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The Rank-One Limit of the Fourier-Mukai Transform

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Abstract. We give a formula for the specialization of the Fourier-Mukai transform on a semi-abelian variety of torus rank 1.

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1. Introduction

Let $\pi : \mathcal{X}^* \to S$ be a semi-abelian variety of relative dimension $g$ over the spectrum $S$ of a discrete valuation ring $R$ with algebraically closed residue field $k$ such that the generic fibre $X_\eta$ is a principally polarized abelian variety. We assume that $\mathcal{X}^*$ is contained in a complete rank-one degeneration $\mathcal{X}$. In particular, the special fibre $X_0$ of $\mathcal{X}$ is a complete variety over $k$ containing as an open part the total space of the $\mathbb{G}_m$-bundle associated to a line bundle $J \to B$ over a $(g-1)$-dimensional abelian variety $B$. The normalization $\nu : \mathbb{P} \to X_0$ of $X_0$ can be identified with the $\mathbb{P}^1$-bundle over $B$ associated to $J$ and $X_0$ is obtained by identifying the zero-section of $\mathbb{P}$ with the infinity-section of $\mathbb{P}$, both isomorphic to $B$, by a translation. Moreover, $X_0$ is provided with a theta divisor that is the specialization of the polarization divisor on the generic fibre.

If $c_\eta$ is an algebraic cycle on $X_\eta$ we can take the Fourier-Mukai transform $\varphi_\eta := F(c_\eta)$ and consider the limit cycle (specialization) $\varphi_0$ of $\varphi_\eta$. A natural question is: What is the limit $\varphi_0$ of $\varphi_\eta$?

If $q : \mathbb{P} \to B$ denotes the natural projection of the $\mathbb{P}^1$-bundle, the Chow ring $A^*(\mathbb{P})$ of $\mathbb{P}$ is the extension $A^*(B)[\eta]/(\eta^2 - \eta \cdot q^* c_1(J))$ of the Chow ring $A^*(B)$ of $B$ with $\eta = c_1(O_{\mathbb{P}}(1))$. We consider now cycles with rational coefficients. We denote by $c_0$ the specialization of the cycle $c_\eta$ on $X_0$. We can write $c_0$ as $\nu_*(\gamma)$ with $\gamma = q^* z + q^* w \cdot \eta$.

Theorem 1.1. Let $c_\eta$ be a cycle on $X_\eta$ with $c_0 = \nu_*(q^* z + q^* w \cdot \eta)$, for $z,w \in A^*(B)$. The limit $\varphi_0$ of the Fourier-Mukai transform $\varphi_\eta = F(c_\eta)$ is given by $\varphi_0 = \nu_*(q^* a + q^* b \cdot \eta)$ with

$$a = F_B(w) + \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B((z + w \cdot c_1(J)) \cdot c_1^n(J) \cdot c_1^{n-m+1}(J)$$

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and
\[ b = \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B([((-1)^{n+1} - 1)z - w \cdot c_1(J)) \cdot c_1^m(J) \cdot c_1^{n-m}(J)], \]
where \( F_B \) is the Fourier-Mukai transform on the abelian variety \( B \).

We denote algebraic equivalence by \( \equiv \). The relation \( c_1(J) \equiv 0 \) implies the following result.

**Theorem 1.2.** With the above notation the limit \( \varphi_0 \) satisfies
\[ \varphi_0 \equiv \nu_\ast (q^\ast F_B(w) - q^\ast F_B(z) \cdot \eta). \]

Note that this is compatible with the fact that for a principally polarized abelian variety \( A \) of dimension \( g \) the Fourier-Mukai transform satisfies \( F_A \circ F_A = (-1)^g(-1)^\ast \).

Beauville introduced in [2] a decomposition on the Chow ring with rational coefficients of an abelian variety using the Fourier-Mukai transform. Theorem 1.2 can be used to deduce non-vanishing results for Beauville components of cycles on the generic fibre of a semi-abelian variety of rank 1; we refer to §7 for examples.

We prove the theorem by constructing a smooth model \( Y \) of \( X \times_S X \) to which the addition map \( X^{\ast} \times_S X^{\ast} \to X^{\ast} \) extends and by choosing an appropriate extension of the Poincaré bundle to \( Y \). The proof is then reduced to a calculation in the special fibre. We refer to Fulton’s book [8] for the intersection theory we use. The theory in that book is built for algebraic schemes over a field. In our case we work over the spectrum of a discrete valuation ring.

But as is stated in §20.1 and 20.2 there, most of the theory in Fulton’s book, including in particular the statements we use in this paper, is valid for schemes of finite type and separated over \( S \). However, for us projective space denotes the space of hyperplanes and not lines, which conflicts with Fulton’s book, but is in accordance with [10].

2. Families of abelian varieties with a rank one degeneration

We now assume that \( R \) is a complete discrete valuation ring with local parameter \( t \), field of quotients \( K \) and algebraically closed residue field \( k \). Suppose that \((X^\ast, L)\) is a semi-abelian variety over \( S = \text{Spec}(R) \) such that the generic fibre \( X_\eta \) is abelian and the special fibre \( X_0^\ast \) has torus rank 1; moreover, we assume that \( L \) is a cubical invertible sheaf (meaning that \( L \) satisfies the theorem of the cube, see [7], p. 2, 8) and \( L_\eta \) is ample. In particular, the special fibre of \( X^\ast \) fits in an exact sequence
\[ 1 \to T_0 \to X_0^\ast \to B \to 0, \]
where \( B \) is an abelian variety over \( k \) and \( T_0 \) the multiplicative group \( G_m \) over \( k \). The torus \( T_0 \) lifts uniquely to a torus \( T_i \) of rank 1 over \( S_i = \text{Spec}(R/(t^{i+1})) \) in \( X_i^\ast = X^\ast \times_S S_i \). The quotient \( X_i^\ast / T_i \) is an abelian variety \( B_i \) over \( S_i \). The system \( \{B_i\}_{i=1}^\infty \) defines a formal abelian variety which is algebraizable, resulting
in an abelian scheme $B$, so that we have an exact sequence of group schemes over $S$

$$1 \to T \to G \overset{\pi}{\to} B \to 0,$$

cf. [F-C, p. 34]. We assume now that we are given a line bundle $M$ on $B$ defining a principal polarization $\lambda : B \to B^t$ and consider $L = \pi^*(M)$. This defines a cubical line bundle on $G$. The extension $G$ is given by a homomorphism $c$ of the character group $\mathbb{Z} \cong \mathbb{Z}$ of $T$ to $B^t$. The semi-abelian group scheme dual to $X^*$ defines a similar extension

$$1 \to T^t \to G^t \to B^t \to 0$$

and the polarization provides an isomorphism $\phi$ of the character group $\mathbb{Z}$ of $T$ with the character group $\mathbb{Z}$ of $T$. Now the degenerating abelian variety (i.e. semi-abelian variety) $X^*$ over $S$ gives rise to the set of degeneration data (cf. [7], p. 51, Thm. 6.2, or [1], Def. 2.3):

(i) an abelian variety $B$ over $S$ and a rank 1 extension $G$. This amounts to a $S$-valued point $b$ of $B = B^t$.

(ii) a $K$-valued point of $G$ lying over $b$.

(iii) a cubical ample sheaf $L$ on $G$ inducing the polarization on $B$ and an action of $\mathbb{Z}$ on $L$.

A section $s \in \Gamma(G, L)$ possesses the analogue of a classical Fourier expansion as explained in [7], p. 37. So $s$ can be written uniquely as $s = \sum_{\chi \in \mathbb{Z}} \sigma_\chi(s)$, where $\sigma_\chi : \Gamma(G, L) \to \Gamma(B, M_\chi)$ is a $R$-linear homomorphism and $M_\chi$ is the twist of $M$ by $\chi$: in fact $\pi_*(O_G) = \oplus \chi O_\chi$ with $O_\chi$ the subsheaf consisting of $\chi$-eigenfunctions. (We refer to [7], p. 43; note also the sign conventions there in the last lines.) We have now by the action $T^*_c(y)M \cong M_{\phi(y)} \cong M \otimes O_{\phi(y)}$, $y \in \mathbb{Z}$.

This satisfies $\sigma_{\chi+1}(s) = \psi(1)\tau(\chi)T^*_c(\sigma_\chi(s))$, where $\tau$ is given by a point of $G(K)$ lying over $b$ and $\psi$ is a cubical trivialization of $T^*_c(y)M^{-1}$ as in [7], p. 44, Thm. 5.1. We refer to Faltings-Chai’s theorem (6.2) of [7], p. 51 for the degeneration data.

The compactification $X$ of $X^*$ is now constructed as a quotient of the action of $Z^t$ on a so-called relatively complete model. Such a relatively complete model $\tilde{P}$ for $G$ can be constructed here in an essentially unique way. If $B$ is trivial (i.e. dim$(B) = 0$) and if the torus is $T = \text{Spec}(R[z, z^{-1}])$ it is given as the toroidal variety obtained by gluing the affine pieces

$$U_n = \text{Spec}(R[x_n, y_n]), \quad \text{with} \quad x_n y_n = t^n$$

where $G \subset \tilde{P}$ is given by $x_n = z/t^n$, $y_n = t^{n+1}/z$, (cf. [13], also in [7], p. 306). By glueing we obtain an infinite chain $\tilde{P}_0$ of $\mathbb{P}^1$‘s in the special fibre. We can ‘divide’ by the action of $Z^t$; this is easy in the analytic case, more involved in the algebraic case, but amounts to the same, cf. [13], also [7], p. 55-56.

In the special fibre we find a rational curve with one ordinary double point. If instead we divide by the action of $nZ^t$ for $n > 1$ we find a cycle consisting of $n$ copies of $\mathbb{P}^1$. 

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In case the abelian part $B$ is not trivial we take as a relatively complete model the contracted (or smashed) product $\tilde{P} \times^T G$ with $\tilde{P}$ the relatively complete model for the case that $B$ is trivial. Call the resulting space $\tilde{P}$. Then $\tilde{P}$ corresponds by Mumford’s [loc. cit., p 29] to a polyhedral decomposition of $Z' \otimes \mathbb{R} = \mathbb{R}$ with $Z'$ the cocharacter group of $T$. Then we essentially quotient by the action of $Z'$ or $nZ'$ as before and obtain a proper $X \to S$.

We describe the central fibre $X_0$ of $X$. Let $b$ be the $k$-valued point of $B \cong B'$ that determines the above $\mathbb{G}_m$-extension. If $M$ denotes a line bundle defining the principal polarization of $B$ we let $M_b$ be the translation of $M$ by $b$ and we set $J = M \otimes M_b^{-1}$ and define the projective bundle $\mathbb{P} = \mathbb{P}(J \oplus O_B)$ with projection $q : \mathbb{P} \to B$. The bundle $\mathbb{P}$ has two natural sections (with images) $\mathbb{P}_1$ and $\mathbb{P}_2$ corresponding to the projections $J \oplus O_B \to J$ and $J \oplus O_B \to O_B$. We have $\mathcal{O}(\mathbb{P}_1) \cong \mathcal{O}(\mathbb{P}_2) \otimes q^* b$ and $\mathcal{O}(1) \cong \mathcal{O}(\mathbb{P}_1)$ with $\mathcal{O}(1)$ the natural line bundle on $\mathbb{P}$. We denote by $\tilde{\mathbb{P}}$ the non-normal variety obtained by gluing the sections $\mathbb{P}_1$ and $\mathbb{P}_2$ under a translation by the point $b$. The singular locus of $\tilde{\mathbb{P}}$ has support isomorphic to $B$. The line bundle $\tilde{L} = \mathcal{O}(\mathbb{P}_1) \otimes q^* M_b \cong \mathcal{O}(\mathbb{P}_2) \otimes q^* M$ descends to a line bundle $\tilde{L}$ on $\tilde{\mathbb{P}}$ with a unique ample divisor $D$, see [14]. The central fibre $X_0$ of the family $\pi : X \to S$ is then equal to $\tilde{\mathbb{P}}$. The cubical invertible sheaf $\mathcal{L}$ on $X^*$ extends (uniquely) to $X$ and its restriction to the central fibre $\tilde{\mathbb{P}}$ is the line bundle $\tilde{L}$, see [15].

3. Extension of the addition map

The addition map $\mu : X^* \times_S X^* \to X^*$ of the semi-abelian scheme $X^*$ does not extend to a morphism $X^* \times_S X \to X$, but it does so after a small blow-up of $X^* \times_S X$ as we shall see.

The degeneration data of $X^*$ defines (product) degeneration data for $X^* \times_S X^*$. Indeed, we can take the fibre product of the relatively complete model $\tilde{P}' = \tilde{P} \times_S \tilde{P}$ and this corresponds (e.g. via [13], Corollary (6.6)) to the standard polyhedral decomposition of $\mathbb{R}^2 = (Z' \otimes \mathbb{R})^2$ by the lines $x = m$ and $y = n$ for $m, n \in \mathbb{Z}$. The special fibre of the model $\tilde{P}'$ is an infinite union of $\mathbb{P}^1 \times \mathbb{P}^1$-bundles over $B \times B$ glued along the fibres over $0$ and $\infty$. The compactified model of $X \times_S X$ is obtained by taking the ‘quotient’ of $\tilde{P}'$ under the action of $Z' \times Z'$. This is not regular; for example the criterion of Mumford ([13], p. 29, point (D1)) is not satisfied. We can remedy this by subdividing. For example, by taking the decomposition of $\mathbb{R}^2$ given by the lines $x = m, y = n$ and $x + y = l$ for $m, n, l \in \mathbb{Z}$.

The special fibre of this model is an infinite union of copies of $\mathbb{P}^1 \times \mathbb{P}^1$-bundles over $B \times B$ blown up in the two anti-diagonal sections $(0, \infty) = \mathbb{P}_1 \times \mathbb{P}_2$ and $(\infty, 0) = \mathbb{P}_2 \times \mathbb{P}_1$. This is regular.

Both polyhedral decompositions are invariant under the action of translations $(x, y) \mapsto (x + a, y + b)$ for fixed $a, b \in \mathbb{Z}$. This means that we can form the ‘quotient’ by $Z' \times Z' \cong \mathbb{Z}^2$ (or a subgroup $nZ' \times nZ'$) and obtain a completed semi-abelian variety $\mathcal{Y}$ of relative dimension $2g$ over $S$. We denote by $\epsilon : \mathcal{Y} \to \mathcal{Y}' = X^* \times_S X$ the natural map. We shall write $V$ for $\mathcal{Y}_0$ and $\sigma : V \to V$ for
its normalization. Then \( \bar{V} \) is an irreducible component of the special fibre of \( \bar{P} \).

We denote by \( \tau : \bar{V} \to \mathbb{P} \times \mathbb{P} \) the blow up map and by \( E_{12} \) and \( E_{21} \) the exceptional divisors over the blowing up loci \( \mathbb{P}_1 \times \mathbb{P}_2 \) and \( \mathbb{P}_2 \times \mathbb{P}_1 \), respectively.

Now consider the addition map \( \mu : X^* \times_S X^* \to X^* \) with \( X^* \) as in \( \S \). This morphism induces (and is induced by) a map \( \bar{\mu} : \bar{X} \times_S G \to G \). However, this map does not extend to a morphism of the relatively complete model \( \bar{P} \) since the corresponding (covariant) map \( (\mathbb{Z}' \otimes \mathbb{R})^2 \to (\mathbb{Z}' \otimes \mathbb{R}) \) does not map cells to cells. After subdividing (by adding the lines \( x+y = l \) with \( l \in \mathbb{Z} \)) this property is satisfied (cf. [11], Thm. 7, p. 25). This means that the map \( \mu \) extends to \( \bar{\mu} : \bar{P} \to \bar{P} \) for the polyhedral decomposition given by this subdivision. It is compatible with the action of \( \mathbb{Z} \) and \( \mathbb{Z} \times \mathbb{Z} \) and hence descends to a morphism \( \bar{\mu} : \bar{Y} \to \bar{X} \).

We summarize:

**Proposition 3.1.** The addition map of group schemes \( \mu : X^* \times_S X^* \to X^* \) extends to a morphism \( \bar{\mu} : \bar{Y} \to \bar{X} \).

We now describe an explicit local construction of the model \( \bar{Y} \) by blowing up the model \( Y \times_S X \). Let \( A_S^{g+1} = \text{Spec}(R[x_1, \ldots, x_{g+1}]) \) denote affine \( S \)-space. In local coordinates, inside \( A_S^{g+1} \), we may assume that the \( g \)-dimensional fibration \( \pi : X^* \to S \) is given by the equation \( x_1 x_2 = t \), where the coordinates \( x_1, \ldots, x_{g+1} \) are not involved, see [14] p. 361-362. We may assume that the zero section of the family is defined by \( x_i = 1 \) for \( i = 1, \ldots, g+1 \).

We form the fibre product \( \pi : \bar{X}^* = X \times_S X \). We denote by \( \Lambda \) the support of the singular locus of \( X_0 \). The \( (2g+1) \)-dimensional family \( \bar{Y}' \) is singular in the special fibre along \( \Sigma = \Lambda \times_k \Lambda = B \times_k B \) of dimension \( 2g-2 \). The generic fibre \( Y'_0 \) is the product \( X_0 \times_k X_0 \) of the abelian variety \( X_0 \), while the zero fibre \( Y'_0 \) is singular. The local equations of \( \bar{Y}' \) in a neighborhood of the singular locus of the family are given in our local coordinates by the system \( x_1 x_2 = t, x'_1 x'_2 = t \).

The singular locus \( \Sigma \) of \( \bar{Y}' \) is given by the equations \( x_1 = x_2 = x'_1 = x'_2 = t = 0 \). We form the fibre product \( \pi : \bar{Y}' = X \times_S X \). We denote by \( \Lambda \) the support of the singular locus of \( X_0 \). The \( (2g+1) \)-dimensional family \( \bar{Y}' \) is singular in the special fibre along \( \Sigma = \Lambda \times_k \Lambda = B \times_k B \) of dimension \( 2g-2 \). The generic fibre \( Y'_0 \) is the product \( X_0 \times_k X_0 \) of the abelian variety \( X_0 \), while the zero fibre \( Y'_0 \) is singular. The local equations of \( \bar{Y}' \) in a neighborhood of the singular locus of the family are given in our local coordinates by the system \( x_1 x_2 = t, x'_1 x'_2 = t \).

For later calculations we write down the morphism \( \bar{\mu} \) explicitly on the special fibre. We start with \( g = 1 \); then \( B \) is trivial and we may restrict the map to an irreducible component of the special fibre of the relatively complete model \( \bar{P} \) and get the map \( n : \mathbb{P} \times \mathbb{P} \to \mathbb{P} \) given by \((a : b), (a' : b') \mapsto (aa' : bb') \). This is not defined in the points \((0, \infty)\) and \((\infty, 0)\). After blowing up these points (which is the blow up \( \bar{Y} \to \bar{Y}' \)) the rational map becomes a regular map \( \bar{m} : \bar{V} \to \mathbb{P} \). It is defined by the two sections \( \text{prop}(p_1^*\{0\}) + \text{prop}(p_2^*\{0\}) \) and \( \text{prop}(p_1^*(\infty)) + \text{prop}(p_2^*(\infty)) \) of the linear system \(|\tau^*(F_1 + F_2) - E_{12} - E_{21}| \) with
Proof. We have the following commutative diagram of maps

\[
\begin{array}{ccc}
V & \overset{m}{\rightarrow} & \mathbb{P} \\
\downarrow{\sigma} & & \downarrow{\nu} \\
\bar{V} & \overset{\bar{m}}{\rightarrow} & \mathbb{P} \\
\downarrow{\tau} & & \\
P & \overset{p_1}{\leftarrow} & \mathbb{P} \times \mathbb{P} \\
\downarrow{q} & & \downarrow{\nu} \\
B & \overset{\mu_B}{\leftarrow} & B \\
\end{array}
\]

Let \( \mathcal{L} \) be the theta line bundle on the family \( \mathcal{X} \) introduced in \( \S \) 2. We define the extension of \( \mathcal{P}_\eta \) by

\[
\mathcal{P} := \bar{\mu}^* \mathcal{L} \otimes \rho_1^* \mathcal{L}^{-1} \otimes \rho_2^* \mathcal{L}^{-1},
\]

where we denote by \( \rho_1, \rho_2 : \mathcal{Y} \rightarrow \mathcal{X} \) the compositions of the natural projections \( \rho_i' : \mathcal{Y}' \rightarrow \mathcal{X} \) with the blowing up map \( \epsilon : \mathcal{Y} \rightarrow \mathcal{Y}' \) of \( \S \) 3. We then have \( \sigma^* \mathcal{P}_0 = \sigma^* (\bar{m}^* j_0^* \mathcal{L}) \otimes \sigma^* i_0^* \rho_1^* \mathcal{L}^{-1} \otimes \sigma^* i_0^* \rho_2^* \mathcal{L}^{-1} \). Now \( \bar{m}^* j_0^* \mathcal{L} = \bar{m}^* L \) and by using
the description of $\bar{L}$ in \S 2 we have $\sigma^*(\bar{m}^*j_0^*L) = \bar{m}^*\nu^*L = \bar{m}^*(O(P_1) \otimes q^*M_b)$. In view of $O(P_1) = O(1)$ we have by the discussion at the end of \S 3 that
\[ \bar{m}^*O(P_1) = \tau^*p_1^*O(P_1) \otimes \tau^*p_2^*O(P_1) \otimes O(-E_{12} - E_{21}) \]
and $\bar{m}^*q^*M_b = \tau^*(q \times q)^*\mu_B^*M_b$. On the other hand we have
\[ \sigma^*(i_0^*\rho_*^*L) = \tau^*p_1^*\nu^*\bar{L} = \tau^*p_1^*O(P_1) \otimes \tau^*(q \times q)^*q^*_bM_b \]
and putting this together we get the result. 

5. THE BASIC CONSTRUCTION

The fibration $\pi : \mathcal{Y} \to S$ is a flat map since $\mathcal{Y}$ is irreducible and $S$ is smooth 1-dimensional, see \cite{10}, Ch. III, Proposition 9.7. The maps $\rho_i : \mathcal{Y} \to \mathcal{X}$, $i = 1, 2$, defined in the proof of Theorem 4.1, are flat maps too since they are maps of smooth irreducible varieties with fibres of constant dimension $g$, see e.g. \cite{12}, Corollary of Thm. 23.1.

We denote by $Y_0$ (resp. $Y_\eta$) the special fibre (resp. the generic fibre) and by $i_0 : Y_0 \to \mathcal{Y}$ (resp. $i_\eta : Y_\eta \to \mathcal{Y}$) the corresponding embedding. According to \cite{8}, Example 10.1.2, $i_0$ is a regular embedding. Similarly, $j_0 : X_0 \to \mathcal{X}$ is a regular embedding. We consider the diagram

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{i_0} & \mathcal{Y} \\
\downarrow{\pi_0} & & \downarrow{\pi} \\
\text{Spec}(k) & \xrightarrow{\delta} & S
\end{array}
$$

Let $i_0^* : A_k(\mathcal{Y}) \to A_{k-1}(Y_0)$ be the Gysin map (see \cite{8}, Example 5.2.1). Since $Y_0$ is an effective Cartier divisor in $\mathcal{Y}$ the Gysin map $i_0^*$ coincides with the Gysin map for divisors (see \cite{8}, Example 5.2.1 (a) and \S 2.6).

We now consider specialization of cycles, see \cite{8}, \S 20.3. Note that according to \cite{8}, Remark 6.2.1., in our case we have $\delta_0^*a = i_0^*a$, $a \in A_*(\mathcal{Y})$. If $Z$ is a flat scheme over the spectrum of a discrete valuation ring $S$ the specialization homomorphism $\sigma_Z : A_k(Z_0) \to A_k(Z_0)$ is defined as follows, see \cite{8}, pg. 399:

If $\beta_0$ is a cycle on $Z_0$ we denote by $\beta$ an extension of $\beta_0$ in $Z$ (e.g. the Zariski closure of $\beta_0$ in $Z$) and then $\sigma_Z(\beta_0) = i_0^*\beta$, where $i_0 : Z_0 \to Z$ is the natural embedding.

Let $c_0$ be a cycle on $X_0$ and let $\varphi_0 = F(c_0)$ be the Fourier-Mukai transform. It is defined by $F(c_0) = \rho_2_*(e^{c_1}(P_0)) \cdot \rho_1^*c_0 \in A_*(X_0)$. Let $\sigma_X : A_k(X_0) \to A_k(X_0)$ be the specialization map. We have to determine $\sigma_X(F(c_0))$.

If $\beta_0$ is a cycle on $Y_0$, we have $\rho_2_\eta \sigma_Y(\beta_0) = \sigma_X \rho_2_*(\beta_0)$ by applying \cite{8} Proposition 20.3 (a) to the proper map $\rho_2 : \mathcal{Y} \to \mathcal{X}$. By choosing $\beta_0 = e^{c_1}(P_0) \cdot \rho_1^*c_0$ we have

\[ \sigma_X(F(c_0)) = \rho_2_\eta \sigma_Y(e^{c_1}(P_0) \cdot \rho_1^*c_0). \]

Therefore, in order to compute $\sigma_X(F(c_0))$ we have to identify $\sigma_Y(e^{c_1}(P_0) \cdot \rho_1^*c_0)$. We take the extension $e^{c_1}(P)$ of $e^{c_1}(P_0)$ and the extension of $\rho_1^*c_0$ given by $\rho_1^*c_0$.  

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where \( c \) is the Zariski closure of \( c_0 \) in \( X \). Since \( i_\eta : Y_\eta \to Y \) is an open embedding and hence a flat map of dimension 0, we have \( i_\eta^* (e^{c_1(P)} \cdot \rho_1^* c) = e^{c_1(P)} \cdot \rho_1^* c_0 \), see [8], Proposition 2.3 (d). In other words, the cycle \( e^{c_1(P)} \cdot \rho_1^* c \) extends the cycle \( e^{c_1(P)} \cdot \rho_1^* c_0 \) and hence \( \sigma_Y (e^{c_1(P)} \cdot \rho_1^* c_0) = i_\eta^* (e^{c_1(P)} \cdot \rho_1^* c) \).

Now, for any \( k \)-cycle \( a \) on \( Y \) we have the identity

\[
i_\eta^* (c_1(P) \cdot a) = c_1(P_0) \cdot i_\eta^*(a)
\]

in \( A_{k-2}(Y_0) \), where \( P_0 = i_\eta^* P \) is the pull back of the line bundle and \( i_\eta^* a \) the Gysin pull back to the divisor \( Y_0 \). This follows from applying the formula in [8], Proposition 2.6 (e) to \( i_\eta : Y_0 \to Y \), with \( D = Y_0 \), \( X = Y \) and \( L = P \) the Poincaré bundle. Hence

\[
\sigma_Y (e^{c_1(P_0)} \cdot \rho_1^* c_0) = e^{c_1(P_0)} \cdot i_\eta^*(\rho_1^* c). 
\]

By the Moving Lemma (see [8], §11.4), we may choose the cycle \( c \) on the regular \( X \) such that it intersects the singular locus \( \Lambda \) of the central fibre properly. Since \( \Lambda \subseteq X_0 \) the cycle \( c_0 = j_0^*(c) \) meets \( \Lambda \) properly by the following dimension argument. We have \( \dim(e \cap \Lambda) = \dim(c_0 \cap \Lambda) \), hence

\[
\dim(c_0 \cap \Lambda) = \dim(c) + \dim(\Lambda) - \dim(X) \\
= (\dim(c) - 1) + \dim(\Lambda) - (\dim(X) - 1) \\
= \dim(c_0) + \dim(\Lambda) - \dim(X_0).
\]

Since \( \Lambda \) is of codimension 1 in \( X_0 = \Pbar \), saying that \( c_0 \) meets \( \Lambda \) properly, is equivalent to saying that no component of \( c_0 \) is contained in \( \Lambda \).

Lemma 5.1. There exists a cycle \( \gamma \) on \( P \) with \( c_0 = \nu_* \gamma \) that meets the sections \( \Pbar_i \) for \( i = 1, 2 \) properly.

Proof. If \( \Lambda \) is the singular locus of \( \Pbar \) and \( A = \Pbar_1 \cup \Pbar_2 \) its preimage in \( \Pbar \), then \( \Pbar \setminus \Lambda \cong \Pbar \setminus A \). We may assume that the cycle \( c_0 \) is irreducible and we consider the support of \( c_0 \cap (\Pbar \setminus \Lambda) \) as a subset \( W \) of \( \Pbar \setminus \Lambda \). Its Zariski closure \( \gamma = \overline{\gamma} \) is an irreducible cycle on \( \Pbar \). Then \( \nu_* \gamma \) is an irreducible cycle on \( \Pbar \) since the map \( \nu \) is a projective map. Also, \( \nu_* \gamma \cap (\Pbar \setminus \Lambda) = c_0 \cap (\Pbar \setminus \Lambda) \), hence \( \nu_* \gamma \) is the Zariski closure of \( c_0 \cap (\Pbar \setminus \Lambda) \) and so, by the irreducibility, we have \( \nu_* \gamma = c_0 \). \( \square \)

Lemma 5.2. If \( c_0 = \nu_* \gamma \), then we have \( i_\eta^* \rho_1^* \gamma = \sigma_* (\tau^* (\rho_1^* \gamma)) \).

Proof. We denote the restriction of \( \rho_1 \) to the special fibre again by \( \rho_1 \). Then we have \( i_\eta^* \rho_1^* c = \rho_1^* c_0 \) since \( \rho_1 \) is a flat map and \( i_\eta, j_0 \) are regular embeddings (see [8], Theorem 6.2 (b) and Remark 6.2.1). We will use the following commutative diagram

\[
\begin{array}{ccc}
\Pbar \times \Pbar & \stackrel{\tau}{\longrightarrow} & \Pbar \\
\downarrow{\rho_1} & & \downarrow{\rho_1'} \\
\Pbar & \stackrel{\nu}{\longrightarrow} & \Pbar
\end{array}
\]

\[
\begin{array}{ccc}
\Pbar \times \Pbar & \stackrel{\sigma}{\longrightarrow} & V \\
\downarrow{\rho_1} & & \downarrow{\rho_1'} \\
\Pbar & \stackrel{\nu}{\longrightarrow} & \Pbar
\end{array}
\]
We may assume that $c_0$ and $\gamma$ are irreducible $k$-cycles. We claim that $\rho_1^*c_0$ is irreducible. Indeed, the map $\rho_1$ is a flat map of relative dimension $g$. The cycle $\rho_1^*c_0$ is then a cycle of pure dimension $k + g$ and contains the proper transform of $(\rho_1')^*c_0$ and that is an irreducible cycle. Any other irreducible component of $\rho_1^*c_0$ must have support on the preimage of $\Lambda$. But since the cycle $c_0$ intersects $\Lambda$ along a $k - 1$-cycle, there is no irreducible component of $\rho_1^*c_0$ on the preimage of $\Lambda$. On the other hand, since $\gamma$ meets the sections $P_i$ properly, the cycle $\tau^*p_1^*\gamma$ is an irreducible cycle, and hence so is $\sigma_*(\tau^*p_1^*\gamma)$. But as $\rho_1^*c_0$ and $\sigma_*(\tau^*p_1^*\gamma)$ coincide outside the exceptional divisor of $V$, they have to coincide everywhere. \hfill \square

**Proposition 5.3.** We have

$$\sigma_X(F(c_0)) = \rho_2*(c_1(P_0) \cdot \sigma_*(\tau^*p_1^*\gamma)).$$

**Proof.** By equation (2) and Lemma 5.2 we have

$$\sigma_Y(c_1(P_0) \cdot \rho_1^*c_0) = c_1(P_0) \cdot \sigma_*(\tau^*p_1^*\gamma).$$

The result follows from equation (1). \hfill \square

In order to calculate the limit of the Fourier-Mukai transform we are thus reduced to a calculation in the special fibre.

6. A Calculation in the Special Fibre - Proof of the Main Theorem

Recall the normalization map $\sigma : \tilde{V} \to V$. Suppose we have a cycle $\rho$ on $\tilde{V}$ with $\sigma_\rho = c_0$. We can consider the intersection $c_1(P_0)^k \cdot c_0$, that is a successive intersection of a cycle with a Cartier divisor on the singular variety $V$. On the other hand we have the cycle $\sigma_*(c_1(\sigma^*P_0)^k \cdot \rho)$ and the projection formula ([8], Proposition 2.5 (c)) implies that

$$c_1(P_0)^k \cdot c_0 = \sigma_*(c_1(\sigma^*P_0)^k \cdot \rho).$$

Now we will use the following diagram of maps.

**Lemma 6.1.** Let $x$ be a cycle on $B \times B$. Then the following holds.

1. $p_{2*}((q \times q)^*x) = 0$.
2. $p_{2*}((q \times q)^*x \cdot p_1^*\eta) = q^*q_{2*}x$.  

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Proof. For (1) we observe that \(p_{2*} = \kappa_{2*} \alpha_{1*}\), and \((q \times q)^* = \alpha_1^* \beta_2^*\) and \(\alpha_{1*} = 0\). For (2) we use the identities
\[
p_{2*}((q \times q)^* x \cdot p_1^* \eta) = p_{2*}(\alpha_2^* \beta_1^* x \cdot \alpha_2^* \kappa_1^* \eta) = p_{2*} \alpha_2^* (\beta_1^* x \cdot \kappa_1^* \eta)
\]
\[
= \kappa_{2*} \alpha_{1*} \alpha_2^* (\beta_1^* x \cdot \kappa_1^* \eta) = \kappa_{2*} \beta_2^* (\beta_1^* x \cdot \kappa_1^* \eta)
\]
\[
= \kappa_{2*} \beta_2^* (x \cdot \beta_1^* \kappa_1^* \eta) = q^* q_{2*} (x \cdot q_1^* q_2 \eta) = q^* q_{2*} x.
\]

Consider the following diagram of maps

\[
\begin{array}{cccc}
P_i & P_i \times P_j & E_{ij} \\
\lambda_i & \pi_{ij} & \epsilon_{ij} \\
P & P \times P & V \\
q & q \times q & \sigma \\
B & q_B & V \\
B & B \times B & V \\
B & B & B \\
\end{array}
\]

where \(p_i, q_i\) are the projections to the \(i\)th factor, \(\pi_{ij}\) the canonical map of the projective bundle \(E_{ij}\), and the maps \(\lambda_i, \lambda_{ij}\) and \(\epsilon_{ij}\) the natural inclusions. The map \((q \times q) \circ \lambda_{ij}\) is an isomorphism.

By the adjunction formula, the normal bundles to \(P_i \to P \) and \(P_j \to P\) are \(N_{P_i}(P) = J = N_{P_j}(P) = J^{-1}\). The exceptional divisors \(E_{12}\) and \(E_{21}\) are projective bundles over the blowing up loci \(P_i \times P_j\). By identifying \(P_i \times P_j\) with \(B \times B\), via the map \((q \times q) \circ \lambda_{ij}\), we have \(E_{12} = P(q_1^* J^{-1} \oplus q_2^* J)\) and \(E_{21} = P(q_1^* J \oplus q_2^* J^{-1})\). We set \(\xi_{ij} = c_1(O(1))\) on \(E_{ij}\). By standard theory \([10],\) ch. II, Theorem 8.24 (c) we have \(\epsilon_{ij}^* E_{ij} = -\xi_{ij}\).

We now introduce the notation
\[
\gamma := c_1(J), \quad \gamma_i = q_i^* \gamma, \quad \eta_i = p_i^* \eta, \quad i = 1, 2.
\]

Note that \(\gamma\) is algebraically equivalent to 0, but not rationally equivalent to 0.

We have the quadratic relations
\[
(\xi_{ij} - \pi_{ij}^* \gamma_j)(\xi_{ij} + \pi_{ij}^* \gamma_i) = 0
\]

where \(\pi_{ij} : E_{ij} \to B \times B\) is the natural map, showing that \(\xi_{ij}^2\) is expressible in lower powers.

**Lemma 6.2.** Suppose that \(\xi\) satisfies the relation \(\xi^2 + (a - b) \xi - ab = 0\). Then, with \(\phi_k = (b^k - (-a)^k)/(b + a)\) we have \(\xi^k = \phi_k \xi + ab \phi_{k-1}\) for any \(k \geq 1\) (where we put \(\phi_0 = 0\)).

**Proof.** Immediate by checking the relation with \(\xi = b\) or \(\xi = -a\). \qed

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Applying the above for the classes $\xi_{ij}$ of the bundles $E_{ij}$, considered as bundles over $B \times B$ via the isomorphism $(q \times q) \circ \lambda_{ij}$, we get, by choosing

$$\phi_k = \sum_{m=0}^{k-1} (-1)^m \gamma_1^{m} \gamma_2^{k-1-m},$$

that

$$\xi_{12}^k = \pi_{12}^* \phi_k \cdot \xi_{12} + \pi_{12}^*(\gamma_1 \gamma_2 \phi_{k-1}).$$

$$\xi_{21}^k = (-1)^{k+1} \pi_{21}^* \phi_k \cdot \xi_{21} + (-1)^k \pi_{21}^*(\gamma_1 \gamma_2 \phi_{k-1}).$$

We view now the bundles $E_{ij}$ as bundles over $\mathbb{P}_i \times \mathbb{P}_j$ and, for any $k \geq 0$, we write $\xi_{ij}^k = \pi_{ij}^* A_{ij}(k) \xi_{ij} + \pi_{ij}^* B_{ij}(k)$, for some cycles $A_{ij}(k)$, $B_{ij}(k)$ on $\mathbb{P}_i \times \mathbb{P}_j$. By the above relations we have

$$(q \times q) \ast \lambda_{ij} \ast A_{ij}(k) = (-1)^{(k+1)/2} \phi_k.$$  

\textbf{Lemma 6.3.} We have

$$\lambda_{ij} \ast A_{ij}(k) = (-1)^{(k+1)/2}[(q \times q) \ast \phi_k \cdot \eta_1 \eta_2 - (q \times q) \ast (\phi_k \gamma_j) \cdot \eta_i].$$

\textbf{Proof.} We let $\psi_{ij} = (q \times q) \circ \lambda_{ij} : \mathbb{P}_i \times \mathbb{P}_j \to B \times B$ be the natural isomorphism.

We then have the identity

$$\lambda_{ij} \ast A_{ij}(k) = \lambda_{ij} \ast (\psi_{ij}^* \psi_{ij} \ast A_{ij}(k)) = (q \times q)^* \psi_{ij} \ast A_{ij}(k) \ast \lambda_{ij} \ast 1_{\mathbb{P}_i \times \mathbb{P}_j}.$$  

But $\lambda_{ij} \ast 1_{\mathbb{P}_i \times \mathbb{P}_j} = p_1^* \mathbb{P}_i \ast p_2^* \mathbb{P}_j = \eta_i (\eta_j - p_j^* q^* \gamma) = \eta_1 \eta_2 - \eta_i \cdot (q \times q)^* \gamma_j$ and the result follows. \hfill $\square$

\textbf{Lemma 6.4.} For a cycle class $x = q^* z + q^* w \cdot \eta$ on $\mathbb{P}$ the cycle class $\tau_*(\tau^* p_1^* x \cdot (E_{12}^k + E_{21}^k))$ for $k \geq 1$ is given by

$$\sum_{m=0}^{k-2} (-1)^m ((q \times q)^* q_1^* [((-1)^{k+1} - 1)z + (-1)^{k+1} w \gamma) \cdot \eta_1 \eta_2$$

$$+ (-1)^k (q \times q)^* q_1^* [z + w \gamma) \gamma^m) \cdot \eta_1 \cdot p_2^* q^* \gamma$$

$$+ (q \times q)^* q_1^* [z \gamma^m + \cdot \eta_1 \cdot p_2^* q^* \gamma^{k-2-m}}.$$  

Note that for $k = 1$ the above sum is zero.

\textbf{Proof.} Since $\epsilon_{ij}^* E_{ij} = -\xi_{ij}$ we have $E_{ij}^k = (-1)^{k+1} \epsilon_{ij} \ast \xi_{ij}^k$. Therefore

$$\tau_*(\tau^* p_1^* x \cdot E_{ij}^k) = (-1)^{k+1} p_1^* x \cdot \tau_* \epsilon_{ij} \ast \xi_{ij}^k$$

$$= (-1)^{k+1} p_1^* x \cdot \lambda_{ij} \ast \pi_{ij} \ast (\pi_{ij}^* A_{ij}(k-1) \xi_{ij} + \pi_{ij}^* B_{ij}(k-1))$$

$$= (-1)^{k+1} p_1^* x \cdot \lambda_{ij} \ast A_{ij}(k-1)$$

since $\pi_{ij} \ast \xi_{ij} = 1_{\mathbb{P}_i \times \mathbb{P}_j}$. Note that since $A_{ij}(0) = 0$ the above calculation shows that $\tau_*(\tau^* p_1^* x \cdot E_{ij}) = 0$. By Lemma 6.3 and by using the relation

$$p_1^* x = (q \times q)^* q_1^* z + (q \times q)^* q_1^* w \cdot \eta_1,$$

we have

$$\tau_*(\tau^* p_1^* x \cdot E_{ij}^k) = (-1)^{k+1} ((q \times q)^* q_1^* z + (q \times q)^* q_1^* w \cdot \eta_1)$$

$$\cdot [(q \times q)^* \phi_{k-1} \cdot \eta_1 \eta_2 - (q \times q)^* (\phi_{k-1} \gamma_j) \cdot \eta_1].$$
We then have, by using the formula $\eta^2 = q^\gamma \cdot \eta$, that
\[
\tau_* (\tau^* p_1^* x \cdot E_{12}^k) = (-1)^{k+1} [(q \times q)^* (q_1^* z \cdot \phi_{k-1}) \cdot \eta_1 \eta_2 - (q \times q)^* (q_1^* w \cdot \phi_{k-1} \gamma_j) \cdot \eta_i] 
\]
We then have, by using the formula $\eta^2 = q^\gamma \cdot \eta$, that
\[
\tau_* (\tau^* p_1^* x \cdot E_{21}^k) = (-1)^{k+1} [(q \times q)^* (q_1^* (z + w \gamma) \cdot \phi_{k-1}) \cdot \eta_1 \eta_2 
- (q \times q)^* (q_1^* (z \gamma) \cdot \phi_{k-1}) \cdot \eta_1 \cdot p_2^* q^* \gamma)]
\]

Proposition 6.5. Let $z, w$ be cycles on $B$. Then we have
\[
p_{2*} \tau_* (e^{c_1(x^* p_0)} \cdot \tau^* (p_1^* (q^* z + q^* w \cdot \eta))) = q^* a + q^* b \cdot \eta,
\]
with $a$ and $b$ as in Theorem 1.1.

Proof. We put $x = q^* z + q^* w \cdot \eta$. We want to calculate
\[
p_{2*} \tau_* (e^{\tau^* (q \times q)^* c_1 (P_B)^*} \cdot E_{12}^k \cdot E_{21}^k \cdot \tau^* (p_1^* x))
\]
which equals
\[
p_{2*} (e^{(q \times q)^* c_1 (P_B)^*} \cdot \tau_* (e^{-E_{12}^k \cdot E_{21}^k} \cdot \tau^* (p_1^* x))).
\]
Since $E_{12} \cdot E_{21} = 0$ we have
\[
e^{-E_{12}^k \cdot E_{21}^k} = 1 + \sum_{k=1}^{2g} \frac{(-1)^k}{k!} (E_{12}^k + E_{21}^k)
\]
and so $\tau_* (e^{-E_{12}^k \cdot E_{21}^k} \cdot \tau^* p_1^* x)$ equals
\[
p_1^* x + \sum_{k=1}^{2g} \frac{(-1)^k}{k!} \tau_* [\tau^* p_1^* x \cdot (E_{12}^k + E_{21}^k)]
\]
We have
\[
p_{2*} ((q \times q)^* e^{c_1 (P_B)^*} \cdot p_1^* x) = 
= p_{2*} (e^{(q \times q)^* c_1 (P_B)^*} \cdot p_1^* (q^* z + q^* w \cdot \eta)) 
= p_{2*} ((q \times q)^* (e^{c_1 (P_B)^*} q_1^* (z + (q \times q)^* c_1 (P_B)^* q_1^* w)) p_1^* \eta) 
= 0 + q^* q_{2*} (e^{c_1 (P_B)^*} q_1^* w) = q^* F_B(w)
\]
by Lemma 6.1. Combining the above with Lemma 6.4 we find that
\[
p_{2*} \tau_* (e^{\tau^* (q \times q)^* c_1 (P_B)^*} \cdot E_{12}^k \cdot E_{21}^k \cdot \tau^* (p_1^* x))
\]
By Proposition 5.3 we have

$$\sum_{k=2}^{2g} \sum_{m=0}^{k-2} \frac{(-1)^{k+m}}{k!} \{p_{2*}[((q \times q)^* e_{c_1(P_n^*)} q_1^* ((z + w_\gamma)^{m+1})] \cdot \eta] \cdot q^* \gamma^{k-2} - m,}

the third term is

$$\sum_{k=2}^{2g} \sum_{m=0}^{k-2} \frac{(-1)^{k+m}}{k!} \{p_{2*}[(q \times q)^* e_{c_1(P_n^*)} q_1^* ((z + w_\gamma)^m] \cdot \eta) \cdot q^* \gamma^{k-1} - m,}

and finally the fourth is

$$\sum_{k=2}^{2g} \sum_{m=0}^{k-2} \frac{(-1)^{k+m}}{k!} \{p_{2*}[(q \times q)^* e_{c_1(P_n^*)} q_1^* ((z+\gamma)^{m+1})] \cdot \eta) \cdot q^* \gamma^{k-2} - m,}

By applying now Lemma 6.1 and by making the substitution $n = k - 2$ we get the desired expression.

Corollary 6.6. Let $z, w$ be cycles on $B$. Then modulo algebraic equivalence we have

$$p_{2*}(e_{c_1(\sigma \circ \sigma)}) \tau^*(p_1^*(q^* z + q^* w \cdot \eta)) = p_{2*}q^* F_B(w) - q^* F_B(z) \cdot \eta.$$

Proof. Indeed, since $c_1(J)^{\alpha} = 0$ it is clear that $a = F_B(w)$ and $b = q^* F_B(z)$ since the only non zero term of the sum corresponds to $m = 0, n = 0$.

We conclude now with the proof of the basic Theorem 1.1 and Theorem 1.2:

Proof. By Proposition 5.3 we have $\bar{\varphi}_0 = \sigma X F(c_{\eta}) = p_{2*}(e_{c_1(\sigma \circ \sigma)} \cdot \tau^* p_1^*(\gamma))$. By the projection formula we have $e_{c_1(\sigma \circ \sigma)} \cdot \tau^* p_1^*(\gamma) = e_{c_1(\sigma \circ \sigma)} \cdot \tau^* p_1^*(\gamma)$. Observe now that $p_2 \circ \sigma = \nu \circ (p_2 \circ \tau) : \bar{V} \to \bar{P}$, see the diagram in the proof of Lemma 5.2. The proof then follows from Proposition 6.5 and Corollary 6.6.

7. Applications

Let $X \to S$ be a completed rank-one degeneration as described in § 2. According to Beauville [2] we have a decomposition of $A^i_0(X_\eta)$ into subspaces which are eigenspaces for the action by multiplication by an integer on $X_\eta:

$$A_0^i(X_\eta) = \bigoplus_j A^i_{(j)}(X_\eta)$$

such that $n^*(x) = n^{2i-j} x$ for $x \in A^i(X_\eta)$.

(Useful to note that $\eta^*(x) = \eta^{2i-j} x$ for $x \in A^i(X_\eta)$.) The multiplication map $n$ acts as multiplication by $n^{2i}$ on homology and therefore all cycles in $A^i_{(j)}(X_\eta)$ are homologically trivial for $j \neq 0$.

Since under the Fourier-Mukai transform we have $F(A^i_{(j)}(X_\eta)) = A^g_{(j-i+j)}(X_\eta)$, the elements of $A^i$ that lie in $A^i_{(j)}$ are characterized by the codimension of their Fourier transform (namely $g - i + j$).
Suppose now that \( c = \sum c^{(j)} \in A^i(X_\eta) \) with \( c^{(j)} \in A^i_{(j)}(X_\eta) \), where the decomposition corresponds to \( \varphi := F(c) = \sum \varphi^{(j)} \) with \( \varphi^{(j)} \in A^{i-j}(X_\eta) \).

**Theorem 7.1.** Let \( c = c_0 = \sum c^{(j)} \in A^i(X_\eta) \) with \( c^{(j)} \in A^i_{(j)}(X_\eta) \). Assume for some \( j' \) that \( \varphi_{(j')}^{(j')} \neq 0 \), where \( \varphi_0 \) is the specialization of \( \varphi \) and \( \varphi_0^{(j')} \) the codimension \( g - i + j' \)-part of \( \varphi_0 \). Then \( c^{(j')} \neq 0 \).

**Proof.** The specialization map preserves the codimension of cycles. Therefore, if \( c^{(j')} = 0 \) then \( \varphi^{(j')} = 0 \), hence \( \varphi_0^{(j')} = 0 \) and this contradicts our assumption. \( \square \)

This theorem, which holds as well for cycles modulo algebraic equivalence, can be used to prove non-vanishing results for cycles. For the rest of this section we work modulo algebraic equivalence. For example, consider a threefold \( \mathcal{Z}/S \) such that \( \mathcal{Z}_0 \) is a smooth cubic threefold and \( \mathcal{Z}_0 \) is a generic nodal cubic threefold. We shall consider the Picard variety of the Fano surface of this degenerating cubic threefold and this will give us a degenerating abelian variety of dimension 5, cf. [5].

As is well-known the nodal cubic threefold \( \mathcal{Z}_0 \), and hence its Fano surface, corresponds to a canonical genus 4 curve \( C \) in \( \mathbb{P}^3 \), see e.g. [9] Section 2. The genericity assumption means that the curve \( C \) is a generic curve and hence we may assume by Ceresa’s result [4] that the class \( C^{(1)} \) does not vanish in the Jacobian \( B \) of the curve \( C \). Since \( C \) is a trigonal curve we have by [6] that \( C^{(j)} \neq 0 \) for \( j \geq 2 \). Hence the Beauville decomposition of \( C \) is \( [C] = C^{(0)} + C^{(1)} \) with \( F_B(C^{(0)}) \in A^1_{(0)}(B) \) and \( F_B(C^{(1)}) \in A^2_{(1)}(B) \).

The Picard variety \( \mathcal{X}/S \) of the Fano surface of \( \mathcal{Z}/S \) defines a principally polarized semi-abelian variety with central fibre a rank-one extension of the Jacobian \( B \) of the curve \( C \), see [9], Corollary 6.3 and Section 10. The principal polarization on \( X_\eta \) is induced by a geometrically defined divisor \( \Theta \). Let \( \Sigma \) be the Fano surface of lines in \( \mathcal{Z}_\eta \). If \( s \in \Sigma \) we denote by \( l_s \) the corresponding line in \( \mathcal{Z}_\eta \). For each \( s \in \Sigma \) we have the divisor

\[
\Delta_s = \{ s' \in \Sigma, l_{s'} \cap l_s \neq \emptyset \}
\]

on \( \Sigma \) as defined in [5]. We then have a natural map

\[
\Sigma \to \text{Pic}^0(\Sigma), \quad s \mapsto \Delta_s - \Delta_{s_0},
\]

with \( s_0 \in \Sigma \) a base point. It is well known that the cohomology class of \( \Sigma \) in \( \text{Pic}^0(\Sigma) \) is equal to that of the cycle \( \Theta^4/3! \), see [5]. By [2], Propositions 3 and 4, we have that \( A^3_{(j)}(X_\eta) = 0 \) for \( j < 0 \) and \( A^5_{(j)}(X_\eta) = 0 \) for \( j \neq 0 \). We have therefore the decomposition

\[
[\Sigma] = \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)} \quad \text{with} \quad \Sigma^{(1)} \in A^3_{(1)}.
\]

Indeed, \( \Sigma^{(1)} \in A^3_{(1)}(X_\eta) \), hence \( F(\Sigma^{(j)}) \in A^{2+j}_{(j)}(X_\eta) \) which is zero for \( j \geq 3 \). Now we show that \( \Sigma^{(1)} \neq 0 \), and we thus obtain a cycle which is homologically
but not algebraically equivalent to zero. Since $\Theta \in A^1_{(0)}(X_\eta)$ this implies that 
$\Sigma$ is homologically, but not algebraically equivalent to $\Theta^1/3!$.
We denote by $X$ the completed rank one degeneration of $X_\eta$. The class $[\Sigma]$ 
degenerates to a cycle $[\Sigma_0] = \nu_*(\gamma)$ on the central fibre $X_0$ of class 
$$\gamma = q^*[C] + \frac{1}{2} q^*[C \cdot C] \cdot \eta,$$
where $C \cdot C$ is the Beauville decompositions. If $c$ 
By using the specialization of the Fourier-Mukai transform we can deduce the 
algebraic equivalence.
Specialization of the Beauville decomposition. We do this working modulo 
$z$ and let $c$ 
Since $C \cdot C$ then to the cycle $X$ on 
Pic

$C → \phi$ 
0 degenerates to a cycle $[Σ_c]$ where 
C

= $0$ the limit of the Fourier-Mukai transform. By Theorem 1.2, we have 
\[ \varphi_0 = \nu_*(\frac{1}{2} q^*[F_B(C) \cdot F_B(C)] - q^* F_B(C) \cdot \eta), \]
hence 
\[ \varphi_0^a = \nu_*(q^*[F_B(C^0) \cdot F_B(C^1)] - q^* F_B(C^1) \cdot \eta). \]
Since $C^1 \neq 0$ we conclude that $\varphi_0^a \neq 0$, and this implies the result.

By using the specialization of the Fourier-Mukai transform we can deduce the 
specialization of the Beauville decomposition. We do this working modulo 

algebraic equivalence.

**Proposition 7.2.** Let $c = c_0 \in A^1(X_\eta)$ with specialization $c_0 = \nu_*(q^* z + q^* w \cdot \eta)$, where $z \in A(B)$ and $w \in A^{-1}(B)$. Let $c = \sum c^{(j)}$ with $c^{(j)} \in A^1_{(j)}(X_\eta)$, and let $z = \sum z^{(j)}$ with $z^{(j)} \in A^1_{(j)}(B)$ and $w = \sum w^{(j)}$ with $w^{(j)} \in A^{1+1}_{(j)}(B)$ be the Beauville decompositions. If $c_0^{(j)}$ is the specialization of $c^{(j)}$, then 
\[ c_0^{(j)} = \nu_*(q^* z^{(j)} + q^* w^{(j)} \cdot \eta). \]

**Proof.** By the proof of the main theorem in [2], the component $c^{(j)}$ is defined as 
\((-1)^g F((-1)^* \phi^{(j)}) \) with $\phi^{(j)} \in A^{g-1+j}(X_\eta)$ (notation as above). The inversion 
on $X_\eta$ leaves the cell decomposition of the toroidal compactification invariant 
and hence extends naturally to $X_0$. Since $c_0^{(j)} = (-1)^g F((-1)^* \phi_0^{(j)})$ with 
$\phi_0^{(j)} \in A^{g-1+j}(X_0)$. Therefore, by Theorem 1.2, we have 
\[ c_0^{(j)} = (-1)^g (-1)^{g-1+j} \nu_*(q^* z^{(j)} - q^* w^{(j)} \cdot \eta) = \nu_*(q^* z^{(j)} + q^* w^{(j)} \cdot \eta). \]

$\square$

For example, let $C \to S$ be a genus $g$ curve with $C_0$ a smooth curve and 
$C_0$ a one-nodal curve with normalization $\tilde{C}_0$. Let $p$ be the node of $C_0$ and 
x_1, x_2 the points of $\tilde{C}_0$ lying over $p$. The compactified Jacobian $\mathcal{X} = \mathcal{F}C/S$ is 
then a complete rank one degeneration with central fibre the $\mathbb{P}^1$-bundle over 
Pic$^0(\tilde{C}_0)$ associated to the line bundle $J = O(x_1 - x_2)$. Let $\tilde{u} : C \to $ 
the compactified Abel-Jacobi map and let $c_0 = [\tilde{u}(C_0)]$. The cycle $c_0$ specializes 
then to the cycle $c_0 = [\tilde{u}(C_0)]$ with $c_0 = \nu_*(q^*[pt] + q^* c_0 \cdot \eta)$, where $[pt]$ is the
class of a point and $\tilde{c}_0$ is the class of the Abel-Jacobi image of the smooth curve $\tilde{C}_0$ in $\text{Pic}^0(\tilde{C}_0)$, see e.g. [9], Proposition 7.1. By Proposition 7.2 we have then

$$c^{(j)}_0 = \begin{cases} q^{*}\tilde{c}^{(j)}_0 \cdot \eta, & j \neq 0, \\ q^{*}[pt] + q^{*}\tilde{c}_0^{(0)} \cdot \eta, & j = 0. \end{cases}$$

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References
