



UvA-DARE (Digital Academic Repository)

A remark on deformations of Hurwitz Frobenius manifolds

Buryak, A.; Shadrin, S.

DOI

[10.1007/s11005-010-0410-8](https://doi.org/10.1007/s11005-010-0410-8)

Publication date

2010

Document Version

Final published version

Published in

Letters in Mathematical Physics

[Link to publication](#)

Citation for published version (APA):

Buryak, A., & Shadrin, S. (2010). A remark on deformations of Hurwitz Frobenius manifolds. *Letters in Mathematical Physics*, 93(3), 243-252. <https://doi.org/10.1007/s11005-010-0410-8>

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

A Remark on Deformations of Hurwitz Frobenius Manifolds

ALEXANDR BURYAK^{1,2} and SERGEY SHADRIN^{1,3}

¹*Department of Mathematics, University of Amsterdam, P. O. Box 94248, 1090 GE Amsterdam, The Netherlands. e-mail: a.y.buryak@uva.nl; s.shadrin@uva.nl*

²*Department of Mathematics, Moscow State University, Leninskie gory, GSP-2, Moscow 19992, Russia. e-mail: buryaksh@mail.ru*

³*Department of Mathematics, Institute of System Research, Nakhimovsky prospekt 36-1, Moscow 117218, Russia. e-mail: shadrin@mccme.ru*

Received: 4 January 2010 / Revised: 24 June 2010 / Accepted: 29 June 2010

Published online: 10 July 2010 – © The Author(s) 2010. This article is published with open access at Springerlink.com

Abstract. In this note, we use the formalism of multi-KP hierarchies in order to give some general formulas for infinitesimal deformations of solutions of the Darboux–Egoroff system. As an application, we explain how Shramchenko’s deformations of Frobenius manifold structures on Hurwitz spaces fit into the general formalism of Givental–van de Leur twisted loop group action on the space of semi-simple Frobenius manifolds.

Mathematics Subject Classification (2000). 37K10 (53D45, 37K30).

Keywords. Frobenius manifold, Hurwitz space, Darboux–Egoroff equations, multi-KP hierarchy, deformations.

1. Introduction

In [1], Dubrovin has associated a structure of Frobenius manifolds to an arbitrary Hurwitz space of meromorphic functions on Riemann surfaces of genus g with simple finite critical values and a prescribed ramification indices over infinity. Shramchenko [10] observed that the structure of Frobenius manifold associated to a Hurwitz space can be included into a family of Frobenius manifold structures parametrized by a symmetric $g \times g$ matrix. There is a beautiful description of this deformation in terms of the values of holomorphic differentials at the critical points and their B -periods matrix.

Meanwhile, Givental [3,4] and, independently, van de Leur [11] have constructed an action of the twisted loop group of GL_n on the space of semi-simple Frobenius

A. Buryak was partially supported by the Grants RFBR-07-01-00593, NSH-709.2008.1. Both A. Buryak and S. Shadrin were partly supported by the Vidi grant of NWO.

manifolds. Moreover, Givental has shown that this group acts transitively on the space semi-simple Frobenius manifolds. This two constructions of the group action were identified in [2] via an identification of the formulas of Y.-P. Lee for the infinitesimal Givental action [7] with the tangent van de Leur action computed in [2] in terms of twisted wave functions of multi-component KP hierarchies.

In this paper, we extend in some way the formulas for the tangent van de Leur action computed in [2]. Namely, we express infinitesimal Lie algebra action on the space of solutions of the Darboux–Egoroff system in terms of the twisted wave functions of multi-component KP. In principle, these formulas are of independent interest. In particular, they allow us to fit Shramchenko’s deformations into a general Givental–van de Leur scheme. In particular, it is interesting to trace a correspondence between geometric ingredients of Shramchenko’s deformation and particular wave functions of the multi-component KP hierarchy that is associated to an arbitrary solution of the Darboux–Egoroff system in van de Leur’s approach.

1.1. ORGANIZATION OF THE PAPER

In Section 2, we recall the constructions of Hurwitz Frobenius manifolds and their deformations. In Section 3, we recall the van de Leur approach to Frobenius manifolds and use it in order to derive explicit formulas for the Givental–van de Leur infinitesimal deformations of solutions of the Darboux–Egoroff equations. In Section 4, we discuss the simplest possible example of such infinitesimal deformations that can be integrated explicitly and show that it is exactly the way one could obtain Shramchenko’s deformations of Hurwitz Frobenius manifolds.

2. Frobenius Structures Associated with Hurwitz Spaces

2.1. DARBOUX–EGOROFF EQUATIONS

In this paper, we consider only semi-simple Frobenius manifolds. There is a way to encode the structure of a semi-simple Frobenius manifold in canonical coordinates as a solution of a system of PDEs that is called Darboux–Egoroff equations [1].

Let $n \geq 1$. We consider functions $\gamma_{ij} = \gamma_{ji}$, $i, j = 1, \dots, n$, $i \neq j$, in variables u_1, \dots, u_n . The Darboux–Egoroff equations read:

$$\begin{aligned} \frac{\partial \gamma_{ij}}{\partial u_k} &= \gamma_{ik} \gamma_{kj}, & i \neq j \neq k \neq i \\ \sum_{k=1}^n \frac{\partial \gamma_{ij}}{\partial u_k} &= 0 & i \neq j \end{aligned} \tag{1}$$

Is it convenient to collect γ_{ij} into a symmetric matrix with the diagonal terms that can be either equal to 0 or just arbitrary. We introduce a special notation for that. Let M be a symmetric matrix. By $n.d.M$ we denote the same matrix with non-specified diagonal terms.

Remark 2.1. A Frobenius manifold in the sense of [1] must be equipped with an Euler vector field. We omit this requirement, since it is not well compatible with the deformation technique that we use this paper.

2.2. HURWITZ SPACES

We fix some integer numbers $d_1, \dots, d_m > 0$ and $g \geq 0$. Let \mathcal{H} be the space of the equivalence classes of the tuples of data $(C_g, \{a_i, b_i\}_{i=1}^g, f: C_g \rightarrow \mathbb{CP}^1)$, where C_g is a Riemann surface of genus g , $\{a_i, b_i\}_{i=1}^g$ is a choice of the canonical basis of cycles on C_g , and $f: C_g \rightarrow \mathbb{CP}^1$ is a meromorphic function of degree $d := \sum_{i=1}^m d_i$ with exactly m poles of multiplicity d_1, \dots, d_m and $n := 2g + d + m - 2$ simple critical points $x_1, \dots, x_n \in C_g$. In addition, we choose local parameters z_1, \dots, z_n at the points $x_1, \dots, x_n \in C_g$ such that $f = z_i^2$ in a neighbourhood of x_i . Two tuples of this data are equivalent if there is a biholomorphic map between two source curves that preserves the rest of the data.

The critical values of meromorphic functions $u_i := f(x_i)$, $i = 1, \dots, n$, are local coordinates on the space \mathcal{H} .

We recall the Kokotov–Korotkin construction [5] of a solution of the Darboux–Egoroff equations. Let $W(P, Q)$ be the canonical meromorphic bidifferential on a Riemann surface C_g . That is, $W(P, Q)$ is specified by the following properties: it is symmetric, it has a quadratic pole on the diagonal $P = Q$ with biresidue 1, and its a -periods with respect to both variables vanish. Then the functions

$$\gamma_{ij} := \frac{1}{2} W(x_i, x_j) := \frac{1}{2} \frac{W(P, Q)}{dz_i(P) dz_j(Q)} \Big|_{P=x_i, Q=x_j} \tag{2}$$

in variables u_1, \dots, u_n satisfy the Darboux–Egoroff equations.

2.3. SHRAMCHENKO’S DEFORMATIONS

Let ω_i , $i = 1, \dots, g$, be the basis of holomorphic differentials on C_g normalized by $\int_{a_i} \omega_j = \delta_{ij}$. Denote by ω the matrix of the values of ω_i at critical points:

$$\omega_{ij} := \omega_i(x_j) := \frac{\omega_i(P)}{dz_j(P)} \Big|_{P=x_j} \tag{3}$$

Denote by B the matrix of b -periods of these differentials divided by $\pi\sqrt{-1}$, $B_{ij} := \frac{1}{\pi\sqrt{-1}} \int_{b_i} \omega_j$. Let M be an arbitrary $g \times g$ symmetric matrix such that $B + M$ is non-degenerate. Shramchenko’s [10] deformations of Hurwitz Frobenius manifolds are given by the formula

$$\text{n.d.}\gamma(M) := \text{n.d.} \left(\gamma - \omega^t (B + M)^{-1} \omega \right). \tag{4}$$

Here $\text{n.d.}\gamma$ is given by Equation (2). Shramchenko proved that $\gamma_{ij}(M)$ are solutions of the Darboux–Egoroff equations in the variables u_1, \dots, u_n in the domain

$\det(B + M) \neq 0$. Observe that $\text{n.d.}\gamma(M)$ tends to $\text{n.d.}\gamma$ when $(B + M)^{-1}$ tends to zero.

The proof that $\text{n.d.}\gamma(M)$ is a solution of the Darboux–Egoroff equations is based on Rauch variational formula and its corollaries:

$$\frac{\partial W(P, Q)}{\partial u_j} = \frac{1}{2} W(P, x_j) W(Q, x_j), \tag{5}$$

$$\frac{\partial \omega_i(P)}{\partial u_j} = \frac{1}{2} \omega_i(x_j) W(P, x_j), \tag{6}$$

$$\frac{\partial B_{kl}}{\partial u_j} = \omega_k(x_j) \omega_l(x_j), \tag{7}$$

where evaluation of differentials at particular points is defined in (2) and (3).

3. Van de Leur’s Formalism for Frobenius Manifolds

In this section, we explain van de Leur’s construction of a Frobenius structure associated with a point in the isotropic semi-infinite Grassmannian.

3.1. BASIC DEFINITIONS

Let $V = \langle e_1, \dots, e_n \rangle$ be an n -dimensional vector space over \mathbb{C} . Let z be a formal variable. We denote by \mathcal{V} the vector space $\Lambda^{\infty/2}(V \otimes \mathbb{C}[[z^{-1}, z]])$ spanned by the semi-infinite wedge products

$$\omega = (e_{i_1} z^{d_1}) \wedge (e_{i_2} z^{d_2}) \wedge (e_{i_3} z^{d_3}) \wedge \dots$$

such that the tail of ω coincides with the tail of vacuum vector

$$|0\rangle := (e_1 z^0) \wedge \dots \wedge (e_n z^0) \wedge (e_1 z^1) \wedge \dots \wedge (e_n z^1) \wedge \dots$$

By tail of ω we call another basis vector in \mathcal{V} that is obtained from ω by removing the first few factors in the wedge product.

Consider a matrix series $A(z) \in \text{End}(V) \otimes \mathbb{C}[[z^{-1}, z]]$ such that $A^t(-z)A(z) = \text{Id} \cdot z^0$ (it is better to imagine it as a finite product of invertible matrix series in $\text{End}(V) \otimes \mathbb{C}[[z]]$ and $\text{End}(V) \otimes \mathbb{C}[[z^{-1}]]$ with the unit constant term satisfying the same symplectic condition).

Let α_i be a local Lie algebra element whose action on $V \otimes \mathbb{C}[[z^{-1}, z]]$ is defined by

$$\alpha_i(e_j z^d) := \begin{cases} e_j z^{d+1} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and is expanded to \mathcal{V} by the Leibnitz rule.

All basic objects that we are going to consider are some matrix elements of the operator

$$\mathcal{A} := \exp\left(\sum_{i=1}^n \alpha_i u_i\right) A(z),$$

where u_1, \dots, u_n are formal variables.

We denote by $\gamma_{ij} = \gamma_{ij}(A)$, $i, j = 1, \dots, n$, $i \neq j$, the following matrix elements of \mathcal{A} :

$$\gamma_{ij} := \pm \frac{\langle |0\rangle | \mathcal{A} | (e_i z^{-1}) \wedge \partial_{(e_j z^0)} |0\rangle \rangle}{\langle |0\rangle | \mathcal{A} ||0\rangle \rangle}.$$

(the vector $(e_i z^{-1}) \wedge \partial_{(e_j z^0)} |0\rangle$ is obtained, up to a sign, from the vacuum vector $|0\rangle$ by the replacement of $(e_j z^0)$ by $(e_i z^{-1})$).

We denote by $(\Psi_d)_{ij} = (\Psi_d)_{ij}(A)$, $i, j = 1, \dots, n$, the following matrix elements of \mathcal{A} :

$$(\Psi_d)_{ij} := \frac{\langle (e_j z^{-1}) \wedge |0\rangle | \mathcal{A} | (e_i z^{-1-d}) \wedge |0\rangle \rangle}{\langle |0\rangle | \mathcal{A} ||0\rangle \rangle}.$$

These matrices can be arranged into a generating series $\Psi(z) := \sum_{d=0}^\infty z^d \Psi_d$ that would be a wave function of multi-KP hierarchy multiplied by $A(z)$ from the right. The property $A^t(-z)A(z) = \text{Id} \cdot z^0$ imply that $\Psi^t(-z)\Psi(z) = \text{Id} \cdot z^0$.

Van de Leur has shown in [11] that a formal locally semi-simple Frobenius structure can expressed in terms of the matrices γ and Ψ_d , $d \geq 0$. In particular, γ_{ij} is a solution of the Darboux-Egoroff system; u_1, \dots, u_n are canonical coordinates; $\Psi'_0 \Psi_1 \mathbf{1}$ is a column of flat coordinates (here $\mathbf{1}$ is the column of units); and $(1/2) \cdot \mathbf{1}^t \Psi'_0 (-\Psi_3 \Psi'_0 + \Psi_2 \Psi'_1) \Psi_0 \mathbf{1}$ is the prepotential of a Frobenius manifold.

3.2. INFINITESIMAL DEFORMATIONS

From the previous section we see that the right multiplication of a matrix $A(z)$ by a matrix $B(z)$ defines the action on the functions $\gamma(A)$, $\Psi_d(A)$ of the groups of matrices $B(z) \in \text{Hom}(V, V) \otimes \mathbb{C}[[z]]$, $B^t(-z)B(z) = \text{Id} \cdot z^0$, and $B(z^{-1}) \in \text{Hom}(V, V) \otimes \mathbb{C}[[z^{-1}]]$, $B^t(-z^{-1})B(z^{-1}) = \text{Id} \cdot z^0$, both with the unit constant term. This group action is crucially important, see, e.g., [2, Introduction] for a list of particular applications and references.

We discuss the corresponding Lie algebra action. Let $k \geq 0$ and $\ell > 0$. Let matrices r and s be symmetric for odd ℓ and skewsymmetric for even ℓ . It is proven in [2] that

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \Psi_k(A \exp \epsilon(rz^{-\ell})) &= \Psi_{\ell+k}(A)r - \sum_{p=1}^{\ell} \sum_{q=0}^{\ell-p} (-1)^{\ell-p-q} \Psi_q(A)r \Psi_{\ell-p-q}^t(A) \Psi_{p+k}(A); \end{aligned} \tag{8}$$

$$\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \Psi_k(A \exp \epsilon(sz^{\ell})) = \begin{cases} \Psi_{k-\ell}(A)s, & \ell \leq k; \\ 0, & \ell > k \end{cases} \tag{9}$$

This allows to compute the action of this Lie algebra on the prepotential of Frobenius manifolds in flat coordinates, since both the prepotential and the flat coordinates are expressed in terms of Ψ_d , $d \geq 0$.

In order to deal with Hurwitz Frobenius manifolds, we need some formulas for the Lie algebra action on $\text{n.d.}\gamma(A)$.

THEOREM 3.1. *Let $\ell \geq 0$. Let matrices r and s be symmetric for odd ℓ and skew-symmetric for even ℓ . We have:*

$$\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \text{n.d.}\gamma(A \exp \epsilon(rz^{-\ell})) = \text{n.d.} \sum_{i+j=\ell-1} (-1)^{j-1} \Psi_i(A)r \Psi_j^t(A) \tag{10}$$

$$\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \text{n.d.}\gamma(A \exp \epsilon(sz^{\ell})) = 0. \tag{11}$$

Proof. This theorem is an easy consequence of formulas (8) and (9). Indeed, there is a relation between Ψ_d , $d \geq 0$ and $\text{n.d.}\gamma$ that is proven in [11]. For any $d \geq 0$, $k = 1, \dots, n$, we have:

$$\frac{\partial}{\partial u_k} \Psi_d = E_{kk} \Psi_{d-1} + [\text{n.d.}\gamma, E_{kk}] \Psi_d \tag{12}$$

Here and below we assume that $\Psi_{-1} = 0$ and we use γ with arbitrary diagonal terms since they disappear in the commutator with E_{kk} . By E_{kk} we denote the matrix unit, that is $(E_{kk})_{ij} := \delta_{ik} \delta_{jk}$. This formula gives an expression for all elements of $\text{n.d.}\gamma$ in terms of Ψ_0 and its derivatives.

From (12) it follows that

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \frac{\partial}{\partial u_k} \Psi_0(A \exp \epsilon(rz^{-\ell})) &= \left[\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \gamma(A \exp \epsilon(rz^{-\ell})), E_{kk} \right] \Psi_0(A) \\ &+ [\gamma(A), E_{kk}] \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \Psi_0(A \exp \epsilon(rz^{-\ell})). \end{aligned}$$

Hence, using (8), we obtain the following equation:

$$\begin{aligned} & \frac{\partial}{\partial u_k} \left(\Psi_{\ell r} + \sum_{q=0}^{\ell-1} (-1)^{\ell-q} \Psi_{qr} \Psi_{\ell-q}^t \Psi_0 \right) \\ &= [\delta\gamma, E_{kk}] \Psi_0 + [\gamma, E_{kk}] \left(\Psi_{\ell r} + \sum_{q=0}^{\ell-1} (-1)^{\ell-q} \Psi_{qr} \Psi_{\ell-q}^t \Psi_0 \right), \end{aligned}$$

where we denote $\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \gamma(A \exp \epsilon(rz^{-\ell}))$ by $\delta\gamma$. Using Equation (12) and $\Psi'(-z)\Psi(z) = \text{Id} \cdot z^0$, we see that

$$[\delta\gamma, E_{kk}] \Psi_0 = \left[E_{kk}, \sum_{i=0}^{\ell-1} (-1)^i \Psi_{\ell-1-i} r \Psi_i^t \right] \Psi_0.$$

Since Ψ_0 is invertible, $\Psi_0 \Psi_0^t = \text{Id}$, we obtain Equation (10). Equation (11) can be proven in the same way, but in fact it is obvious from the definition of γ . \square

EXAMPLE 3.2. The simplest non-trivial deformation would be by an element rz^{-1} , where r is an arbitrary symmetric matrix. Denote by $\delta_r \gamma$ and $\delta_r \Psi_d$ the corresponding infinitesimal deformations. We have the following system of equations:

$$\begin{aligned} \text{n.d.} \delta_r \gamma &= -\text{n.d.} \Psi_0 r \Psi_0^t \\ \delta_r \Psi_0 &= \Psi_1 r - \Psi_0 r \Psi_0^t \Psi_1 \\ \delta_r \Psi_1 &= \Psi_2 r - \Psi_0 r \Psi_0^t \Psi_2 \\ &\text{and so on.} \end{aligned} \tag{13}$$

4. Special Deformations

Shramchenko’s deformations of Hurwitz Frobenius manifolds discussed in Section 2.3 fits into a special case of Example 3.2 that can be integrated explicitly.

4.1. THE INPUT

Consider a symmetric $n \times n$ matrix $\text{n.d.} \gamma$ whose elements are functions in u_1, \dots, u_n . Let $\text{n.d.} \gamma$ be a solution of the Darboux–Egoroff equations (1). There are two matrix-valued functions with clear geometric meanings associated with the Frobenius structures corresponding to $\text{n.d.} \gamma$.

First, there is a solution of the commutativity equations [8], which is a symmetric $n \times n$ matrix $C = C(u_1, \dots, u_n)$ such that $dC \wedge dC = 0$. In terms of multi-KP tau-functions, $C = \Psi_0^t \Psi_1$.

Second, one can consider Ψ_0 itself. In geometric terms Ψ_0 is defined by the equation $dC = \Psi_0^t \cdot \text{diag}(du_1, \dots, du_n) \cdot \Psi_0$. That is, it is the matrix that relates

the canonical and the flat coordinates for the underlying Frobenius structures, see [1,11]. An alternative way to define Ψ_0 is the following. Consider the system of equations (it is equivalent to Equation (12) with $d = 0$):

$$\frac{\partial(\Psi_0)_{ij}}{\partial u_k} = \gamma_{ik}(\Psi_0)_{kj}, \quad i \neq k, \tag{14}$$

$$\sum_{k=1}^n \frac{\partial(\Psi_0)_{ij}}{\partial u_k} = 0. \tag{15}$$

Compatibility of this system of equations follows from the Darboux–Egoroff equations for $\text{n.d.}\gamma$. This system of equations implies that $\partial(\Psi_0^t \Psi_0)/\partial u_k = 0$, $k = 1, \dots, n$, and Ψ_0 that we need is a particular solution of this system of equations such that $\Psi_0^t \Psi_0 = \text{id}$.

4.2. SPECIAL DEFORMATIONS

We consider a distribution of vector subspaces in the tangent bundle of the space of the solutions of the Darboux–Egoroff equations. It is given by the Givental–van de Leur tangent vectors of the type (13) described in Example 3.2. It is easy to see that, roughly speaking, a deformation of a particular solution of the Darboux–Egoroff equations is given by an ordinary differential equations of the infinite order.

However, there is a special class of infinitesimal deformations that can be reduced to a finite order ODEs. We fix a positive integer $g \leq n/2$. Let D be a $g \times n$ constant matrix of rank g such that $DD^t = 0$. Let us consider the distribution in the tangent bundle of the moduli space of the solutions of the Darboux–Egoroff equations given by the Givental–van de Leur tangent vectors of the type (13) described in Example 3.2 with the matrix r that can be represented as $r = D^t M D$, where M is an arbitrary symmetric $g \times g$ matrix. In that case, Equation (13) can be reduced to an ODE of finite order.

PROPOSITION 4.1 *Equation (13) for the matrix $r = D^t M D$ implies the following system of ODEs for $\text{n.d.}\gamma$, $\omega := D\Psi_0^t$, and $B := D C D^t$:*

$$\begin{aligned} \text{n.d.}\delta_M \gamma &= -\text{n.d.}\omega^t M \omega; \\ \delta_M \omega &= -B M \omega; \\ \delta_M B &= -B M B. \end{aligned} \tag{16}$$

Proof Direct computation. □

In order to use this proposition for a particular $\text{n.d.}\gamma$ without going back to the full multi-KP framework, we need an independent definitions of ω and B in terms of γ and D . We define ω as a $g \times n$ -matrix-valued solutions of the equation

$$d\omega = \omega \cdot [\text{diag}(du_1, \dots, du_n), \text{n.d.}\gamma] \tag{17}$$

with the constant term $\omega|_{u=0} = D\Psi_0^t|_{u=0}$. We define B to be a $g \times g$ -matrix-valued solution of the equation

$$dB = \omega \cdot \text{diag}(du_1, \dots, du_n) \cdot \omega^t \tag{18}$$

with the constant term $B|_{u=0} = DC D^t|_{u=0}$.

Equations (16) can be integrated explicitly in the case when M is a constant matrix independent of n.d. γ , ω , and B . Indeed, let us define n.d. $\gamma(\epsilon)$, $\omega(\epsilon)$, and $B(\epsilon)$ by the following formulas:

$$\begin{aligned} \text{n.d.}\gamma(\epsilon) &:= \text{n.d.}\gamma - \text{n.d.}\omega^t \epsilon M (1 + \epsilon BM)^{-1} \omega; \\ \omega(\epsilon) &= (1 + \epsilon BM)^{-1} \omega; \\ B(\epsilon) &= (1 + \epsilon BM)^{-1} B. \end{aligned} \tag{19}$$

(these formulas are defined in the domain where $(1 + \epsilon BM)$ is invertible).

PROPOSITION 4.2 *The matrices n.d. $\gamma(\epsilon)$, $\omega(\epsilon)$, and $B(\epsilon)$ satisfy Equations (17) and (18) for any $\epsilon \geq 0$. They integrate the constant vector field determined by the matrix M , that is,*

$$\begin{aligned} \text{n.d.} \frac{\partial \gamma(\epsilon)}{\partial \epsilon} &= -\text{n.d.}\omega(\epsilon)^t M \omega(\epsilon); & \text{n.d.}\gamma(\epsilon)|_{\epsilon=0} &= \text{n.d.}\gamma; \\ \frac{\partial \omega(\epsilon)}{\partial \epsilon} &= -B(\epsilon) M \omega(\epsilon); & \omega(\epsilon)|_{\epsilon=0} &= \omega; \\ \frac{\partial B(\epsilon)}{\partial \epsilon} &= -B(\epsilon) M B(\epsilon); & B(\epsilon)|_{\epsilon=0} &= B; \end{aligned}$$

Proof Direct computation. □

4.3. SHRAMCHENKO'S FORMULAS

In this context, Shramchenko's formulas are a version of formulas (19) for $\epsilon = 1$, with some appropriate changes. Let us use n.d. γ , ω and B defined in Section 2.3. Equations (5)–(7) imply Darboux–Egoroff equations for n.d. γ and Equations (17)–(18). Therefore, we have a system suitable for deformation given by (16) (the initial conditions for ω and B depend on the choice of a particular point of a formal expansion). Indeed, let us substitute $\epsilon = 1$ in Equation (19). We have:

$$\text{n.d.}\gamma(\epsilon)|_{\epsilon=1} = \text{n.d.}\gamma(\epsilon)|_{\epsilon=0} - \text{n.d.}\omega^t (M^{-1} + B)^{-1} \omega.$$

If we change the notations in order to replace M^{-1} with M , we obtain exactly formula (4).

Remark 4.3. One could obtain the same solution of the Darboux-Egoroff equations from a special deformation of the following triple: n.d. $\tilde{\gamma} := \gamma - \omega^t B \omega$, $\tilde{\omega} := B^{-1} \omega$, and $\tilde{B} := B^{-1}$. In that case some formulas would look a bit simpler.

Remark 4.4. Deformations of “real doubles” [9] of Hurwitz Frobenius manifolds fit into exactly the same scheme as we discuss in Section 4.2.

Moreover, in our argument there is still some freedom for the choice of the matrix D . For example, one can look for the deformations corresponding to a matrix D represented as $D := UD_0$, where U is an arbitrary $g \times g$ matrix and D_0 is some fixed $g \times n$ matrix that satisfies $D_0 D_0^t = 0$. It would give us a family of special deformations parametrized by two matrices, and the deformations of Hurwitz Frobenius manifolds considered in [6, Remark 4] fit into this scheme after a change of parametrization.

Acknowledgements

The authors are grateful to J. van de Leur and H. Posthuma for useful remarks and discussions and to an anonymous referee for a number of suggestions that helped us to improve the exposition.

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable Systems and Quantum Groups (Montecatini Terme, 1993). Lecture Notes in Mathematics, vol. 1620, pp. 120–348. Springer, Berlin (1996)
2. Feigin, E., van de Leur, J., Shadrin, S.: Givental symmetries of Frobenius manifolds and multi-component KP tau-functions. *Adv. Math.* **224**(3), 1031–1056 (2010)
3. Givental, A.: Semisimple Frobenius structures at higher genus. *Int. Math. Res. Notices* **2001**(23), 1265–1286 (2001)
4. Givental, A.: Gromov–Witten invariants and quantization of quadratic Hamiltonians. *Mosc. Math. J.* **1**(4), 551–568 (2001)
5. Kokotov, A., Korotkin, D.: A new hierarchy of integrable systems associated to Hurwitz spaces. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **66**(1867), 1055–1088 (2008)
6. Korotkin, D., Shramchenko, V.: Riemann–Hilbert problem for Hurwitz Frobenius manifolds: regular singularities. [arXiv:0909.0543v2](https://arxiv.org/abs/0909.0543v2), 1–61
7. Lee, Y.-P.: Invariance of tautological equations I: conjectures and applications. *J. Eur. Math. Soc. (JEMS)* **10**(2), 399–413 (2008)
8. Shadrin, S., Zvonkine, D.: A group action on Losev–Manin cohomological field theories. [arXiv:0909.0800v1](https://arxiv.org/abs/0909.0800v1), 1–21
9. Shramchenko, V.: “Real doubles” of Hurwitz Frobenius manifolds. *Comm. Math. Phys.* **256**(3), 635–680 (2005)
10. Shramchenko, V.: Deformations of Frobenius structures on Hurwitz spaces. *Int. Math. Res. Notices* **2005**(6), 339–387 (2005)
11. van de Leur, J.: Twisted GL_n loop group orbit and solutions of the WDVV equations. *Int. Math. Res. Notices* **2001**(11), 551–573