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Division by Zero in Non-involutive Meadows

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Abstract. Meadows have been proposed as alternatives for fields with a purely equational axiomatization. At the basis of meadows lies the decision to make the multiplicative inverse operation total by imposing that the multiplicative inverse of zero is zero. Thus, the multiplicative inverse operation of a meadow is an involution. In this paper, we study 'non-involutive meadows', i.e. variants of meadows in which the multiplicative inverse of zero is not zero, and pay special attention to non-involutive meadows in which the multiplicative inverse of zero is one.

Keywords: non-involutive meadow, one-based non-involutive meadow, one-totalized field, one-totalized field of rational numbers, equational specification, initial algebra specification

MSC2000 codes: 12E12, 12L12, 68Q65

1 Introduction

The primary mathematical structure for measurement and computation is unquestionably a field. However, fields do not have a purely equational axiomatization, not all fields are total algebras, and the class of all total algebras that satisfy the axioms of a field is not a variety. This means that the theory of abstract data types cannot use the axioms of a field in applications to number systems based on rational, real or complex numbers.

In [7], meadows are proposed as alternatives for fields with a purely equational axiomatization. A meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero. Thus, all meadows are total algebras and the class of all meadows is a variety. At the basis of meadows lies the decision to make the multiplicative inverse operation total by imposing that the multiplicative inverse of zero is zero. All fields in which the multiplicative inverse of zero is zero, called zero-totalized fields, are meadows, but not conversely. In 2009, we found in [13] that meadows were already introduced almost 35 years earlier in [11], where they go by the name of desirable pseudo-fields. This discovery was first reported in [4].

We expect the total multiplicative inverse operation of zero-totalized fields, which is conceptually and technically simpler than the conventional partial multiplicative inverse operation, to be useful in among other things mathematics education. Recently, in a discussion about research and development in mathematics education (M. van den Heuvel-Panhuizen, personal communication, March 25, 2014), we came across the alternative where the multiplicative inverse operation is made total by imposing that the multiplicative inverse of zero is one (see [12, pp. 158–160]). At first sight, this seems a poor alternative. However, it turns out to be difficult to substantiate this without working out the details of the variants of meadows in which the multiplicative inverse of zero is one.

By imposing that the multiplicative inverse of zero is zero, the multiplicative inverse operation is made an involution. Therefore, we coined the name non-involutive meadow for a variant of a meadow in which the multiplicative inverse of zero is not zero and the name n-based non-involutive meadow ($n > 0$) for a non-involutive meadow in which the multiplicative inverse of zero is n. We consider both zero-based meadow and involutive meadow to be alternative names for a meadow. In this paper, we work out the details of one-based non-involutive meadows and non-involutive meadows.

We will among other things give equational axiomatizations of one-based non-involutive meadows and non-involutive meadows. The axiomatization of non-involutive meadows allows of a uniform treatment of n-based non-involutive meadows for all $n > 0$. It remains an open question whether there exists an equational axiomatization of the total algebras that are either involutive meadows or non-involutive meadows.

This paper is organized as follows. First, we survey the axioms for meadows and related results (Section 2). Next, we give the axioms for one-based non-involutive meadows and present results concerning the connections of one-based non-involutive meadows with meadows, one-totalized fields in general, and the one-totalized field of rational numbers (Section 3). Then, we give the axioms for non-involutive meadows and present generalizations of the main results from the previous section to n-based non-involutive meadows (Section 4). Finally, we make some concluding remarks (Section 5).

2 Meadows

In this section, we give a finite equational specification of the class of all meadows and present related results. For proofs, the reader is referred to earlier papers in which meadows have been investigated. Meadows has been proposed as alternatives for fields with a purely equational axiomatization in [7]. They have been further investigated in e.g. [2,3,4,8] and applied in e.g. [1,5,6].

A meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying two equations which imply that the multiplicative inverse of zero is zero. Hence, the signature of meadows includes the signature of a commutative ring with a multiplicative identity element.

The signature of commutative rings with a multiplicative identity element consists of the following constants and operators:

Table 1. Axioms of a commutative ring with a multiplicative identity element

$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x + y = y + x$	$x \cdot y = y \cdot x$
$x + 0 = x$	$x \cdot 1 = x$
$x + (-x) = 0$	$x \cdot (y + z) = x \cdot y + x \cdot z$

Table 2. Additional axioms for a meadow

$(x^{-1})^{-1} = x$	(2.1)
$x \cdot (x \cdot x^{-1}) = x$	(2.2)

- the *additive identity* constant 0;
- the *multiplicative identity* constant 1;
- the binary *addition* operator $+$;
- the binary *multiplication* operator $\cdot$;
- the unary *additive inverse* operator $-$;

The signature of meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity element and in addition:

- the unary *zero-totalized multiplicative inverse* operator ${}^{-1}$.

We write:

$$\Sigma_{\mathrm{CR}} \text{ for } \{0, 1, +, \cdot, -\} ,$$
$$\Sigma_{\mathrm{Md}} \text{ for } \Sigma_{\mathrm{CR}} \cup \{{}^{-1}\} .$$

We assume that there are infinitely many variables, including x, y and z. Terms are build as usual. We use infix notation for the binary operators, prefix notation for the unary operator $-$, and postfix notation for the unary operator ${}^{-1}$. We use the usual precedence convention to reduce the need for parentheses. We introduce subtraction, division, and squaring as abbreviations: $p - q$ abbreviates $p + (-q)$, $p \,/\, q$ abbreviates $p \cdot q^{-1}$, and p^2 abbreviates $p \cdot p$. For each non-negative natural number n, we write $\underline{n}$ for the numeral for n. That is, the term $\underline{n}$ is defined by induction on n as follows: $\underline{0} = 0$ and $\underline{n+1} = \underline{n} + 1$.

The constants and operators from the signature of meadows are adopted from rational arithmetic, which gives an appropriate intuition about these constants and operators.

A commutative ring with a multiplicative identity element is a total algebra over the signature Σ_{CR} that satisfies the equations given in Table 1. A *meadow* is a total algebra over the signature Σ_{Md} that satisfies the equations given in Tables 1 and 2.[1] We write:

[1] Throughout the paper, we use the term total algebra instead of algebra to emphasis that we mean an algebra without partial operations.

E_{CR} for the set of all equations in Table 1 ,
E_{mi^0} for the set of all equations in Table 2 ,
E_{Md} for $E_{\mathrm{CR}} \cup E_{\mathrm{mi}^0}$.

Equation (2.1) is called *Ref*, for reflection, and equation (2.2) is called *Ril*, for restricted inverse law.

Equations making the nature of the multiplicative inverse operation in meadows more clear are derivable from the equations E_{Md}.

Proposition 1. *The equations*

$$0^{-1} = 0 \ , \quad 1^{-1} = 1 \ , \quad (-x)^{-1} = -(x^{-1}) \ , \quad (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$$

are derivable from the equations E_{Md}.

Proof. Theorem 2.2 from [7] is concerned with the derivability of the first equation and Proposition 2.8 from [3] is concerned with the derivability of the last two equations. The derivability of the second equation is trivial. □

The advantage of working with a total multiplicative inverse operation lies in the fact that conditions like $x \neq 0$ in $x \neq 0 \Rightarrow x \cdot x^{-1} = 1$ are not needed to guarantee meaning.

A *non-trivial meadow* is a meadow that satisfies the *separation axiom*

$$0 \neq 1 \ ;$$

and a *cancellation meadow* is a meadow that satisfies the *cancellation axiom*

$$x \neq 0 \wedge x \cdot y = x \cdot z \Rightarrow y = z$$

or, equivalently, the *general inverse law*

$$x \neq 0 \Rightarrow x \cdot x^{-1} = 1 \ .$$

A *totalized field* is a total algebra over the signature Σ_{Md} that satisfies the equations E_{CR}, the separation axiom, and the general inverse law. A *zero-totalized field* is a totalized field that satisfies in addition the equation $0^{-1} = 0$.

Proposition 2. *The equations* E_{mi^0} *are derivable from the axiomatization of zero-totalized fields given above.*

Proof. This is Lemma 2.5 from [7]. □

The following is a corollary of Proposition 2.

Corollary 1. *The class of all non-trivial cancellation meadows and the class of all zero-totalized fields are the same.*

Not all non-trivial meadows are zero-totalized fields, e.g. the initial meadow is not a zero-totalized field. Nevertheless, we have the following theorem.

Theorem 1. *The equational theory of meadows and the equational theory of zero-totalized fields are the same.*

Proof. This is Theorem 3.10 from [3]. □

Theorem 1 can be read as follows: E_{Md} is a finite basis for the equational theory of cancellation meadows.

As a consequence of Theorem 1, the separation axiom and the cancellation axiom may be used to show that an equation is derivable from the equations E_{Md}.

The cancellation meadow that we are most interested in is $\mathcal{Q}_0$, the zero-totalized field of rational numbers. $\mathcal{Q}_0$ differs from the field of rational numbers only in that the multiplicative inverse of zero is zero.

Theorem 2. *$\mathcal{Q}_0$ is the initial algebra among the total algebras over the signature Σ_{Md} that satisfy the equations*

$$E_{\mathrm{Md}} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{-1} = 1\} .$$

Proof. This is Theorem 9 from [4]. □

The following is an outstanding question with regard to meadows: does there exist an equational specification of the class of all meadows with less than 10 equations?

3 One-Based Non-involutive Meadows

By imposing that the multiplicative inverse of zero is zero, the multiplicative inverse operation of a meadow is made an involution. Therefore, we coined the name non-involutive meadow for a variant of a meadow in which the multiplicative inverse of zero is not zero and the name one-based non-involutive meadow for a non-involutive meadow in which the multiplicative inverse of zero is one. In this section, we give a finite equational specification of the class of all one-based non-involutive meadows. Moreover, we present results concerning the connections of one-based non-involutive meadows with meadows, one-totalized fields in general, and the one-totalized field of rational numbers.

A one-based non-involutive meadow is a commutative ring with a multiplicative identity element and a total multiplicative inverse operation satisfying four equations which imply that the multiplicative inverse of zero is one.

The signature of one-based non-involutive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity element and in addition:

– the unary *one-totalized multiplicative inverse* operator ${}^{\backsim 1}$.[2]

[2] We use different symbols for the zero-totalized and one-totalized multiplicative inverse operations to allow of defining these operations in terms of each other.

Table 3. Additional axioms for a one-based non-involutive meadow

$(x^{\backsim 1})^{\backsim 1} = x + (1 - x \cdot x^{\backsim 1})$	(3.1)
$x \cdot (x \cdot x^{\backsim 1}) = x$	(3.2)
$x^{\backsim 1} \cdot (x^{\backsim 1})^{\backsim 1} = 1$	(3.3)
$(x \cdot (x^{\backsim 1} \cdot x^{\backsim 1}))^{\backsim 1} \cdot (x \cdot x^{\backsim 1}) = x$	(3.4)

We write:

$$\Sigma_{\mathrm{NiMd}^1} \text{ for } \Sigma_{\mathrm{CR}} \cup \{{}^{\backsim 1}\} \ .$$

We use postfix notation for the unary operator ${}^{\backsim 1}$.

A *one-based non-involutive meadow* is a total algebra over the signature Σ_{NiMd^1} that satisfies the equations given in Tables 1 and 3. We write:

$$\begin{array}{ll} E_{\mathrm{mi}^1} & \text{for the set of all equations in Table 3 ,} \\ E_{\mathrm{NiMd}^1} & \text{for } E_{\mathrm{CR}} \cup E_{\mathrm{mi}^1} \ . \end{array}$$

Apart from the different symbols used for the multiplicative inverse operation, equation (3.1) is *Ref* adapted to one-totalization of the multiplicative inverse operation and equation (3.2) is simply *Ril*. The counterpart of equation (3.4), viz. $(x \cdot (x^{-1} \cdot x^{-1}))^{-1} \cdot (x \cdot x^{-1}) = x$, is derivable from (2.1) and (2.2) and consequently does hold in meadows as well. However, the counterpart of equation (3.3), viz. $x^{-1} \cdot (x^{-1})^{-1} = 1$, does not hold in meadows.

Proposition 3. *The equations $0^{\backsim 1} = 1$ and $1^{\backsim 1} = 1$ are derivable from the equations E_{NiMd^1}.*

Proof. We have $0^{\backsim 1} \cdot (0^{\backsim 1})^{\backsim 1} = 1$ by (3.3) and $(0^{\backsim 1})^{\backsim 1} = 1$ by (3.1). From these equations, it follows immediately that $0^{\backsim 1} = 1$. We have $1^{\backsim 1} = 1$ by (3.2). □

The zero-totalized multiplicative inverse operator can be explicitly defined in terms of the one-totalized multiplicative inverse operator by the equation $x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})$ and the one-totalized multiplicative inverse operator can be explicitly defined in terms of the zero-totalized multiplicative inverse operator by the equation $x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})$.

The following two lemmas will be used in proofs of subsequent theorems.

Lemma 1. *The following is derivable from $E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\}$ as well as $E_{\mathrm{NiMd}^1} \cup \{x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})\}$:*

$$x \cdot x^{\backsim 1} = x \cdot x^{-1} \ .$$

Proof. We have $x \cdot x^{\backsim 1} = x \cdot x^{-1} + (x - x \cdot (x \cdot x^{-1}))$ by $x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})$. From this equation, it follows by (2.2) that $x \cdot x^{\backsim 1} = x \cdot x^{-1}$. We have $x \cdot x^{-1} = (x \cdot (x \cdot x^{\backsim 1})) \cdot x^{\backsim 1}$ by $x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})$. From this equation, it follows by (3.2) that $x \cdot x^{-1} = x \cdot x^{\backsim 1}$. □

Table 4. Formulas concerning ${}^{\backsim 1}$ and ${}^{-1}$

$x \neq 0 \Rightarrow x^{\backsim 1} = x^{-1}$	(4.1)
$x \neq 0 \Rightarrow (x^{\backsim 1})^{\backsim 1} = (x^{-1})^{-1}$	(4.2)
$x \neq 0 \Rightarrow x \cdot (x^{\backsim 1} \cdot x^{\backsim 1}) = x^{-1}$	(4.3)
$x \neq 0 \Rightarrow (x \cdot (x^{\backsim 1} \cdot x^{\backsim 1}))^{\backsim 1} = x$	(4.4)
$x = 0 \Rightarrow x^{\backsim 1} = 1$	(4.5)
$x = 0 \Rightarrow (x^{\backsim 1})^{\backsim 1} = 1$	(4.6)

Lemma 2. *The conditional equations given in Table 4 are derivable from* $E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\}$*:*

Proof. Recall that the equations $0^{-1} = 0$ and $1^{-1} = 1$ are derivable from E_{Md}. By Theorem 1, we may use the general inverse law (*Gil*) and the separation axiom (*Sep*) to prove derivability from E_{Md}. It follows from (2.2) and *Sep* that $x \neq 0 \Rightarrow x^{-1} \neq 0$ (*).

The derivability of (4.1)–(4.6) from E_{Md} and the defining equation of ${}^{\backsim 1}$ is proved as follows:

- (4.1) follows immediately from the defining equation of ${}^{\backsim 1}$ and *Gil*;
- (4.2) follows immediately from (*) and (4.1);
- (4.3) follows immediately from (4.1), (2.1), and (2.2);
- (4.4) follows immediately from (4.3), (*), (4.1), and (2.1);
- (4.5) follows immediately from the defining equation of ${}^{\backsim 1}$ and $0^{-1} = 0$;
- (4.6) follows immediately from (4.5), the defining equation of ${}^{\backsim 1}$, and $1^{-1} = 1$. □

Despite the different multiplicative inverse operators, E_{Md} and E_{NiMd^1} are essentially the same in a well-defined sense.

Theorem 3. E_{Md} *is definitionally equivalent to* E_{NiMd^1}*,*[3] *i.e.*

$$E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\} \vdash E_{\mathrm{NiMd}^1} \cup \{x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})\}$$

and

$$E_{\mathrm{NiMd}^1} \cup \{x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})\} \vdash E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\}\ .$$

Proof. By Theorem 1, we may use the general inverse law (*Gil*) to prove derivability from E_{Md}. Recall that the equation $0^{-1} = 0$ is derivable from E_{Md}. By Lemma 1, the equation $x \cdot x^{\backsim 1} = x \cdot x^{-1}$ (**) is derivable from E_{Md} and the defining equation of ${}^{\backsim 1}$.

The derivability of (3.1)–(3.4) and the defining equation of ${}^{-1}$ from (2.1)–(2.2) and the defining equation of ${}^{\backsim 1}$ is proved as follows:

[3] The notion of definitional equivalence originates from [9], where it was introduced, in the setting of first-order theories, under the name of synonymy. In [14], the notion of definitional equivalence was introduced in the setting of equational theories under the ambiguous name of equivalence. An abridged version of [14] appears in [10].

– if $x \neq 0$, then (3.1) follows immediately from (4.2), (2.1), *Gil*, and (**);
  if $x = 0$, then (3.1) follows immediately from (4.6);
– (3.2) follows immediately from (**) and (2.2);
– if $x \neq 0$, then (3.3) follows immediately from (4.1), (4.2), (2.1), and *Gil*;
  if $x = 0$, then (3.3) follows immediately from (4.5) and (4.6);
– if $x \neq 0$, then (3.4) follows immediately from (4.4), (**), and (2.2);
  if $x = 0$, then (3.4) follows trivially;
– if $x \neq 0$, then the defining equation of ${}^{-1}$ follows immediately from (4.3);
  if $x = 0$, then the defining equation of ${}^{-1}$ follows immediately from $0^{-1} = 0$.

The derivability of (2.1)–(2.2) and the defining equation of ${}^{\backsim 1}$ from (3.1)–(3.4) and the defining equation of ${}^{-1}$ is proved as follows:

– (2.1) follows immediately from the defining equation of ${}^{-1}$ and (3.4) (twice);
– (2.2) follows immediately from the defining equation of ${}^{-1}$ and (3.2) (twice);
– the defining equation of ${}^{\backsim 1}$ follows immediately from the the defining equation of ${}^{-1}$, (3.2), (3.3), and (3.1). □

A *non-trivial one-based non-involutive meadow* is a one-based non-involutive meadow that satisfies the separation axiom and a *one-based non-involutive cancellation meadow* is a one-based non-involutive meadow that satisfies the cancellation axiom or, equivalently, $x \neq 0 \Rightarrow x \cdot x^{\backsim 1} = 1$.

The following two lemmas will be used in the proof of a subsequent theorem.

Lemma 3. *Let α be the mapping from the class of all meadows to the class of all one-based non-involutive meadows that maps each meadow $\mathcal{A}$ to the restriction to Σ_{NiMd^1} of the unique expansion of $\mathcal{A}$ for which $E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\}$ holds. Then:*

1. *α is a bijection;*
2. *the restriction of α to the class of all cancellation meadows is a bijection.*

Proof. Let α' be the mapping from the class of all one-based non-involutive meadows to the class of all meadows that maps each one-based non-involutive meadow $\mathcal{A}'$ to the restriction to Σ_{Md} of the unique expansion of $\mathcal{A}'$ for which $E_{\mathrm{NiMd}^1} \cup \{x^{-1} = x \cdot (x^{\backsim 1} \cdot x^{\backsim 1})\}$ holds. Then $\alpha \circ \alpha'$ and $\alpha' \circ \alpha$ are identity mappings by Theorem 3. Hence, α is a bijection.

By Lemma 1, for each meadow $\mathcal{A}$, $x \cdot x^{\backsim 1} = x \cdot x^{-1}$ holds in the unique expansion of $\mathcal{A}$ for which $E_{\mathrm{Md}} \cup \{x^{\backsim 1} = x^{-1} + (1 - x \cdot x^{-1})\}$ holds. This implies that, for each meadow $\mathcal{A}$, $\alpha(\mathcal{A})$ satisfies the cancellation axiom if $\mathcal{A}$ satisfies it. In other words, α maps each cancellation meadow to a one-based non-involutive cancellation meadow. Similar remarks apply to the inverse of α. Hence, the restriction of α to the class of all cancellation meadows is a bijection. □

Lemma 4. *Let ϵ be the mapping from the set of all equations between terms over Σ_{NiMd^1} to the set of all equations between terms over Σ_{Md} that is induced by the defining equation of ${}^{\backsim 1}$ and let α be as in Lemma 3. Then:*

1. *for each equation ϕ between terms over Σ_{NiMd^1}, $E_{\mathrm{NiMd}^1} \vdash \phi$ iff $E_{\mathrm{Md}} \vdash \epsilon(\phi)$;*
2. *for each meadow $\mathcal{A}$ and equation ϕ between terms over Σ_{NiMd^1}, $\alpha(\mathcal{A}) \models \phi$ iff $\mathcal{A} \models \epsilon(\phi)$.*

Proof. Let ϵ' be the mapping from the set of all equations between terms over Σ_{Md} to the set of all equations between terms over Σ_{NiMd^1} that is induced by the defining equation of ${}^{-1}$. Then, by Theorem 3:

– for each equation ϕ between terms over Σ_{NiMd^1}, $E_{\mathrm{Md}} \vdash \epsilon(\phi)$ if $E_{\mathrm{NiMd}^1} \vdash \phi$;
– for each equation ϕ' between terms over Σ_{Md}, $E_{\mathrm{NiMd}^1} \vdash \epsilon'(\phi')$ if $E_{\mathrm{Md}} \vdash \phi'$;
– for each equation ϕ between terms over Σ_{NiMd^1}, $E_{\mathrm{NiMd}^1} \vdash \epsilon'(\epsilon(\phi)) \Leftrightarrow \phi$.

From this it follows immediately that, for each equation ϕ between terms over Σ_{NiMd^1}, $E_{\mathrm{NiMd}^1} \vdash \phi$ iff $E_{\mathrm{Md}} \vdash \epsilon(\phi)$.

Let α' be as in the proof of Lemma 3. Then for each one-based non-involutive meadow $\mathcal{A}'$ and equation ϕ between terms over Σ_{NiMd^1}, $\mathcal{A}' \models \phi$ iff $\alpha'(\mathcal{A}') \models \epsilon(\phi)$ by the construction of $\alpha'(\mathcal{A}')$. From this and the fact that $\alpha'(\alpha(\mathcal{A})) = \mathcal{A}$ for each meadow $\mathcal{A}$, it follows that, for each meadow $\mathcal{A}$ and equation ϕ between terms over Σ_{NiMd^1}, $\alpha(\mathcal{A}) \models \phi$ iff $\mathcal{A} \models \epsilon(\phi)$. □

Recall that a totalized field is a total algebra over the signature Σ_{Md} that satisfies the equations E_{CR}, the separation axiom, and the general inverse law. A *one-totalized field* is a totalized field that satisfies in addition the equation $0^{-1} = 1$.

Proposition 4. *After replacing all occurrences of the operator ${}^{\backsim 1}$ with ${}^{-1}$, the equations E_{mi^1} are derivable from the axiomatization of one-totalized fields given above.*

Proof. It follows from the general inverse law and the separation axiom that $x \neq 0 \Rightarrow x^{-1} \neq 0$. It follows from this and the general inverse law that $x \neq 0 \Rightarrow (x^{-1})^{-1} = x$ (†).

The derivability of (3.1)–(3.4) from E_{CR}, the separation axiom, the general inverse law, and $0^{-1} = 1$ is proved as follows:

– if $x \neq 0$, then (3.1) follows immediately from (†) and *Gil*;
  if $x = 0$, then (3.1) follows immediately from $0^{-1} = 1$ and *Gil*;
– if $x \neq 0$, then (3.2) follows immediately from *Gil*;
  if $x = 0$, then (3.2) follow trivially;
– if $x \neq 0$, then (3.3) follows immediately from (†) and *Gil*;
  if $x = 0$, then (3.3) follows immediately from $0^{-1} = 1$ and *Gil*;
– if $x \neq 0$, then (3.4) follows immediately from *Gil* and (†);
  if $x = 0$, then (3.4) follows trivially. □

The following is a corollary of Proposition 4.

Corollary 2. *Up to naming of the multiplicative inverse operation, the class of all non-trivial one-based non-involutive cancellation meadows and the class of all one-totalized fields are the same.*

Not all non-trivial one-based non-involutive meadows are one-totalized fields, e.g. the initial one-based non-involutive meadow is not a one-totalized field. Nevertheless, we have the following theorem.

Theorem 4. *Up to naming of the multiplicative inverse operation, the equational theory of one-based non-involutive meadows and the equational theory of one-totalized fields are the same.*

Proof. Let ϵ be as in Lemma 4. By Lemmas 3.2 and 4.2, we have that, for each equation ϕ between terms over Σ_{NiMd^1}, ϕ holds in all one-based non-involutive cancellation meadows only if $\epsilon(\phi)$ holds in all cancellation meadows. From this, Theorem 1 and Corollary 1 it follows that, for each equation ϕ between terms over Σ_{NiMd^1}, ϕ holds in all one-based non-involutive cancellation meadows only if $\epsilon(\phi)$ is derivable from E_{Md}. From this and Lemma 4.1 it follows that, for each equation ϕ between terms over Σ_{NiMd^1}, ϕ holds in all one-based non-involutive cancellation meadows only if ϕ is derivable from E_{NiMd^1}. Hence, the equational theory of one-based non-involutive meadows and the equational theory of one-based non-involutive cancellation meadows are the same. From this and Corollary 2 it follows that the equational theory of one-based non-involutive meadows and the equational theory of one-totalized fields are the same. □

Theorem 4 can be read as follows: E_{NiMd^1} is a finite basis for the equational theory of one-based non-involutive cancellation meadows.

As a consequence of Theorem 4, the separation axiom and the cancellation axiom may be used to show that an equation is derivable from the equations E_{NiMd^1}.

Proposition 5. *The equations*

$$(-x)^{\backsim 1} = -(x^{\backsim 1}) \cdot (x \cdot x^{\backsim 1}) + (1 - x \cdot x^{\backsim 1}) \;,$$
$$(x \cdot y)^{\backsim 1} = (x^{\backsim 1} \cdot y^{\backsim 1}) \cdot ((x \cdot x^{\backsim 1}) \cdot (y \cdot y^{\backsim 1})) + (1 - (x \cdot x^{\backsim 1}) \cdot (y \cdot y^{\backsim 1}))$$

are derivable from the equations E_{NiMd^1}.

Proof. Recall that the equation $0^{\backsim 1} = 1$ is derivable from E_{NiMd^1}. The conditional equation $x \neq 0 \Rightarrow x \cdot x^{\backsim 1} = 1$, which will be called *Gil′* below, is a variant of the general inverse law derivable from (3.2) and the cancellation axiom.

- if $x \neq 0$, then $-x \cdot (-x \cdot (-x)^{\backsim 1}) = -x$ by (3.2) and $-x \cdot (-x \cdot -(x^{\backsim 1})) = -x$ by (3.2), hence $(-x)^{\backsim 1} = -(x^{\backsim 1})$ by the cancellation axiom, hence $(-x)^{\backsim 1} = -(x^{\backsim 1}) \cdot (x \cdot x^{\backsim 1}) + (1 - x \cdot x^{\backsim 1})$ by *Gil′*;
  if $x = 0$, then the equation reduces to $0^{\backsim 1} = 1$;
- if $x \neq 0$ and $y \neq 0$, then $(x \cdot y) \cdot ((x \cdot y) \cdot (x \cdot y)^{\backsim 1}) = x \cdot y$ by (3.2) and $(x \cdot y) \cdot ((x \cdot y) \cdot (x^{\backsim 1} \cdot y^{\backsim 1})) = x \cdot y$ by (3.2), hence $(x \cdot y)^{\backsim 1} = x^{\backsim 1} \cdot y^{\backsim 1}$ by the cancellation axiom, hence $(x \cdot y)^{\backsim 1} = (x^{\backsim 1} \cdot y^{\backsim 1}) \cdot ((x \cdot x^{\backsim 1}) \cdot (y \cdot y^{\backsim 1})) + (1 - (x \cdot x^{\backsim 1}) \cdot (y \cdot y^{\backsim 1}))$ by *Gil′*;
  if $x = 0$ or $y = 0$, then the equation reduces to $0^{\backsim 1} = 1$. □

Table 5. Additional axioms for a non-involutive meadow

$0^{\sim 1} \cdot (x^{\sim 1})^{\sim 1} = 0^{\sim 1} \cdot x + (1 - x \cdot x^{\sim 1})$	(5.1)
$x \cdot (x \cdot x^{\sim 1}) = x$	(5.2)
$x^{\sim 1} \cdot (x^{\sim 1})^{\sim 1} = 1$	(5.3)
$(x \cdot (x^{\sim 1} \cdot x^{\sim 1}))^{\sim 1} \cdot (x \cdot x^{\sim 1}) = x$	(5.4)

The one-based non-involutive cancellation meadow that we are most interested in is $\mathcal{Q}_1$, the one-totalized field of rational numbers. $\mathcal{Q}_1$ differs from the field of rational numbers only in that the multiplicative inverse of zero is one.

Theorem 5. *$\mathcal{Q}_1$ is the initial algebra among the total algebras over the signature Σ_{NiMd^1} that satisfy the equations*

$$E_{\mathrm{NiMd}^1} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{\smile 1} = 1\} .$$

Proof. The proof goes as for Theorem 9 from [4]. □

4 Non-involutive Meadows

Recall that we coined the name non-involutive meadow for a variant of a meadow in which the multiplicative inverse of zero is not zero. Thus, in a non-involutive meadow, the multiplicative inverse of zero can be anything. In this section, we give a finite equational specification of the class of all non-involutive meadows. Moreover, we present generalizations of the main results from Section 3 to n-based non-involutive meadows. Because these generalizations turn out to present no additional complications, for most proofs, the reader is only informed about the main differences with the corresponding proofs from Section 3.

The signature of non-involutive meadows consists of the constants and operators from the signature of commutative rings with a multiplicative identity element and in addition:

- the unary *totalized multiplicative inverse* operator ${}^{\sim 1}$.

We write:

$$\Sigma_{\mathrm{NiMd}} \text{ for } \Sigma_{\mathrm{CR}} \cup \{{}^{\sim 1}\} .$$

We use postfix notation for the unary operator ${}^{\sim 1}$.

A *non-involutive meadow* is a total algebra over the signature Σ_{NiMd} that satisfies the equations given in Tables 1 and 5. An *n-based non-involutive meadow* is a non-involutive meadow that satisfies the equation $0^{\sim 1} = \underline{n}$. We write:

$$\begin{array}{ll} E_{\mathrm{mi}} & \text{for the set of all equations in Table 5 ,} \\ E_{\mathrm{NiMd}} & \text{for } E_{\mathrm{CR}} \cup E_{\mathrm{mi}} \text{ ,} \\ E_{\mathrm{NiMd}_n} & \text{for } E_{\mathrm{NiMd}} \cup \{0^{\sim 1} = \underline{n}\} \text{ .} \end{array}$$

Apart from the different symbols used for the multiplicative inverse operation, equation (5.1) is *Ref* adapted to the arbitrary totalization of the multiplicative inverse operation and equation (5.2) is simply *Ril*.

Notice that E_{NiMd^1} and E_{NiMd_1} are different sets of equations. However, both E_{NiMd^1} and E_{NiMd_1} equationally define the class of all one-based non-involutive meadows.

Proposition 6. *E_{NiMd^1} and E_{NiMd_1} are deductively equivalent, i.e.*

$$E_{\mathrm{NiMd}^1} \vdash E_{\mathrm{NiMd}_1} \quad \text{and} \quad E_{\mathrm{NiMd}_1} \vdash E_{\mathrm{NiMd}^1} \ .$$

Proof. To prove that $E_{\mathrm{NiMd}^1} \vdash E_{\mathrm{NiMd}_1}$, it is sufficient to prove that $0^{\smile 1} = 1$ and (5.1) are derivable from E_{NiMd^1}. By Proposition 3, we have that $E_{\mathrm{NiMd}^1} \vdash 0^{\smile 1} = 1$. From (3.1) and $0^{\smile 1} = 1$, (5.1) follows immediately. To prove that $E_{\mathrm{NiMd}_n} \vdash E_{\mathrm{NiMd}^1}$, it is sufficient to prove that (3.1) is derivable from E_{NiMd_n}. From (5.1) and $0^{\sim 1} = 1$, (3.1) follows immediately. □

A *non-trivial* (n*-based*) *non-involutive meadow* is an (n-based) non-involutive meadow that satisfies the separation axiom and an (n*-based*) *non-involutive cancellation meadow* is an (n-based) non-involutive meadow that satisfies the cancellation axiom or, equivalently, $x \neq 0 \Rightarrow x \cdot x^{\sim 1} = 1$.

Recall that a totalized field is a total algebra over the signature Σ_{Md} that satisfies the equations E_{CR}, the separation axiom, and the general inverse law. A *non-zero-totalized field* is a totalized field that satisfies the inequation $0^{-1} \neq 0$ and a n*-totalized field* is a totalized field that satisfies the equation $0^{-1} = \underline{n}$.

Proposition 7. *After replacing all occurrences of the operator ${}^{\sim 1}$ by ${}^{-1}$, the equations E_{mi} are derivable from the axiomatization of non-zero-totalized fields given above.*

Proof. The proof goes as for Proposition 4, with $0^{-1} = 1$ everywhere replaced by $0^{-1} \neq 0$. □

For each $n > 0$, n-based non-involutive meadows have a lot of properties in common with zero-based non-involutive meadows.

Theorem 6. *For each $n > 0$, E_{Md} is definitionally equivalent to E_{NiMd_n}, i.e.*

$$E_{\mathrm{Md}} \cup \{x^{\sim 1} = x^{-1} + \underline{n} \cdot (1 - x \cdot x^{-1})\} \vdash E_{\mathrm{NiMd}_n} \cup \{x^{-1} = x \cdot (x^{\sim 1} \cdot x^{\sim 1})\}$$
$$\text{and}$$
$$E_{\mathrm{NiMd}_n} \cup \{x^{-1} = x \cdot (x^{\sim 1} \cdot x^{\sim 1})\} \vdash E_{\mathrm{Md}} \cup \{x^{\sim 1} = x^{-1} + \underline{n} \cdot (1 - x \cdot x^{-1})\} \ .$$

Proof. The proof goes essentially as for Theorem 3. The proof of the derivability of (5.1)–(5.4) and the defining equation of ${}^{-1}$ from (2.1)–(2.2) and the defining equation of ${}^{\sim 1}$ goes slightly different for each of these equations in the case $x = 0$. □

Not all non-trivial n-based non-involutive meadows are n-totalized fields, e.g. the initial n-based non-involutive meadow is not a n-totalized field. Nevertheless, we have the following theorem.

Theorem 7. *For each $n > 0$, up to naming of the multiplicative inverse operation, the equational theory of n-based non-involutive meadows and the equational theory of n-totalized fields are the same.*

Proof. The proof goes as for Theorem 4. □

Theorem 7 can be read as follows: E_{NiMd_n} is a finite basis for the equational theory of n-based non-involutive cancellation meadows.

As a consequence of Theorem 7, the separation axiom and the cancellation axiom may be used to show that an equation is derivable from the equations E_{NiMd_n}.

Proposition 8. *For each $n > 0$, the equations*

$$0^{\sim 1} = \underline{n}\ , \qquad 1^{\sim 1} = 1\ , \qquad (-x)^{\sim 1} = -(x^{\sim 1}) \cdot (x \cdot x^{\sim 1}) + \underline{n} \cdot (1 - x \cdot x^{\sim 1})\ ,$$
$$(x \cdot y)^{\sim 1} = x^{\sim 1} \cdot y^{\sim 1} \cdot ((x \cdot x^{\sim 1}) \cdot (y \cdot y^{\sim 1})) + \underline{n} \cdot (1 - (x \cdot x^{\sim 1}) \cdot (y \cdot y^{\sim 1}))$$

are derivable from the equations E_{NiMd_n}.

Proof. The equation $0^{\sim 1} = \underline{n}$ belongs to E_{NiMd_n}. We have $1^{\sim 1} = 1$ by (5.2). The proof for the last two equations goes as for Proposition 5. □

The n-based non-involutive cancellation meadow that we are most interested in is $\mathcal{Q}_n$, the n-totalized field of rational numbers. $\mathcal{Q}_n$ differs from the field of rational numbers only in that the multiplicative inverse of zero is n.

Theorem 8. *$\mathcal{Q}_n$ is the initial algebra among the total algebras over the signature Σ_{NiMd} that satisfy the equations*

$$E_{\mathrm{NiMd}_n} \cup \{(1 + x^2 + y^2) \cdot (1 + x^2 + y^2)^{\sim 1} = 1\}\ .$$

Proof. The proof goes as for Theorem 9 from [4]. □

The following is an outstanding question with regard to non-involutive meadows: are the equational theory of non-involutive meadows and the equational theory of non-zero-totalized fields the same up to naming of the multiplicative inverse operation?

5 Concluding Remarks

We have worked out the details of one-based non-involutive meadows and non-involutive meadows. We have given finite equational specifications of the class of all one-based non-involutive meadows and the class of all non-involutive meadows. We have presented results concerning the connections of one-based non-involutive meadows with meadows, one-totalized fields in general, and the one-totalized field of rational numbers and also generalizations of these results to n-based non-involutive meadows.

One-based non-involutive meadows and non-involutive meadows require more axioms than (zero-based/involutive) meadows. We believe that the axioms of meadows are more easily memorized than the axioms of one-based non-involutive meadows and the axioms of non-involutive meadows. Despite the differences, the axiomatizations of (zero-based/involutive) meadows and n-based non-involutive meadows ($n > 0$) are essentially the same (i.e. they are definitionally equivalent). Moreover, the connections of n-based non-involutive meadows ($n > 0$) with n-totalized fields in general and the n-totalized field of rational numbers are essentially the same as the connections of (zero-based/involutive) meadows with zero-totalized fields in general and the zero-totalized field of rational numbers.

The equational specification of the class of all non-involutive meadows allows of a uniform treatment of n-based non-involutive meadows for all $n > 0$. It is an open question whether there exists a finite equational specification of the class of all involutive and non-involutive meadows.

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