A supersymmetric model for lattice fermions

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Chapter 3
Superconformal field theory

3.1 Continuum theory

Identifying critical behavior in the supersymmetric lattice models is an important part of this thesis. Since the models we describe in this thesis have an exact supersymmetry by definition, this property will persist in the continuum limit. For quantum critical lattice models the continuum limit typically exhibits conformal invariance. This can roughly be understood from the fact that a critical lattice model has no scale except for the lattice spacing. At a critical point the gap or mass vanishes or, equivalently, the correlation length diverges. Upon sending the lattice spacing to zero in the continuum limit the conformal invariance emerges. Consequently, one expects the continuum limit of a *supersymmetric, quantum critical* lattice model to be described by a supersymmetric conformal field theory (SCFT). For various (quasi) one dimensional systems we compare the properties of finite size systems to the properties one expects for a system that is described by a supersymmetric conformal field theory in the continuum limit.

In this chapter we briefly introduce supersymmetric conformal field theory (for a nice review see [41]), where we will emphasize that it is an extension of the more familiar conformal field theory [42].

3.2 Conformal field theory

A conformal field theory is a theory that is invariant under the conformal transformations: translations, rotations, scale transformations and the so-called special conformal transformations. These transformations are generated by the stress-energy tensor T. Restricting the discussion to $d = 2$ dimensions, the holomorphic and anti-holomorphic parts of the stress-energy tensor have the following operator product expansions (OPE) on the complex plane:

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) + \ldots \\
\overline{T}(\overline{z})\overline{T}(\overline{w}) &= \frac{c/2}{(\overline{z}-\overline{w})^4} + \frac{2}{(\overline{z}-\overline{w})^2}\overline{T}(\overline{w}) + \frac{1}{(\overline{z}-\overline{w})}\overline{\partial T}(\overline{w}) + \ldots
\end{aligned}
$$

here c is the central charge or conformal anomaly of the theory. As usual in OPEs, we indicated the terms that are finite in the limit that $z \to w$ by $+\ldots$. In the following we will restrict the discussion to the holomorphic part of the theory and assume that the anti-holomorphic sector has a structure directly parallel to the holomorphic sector. The stress-energy tensor is a quasi-primary field of conformal dimension 2 and its mode expansion reads

$$
T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}.
$$

From the OPE and the mode expansion one derives the following commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}c(m^3 - m)\delta_{m+n,0}. \tag{3.1}$$

This is called the Virasoro algebra and the modes, L_m, are called the Virasoro generators. Representations of the Virasoro algebra are given by highest weight states defined by primary fields and their descendants. A primary field $\phi(z)$ of conformal dimension h is a field that under the transformation $z \to f(z)$, transforms as

$$\phi(z) \to \Big(\frac{\partial f(z)}{\partial z}\Big)^h \phi(f(z)). \tag{3.2}$$

Its OPE with the stress-energy tensor reads

$$T(z)\phi(w) = \frac{h}{(z-w)^2}\phi(w) + \frac{1}{(z-w)}\partial\phi(w) + \dots \tag{3.3}$$

If we define the vacuum as the state that has $L_m|0\rangle = 0$ for $m \geq -1$, then the state $|h\rangle = \phi(0)|0\rangle$ created from the vacuum by this primary field is called a highest weight state. It satisfies

$$L_0|h\rangle = h|h\rangle \quad L_m|h\rangle = 0, \ m > 0. \tag{3.4}$$

The descendants of the highest weight state are generated by the Virasoro generators L_m with $m < 0$. Since $L_0 + \overline{L}_0$ generates translations, the eigenvalue of L_0 is related to the energy of the state. From the Virasoro algebra it follows that the highest weight state has the lowest energy and the descendant states form a so-called tower of higher energy states.

3.3 $\mathcal{N} = 2$ Superconformal field theory

In an $\mathcal{N} = 2$ superconformal field theory [43, 44] there are three generators besides the stress-energy tensor: two supercharges, $G^+(z)$ and $G^-(z)$, and a $U(1)$ current, $J(z)$. They satisfy the following OPEs

$$G^\pm(z)G^\mp(w) = \frac{2c/3}{(z-w)^3} \pm \frac{2}{(z-w)^2}J(w) + \frac{1}{(z-w)}(2T(w) \pm \partial J(w)) + \dots$$

$$J(z)G^\pm(w) = \pm\frac{1}{(z-w)}G^\pm(w) + \dots$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + \dots$$

$$T(z)J(w) = \frac{1}{(z-w)^2}J(w) + \frac{1}{(z-w)}\partial J(w) + \dots$$

$$T(z)G^\pm(w) = \frac{3/2}{(z-w)^2}G^\pm(w) + \frac{1}{(z-w)}\partial G^\pm(w) + \dots \tag{3.5}$$

In this thesis, we will mostly consider models with an $\mathcal{N} = (2,2)$ type supersymmetry, which means that there is an analog of (3.5) for the anti-holomorphic fields. It is custom

to concentrate only on the holomorphic fields, since many of the arguments are identical for the anti-holomorphic fields. This strategy is also adopted here. We will, however, also encounter models with $\mathcal{N} = (2, 0)$ (or simply $\mathcal{N} = 2$) supersymmetry in which case we will mention this explicitly.

By comparing the latter two OPEs in (3.5) with the OPE of a primary field (3.3), we conclude that the supercharges and the current are primary fields with the conformal dimensions $\frac{3}{2}$, $\frac{3}{2}$ and 1 respectively. As for the stress-energy tensor, we can write the mode expansions for the supercharges and the current

$$
\begin{aligned}
G^{\pm}(z) &= \sum_r G_r^{\pm} z^{-r-3/2} \\
J(z) &= \sum_{n=-\infty}^{\infty} J_n z^{-n-1},
\end{aligned}
\tag{3.6}
$$

where r runs over all values in $\mathbb{Z} + \alpha$, with α a real number which determines the branch cut properties of $G^{\pm}(z)$. For $\alpha = 0$ the theory is said to be in the Ramond sector and for $\alpha = 1/2$ it is said to be in the Neveu-Schwarz sector. This will be discussed in more detail in section 4.8.

From the mode expansions, we obtain the superconformal algebra. The superconformal algebra is defined by the Virasoro algebra (3.1) together with a $U(1)$ Kac-Moody algebra for the current

$$
[J_m, J_n] = \frac{c}{3} m \delta_{m+n,0} \quad [L_m, J_n] = -n J_{m+n},
\tag{3.7}
$$

and the algebra of the supercharges

$$
\begin{aligned}
\left[L_m, G_r^{\pm}\right] &= (\tfrac{1}{2}m - r) G_{m+r}^{\pm}, \\
\left[J_m, G_r^{\pm}\right] &= \pm G_{m+r}^{\pm}, \\
\{G_r^{\pm}, G_s^{\mp}\} &= 2 L_{r+s} \pm (r - s) J_{r+s} + \frac{1}{3} c (r^2 - \frac{1}{4}) \delta_{r+s,0}.
\end{aligned}
\tag{3.8} \tag{3.9} \tag{3.10}
$$

Like in CFT, representations of the superconformal algebra are formed by the primary fields and their descendants. A primary field $\psi(z)$ satisfies

$$
\begin{aligned}
J(z)\psi(w) &= \frac{q}{(z-w)} \psi(w) + \dots \\
G^{\pm}\psi(w) &= \frac{1}{(z-w)} \Lambda^{\pm}(w) + \dots
\end{aligned}
$$

where the fields $\Lambda^{\pm}(w)$ are the superpartners of $\psi(w)$. In terms of modes, this gives for the highest weight state corresponding to $\psi(z)$ ($|h, q\rangle = \psi(0)|0\rangle$)

$$
\begin{aligned}
L_n|h, q\rangle &= J_n|h, q\rangle = 0, \ n > 0 \\
L_0|h, q\rangle &= h|h, q\rangle, \ J_0|h, q\rangle = q|h, q\rangle, \\
G_r^{\pm}|h, q\rangle &= 0, \ r \geq 1/2 \\
G_{-\frac{1}{2}}^{\pm}|h, q\rangle &= \Lambda^{\pm}(0)|0\rangle.
\end{aligned}
\tag{3.11}
$$

So h is related to the energy of the state and q is related to the charge of the state under the $U(1)$ current.

3.4 Minimal series

In conformal field theories with central charge $c < 1$, the unitarity constraint leads to a discrete series called the minimal series [45, 46]. Below $c = 1$ one can only have a unitary theory for

$$c = 1 - \frac{6}{m(m+1)}, \ m = 3, 4, \ldots \tag{3.12}$$

Furthermore, there is a finite set of primary fields or highest weight states with conformal dimensions

$$h_{p,r}(m) = \frac{[(m+1)p - mr]^2 - 1}{4m(m+1)} \tag{3.13}$$

where p, r are integers satisfying $1 \le p \le m - 1$, $1 \le r \le p$.
In supersymmetric theories, the unitarity constraint leads to a discrete series for theories with $c < 3$ [47, 48]. The allowed values for the central charge are

$$c = 3 - \frac{6}{(k+2)}, \ k = 1, 2, \ldots \tag{3.14}$$

For each of these supersymmetric minimal models there is again a finite set of primary fields. Their conformal dimensions and corresponding $U(1)$ charges are

$$h_{p,r}(k) = \frac{p(p+2) - r(r-2) - 4r\alpha + 2k(1/2 - \alpha)^2}{4(k+2)} \quad q_r(k) = \frac{r + k/2 - k\alpha}{k+2}, \quad (3.15)$$

with $p = 0, 1, \ldots, k$ and $r = -p, -p+2, \ldots, p-2, p$. Remember that α is a real number connected to the boundary conditions imposed on the supercharges. For $\alpha = 0$ ($\alpha = 1/2$) the theory the Ramond (Neveu-Schwarz) sector.

3.5 Superpartners and Witten index

Let us now try to relate the above to the properties of our models that we derived from supersymmetry. That is, a positive definite energy spectrum, and a decomposition of the spectrum in singlets and doublets, the notion of a Witten index, etc. We start by noting that the hamiltonian H, i.e. the energy operator, is the generator of translations in the time direction. On the plane we have $H = L_0 + \overline{L}_0$ and on the cylinder this is $H = L_0 + \overline{L}_0 - \frac{c}{12}$. If we now define two supercharges as

$$G = \frac{1}{\sqrt{2}}(G_0^+ - \overline{G}_0^+) \quad G^\dagger = \frac{1}{\sqrt{2}}(G_0^- - \overline{G}_0^-), \tag{3.16}$$

we find that we can write the hamiltonian on the cylinder as

$$H = L_0 + \overline{L}_0 - \frac{c}{12} = \{G, G^\dagger\}. \tag{3.17}$$

Note that these supercharges are defined in the Ramond sector. Indeed it turns out that the supersymmetric structure as we discussed it in section 2.1.1 is only fully realized in the

Ramond sector of the superconformal algebra. From the commutator of the supercharges with the Virasoro generators (3.8), one easily verifies that the hamiltonian commutes with G and $G^\dagger$. Furthermore, from the commutation relation with the U(1) current, we find that the hamiltonian also commutes with $F = J_0 - \overline{J}_0$. Note that for F to be a well defined fermion number operator we must have $q - \overline{q} \in \mathbb{Z}$. The supercharges satisfy

$$[F, G] = -G \quad \left[F, G^\dagger\right] = G^\dagger. \tag{3.18}$$

These are precisely the relations we found for the supersymmetric model (see section 2.1.1). It thus follows that the spectrum of the hamiltonian in the Ramond sector will indeed have the properties mentioned above.

We now focus on the minimal models for a moment. One can see directly from the definition of the hamiltonian that the state with $h = \overline{h} = 0$ has negative energy. From the allowed conformal dimensions of the minimal models, however, one quickly finds that this state, with energy $-\frac{c}{12}$, is in the Neveu-Schwarz sector. Let us now check that for the minimal models the energy is positive definite for all states in the Ramond sector. In the Ramond sector the highest weight states have conformal dimension

$$h_{p,r}(k) = \frac{p(p+2) - r(r-2) + k/2}{4(k+2)}, \tag{3.19}$$

which is minimized if $p(p+2) - r(r-2)$ is minimized. If we write $r = -p + 2m$, where $m = 0, 1, \ldots, p$, we find

$$h_{p,-p+2m}(k) = \frac{4m(p-m+1) + k/2}{4(k+2)}, \tag{3.20}$$

which is minimal for $m = 0$. In this case the conformal dimension does not depend on p so there are $k + 1$ fields with the lowest conformal dimension given by

$$h_{p,-p}(k) = \frac{k}{8(k+2)} = \frac{c}{24}. \tag{3.21}$$

One can check that the corresponding values of the $U(1)$ charges satisfy $q_{-p}(k) = \frac{k/2-p}{k+2}$. Consequently, $q \in (-\frac{1}{2}, \frac{1}{2})$ and the condition that $q - \overline{q} \in \mathbb{Z}$ then implies $q = \overline{q}$. It follows that in the Ramond sector there are $k+1$ states with $h = \overline{h} = c/24$ and thus zero energy. All other states have an energy larger than zero.

As we have seen before, the Witten index is simply the partition sum with an extra factor $(-1)^F$ in the trace. We have to define however in which sector we take the trace. In the Ramond sector, we have found that the manifestly supersymmetric structure is realized. It follows that the states with energy greater than zero do not contribute to the trace. The number of zero energy states is $k+1$ for the k-th minimal model. Since these states all have $q - \overline{q} = 0$, their contribution to the Witten index is positive: $(-1)^F = (-1)^{J_0 - \overline{J}_0} = +1$. It thus follows that we find for the Witten index of the k-th minimal model

$$W_k = k + 1. \tag{3.22}$$

3.6 Spectral flow

An important tool in the analysis of superconformal field theories is the spectral flow [49] (see also [50, 41]). It follows from an operator that shifts the $U(1)$ charge of any field. Since the current is a bosonic field of conformal dimension 1 we can write it as

$$J(z) = \imath \sqrt{\frac{2gc}{3}} \partial \Phi(z), \tag{3.23}$$

where $\Phi(z, \overline{z}) = \Phi(z) + \overline{\Phi}(\overline{z})$ is just the free boson with action $S = \frac{g}{\pi} \int \partial \Phi \overline{\partial} \Phi$, where g is the coupling. We can now define the spectral flow operator as

$$U(z, \overline{z}) = \exp\left[\imath \sqrt{\frac{2gc}{3}} \theta(\Phi - \overline{\Phi})\right]. \tag{3.24}$$

The action of this operator on a field with charge q is that it changes it to a field with charge $q + \theta c/3$. It can be shown that the generators of the superconformal algebra transform as follows under this map [49]

$$
\begin{aligned}
L_n^\theta &= L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0} \\
J_n^\theta &= J_n + \frac{c}{3}\theta \delta_{n,0} \\
G_r^{\theta,+} &= G_{r-\theta}^+ \\
G_r^{\theta,-} &= G_{r+\theta}^-.
\end{aligned}
$$

First of all, one can check that for θ integer the algebra maps back to itself. Second, we find that if r is integer, the spectral flow maps the superconformal algebra of the Ramond sector onto a representation with $\alpha = \theta$. In particular, for $\theta = \frac{1}{2}$ it maps the Ramond sector onto the Neveu-Schwarz sector.

This mapping will turn out to be particularly powerful in the study of the supersymmetric lattice models. As we mentioned before, representations with different values of α correspond to theories with different boundary conditions. It turns out that these boundary conditions are directly related to the boundary conditions in the lattice model.