A supersymmetric model for lattice fermions

Huijse, L.

Chapter 2
A supersymmetric model for lattice fermions

In this chapter we define the supersymmetric model for lattice fermions which was first introduced in [22] and which is the subject of study in this thesis. In the first section, we discuss the general properties of the spectrum of this model that follow from imposing supersymmetry. In section 2.2, we define an independence complex and its cohomology and show how these are related to the Hilbert space and the ground states of the model.

2.1 The model

In this section, we first introduce quantum mechanical supersymmetry and derive some basic properties for the spectrum of a supersymmetric theory. We then introduce hard-core lattice fermions and write down the model. Finally, we illustrate the basic properties of this model with a simple example.

2.1.1 Supersymmetry

An $\mathcal{N} = 2$ supersymmetric quantum mechanical theory is constructed from a basic algebra, defined by two nilpotent supercharges Q and $Q^\dagger$ (complex conjugation is implied) [32],

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \tag{2.1}$$

and the hamiltonian given by

$$H = \{Q^\dagger, Q\}. \tag{2.2}$$

It satisfies

$$[H, Q] = [H, Q^\dagger] = 0. \tag{2.3}$$

The eigenvalues and eigenvectors of the hamiltonian give the energy spectrum and the corresponding quantum states. The definition of the hamiltonian has some immediate consequences for the energy spectrum. First of all, it is positive definite:

$$\begin{aligned} \langle\psi|H|\psi\rangle &= \langle\psi|(Q^\dagger Q + QQ^\dagger)|\psi\rangle \\ &= |Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0 \end{aligned} \tag{2.4}$$

for all choices of the quantum state $|\psi\rangle$. Second of all, the fact that both Q and $Q^\dagger$ commute with the hamiltonian, gives rise to a twofold degeneracy in the energy spectrum. In other words, all eigenstates of the hamiltonian with an energy $E_s > 0$ form doublet representations of the supersymmetry algebra. To see this we first note that due to the

nilpotency of the supercharges the eigenstates of the hamiltonian decompose into quadruplets $(|s'\rangle, Q|s'\rangle, Q^\dagger|s'\rangle, QQ^\dagger|s'\rangle)$ (the state $Q^\dagger Q|s'\rangle$ is not linearly independent, since it can be written as $Q^\dagger Q|s'\rangle = (H - QQ^\dagger)|s'\rangle = E_s|s'\rangle + QQ^\dagger|s'\rangle$). However, this four dimensional representation is reducible. Let us define

$$|s\rangle \equiv |s'\rangle - \frac{1}{E_s} QQ^\dagger|s'\rangle. \tag{2.5}$$

It follows that $Q^\dagger|s\rangle = 0$, so the quadruplet reduces to the doublets $(|s\rangle, Q|s\rangle)$ and $(Q^\dagger|s'\rangle, QQ^\dagger|s'\rangle)$. A doublet thus consists of two states, $|s\rangle$ and $Q|s\rangle$, such that $Q^\dagger|s\rangle = 0$[1]. The states $|s\rangle$ and $Q|s\rangle$ are said to be superpartners. Finally, all states with zero energy must be singlets: $Q|g\rangle = Q^\dagger|g\rangle = 0$ and conversely, all singlets must be zero energy states [32]. In addition to supersymmetry our models will also have a particle-number symmetry generated by the operator F with

$$[F, Q^\dagger] = -Q^\dagger \quad \text{and} \quad [F, Q] = Q. \tag{2.6}$$

Consequently, F commutes with the hamiltonian. Furthermore, this tells us that superpartners differ in their fermion number by one (let $F|s\rangle = f_s|s\rangle$, then $F(Q|s\rangle) = Q(F+1)|s\rangle = (f_s+1)(Q|s\rangle)$).

2.1.2 Witten index

An important issue is whether or not supersymmetric ground states at zero energy occur, that is, whether there are singlet representations of the algebra. For this one considers the Witten index [32]

$$W = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \ , \tag{2.7}$$

where the trace is over the entire Hilbert space. Remember that all excited states come in doublets with the same energy and differing in their fermion-number by one. This means that in the trace all contributions of excited states will cancel pairwise, and that the only states contributing are the zero energy ground states. We can thus evaluate W in the limit of $\beta \to 0$, where all states contribute $(-1)^F$. It also follows that $|W|$ is a lower bound to the number of zero energy ground states.

2.1.3 Lattice fermions

We now make the model more concrete by defining the supercharges in terms of lattice particles. The particles we will consider are spin-less electrons, also called spin-less fermions. Their key property is that the wavefunction is antisymmetric under the exchange of two fermions. It follows that the operator $c_i^\dagger$ that creates a fermion on site i in the lattice and the operator c_j that annihilates a fermion on site j in the lattice, satisfy the following anti-commutation relations:

$$\{c_i^\dagger, c_j\} = \delta_{ij}$$
$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0.$$

[1] Note that the choice $(|s\rangle, Q^\dagger|s\rangle)$, with $Q|s\rangle = 0$ is completely equivalent.

The particle-number operator for fermions is defined as $F = \sum_i c_i^\dagger c_i$, where the sum is over all lattice sites. This operator counts the total number of particles in a state. A simple choice for the supercharges would be $Q = \sum_i c_i^\dagger$ and $Q^\dagger = \sum_i c_i$. It is readily verified that both obey the nilpotency condition and that the commutation relations with F (2.6) are satisfied. However, this choice leads to a trivial hamiltonian: $H = L$, where L is the total number of sites of the lattice. To obtain a non-trivial hamiltonian, we dress the fermion with a projection operator: $P_{<i>} = \prod_{j \text{ next to } i}(1 - c_j^\dagger c_j)$, which requires all sites adjacent to site i to be empty. We can now formulate the supercharges in terms of these hard-core fermions: $Q = \sum c_i^\dagger P_{<i>}$ and $Q^\dagger = \sum c_i P_{<i>}$. Again the nilpotency condition and the commutation relations (2.6) are satisfied, but now the hamiltonian of these hard-core fermions reads

$$H = \{Q^\dagger, Q\} = \sum_i \sum_{j \text{ next to } i} P_{<i>} c_i^\dagger c_j P_{<j>} + \sum_i P_{<i>}.$$

The first term is a nearest neighbor hopping term, that is, the fermions can hop from site j to site i as long as i and j are connected by an edge and provided that the neighboring sites are empty. The second term contains a next-nearest neighbor repulsion, a chemical potential and a constant. The details of the latter terms will depend on the lattice we choose.
Note that all the parameters in the hamiltonian (the hopping t, the nearest neighbor repulsion V_1, the next-nearest neighbor repulsion V_2 and the chemical potential μ) are fixed by the choice of the supercharges and the requirement of supersymmetry and eventually the lattice.

2.1.4 Example: 6-site chain

Let us consider as an example of all the above, the chain of six sites with periodic boundary conditions. The first thing we note is that the hamiltonian for an L-site chain with periodic boundary conditions can be rewritten in the following form:

$$H = H_{\text{kin}} + H_{\text{pot}}, \tag{2.8}$$

where

$$H_{\text{kin}} = \sum_{i=1}^{L} \left[P_{i-1} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) P_{i+2} \right], \tag{2.9}$$

$$\begin{aligned} H_{\text{pot}} &= \sum_{i=1}^{L} P_{i-1} P_{i+1} \\ &= \sum_{i=1}^{L} \left[1 - 2n_i + n_i n_{i+2} \right] \\ &= \sum_{i=1}^{L} (n_i n_{i+2}) + L - 2F. \end{aligned}$$

Here $P_i = 1 - n_i$, $n_i = c_i^\dagger c_i$ is the usual number operator and $F = \sum_i n_i$ is the total number of fermions. (We shall denote eigenvalues of this operator by f and write the

fermion density or filling fraction as $\nu = f/L$.) The form of the hamiltonian makes clear that the hopping parameter t is tuned to be equal to the next-nearest neighbor repulsion V_2, which is tuned to unity. The nearest neighbor repulsion V_1 is by definition infinite and the chemical potential μ is 2. Finally, there is a constant contribution L to the hamiltonian. Note that the second term in the hamiltonian $H_{\rm pot}$ suggests that the energy is minimized when the hard-core fermions are three sites apart.

Let us consider the possible configurations of the 6-site chain. In addition to the empty state, there are six configurations with one fermion, nine with two fermions and two with three fermions (see Fig. 2.1). Because of the hard-core character of the fermions, half-filling is the maximal density. Clearly, the operator Q gives zero on these maximally filled states. On the other hand, $Q^\dagger$ acts non-trivially on these states, so two of the nine states with two fermions are superpartners of the maximally filled states. The empty state $|0\rangle$ has an energy $E = 6$ and $Q^\dagger|0\rangle = 0$, whereas $Q|0\rangle = \sum_i c_i^\dagger|0\rangle$, so $(|0\rangle, Q|0\rangle)$ make up a doublet. The other five states with one fermion are annihilated by $Q^\dagger$ and Q acts non-trivially on them, so they form supersymmetry doublets with five two-fermion-states. At this point, seven of the nine two-fermion-states are paired up in doublets, either with one- or three-fermion-states. The remaining two states cannot be part of a doublet, which implies that they must be singlet states and thus have zero energy. So we find that the 6-site chain has a twofold degenerate zero energy ground state at filling $\nu = f/L = 1/3$. The full spectrum of the 6-site chain is shown in Fig. 2.1.

We observe that the ground state filling fraction of 1/3 agrees with the expectation that fermions tend to be three sites apart. This geometric rule suggests three possible ground states; in the full quantum theory two are realized as zero-energy states. Note that the actual ground state wavefunctions are superpositions of many different configurations. In particular, the ground states of the 6-site periodic chain can be written as

$$\begin{aligned} |\psi_1\rangle &= |14\rangle - |15\rangle - |24\rangle + 2|25\rangle - |26\rangle - |35\rangle + |36\rangle, \\ |\psi_2\rangle &= -|13\rangle + 3|14\rangle - 2|15\rangle - 2|24\rangle + 3|25\rangle - |26\rangle - |35\rangle + |46\rangle, \end{aligned}$$

where $|ij\rangle \equiv c_i^\dagger c_j^\dagger|0\rangle$ denotes a configuration with sites i and j occupied.

Since the hamiltonian commutes with the fermion number operator F and the translation operator T, we can block diagonalize the hamiltonian. In the two particle sector, diagonalizing the translation operator gives $|\phi_k\rangle = \sum_{l=0}^5 t_k^{-l} T^l|13\rangle$, where $t_k = \exp(2\pi\imath k/6)$, with $k = 1, \ldots, 6$ and $|\chi_k\rangle = \sum_{l=0}^5 t_k^{-l} T^l|14\rangle$, where $t_k = \exp(2\pi\imath k/6)$, with $k = 1, 3, 5$. It follows that the hamiltonian reads $H = 3$ in the sectors with $k = 2, 4, 6$ and

$$H = \begin{pmatrix} 3 & \sqrt{2}(1-t^2) \\ \sqrt{2}(1-t^{-2}) & 2 \end{pmatrix}, \tag{2.10}$$

with eigenvalues $E_\pm = (5 \pm \sqrt{17 - 16\cos(2\pi k/3)})/2$. For $k = 3$ this gives $E_+ = 3$ and $E_- = 2$ and for $k = 1$ and $k = 5$ we find $E_+ = 5$ and $E_- = 0$. It follows that the ground states have eigenvalues $\exp(\pm\pi i/3)$ under translation by one site.

Now let us compute the Witten index for this example. Remember that for a supersymmetric theory it simply reads

$$W = \mathrm{tr}(-1)^F. \tag{2.11}$$

Note that we can take any basis of states we like to compute the trace. Above we have specified a basis by considering all the possible configurations of up to three fermions on

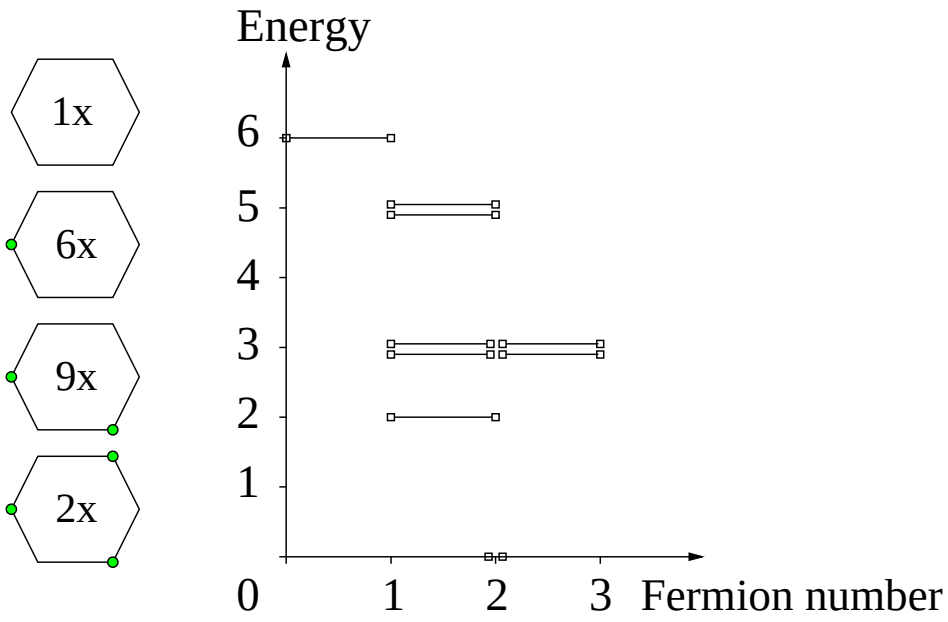

Figure 2.1: On the left we show the 6-site chain and the possible configurations of hard-core fermions on this lattice. The numbers give the number of possible configurations with a given number of fermions. On the right we show the spectrum, energy versus fermion number, of the 6-site chain.

the chain. It immediately gives $W = 1 - 6 + 9 - 2 = 2$ in agreement with the existence of the two ground states that we found.

We close this section with two comments. First, we stress that the extremely simple computation of W alone guarantees the existence of at least two ground states at zero energy. Similar results are easily established for much larger systems, where a direct evaluation of the ground state energies is way out of reach, showing the power of supersymmetry. Second, we observe that here the Witten index is exactly equal to the number of ground states. We will encounter examples where ground states exist at more than one fermion number f, leading to cancellations in the Witten index so that $|W|$ is strictly smaller than the number of ground states.

2.2 Relation to independence complex and (co)homology theory

In this section we define independent sets, an independence complex and its (co)homology. We show that these are related to hard-core fermion configurations, the Hilbert space and the zero energy ground states of the lattice model. These relations have proven to be extremely powerful in the study of the ground state structure of the lattice model. Central in these studies is the 'tic-tac-toe' lemma of [33]. This technique will be illustrated at the end of this section in a few simple examples.

2.2.1 Independence complex

An independent set on a graph is a subset of the vertex set of the graph with the property that no two vertices are adjacent. Since hard-core fermions cannot occupy adjacent sites, it is clear that each allowed configuration of hard-core fermions forms an independent set. In the following we will use the term lattice (i.e. a grid) instead of the more general term graph, since in the physics context it is most natural to study fermions on a lattice. However, the correspondences we establish in this section hold for graphs in general. The family of independent sets of a lattice forms the independence complex Σ of the lattice. We can define the partition sum in the asymptotic (thermodynamic) limit for the independence complex Σ as

$$Z(\Sigma, z) \equiv \sum_{\sigma \in \Sigma} z^{|\sigma|}, \tag{2.12}$$

where z is called the activity. Until recently there were essentially no exact results for independence complexes on two dimensional lattices, with one important exception. Baxter [40] gave an analytic expression for the partition sum of the independence complex on the triangular lattice with positive activity in the thermodynamic limit. This is also referred to as the exact solution for hard hexagons (hard-core fermions on the triangular lattice can, in this context, be viewed as hexagons that share, at most, a side or a corner).
Now observe that the coefficient of z^k in (2.12) is the number of sets in Σ with size k or, in other words, the number of configurations with k hard-core fermions. Consequently, $Z(\Sigma, 1)$ gives the dimension of the full Hilbert space $\mathcal{H}$; the space spanned by all possible hard-core fermion configurations. What is even more interesting, however, is that $Z(\Sigma, -1)$ coincides with the Witten index (2.7)

$$Z(\Sigma, -1) = \sum_{\sigma \in \Sigma} (-1)^{|\sigma|} = \mathrm{tr}(-1)^F.$$

2.2.2 Cohomology and homology theory

It should be clear from the previous that the Hilbert space is a graded vector space, where the grading is defined by the particle-number operator F. That is, the Hilbert space can be written as $\mathcal{H} = \oplus C_n$, where C_n is a subspace spanned by all the possible configurations with n particles. From the definitions of F and Q and their commutation relations (2.6) it is clear that Q is a map from C_n to C_{n+1}. Since Q squares to zero, we can define its cohomology. On the other hand, $Q^\dagger$ is a map from C_n to C_{n-1} and also nilpotent, so we can define the homology of $Q^\dagger$.

$$C_0 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_1 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_2 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_3 \dots$$

It turns out that the zero energy ground states of the model are in one-to-one correspondence with the non-trivial classes of the cohomology of Q and the homology of $Q^\dagger$. Remember that all states with zero energy must be singlets: $Q|g\rangle = Q^\dagger|g\rangle = 0$ and conversely, all singlets must be zero energy states. Clearly, all singlets, and thus all (zero

energy) ground states, are in the kernel of Q: $Q|g\rangle = 0$ and not in the image of Q, because if we could write $|g\rangle = Q|f\rangle$, then $(|f\rangle, |g\rangle)$, would be a doublet. Equivalently, we can say that a ground state with n fermions is a cycle but not a boundary in C_n. This is precisely the definition of an element of the n-th cohomology of Q, $H_Q^{(n)} = \ker Q/\mathrm{Im}\ Q$ within C_n. Two states $|s_1\rangle$ and $|s_2\rangle$ are said to be in the same cohomology-class if $|s_1\rangle = |s_2\rangle + Q|s_3\rangle$ for some state $|s_3\rangle$. Since a ground state is annihilated by both Q and $Q^\dagger$, different (i.e. linearly independent) ground states must be in different cohomology-classes[2] of Q. Finally, the number of independent ground states is precisely the dimension of the cohomology of Q and the fermion-number of a ground state is the same as the grade of the corresponding cohomology-class. Thus the ground states of a supersymmetric theory are in one-to-one correspondence with the cohomology of Q. With the same line of reasoning we may also conclude that the ground states are in one-to-one correspondence with the homology of $Q^\dagger$. Finally, the Euler characteristic, defined in cohomology theory as

$$\chi \equiv \sum_n \left[(-1)^n \mathrm{dim} H_Q^{(n)}\right],$$

is precisely the Witten index.

2.2.3 The 'tic-tac-toe' lemma

Central to the computation of the cohomology of Q for the lattice models presented in this thesis is the 'tic-tac-toe' lemma of [33]. Let us decompose the lattice S into two sublattices S_1 and $S_2 = S \setminus S_1$ and we write $Q = Q_1 + Q_2$, where Q_1 and Q_2 act on S_1 and S_2 respectively. We can then consider the double complex $\oplus_n C_n = \oplus_n \oplus_{p+q=n} K_{p,q}$, where p (q) is the size of the vertex set on S_1 (S_2). Equivalently, if we define f_i as the number of particles on S_i, we have $f_1 = p$ and $f_2 = q$. Finally, we have $Q_1 : K_{p,q} \to K_{p+1,q}$ and $Q_2 : K_{p,q} \to K_{p,q+1}$. The 'tic-tac-toe' lemma now tells us that the cohomology of Q, H_Q, is the same as the cohomology of Q_1 acting on the cohomology of Q_2, i.e. $H_Q = H_{Q_1}(H_{Q_2}) \equiv H_{12}$, provided that H_{12} has entries only in one row. That is, H_{12} is non-vanishing only for one value of q (or f_2). So a sufficient condition for the lemma to hold is that all non-trivial elements of H_{12} have the same f_2 (the fermion-number on S_2).

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,2} & \xrightarrow{Q_1} & K_{1,2} & \xrightarrow{Q_1} & K_{2,2} & \xrightarrow{Q_1} & \cdots \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,1} & \xrightarrow{Q_1} & K_{1,1} & \xrightarrow{Q_1} & K_{2,1} & \xrightarrow{Q_1} & \cdots \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,0} & \xrightarrow{Q_1} & K_{1,0} & \xrightarrow{Q_1} & K_{2,0} & \xrightarrow{Q_1} & \cdots
\end{array}$$

[2] Let $|s_1\rangle$ and $|s_2\rangle$ be two linearly independent ground states. It follows that $Q|s_1\rangle = Q|s_2\rangle = Q^\dagger|s_1\rangle = Q^\dagger|s_2\rangle = 0$. If we now write $|s_1\rangle = |s_2\rangle + Q|s_3\rangle$, we find that $Q^\dagger|s_1\rangle = Q^\dagger|s_2\rangle + Q^\dagger Q|s_3\rangle$ and thus $Q^\dagger Q|s_3\rangle = 0$. From this we find $\langle s_3|Q^\dagger Q|s_3\rangle = |Q|s_3\rangle|^2 = 0$ and thus $Q|s_3\rangle = 0$. With this we obtain $|s_1\rangle = |s_2\rangle$, which contradicts our assumption that $|s_1\rangle$ and $|s_2\rangle$ are linearly independent, so we conclude that $|s_1\rangle$ and $|s_2\rangle$ must be in different cohomology classes.

2.2.4 Examples: 6-site periodic and 4-site open chain

As an example we compute the cohomology of the 6-site periodic and 4-site open chain. The first example is relatively simple: We will find that the non-trivial elements in the cohomology H_{12} will all have the same number of fermions on the sublattice S_2. This is a sufficient condition for the 'tic-tac-toe' lemma to hold, that is, we directly find the full cohomology $H_Q = H_{12}$. In the second example this condition will not be satisfied. However, with some extra work one can still obtain the full cohomology from H_{12}. In particular we will see that H_Q is contained in H_{12}, but they are not equal.
For the 6-site chain we choose sites 3 and 6 as sublattice S_2 and the rest as sublattice S_1. We now consider H_{Q_2}. If both sites next to site 3 are empty, H_{Q_2} is trivial: Q_2 acting on the empty site 3 does not vanish, while the filled site 3 is Q_2 acting on the empty site. The same holds for site 6 independently. It follows that the only non-trivial elements of H_{Q_2} have at least one site next to site 3 and one site next to site 6 occupied. There are only 2 such configurations: the one with site 2 and 5 occupied and the one with site 4 and 1 occupied. Both states belong to H_{12}: they are closed because Q_1 gives zero on them, and not exact because there are no elements in H_{Q_2} with 1 fermion. By the tic-tac-toe lemma, there must be precisely two different cohomology classes in H_Q, and therefore exactly two ground states with $f = 2$.
We now consider the 4-site chain with open boundary conditions chain as a simple example of the case where H_Q is contained in H_{12} but not exactly equal to it. Take site 2 and 3 as the sublattice S_2. This is a 2-site chain and it is easily verified that it has one ground state with one particle (the state $|2\rangle - |3\rangle$, where $|i\rangle$ is the configuration with the i-th site occupied). Solving H_{Q_2} thus gives two configurations: the one with site 1 and 4 empty and the configuration $|2\rangle - |3\rangle$ on S_2 and the one with both sites 1 and 4 occupied and S_2 empty.
Now, *within* H_{Q_2}, Q_1 gives zero on both configurations, so they are both closed and consequently not exact. So we find two different cohomology classes in H_{12}. However, their particle number on S_2 differs and thus $H_Q = H_{12}$ does not necessarily hold. That is, within H_{12}, there may be configurations that are not in the kernel of Q and there may be configurations that are in the image of Q. To check this we must perform the tic-tac-toe procedure. We start from the configuration with sites 1 and 4 empty. We act on this configuration with Q without restricting ourselves to H_{Q_2}. We find the following: $Q(|2\rangle - |3\rangle) = Q_1(|2\rangle - |3\rangle) = -|24\rangle - |13\rangle$. Remember that the fermionic creation operators anticommute, hence $|42\rangle = -|24\rangle$. Now since H_{Q_2} was empty at $f_1 = f_2 = 1$ and the newly obtained state belongs to the kernel of Q_2, it must also belong to the image of Q_2. Indeed we can write $-|24\rangle - |13\rangle = Q_2(-|4\rangle + |1\rangle)$. So we find that $(|2\rangle - |3\rangle)$ does not belong to the kernel of Q, but maybe $(|2\rangle - |3\rangle) - (-|4\rangle + |1\rangle)$ does. Let us check this:

$$\begin{aligned} Q(|2\rangle - |3\rangle + |4\rangle - |1\rangle) &= Q_1(|2\rangle - |3\rangle) - Q_2(-|4\rangle + |1\rangle) - Q_1(-|4\rangle + |1\rangle) \\ &= -Q_1(-|4\rangle + |1\rangle) \\ &= -(-|14\rangle - |14\rangle) = 2|14\rangle. \end{aligned}$$

Remember that $|14\rangle$ also belongs to H_{12}. Thus we have found that none of the elements in H_{12} belong to H_Q, since $|2\rangle - |3\rangle$ is not closed and $|14\rangle$ is exact. Thus we have found that the number of ground states on the 4-site open chain is zero. It can be shown that this result generalizes to all open chains with length $3s + 1$.