalous, fat-tailed distribution related to the universal $\omega^\alpha$ density of states of quasilocalized, nonphononic excitations in glasses. Second, a structural predictor in the product form $c_\alpha d c_\alpha/d\gamma$ is physically motivated and shown to filter out soft spots that are not coupled to the external deformation of interest. Finally, a metric for quantifying the predictive power of structural predictors is proposed and extensive computer simulations are used to show that $c_\alpha d c_\alpha/d\gamma$ offers an enhanced predictability of plastic rearrangements under deformation in different directions, compared to the LHC $c_\alpha$ alone.

Linear response coupling of the LHC to external deformation. The starting point for our development is the zero-temperature local heat capacity [6,25]

$$c_\alpha = \frac{1}{k_B} \frac{\partial \langle \varphi_\alpha \rangle}{\partial T} \bigg|_{T=0}, \tag{1}$$

where $\langle \varphi_\alpha \rangle = \int \varphi_\alpha(x) \exp \left( \frac{-U(x)}{k_B T} \right) d^d x / \int \exp \left( -\frac{U(x)}{k_B T} \right) d^d x$, $x$ is a vector of the positions of all particles, $\varphi_\alpha$ is the potential energy of any pair of interacting particles, $U(x) = \sum_\alpha \varphi_\alpha$, and $k_B$ is Boltzmann’s constant. The sum over the LHC, $1/k_B \sum_\alpha c_\alpha = \partial U/\partial T|_{T=0}$, is the thermodynamic, zero-temperature heat capacity $C_V$.

An analytic low-temperature expansion of $\langle \varphi_\alpha \rangle$, allows us to explicitly calculate $c_\alpha$ [6], which takes the form $c_\alpha = \varphi_\alpha^\alpha : \mathcal{M}^{-1} - f_\alpha : \mathcal{M}^{-1} : \mathcal{U}^\alpha : \mathcal{M}^{-1}$, where $\cdot$ denotes a contraction over a single index of the relevant tensors and $: \cdot$ over two indices. A prime, here and hereafter, denotes a partial derivative with respect to $x$, $f_\alpha = \varphi_\alpha^\alpha$ are frustration-induced internal forces, and $\mathcal{M} \equiv \partial^2 U/\partial x^\alpha \partial x^\beta$ is the Hessian matrix whose
The LHC $c_a$ is far more sensitive to soft quasilocalized modes than to extended phonons, i.e., it filters out the effect of phonons, and is dominated by its second contribution, which is proportional to the frustration-induced internal forces $f_a$ and to $(M^{-1})^2$ (scalingwise) [6]. The spatial distribution of $c_a$ reveals soft spots (see Fig. 1) which are highly correlated with the loci of plastic rearrangements under external driving (marked by the superimposed triangles, to be further discussed below).

The soft spots are characterized by a degree of softness determined by the typical magnitude of $c_a$, $|c_a|$, in its vicinity.

$$dc_a/d\gamma \simeq -U'' : M^{-1} : (U'' : M^{-1} : U' : M^{-1}) \cdot (f_a : M^{-1})$$

$$- (U'' : M^{-1} : U'' : M^{-1}) : (M^{-1} : U'' : M^{-1} : f_a).$$

Equation (2), valid for the largest values of $dc_a/d\gamma$, shows that these emerge from a fourth power of $M^{-1} \sim \omega^{-2}$ (scalingwise), coupled to the energy anharmonicity tensor $U''$, to the internal force vector $f_a$ and to the mismatch force vector $U''$. Note that similarly to $c_a$ (see the expression above), the existence of frustration-induced internal forces $f_a$—an intrinsic signature of glassy disorder—is essential for the emergence of abnormally large values of $dc_a/d\gamma$. While the expression for $dc_a/d\gamma$ [in Eq. (2) or its exact counterpart in Ref. [27]] is universal, the specific information regarding
the applied deformation $\mathcal{H}(\gamma)$ for which the linear response is calculated is encapsulated in the partial derivative $\delta/\delta \gamma$ [27], here through the mismatch force $\mathcal{U}$. The validity of the analytic expression for $dc_a/d\gamma$ has been directly verified using numerical simulations [27].

**Universal anomalous statistics.** To further establish the linear responses $dc_a/d\gamma$ as a fundamental physical quantity that is intrinsically related to quasilocalized soft glassy modes, we consider next the large tail of its statistical distribution. The latter can be predicted based on Eq. (2) and the universal DOS of soft glassy modes, $\mathcal{D}(\omega) \sim \omega^4$. Considering the eigenrepresentation of $dc_a/d\gamma$ and invoking the same considerations as in Ref. [6], one can show that objects such as those appearing on the right-hand side of Eq. (2) are far more sensitive to quasilocalized glassy modes than to extended phonons as $\omega \to 0$ and that the $\omega$ dependence emerges only from $\mathcal{M}^{-1} \sim \omega^{-2}$. Consequently, we have $dc_a/d\gamma \sim \omega^{-8}$ and $p(dc_a/d\gamma)$ is predicted to satisfy $p(dc_a/d\gamma) = D_G(\omega)d\omega/d(dc_a/d\gamma) \sim (dc_a/d\gamma)^{-13/8}$ in the large $dc_a/d\gamma$ limit.

To test this prediction, we performed extensive numerical simulations of a conventional computer glass former for both simple and pure shear [27] and extracted the statistics of $dc_a/d\gamma$. The results are presented in Fig. 2(a) and are in great quantitative agreement with the theoretical prediction. We thus conclude that $dc_a/d\gamma$ attains anomalously large values described by universal fat-tailed statistics related to the universal DOS of soft quasilocalized glassy modes, $\mathcal{D}(\omega) \sim \omega^4$. The relation between $dc_a/d\gamma$ and quasilocalized modes suggests that the spatial distribution of the former features localized structures, which will be used next to construct a generalized structural predictor in glasses.

**A structural predictor.** We have at hand two quantities that appear to capture the essential physical properties of soft spots in glassy materials. First, the LHC $c_a$ is a signed scalar whose magnitude $|c_a|$ quantifies the degree of softness of soft spots, i.e., it provides a measure for how small the activation barrier for irreversible rearrangements is in some unknown direction. Second, the linear response coupling of the LHC to deformation in a certain direction $dc_a/d\gamma$ is a signed quantity that provides a measure for the degree by which externally applied forces affect the activation barrier in the direction in which they are applied. How do the two quantities combine to form a generalized anisotropic structural predictor in glasses? As both $c_a$ and $dc_a/d\gamma$ are signed quantities and as both are predicted to attain anomalously large values at the loci of soft quasilocalized modes, we expect large positive values of the product $c_a dc_a/d\gamma$ to single out a subpopulation of the soft spots (previously defined by $|c_a|$) that is most relevant for the imposed deformation in a certain direction. Consequently, we propose it as a generalized anisotropic structural predictor in glasses.

As a first test of this idea, we invoke it to predict the large tail statistics of $c_a dc_a/d\gamma$. As we have $c_a \sim \omega^{-4}$ and $dc_a/d\gamma \sim \omega^{-8}$ in the small $\omega$ limit, the spatial overlap prediction implies $c_a dc_a/d\gamma \sim \omega^{-12}$, which leads to $p(c_a dc_a/d\gamma) \sim (c_a dc_a/d\gamma)^{-17/12}$ in the large $dc_a/d\gamma$ limit [using $\mathcal{D}_G(\omega) \sim \omega^4$]. This prediction is quantitatively verified in Fig. 2(b) for both simple and pure shear, lending strong support to the idea that the product $c_a dc_a/d\gamma$ indeed characterizes well-defined soft spots.

We next turn to the spatial properties of $c_a dc_a/d\gamma$, and first consider the glass realization shown in Fig. 1, which is shown again in Fig. 3(a). The product $c_a dc_a/d\gamma$ under both simple and pure shear in the positive direction is shown in Figs. 3(b) and 3(c). Here, black and red correspond respectively to positive and negative values of $c_a dc_a/d\gamma$ (the thickness of the lines quantifies their magnitude). Two major observations can be made: (i) Soft spots that are revealed by $c_a dc_a/d\gamma$ indeed overlap those revealed by $c_a$, alone, and in fact they are more pronounced. (ii) There exist two subspecies of soft spots, one that is positively coupled to deformation in a given direction (black) and one that is negatively coupled to it (red), and these subspecies depend on the direction of the deformation [cf. Figs. 3(b) and 3(c)]. Consequently, the product $c_a dc_a/d\gamma$ reveals orientation-dependent soft spots that offer enhanced predictive power compared to scalar indicators, which will be tested next.

**Quantifying the predictive power of the structural predictor.** We first demonstrate the predictive power of $c_a dc_a/d\gamma$ using the example in Fig. 3; we expect plastic events to occur at one of the softest black (red) spots in Fig. 3(b) when the glass undergoes simple shear deformation in the positive (negative) directions, and similarly for Fig. 3(c) in relation to pure shear in the positive and negative directions. This expectation is fully supported by the results of AQS deformation simulations [27] in the four different directions, as shown by the triangles in Figs. 3(b) and 3(c).

To systematically quantify the predictive power of the proposed structural predictor, we performed extensive computer simulations of a large ensemble of glass realizations deformed in the four different directions and tracked the location of the first plastic event in each one of them. To quantify the degree of predictability, we used the following metric: The system is divided into bins of linear size $\xi = 5$ particle diameters, comparable to the localization length of soft quasilocalized modes [21,23], and assigned a value obtained from the average of the structural indicator inside the bin and all of its neighboring bins (implying that the actual coarse-graining length is in fact larger than $\xi$). A plastic event is assigned a rank $\lambda$ that corresponds to the fraction of the bins with a higher value than that of the bin in which it
In Fig.4(a) we consider simple shear in the positive direction, with respect to plastic events under AQS deformation through the absolute value of the LHC $\alpha/\gamma$ (the quantitative aspect of soft spots inside disordered glass states). The transparent analytic structure of $c_a \, dc_a/d\gamma$ and its relation to quasilocalized soft excitations [6], allows us to gain physical insight into the origin of localized soft spots in glasses and their universal statistical properties. Our structural predictor involves only snapshots of nondeformed glasses and the interparticle interactions. The emerging properties of soft spots strongly echo the original Falk-Langer concept of shear transformation zones (STZs) [32] and should help advancing the development of predictive elastoplastic models. Finally, we believe that our results offer a tool to probe the basic physics of glasses including structural relaxation, aging, memory effects, and nonlinear yielding transitions.

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FIG. 3. (a) The same as Fig. 1. (b) $c_a \, dc_a/d\gamma$ (black (red) corresponds to positive (negative) values, and the line thickness represents the magnitude) for simple shear in the positive direction [the right (left) triangles correspond to the first plastic events under positive (negative) simple shear AQS deformation]. (c) The same as (b), but for pure shear in the positive direction [the up (down) triangles correspond to the first plastic events under positive (negative) pure shear AQS deformation].

FIG. 4. Quantifying the predictive power of structural indicators with respect to plastic events under AQS deformation through the function $C(\lambda)$ (see text for definitions), where $C(\lambda) = \theta(\lambda)$ (Heaviside step function) corresponds to perfect predictability and $C(\lambda) = \lambda$ to no predictability. (a) Results for positive values (negative ones are set to zero) of $c_a \, dc_a/d\gamma$ (diamonds), for negative values (positive ones are set to zero) of $|c_a|$ (squares) and for $|c_a|$ (circles) under simple shear AQS deformation in the **positive** direction. (b) The same as (a), but under simple shear AQS deformation in the **negative** direction.