A supersymmetric model for lattice fermions

Huijse, L.

Chapter 1
Introduction

1.1 Strongly interacting electron systems

The complexity of many-body systems and the richness of the emerging physics makes condensed matter physics a highly inspiring and challenging field. The constituent particles of condensed matter materials, electrons, protons and neutrons, are well-known. They interact via the electro-magnetic Coulomb force and are governed by the laws of quantum mechanics. In principle we can formulate matter in terms of a many-particle hamiltonian describing the interactions of the constituent particles. Nevertheless, it is hopelessly naive to expect that any relevant information about the material can be extracted from this hamiltonian. Instead many a simplification should, and as we shall see, can be made to describe condensed matter materials.
The success of band theory for conventional conductors and insulators, shows that the physics of condensed matter is governed by the valence electrons. The valence electrons feel a lattice potential coming from the attractive interaction with the crystallized ions. Where band theory concerns non-interacting electrons, Landau's Fermi liquid theory [1] enjoys unparalleled success in describing interacting electron systems. In fact, it is said that Fermi liquid theory is unreasonably successful, since despite its perturbative nature it does very well, way outside the perturbative regime. The reason for this lies in *adiabatic continuity* [2], the principle that the quantum numbers that characterize a system are determined by fundamental symmetries. It follows that the physics of the interacting system can be found from a non-interacting system as long as they share the same symmetries. This will typically not be true when the interactions are strong enough to induce a phase transition. As is well-known, nature presents us with numerous examples of materials in which the quasi-particle excitations have little or nothing to do with the microscopic degrees of freedom (the valence electrons). A beautiful example is the fractional quantum Hall system [3, 4, 5], where the quasi-particles carry a charge which is a fraction of the electron charge. Another famous example of a strongly interacting electron system showing 'non-Fermi liquid' behavior is formed by the high temperature superconductors. Examples of the poorly understood phases present in these systems are the pseudo gap phase [6, 7, 8, 9, 10] and the strange metal phase [11, 12, 13].

1.2 The Hubbard model

A widely used simplification in condensed matter physics is to study quantum many-body systems on a lattice. The simplification is motivated by the tight-binding approximation, which describes the valence electrons in a crystal by approximating the electron orbitals around the ions by points located at the positions of the ions in the crystal: the lattice sites. The tight-binding model, which is the simplest lattice model since the electrons are non-interacting, is used in band theory to derive the band structure of conducting solids. This model can be improved by incorporating effects of the electrostatic repulsion

between the electrons. In the Hubbard model this repulsive interaction is approximated by an on-site repulsion assuming that the ions are well separated and the overlap between neighboring orbitals is negligible. The hamiltonian consists of a nearest-neighbor hopping term parameterized by t and an on-site interaction for electrons with opposite spin with interaction strength U. Despite its simplicity, the Hubbard model poses theoretical physicists with a challenge for over 40 years. The model is believed to have a rich phase diagram which shows remarkable resemblance with that of the high T_c cuprates.
For low particle density or filling fraction n the inter-particle distance is large and the system is dominated by the kinetic term in the hamiltonian. In this regime, Fermi liquid theory is valid and the Hubbard model is believed to exhibit a metallic phase. For the half-filled system, that is on average one particle per site, with strong interactions $U/t >> 1$, the electrons get jammed, since double occupancy is energetically highly unfavorable. This results in an unconventional insulating phase, known as the Mott insulator. This phase is characterized by an anti-ferromagnetic ordering of the electron spins and is believed to be present in the undoped cuprate systems. Upon doping away from half-filling the anti-ferromagnetic order will be destroyed and the system is found to form stripe-like domains with spin- and charge order [14].
As of today the question whether the rich phenomenology of the cuprates is captured by the Hubbard model remains a subject of debate. The strongly interacting regime at intermediate filling is still a black box. Here both analytic as well as numerical techniques face the challenges posed by the Fermi sign problem. The Fermi statistics of the electrons renders both quantum field theoretical methods as well as quantum Monte-Carlo techniques into trouble. The fact that the Fermi sign problem falls into the class of NP hard problems [15], is a great incentive to propose drastically different approaches to the study of strongly interacting fermion systems (see for example [16, 17]).

1.3 Hard-core spinless fermions

Where the Hubbard model is already a dramatic simplification of the true interactions in the system, we now propose an even stronger simplification. Instead of electrons we consider hard-core spinless fermions. By the Pauli exclusion principle double occupancy is forbidden for spinless fermions. This is extended by the hard-core character of the particles, which imposes nearest neighbor exclusion. At low enough densities this system will exhibit a metallic phase described by (spinless) Fermi liquid theory. For this system on the square lattice the half-filled phase, that is one particle on every two sites, is trivial: there are two ordered patterns in which the fermions are completely jammed. However, for small hole-doping there is strong evidence for a stripe phase, where domain walls between the different orders favor being parallel [18, 19, 20]. So even though this system seems to violently oversimplify the problem, it can still exhibit non-trivial physics. Moreover, it has proven to be one of the few higher than $D = 1$ dimensional lattice models for which exact results can be obtained in the strongly interacting regime at intermediate filling [21]. Instrumental for these results was the idea to incorporate supersymmetry in the system [22].
In a remarkable series of work [22, 23, 21] P. Fendley and K. Schoutens together with J. de Boer and B. Nienhuis introduce and investigate a specific model for hard-core spinless fermions (see [24] for a review). They show that by turning on a next-nearest neighbor

repulsive term for the hard-core fermions, the model can be tuned to a supersymmetric point. The supersymmetry turns out to induce a subtle interplay between the kinetic and interaction terms at intermediate filling, leading to quantum criticality in D = 1 and a novel feature, called superfrustration, in D = 2 and D = 3.
Quantum criticality typically refers to a second order phase transition, not due to temperature fluctuations, but caused by quantum fluctuations [25]. Even though a quantum phase transition strictly speaking occurs at absolute zero temperature, it is often accompanied by a line of second order classical phase transitions at low, but non-zero, temperatures. Experiments on the superconducting cuprates [26] and heavy fermion materials [27], have induced a tremendous effort on the theory side, to try and understand the nature of quantum criticality for strongly interacting systems. It is widely believed that the physics of unconventional phases, such as the pseudo gap phase, is dominated by the nearby presence of a quantum critical point. The past decades has seen substantial progress in understanding and classifying quantum criticality in D = 1 [28] and the hard-core spinless fermions at the supersymmetric point nicely fit into this framework.
The phenomenon of frustration in lattice systems is best known in the context of spin degrees of freedom [29], the standard example being the anti-ferromagnet on the triangular lattice. As a result of the competition between different terms in the hamiltonian, the system cannot find one way to minimize its energy, instead the lowest energy state is (highly) degenerate. Although less familiar, frustration can also arise in systems with spinless charge degrees of freedom. An example is given in [30], where spinless fermions on a checkerboard lattice with repulsive interactions are shown to have a ground state degeneracy, exponential in the system size, in the absence of kinetic terms. In [21] P. Fendley and K. Schoutens show that the supersymmetric model on two dimensional lattices exhibits a strong form of quantum charge frustration. The competition between the kinetic and interaction terms in the model leads to an exponential number of quantum ground states, resulting in an extensive ground state entropy. This phenomenon is called superfrustration.

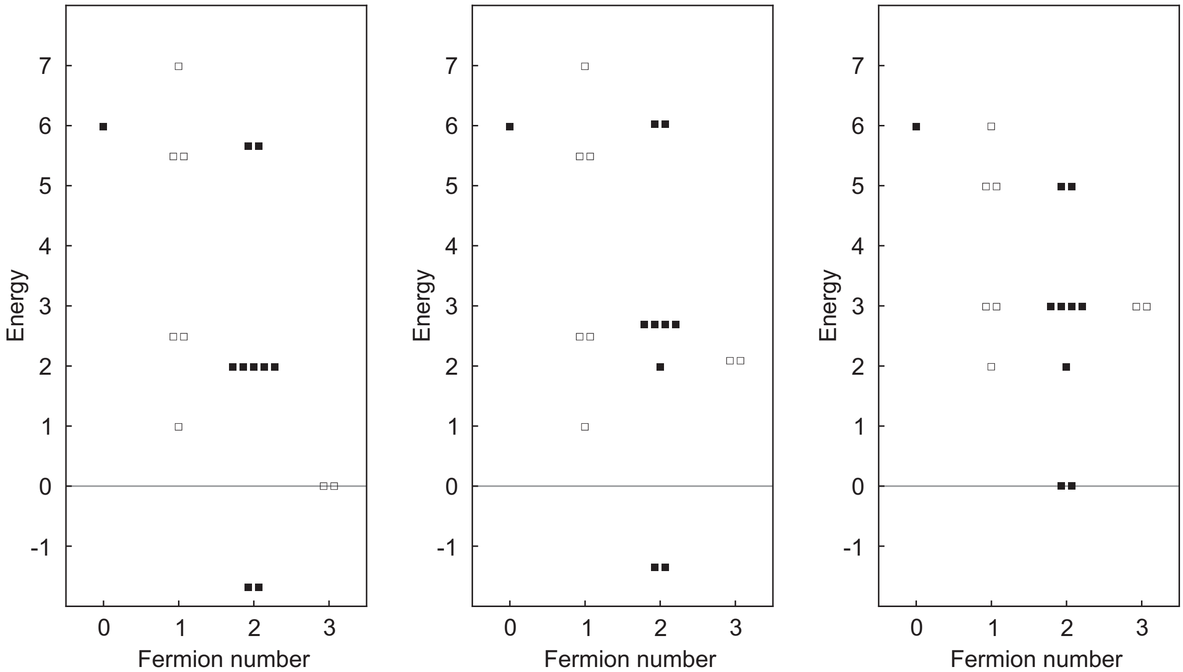
1.4 Supersymmetric models for lattice fermions

Supersymmetry has been instrumental in unravelling the properties described above and will also play an important role in this thesis. Supersymmetry is a symmetry between fermionic and bosonic degrees of freedom (see [31] for a general reference). It plays an important role in theoretical high energy physics, where various theories that go beyond the standard model require supersymmetry for a consistent formulation. In these theories all the known elementary particles are accompanied by yet to be discovered superpartners. In the lattice model discussed here the physical particles are spin-less lattice fermions and the supersymmetry relates fermionic and bosonic many particle states with an odd and even number of the lattice fermions respectively. A central role is played by the operators Q and $Q^\dagger$, called supercharges, which have the following properties

- Q adds one fermion to the system and $Q^\dagger$ takes out one fermion from the system.
- The supercharges are fermionic operators and thus nilpotent: $Q^2 = (Q^\dagger)^2 = 0$.
- The hamiltonian is the anti-commutator of the supercharges, $H = \{Q, Q^\dagger\}$, and

as a consequence it commutes with the supercharges and conserves the number of fermions in the system.

Imposing this structure has some immediate consequences for the spectrum. This is illustrated in figure 1.1, where we show the spectrum for the hard-core fermions on a 6-site chain with no interactions, arbitrary next-nearest neighbor interactions and tuned interactions to the supersymmetric point. The key features at the supersymmetric point, which hold by definition are the following. The energy is positive definite, $E > 0$. States with a positive energy are paired into doublets: each state with energy $E > 0$ has a superpartner with the same energy with one fermion less (or more), that is obtained from the original state by acting with $Q^\dagger$ (or Q). The unpaired states are annihilated by both supercharges, which implies that they have zero energy. If zero energy states exist, supersymmetry is said to be unbroken.



(a) $V_2 = 0$ (b) $t/V_2 = 2.14$ (c) $t/V_2 = 1$

Figure 1.1: We show the spectra (energy versus fermion number) for hard-core spinless fermions on a periodic chain with $L = 6$ sites. The filled squares denote bosonic states and the empty squares denote fermionic states. From left to right we show the spectra for the fermions with no next-nearest neighbor interactions ($V_2 = 0$), arbitrary next-nearest neighbor interactions and tuned interactions to the supersymmetric point. At the supersymmetric point $t/V_2 = 1$, there are two $E = 0$ ground states, and every state with energy $E > 0$ and fermion number f can be paired to a state with the same energy E and with fermion number $f \pm 1$.

As was pointed out above, supersymmetric models allow for a certain degree of analytic control. In particular, there are two important tools to study the ground state structure of these models. One is the Witten index [32] which defines a lower bound to the number of zero-energy ground states. Precisely, it is the number of bosonic ground states minus the number of fermionic ground states. This tool is very powerful, because the doublet structure of the positive energy states allows one to compute this index for the lattice model in a purely combinatorial way. Further analytic control follows from the fact that zero energy ground states are in one-to-one correspondence with elements of the cohomology of the supercharges. Using a spectral sequence technique [33] the cohomology of Q can be obtained analytically for various one and two dimensional lattices. From such computations one obtains the total number of ground states (bosonic plus fermionic) and the number of particles in each ground state.

1.5 This thesis

In this thesis we present a follow-up study of the supersymmetric model for hard-core spinless fermions [34, 35]. At the time this project started, a variety of results for the model on one and two dimensional lattices was known [21, 36, 37, 38].
As was mentioned before, in D = 2 the system typically shows superfrustration, leading to an extensive ground state entropy. A heuristic understanding of this frustration was given by the "3-rule" [21]: to minimize the energy, fermions prefer to be mostly 3 sites apart (with details depending on the lattice). For generic two dimensional lattices the 3-rule can be satisfied in an exponential number of ways. This picture was supported by a remarkable relation found by J. Jonsson [39, 37] between zero energy ground states of the supersymmetric model and tilings of the lattice.
One of the main results presented in this thesis, is the proof of an explicit one-to-one relation between the quantum mechanical ground states of the supersymmetric model on the square lattice and certain tilings of that lattice. As we will see, this relation implies that the number of ground states grows exponentially with the linear size of the system, leading to a sub-extensive ground state entropy.
In D = 1 it was shown that the supersymmetric model on the chain is quantum critical, with the low energy continuum limit described by an $\mathcal{N} = (2, 2)$ superconformal field theory. In this thesis we present a detailed study of this system providing further insight in the relation between the lattice model and the continuum theory. In particular, this insight allows us to develop a technique to probe for critical modes in the supersymmetric lattice model on general one dimensional lattices. By applying this analysis to ladder realizations of the square lattice, we find convincing evidence for criticality in these systems.
The main goal of studying ladder models, although interesting in their own right, is to gather insight in the physics of the full two dimensional system. By combining the ground state-tiling relation with our findings for the ladder models, we arrive at a picture in which the tilings not only count the number of ground states, but actually dominate the ground state wavefunctions. This picture allows us to propose the existence of critical edge modes in the supersymmetric model on the two dimensional square lattice with a boundary.

1.6 Outline

In chapter 2 we introduce the supersymmetric model in detail. We elaborate on the general properties of the spectrum that follow from imposing supersymmetry and make the relation between the quantum ground states and cohomology elements precise by relating the model to an independence complex. In the next chapter, we briefly summarize some aspects of (super) conformal field theory, necessary for the discussion of criticality in the supersymmetric model on one dimensional lattices. In chapter 4 we present a detailed

discussion of the supersymmetric model for spinless fermions on the one dimensional chain. We give a (complete) overview of known results and the relation of this model to other models, before we move on to an in depth analysis relating the lattice model spectrum and operators to the operator content of the continuum theory.
In chapter 5, we shift gears and discuss the model for two dimensional lattices. Again we aim at giving a more or less complete overview of the known results and important open questions. We close by interpreting the theorem that relates the quantum mechanical ground states of the supersymmetric model on the square lattice to tiling configurations. The proof of the theorem can be found in chapter 6.
In chapter 7 we discuss a variety of ladder models, whose ground state structure is also related to tilings. Employing the spectral flow analysis developed in chapter 4 we present convincing evidence for critical modes in these ladder systems. We continue by presenting analytic results for the ground states of certain ladders and numerical and analytic results on entanglement entropy and one-point functions to support the idea that the tilings not only count the number of ground states, but actually dominate the ground state wavefunctions. Also here, we will discuss various interesting open problems. Finally, we close this thesis with a discussion of the presented results in chapter 8. Apart from a brief summary and interpretation, we also allow ourselves to speculate on a variety of topics, such as the existence of a supertopological phase, characterized by a sub-extensive ground state entropy and edge modes, for certain two dimensional systems and possible bulk criticality for other two dimensional systems, the relation between criticality and the imposed supersymmetry and finally a possible connection between superfrustration and zero temperature entropy of extremal black holes.