A supersymmetric model for lattice fermions

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Chapter 5
The supersymmetric model on two dimensional lattices

5.1 Superfrustration

In the previous chapter we studied the supersymmetric model on the chain in great detail. In particular, we investigated the continuum limit properties of the system. For two-dimensional systems this kind of analysis is still out of reach, however, one can obtain (exact) results for the Witten index and the total number of ground states for various two-dimensional systems. In recent years, there has been quite some activity in this field, both in the physics as well as the mathematics literature. On the physics side, these questions arise not only in the context of the supersymmetric lattice model, but also in the context of statistical mechanics. In the latter case one studies the partition sum of, for example, hard squares or hard hexagons [74, 36, 37, 38, 40]. For negative activity there is a direct relation with the Witten index of the supersymmetric model on the square and triangular lattice respectively (see section 2.2.1 for the details). On the mathematics side, these questions arise in the studies of partition sums, (co)homology and homotopy of independence complexes [39, 75, 76, 77]. In sections 2.2.1 and 2.2.2 we discuss the relation between these studies and the supersymmetric lattice model in some detail.

In this chapter we will summarize the results that have been obtained for the supersymmetric model on two-dimensional lattices [21, 36, 78, 37, 38, 35]. These results all show that ground state degeneracy is a generic feature of the model. In fact, in two spatial dimensions the number of ground states typically grows exponentially with the system size. This characteristic of having an extensive ground state entropy S_{GS} goes under the name of superfrustration [21].

A heuristic way of understanding the superfrustration is from the "3-rule": to minimize the energy, fermions prefer to be mostly 3 sites apart (with details depending on the lattice). For generic two-dimensional lattices the 3-rule can be satisfied in an exponential number of ways.

Superfrustration, in particular if it leads to an extensive ground state entropy, is in contradiction with the third law of thermodynamics, which says that as the temperature goes to zero, the entropy vanishes. A well-known counter example is formed by glasses, which have a frozen-in disorder that can lead to a zero temperature entropy of the order of the number of atoms. More recently this feature has also been observed experimentally in highly pure $Sr_3Ru_2O_7$ single crystals at magnetic field strengths for which the compound is believed to have a zero temperature quantum critical point [79] (see also [80] for a nice perspective). For the supersymmetric model, the exact ground state degeneracy is an artifact of the imposed supersymmetry. That is, all parameters in the hamiltonian are tuned, such that the ground states have zero energy precisely and are thus exactly degenerate. The frustration, however, arises due to a subtle interplay between the kinetic and interaction terms and could persist even for modest perturbations around the supersymmetric point. We expect that the exact degeneracy will be lifted, but that for sufficiently small perturbations the ground states will form a low lying band with a non-trivial contribution to the entropy at small temperatures. We thus like to take the point of view that fine tuning to the supersymmetric point allows us to analyze the model, but that the features of the supersymmetric system may well be exhibited by more generic strongly correlated itinerant fermion systems.

On a side note, the term superfrustration also arises in the context of frustrated spin models [81], which also exhibit an extensive ground state entropy. For the spin models, however, the term superfrustration is reserved for models in which the frustration is so strong that it explicitly causes all correlations to decay exponentially at any temperature. For the models discussed here, we believe that this is typically not the case. Apart from explicit examples such as the chain and various ladder models (chapters 4 and 7), the intuition is in general that the supersymmetry is not only related to superfrustration but also to quantum criticality.

In this chapter we aim at giving an up to date overview of the known results for the supersymmetric model on two-dimensional lattices and graphs. Exact results for the number of ground states were obtained for various lattices [21, 35], which will be summarized in sections 5.2 and 5.4. Numerical studies of the Witten index have shown that even this lower bound is typically extensive [78] (see section 5.3.1). For certain two-dimensional lattices it was proven that zero-energy ground states exist at various fillings [39] (see section 5.3), indicating a flat band in the energy as a function of the fermion density. The filling is defined as the number of particles per lattice site. For the square, triangular and hexagonal lattice there exist zero-energy ground states for all rational fillings ν within the range $[1/5, 1/4]$, $[1/7, 1/5]$ and $[1/4, 5/18]$, respectively.

The final section discusses the square lattice [37, 35]. The solution of the cohomology problem for this lattice on the torus is one of the main results of this thesis. We find a theorem that relates elements of the cohomology to tilings of the lattice [35]. This relation between ground states and tilings seems to be more generic [39]. The implications of the theorem are discussed here, the proof, however, is postponed to chapter 6.

5.2 Exact results for the cohomology of Q on two-dimensional lattices

In chapter 2, we discussed the relation between ground states of the supersymmetric model and elements of the cohomology of the supercharge Q. In particular, we stated the 'tic-tac-toe'-lemma for a double complex (2.2.3). This lemma is very powerful for the supersymmetric model, since one can easily construct a double complex by dividing the lattice into two sublattices. Using this method, Fendley and Schoutens computed the cohomology and thus the number of ground states analytically for various two-dimensional lattices [21]. In figure 5.1 we show the enneagon-triangle lattice, also known as the martini lattice, the octagon-square lattice and the square lattice with two additional sites on every link. In the following sections, we will explicitly compute the cohomology of Q for these lattices. Here we first summarize the results. For the martini lattice, the number of ground states is found to grow exponentially with the number of lattice sites. For the octagon-square lattice with $4MN$ sites, the number of ground states grows as $2^M + 2^N - 1$. For the square lattice with two additional sites on every link, one finds that there are just

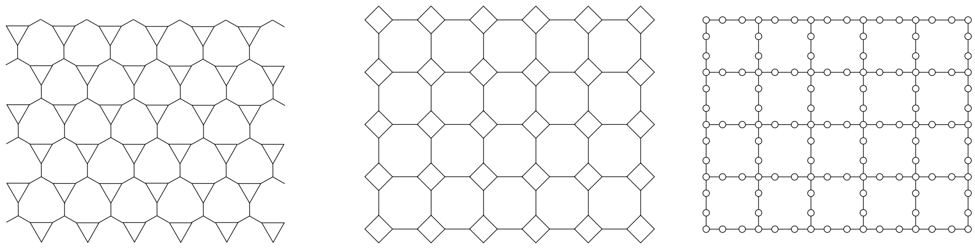

Figure 5.1: From left to right we show the martini lattice, the octagon-square lattice and the square lattice with two additional sites per link.

two ground states. Finally, in section 5.4 we will see that for the square lattice of MN sites, for which an exact solution of the cohomology of Q also exists, the ground state degeneracy roughly grows as 2^{M+N}.

It follows that for all lattices, for which the exact ground state degeneracy was found, only the martini lattice has an extensive ground state entropy. This may seem in contradiction with our claim, at the start of this chapter, that an extensive ground state entropy is a generic property of two-dimensional lattices. However, solving the cohomology problem is in general very difficult and the fact that an exact solution can be found for these lattices is probably related to the fact that they have a relatively simple ground state structure. For the martini lattice and the octagon-square lattice, for example, all ground states have the same particle number. The square lattice with two additional sites on every link has ground states at different fillings, but there are just one or two at each filling. A way to interpret these simplifications is that the lattices for which an exact solution exists, typically accommodate the 3-rule better. This property, in turn, often leads to a reduction in the frustration.

5.2.1 Martini lattice

For the martini lattice, which is formed by replacing every other site on a hexagonal lattice by a triangle (see 5.1), the number of zero-energy ground states was found to equal the number of dimer coverings of the hexagonal lattice [21]. The argument is as follows. Let us define sublattice S_1 as the collection of sites on the corners of the triangles. The remaining sites belong to sublattice S_2. We define $Q = Q_1 + Q_2$, where Q_1 and Q_2 act on different sublattices S_1 and S_2. The 'tic-tac-toe'-lemma says that the cohomology, H_Q, is the same as the cohomology of Q_1 acting on the cohomology of Q_2, that is $H_Q = H_{Q_1}(H_{Q_2})$, given that all non-trivial elements of H_{12} have the same fermion-number, f_2, on S_2. To compute the cohomology of Q_2 we consider a single site on S_2. If both of the adjacent S_1 sites are empty, H_{Q_2} is trivial: Q_2 acting on the empty site does not vanish, while the filled site is Q_2 acting on the empty site. Thus H_{Q_2} is non-trivial only when every site on S_2 is forced to be empty by being adjacent to an occupied site. Note that there can be at most one particle on each triangle in S_1 and that there are as many sites in S_2 as there are triangles in S_1 (for appropriate boundary conditions). It follows that an element in H_{Q_2} must have precisely one particle per triangle, each adjacent to a different site in S_2. The number of such configurations equals the number of dimer coverings on the original hexagonal lattice. This can be seen by thinking of the dimers as stretching between the S_2 site and the site replaced by the triangle whose occupied site is adjacent to the S_2 site. A dimer covering is such that each site is connected to precisely one link containing a dimer. This is equivalent to having precisely one particle per triangle in the martini lattice. Since all elements in H_{Q_2} have the same number of particles on S_1, they are automatically also in H_{12}. Finally, since all elements have $f_2 = 0$, we find that $H_Q = H_{12}$. The number of ground states for the supersymmetric model on the martini lattice, thus equals the number of dimer coverings on the original hexagonal lattice. This problem was solved in the context of statistical mechanics [82, 83]. For large systems (number of sites $N \to \infty$) this gives a closed expression for the ground state entropy, S_{GS}, per site

$$\frac{S_{\text{GS}}}{N} = \frac{1}{\pi}\int_0^{\pi/3} d\theta \ln[2\cos\theta] = 0.16153\ldots$$

We conclude that the ground state entropy is an extensive quantity. This is closely related to the fact that for the martini lattice, due to its structure, the 3-rule can be implemented in many ways.

5.2.2 Octagon-square lattice

The $M \times N$ octagon-square lattice is obtained from the $M \times N$ square lattice by replacing every site by a square (see figure 5.1). The $M \times N$ octagon-square lattice thus contains $4MN$ sites. For this lattice the total number of ground states can be obtained analytically [21], again by using the 'tic-tac-toe' lemma. We will consider the case with doubly periodic boundary conditions. For this case, we take S_1 to consist of all left-most sites of the squares. Consequently, S_2 is a collection of N periodic chains of length $3M$. Now remember that the periodic chain of length $3M$ has two ground states with M fermions. It follows that if we leave S_1 completely empty, we obtain 2^N non-trivial elements of H_2 at grade MN, that is, with MN fermions. To find the other elements of H_2 we note that if we occupy a site on S_1, we block two sites on the S_2 chain to its right. Consequently, the S_2 chain effectively reduces to an isolated site and an open chain of length $3(M-1)$. It follows that a configuration with a single site on S_1 occupied, does not belong to H_2, since it will contain an isolated site that can be both empty and occupied. Upon inspection of the lattice, one quickly sees that if all S_1 sites on a row are occupied, the S_2 sublattice effectively becomes a collection of open chains of length $3(M-1)$. Consequently, this gives an element of H_2 at grade $N(M-1)+N = MN$. It follows that H_2 is trivial unless *all* S_1 sites on a row are either empty or occupied. This gives $2^N - 1$ non-trivial elements of H_2 corresponding to the configurations with at least one row of S_1 sites occupied and finally 2^M non-trivial elements of H_2 corresponding to the configuration with all S_1 sites empty. One easily checks that all non-trivial elements in H_2 are at grade MN and thus with MN fermions. From this it follows that all elements in H_2 are automatically also in H_{12}. Finally, the fact that all elements in H_{12} have the same fermion number $f = MN$, implies that Q cannot map one element to another. From this it follows that $H_Q = H_{12}$, even though, the condition that all elements have equal f_2, does not hold for this case. We conclude that the supersymmetric model on the octagon-square lattice has $2^N + 2^M - 1$ zero energy ground states with MN particles.

For completeness, we mention that using the same arguments as above, we find that the number of ground states for the octagon-square lattice with periodic boundary conditions only in the horizontal (vertical) direction is 2^N (2^M). For open boundary conditions in both directions the ground state is unique.

5.2.3 Graphs with extra sites on the links

In the previous two examples, we clearly saw that the 'tic-tac-toe' lemma could be exploited because the lattices naturally break up into sublattices. In this section, we discuss a class of graphs where this property is again crucial. We introduce graphs of type Λ_n, which are obtained from the original graph Λ by adding $n-1$ additional vertices on every link. In the following the only restriction on the graph Λ is that it does not contain any isolated sites. This is because for that type of graph the cohomology of Q is always trivial.
Let us start with the case Λ_3, which was discussed in [21]. That is, we put two additional sites on every link. Now consider the original graph Λ as the subgraph S_1 and the additional sites as the subgraph S_2. The graph S_2 is a collection of two site chains. The non-trivial elements in H_2 have the S_1 sites neighboring a two site chain on S_2 either both empty or both occupied. It follows that there are only two non-trivial elements in H_2, one with S_1 completely empty and one with S_1 completely filled. If we leave S_1 completely empty, we obtain one non-trivial element of H_2 at grade L_Λ, where L_Λ denotes the number of links in the original graph Λ. The element with S_1 completely filled, clearly sits at grade N_Λ, where N_Λ denotes the number of vertices in the original graph Λ. It now quickly follows that these two elements are also in H_{12} and in H_Q provided that $N_\Lambda \neq L_\Lambda \pm 1$.
As an example consider the square lattice with doubly periodic boundary conditions as the original graph Λ. The lattice Λ_3 is shown in figure 5.1. We find that $L_\Lambda = 2N_\Lambda$ and the total number of sites in Λ_3 is $N = 2L_\Lambda + N_\Lambda$. Consequently, this lattice has one ground state at 1/5 filling and one at 2/5 filling.
We can extend this result for the graphs of type Λ_3 to the general case of Λ_{3m}, with m integer, using the cohomology results for the chain with open boundary conditions (see section 4.4). Remember that open chains of length $3m$ and $3m-1$ have one ground state with $f = m$ fermions and an open chain of length $3m+1$ has no zero-energy ground state. If we take S_2 again to consists of all the added sites in Λ_{3m}, it is a collection of chains of length $3m-1$. It follows that there are again two non-trivial elements in H_2, one at grade mL_Λ and one at grade $N_\Lambda + (m-1)L_\Lambda$. Again we find that these two elements are also in H_{12} and in H_Q provided that $N_\Lambda \neq L_\Lambda \pm 1$.
For graphs of type $\Lambda_{3m\pm1}$ there is no general solution to the cohomology problem, however, one can relate it to the cohomology problem of the original lattice Λ.
For the graph Λ_{3m+1} the subgraph, S_2, is a collection of chains of length $3m$. For this case, two S_1 sites that are neighbors in the original graph, cannot be both occupied, because then the S_2 chain between these two sites effectively reduces to an open chain of length $3m-2$, which has no zero-energy ground state. However, if the two S_1 sites are both empty or if one of them is occupied, the S_2 chain accommodates one ground state with m fermions in all cases. It follows that all allowed configurations on the original graph Λ are non-trivial elements of H_2. Clearly, we then find that H_{12} is directly related to the cohomology of the original graph. Finally, since all elements in H_{12} have $f_2 = mL_\Lambda$ we find that $H_Q = H_{12}$ holds. We thus find that the cohomology of Q on Λ_{3m+1} at grade $n + mL_\Lambda$ is directly related to the cohomology of Q on Λ at grade n. In particular, we find that the Witten indices are related via $W_{\Lambda_{3m+1}} = (-1)^{mL_\Lambda} W_\Lambda$.
For the graph Λ_{3m-1} the subgraph, S_2, is a collection of chains of length $3m-2$. For this case, two S_1 sites that are neighbors in the original graph, cannot be both *empty*, because then the S_2 chain between these two sites has no zero-energy ground state. However, if the two S_1 sites are both occupied or if one of them is occupied, the S_2 chain accommodates

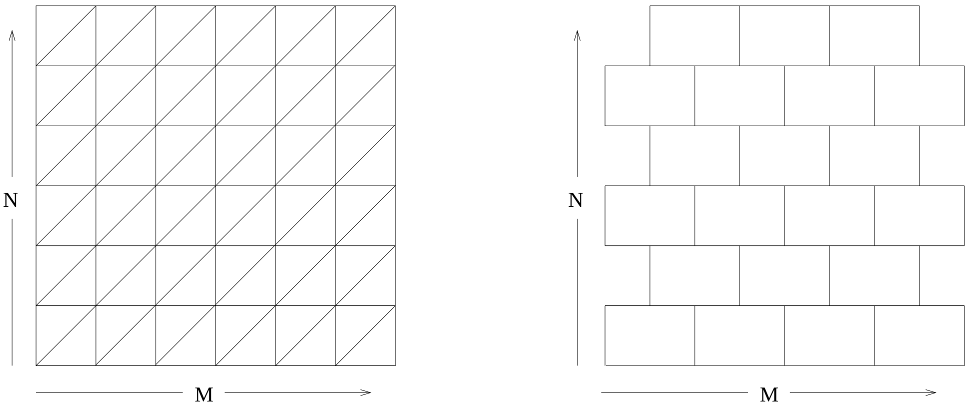

Figure 5.2: The $M \times N$ triangular (left) and hexagonal (right) lattice with periodic boundary conditions along the directions of the two arrows.

one ground state with $m-1$ fermions in all cases. We can again relate all the elements in H_2 to configurations on the original graph Λ, but now we have to use a particle-hole symmetry. Following the same arguments as for the previous case, we conclude that the cohomology of Q on Λ_{3m-1} at grade $n+mL_\Lambda$ is directly related to the cohomology of Q on Λ at grade $N_\Lambda - n$. In particular, the Witten indices are related via $W_{\Lambda_{3m-1}} = (-1)^{mL_\Lambda - N_\Lambda} W_\Lambda$.
The results we derived in this section for graphs of type Λ_n with general n were also obtained on a homotopy level using Alexander dualities in [77].

5.3 Triangular and hexagonal lattice

In this section, we discuss the triangular and hexagonal lattice. For these lattices, the ground state structure is not fully understood. Nevertheless, it is clear that ground states occur in a finite window of filling fractions $\nu = f/L$ [39] and that there is extensive ground state entropy [78]. The latter result, stems from a numerical analysis of the Witten index for finite size systems (see 5.3.1). The former result, is an analytic result, which we discuss in section 5.3.2, where we also discuss an analytic upper bound on the number of ground states [76].

5.3.1 Numerical results for Witten index

In tables 5.1 and 5.2 respectively, we show the Witten indices for the $M \times N$ triangular and hexagonal lattices, with periodic boundary conditions applied along two axes of the lattice [78] (see figure 5.2). The exponential growth of the index is clear from the table. To quantify the growth behavior, one may determine the largest eigenvalue λ_N of the row-to-row transfer matrix for the Witten index on size $M \times N$. For the triangular lattice this gives [78]

$$|W_{M,N}| \sim (\lambda_N)^M + (\bar{\lambda}_N)^M \ , \quad \lambda_N \sim \lambda^N$$
$$|\lambda| \sim 1.14 \ , \quad \arg(\lambda) \sim 0.18 \times \pi$$

leading to a ground state entropy per site of

$$\frac{S_{\mathrm{GS}}}{MN} \geq \frac{1}{MN} \log |W_{M,N}| \sim \log |\lambda| \sim 0.13 \ .$$

The argument of λ indicates that the asymptotic behavior of the index is dominated by configurations with filling fraction around $\nu = 0.18$. Similarly, one obtains for the hexagonal lattice [78]

$$\frac{S_{\mathrm{GS}}}{MN} \gtrsim 0.18 \ .$$

For this case the data is insufficient to determine the argument of λ.

Table 5.1: Witten index for $M \times N$ triangular lattice (taken from [78]).

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

Table 5.2: Witten index for $M \times N$ hexagonal lattice (taken from [78]).

	2	4	6	8	10	12	14	16	18
2	-1	-1	2	-1	-1	2	-1	-1	2
4	3	7	18	47	123	322	843	2207	5778
6	-1	-1	32	-73	44	356	-1387	2087	2435
8	3	7	18	55	123	322	843	2215	5778
10	-1	-1	152	-321	-171	7412	-26496	10079	393767
12	3	7	156	1511	6648	29224	150069	1039991	6208815
14	-1	-1	338	727	-5671	1850	183560	-279497	-4542907
16	3	7	1362	12183	31803	379810	5970107	55449303	327070578

5.3.2 Bounds on the size and dimension of the homology of Q

There are two main results for bounds on the full cohomology problem for the triangular and hexagonal lattices. Here we briefly sketch how they were obtained. The first result, obtained by Jonsson [39], is a certain type of homology cycle for these lattices called cross-cycles. The size of a cross-cycle refers to the number of vertices in the cycle, that is the number of occupied sites. A bound on the size of these cross-cycles is obtained and this results in a bound on the grade of the vector spaces for which the homology is non-vanishing. That is, there is a set of rational numbers r, such that there exist cross-cycles of size rN, where N is the number of vertices of the two-dimensional lattice. For the triangular lattice it is found that $r \in [\frac{1}{7}, \frac{1}{5}] \cap \mathbb{Q}$ and for the hexagonal lattice $r \in [\frac{1}{4}, \frac{5}{18}] \cap \mathbb{Q}$ (and $r \in [\frac{1}{5}, \frac{1}{4}] \cap \mathbb{Q}$ for the square lattice, see section 5.4). Let us give the specific form of a cross-cycle z of size k:

- $z = \prod_{i=1}^{k}(|a_i\rangle - |b_i\rangle)$ such that $z \in C_k$, that is the a_i and b_i obey the hard-core condition.
- Furthermore, there is at least one configuration in z such that all sites in the lattice are either occupied or adjacent to at least one occupied site. This is called a maximal independent set.
- Finally, a_i is adjacent to b_i.

Note that, in this case, we consider the homology and not the cohomology. It is easily verified that z belongs to the kernel of $Q^\dagger$, since $Q^\dagger$ gives zero on each term in the product:

$$Q^\dagger(|a_i\rangle - |b_i\rangle) = |\emptyset\rangle - |\emptyset\rangle = 0.$$

The latter two conditions ensure that z is not exact. The second condition ensures that there is no site c such that $Q^\dagger|c\rangle \prod_{i=1}^{k}(|a_i\rangle - |b_i\rangle) = z$ and the third condition ensures that $|a_j\rangle|b_j\rangle \prod_{i\neq j}(|a_i\rangle - |b_i\rangle)$ violates the hard-core condition.
Clearly, the latter two conditions, combined with the hard-core condition, also impose certain bounds on the size of a cross-cycle. For the triangular lattice the size of the cross-cycles is at most a fifth of all the sites in the lattice and at least a seventh [39]. In fact, the cross-cycles are found to induce tilings of the triangular lattice with paralellogram-shaped tiles of area 5 and area 7. See figure 5.3 for a specific tiling and its corresponding cross-cycle. The black dots are the a_i sites, whereas the white dots are the b_i sites. Note that both $\{a_i\}$ and $\{b_i\}$ are a maximal independent set. For the square lattice the cross-cycles also induce a tiling (see figure 5.7). This will be discussed in more detail in the next section. For further details and the specific form of the cross-cycles for the hexagonal lattice we refer to [39].
The second result that imposes a bound on the (co)homology on the triangular and hexagonal lattices, was obtained by Engström [76]. He finds an upper bound to the total dimension of the cohomology and thus to the Euler characteristic for general graphs G using discrete Morse theory: if G is a graph and D a subset of its vertex set such that $G \setminus D$ is a forest, then $\sum_i \dim(H_Q^{(n)}) \leq |\mathrm{Ind}(G[D])|$. Here $H_Q^{(n)}$ is the n-th cohomology class of Q on the independence complex on the graph G and $\mathrm{Ind}(G[D])$ is the induced independence complex on the subset D. Thus finding the minimal set of vertices that should be removed from G to obtain a forest, gives an upper bound on the total dimension of H_Q and thus on the total number of zero energy ground states for the supersymmetric model on the graph G.
As an example we show the hexagonal lattice of size $2m \times 2n$. The subset D contains mn disconnected sites (see figure 5.4). Each site can be empty or occupied in $\mathrm{Ind}(G[D])$ and thus $|\mathrm{Ind}(G[D])| = 2^{mn}$. Finally, also the triangular lattice is shown (figure 5.5). For the triangular lattice of size $2m \times n$ the upper bound was found to be approximately ϕ^{mn}, with $\phi = \frac{1}{2}(1 + \sqrt{5})$, the golden ratio.

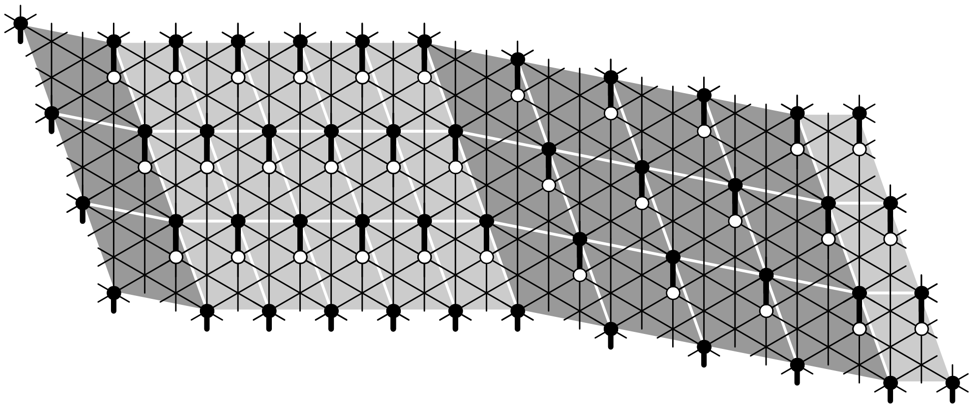

Figure 5.3: An example of a cross-cycle on the triangular lattice (taken from [39]). The black dots are the a_i sites, the white dots are the b_i sites. Both sets $\{a_i\}$ and $\{b_i\}$ are maximal independent sets. Notice the induced tiling.

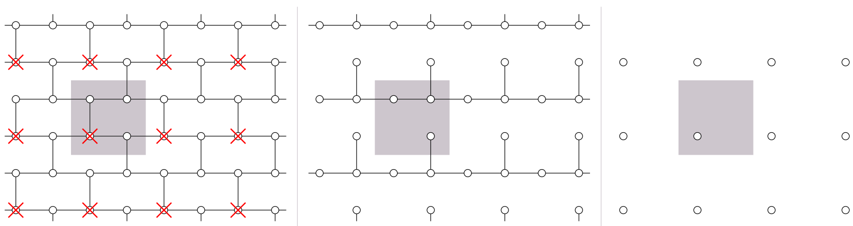

Figure 5.4: For the hexagonal lattice we show from left to right: the graph G, the forest $G \setminus D$ and the subset D (taken from [76]).

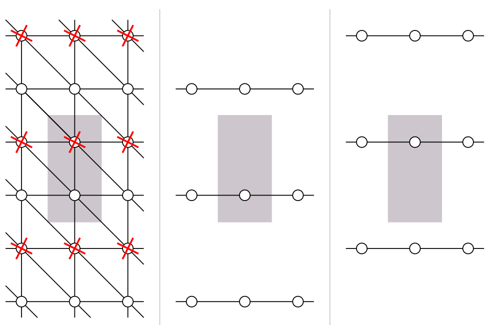

Figure 5.5: For the triangular lattice we show from left to right: the graph G, the forest $G \setminus D$ and the subset D (taken from [76]).

Table 5.3: Witten Index for $M \times N$ square lattice with periodic boundary conditions along the horizontal and vertical direction (taken from [36]).

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	-1	1	3	1	-1	1	3	1	-1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4
4	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	-9	1	1	1	1	11	1	1
6	1	-1	4	3	1	14	1	3	4	-1	1	18
7	1	1	1	1	1	1	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7
9	1	1	4	1	1	4	1	1	40	1	1	4
10	1	-1	1	3	11	-1	1	43	1	9	1	3
11	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166
13	1	1	1	1	1	1	1	1	1	1	1	1
14	1	-1	1	3	1	-1	-27	3	1	69	1	3
15	1	1	4	1	-9	4	1	1	4	11	1	4

5.4 Square lattice

In this section, we discuss the square lattice which plays a central role in this thesis. For this lattice we really tried to push our understanding of both the ground state properties as well as the low lying excitations for the two-dimensional system. In this section we summarize the results that were obtained for the ground state counting. For the square lattice with various types of boundary conditions the cohomology problem can be fully solved [35, 75, 84], however, the proofs, especially for the square lattice wrapped around the torus, are rather involved and are presented separately in chapter 6. The discussion of the low lying excitations is postponed to chapter 8.
A first study of the Witten index for the square lattice with doubly periodic boundary conditions revealed various interesting properties [36]. First of all, numerical studies of the Witten index does not show the exponential growth as was found for various other lattices. Instead, at first glance it clearly shows a very different behavior (see table 5.3). A more detailed investigation of these results led to two conjectures [36] for which a proof was found by Jonsson [37].
We state these results here:

- The eigenvalues of the transfer matrix with periodic boundary conditions are all roots of unity.
- For an $M \times N$ square lattice with periodic boundary conditions in both directions, $W = 1$ when M and N are coprime.

Extending this work, Jonsson found a general expression for the Witten index $W_{u,v}$ of hard-core fermions on the square lattice with periodic boundary conditions given by the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$. The $M \times N$ square lattice is now a specific case with $u = (M, 0)$ and $v = (0, N)$ [37]. A crucial step in [37] is the introduction of rhombus tilings

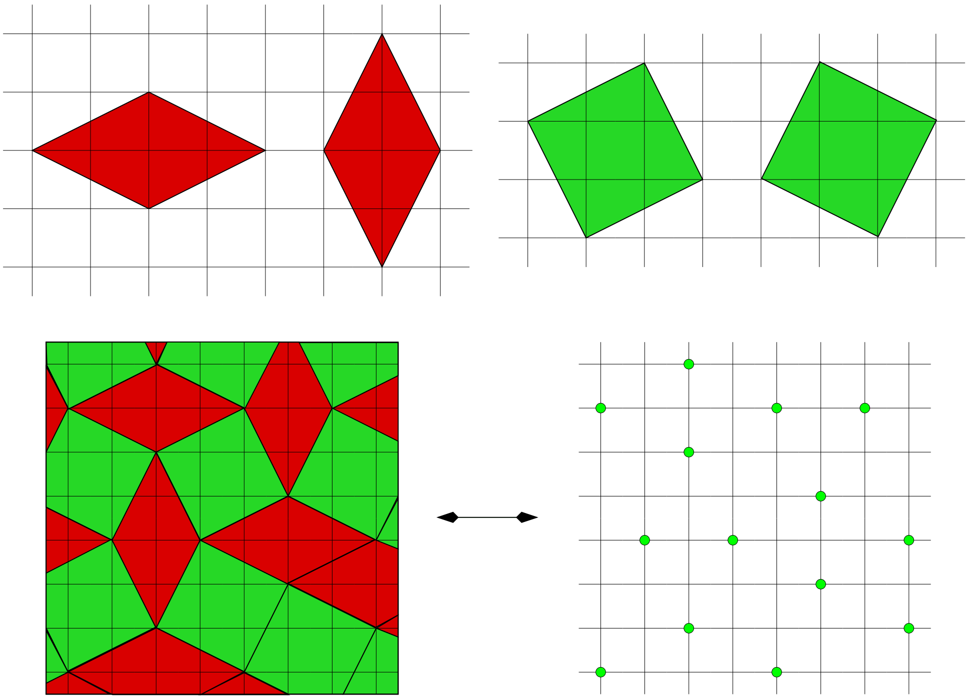

Figure 5.6: Tilings of the 2D square lattice. Above: the four different rhombi. Below: mapping between tiles and hard-core fermions.

of the square lattice. It is shown that the trace in the Witten index can be restricted to configurations that can be mapped to coverings of the plane with the four rhombi or tiles shown in figure 5.6. A rhombus tiling is obtained by tiling the plane with the rhombi depicted in figure 5.6, such that the entire plane is tiled and the rhombi do not overlap (they can have only a corner or a side in common). We call the tiles with area 4 diamonds and the ones with area 5 squares. Note that the sides of these rhombi, which connect the hard-core fermions, are in agreement with the heuristic 3-rule.
To state Jonsson's results for the Witten index we introduce the following notations. We denote by $R_{u,v}$ the family of tilings of the plane with boundary conditions given by $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Furthermore $|R^+_{u,v}|$ and $|R^-_{u,v}|$ are the number of tilings of this plane with an even and an odd number of tiles, respectively. Finally, we define

$$\theta_d \equiv \begin{cases} 2 & \text{if } d = 3k, \text{ with } k \text{ integer} \\ -1 & \text{otherwise.} \end{cases}$$

Jonsson's result then reads

Theorem 1 (Jonsson, 2006) *The Witten index for the square lattice with periodicities* $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *is given by*

$$W_{u,v} = -(-1)^d \theta_d \theta_{d^*} + |R^+_{u,v}| - |R^-_{u,v}|,$$

where $d \equiv \gcd(u_1 - u_2, v_1 - v_2)$ *and* $d^* \equiv \gcd(u_1 + u_2, v_1 + v_2)$.

It can be shown that the Witten index grows exponentially with the linear size (not the area) of the 2D lattice. Detailed results for the case of diagonal boundary conditions have been given in [38]. Further studies of the Witten index transfermatrix for the square lattice

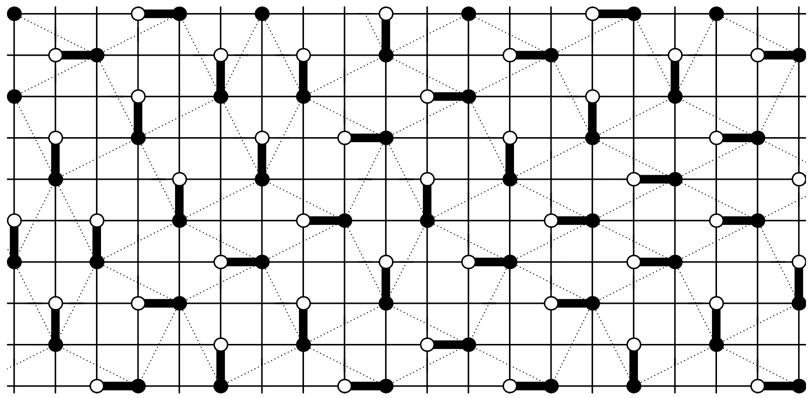

Figure 5.7: An example of a tiling on the square lattice and its corresponding cross-cycle. The black dots are the a_i sites, the white dots are the b_i sites. Both sets $\{a_i\}$ and $\{b_i\}$ are maximal independent sets. (Figure source: [39])

with diagonal and free boundary conditions by Baxter [74] have led to an additional set of conjectures.
We should stress that this result was obtained by studying the purely combinatorial problem of configurations of hard-core fermions or, equivalently, of hard squares with negative activity. For the square lattice the condition that two particles cannot occupy two adjacent sites readily translates to the hard square condition if we define the squares to be tilted by 45° and to have a particle at their center. It follows that the squares cannot overlap, however they can have a corner or a side in common.
In a follow-up study Jonsson presents the first results on the homology problem for the square lattice with doubly periodic boundary conditions [39]. He shows that at certain grades the homology must be non-vanishing since there exist non-trivial elements of the homology. These cross-cycles were discussed in section 5.3.2. Jonsson finds that for the square lattice the size of the cross-cycles is at most a quarter of all the sites in the lattice and at least a fifth. In fact, he finds that all the possible cross-cycles are one-to-one with the tiling configurations of the rhombi of figure 5.6. See figure 5.7 for a specific tiling and its corresponding cross-cycle $z = \prod_{i=1}^{k}(|a_i\rangle - |b_i\rangle)$. The black dots are the a_i sites, whereas the white dots are the b_i sites. Note that both $\{a_i\}$ and $\{b_i\}$ are a maximal independent set.
A natural extension of the results by Jonsson, is to relate the full cohomology problem to tiling configurations. This relation was first conjectured by Fendley [34]. Using the 'tic-tac-toe' lemma and a spectral sequence, we were able to prove this relation explicitly when $\vec{u} = (m, -m)$ and $v_1 + v_2 = 3p$ [35] (see chapter 6).

Theorem 2 *For the square lattice with periodicities $\vec{v} = (v_1, v_2)$, $v_1 + v_2 = 3p$ with p a positive integer and $\vec{u} = (m, -m)$, we find for the cohomology H_Q*

$$N_n = \dim(H_Q^{(n)}) = t_n + \Delta_n \tag{5.1}$$

where N_n is the number of zero energy ground states with n fermions, t_n is the number of rhombus tilings with n tiles, and

$$\Delta_n = \begin{cases} \Delta \equiv -(-1)^{(\theta_m+1)p}\theta_d\theta_{d*} & \text{if } n = [2m/3]p \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

with $[a]$ *the nearest integer to* a*. Finally,* $d = \gcd(u_1 - u_2, v_1 - v_2)$*,* $d^* = \gcd(u_1 + u_2, v_1 + v_2)$ *and*

$$\theta_d \equiv \begin{cases} 2 & \textit{if } d = 3k, \textit{ with } k \textit{ integer} \\ -1 & \textit{otherwise.} \end{cases} \tag{5.3}$$

Although, the proof is restricted to a certain set of periodicities, the theorem is expected to hold for general $\vec{u}$ and $\vec{v}$. Compelling evidence stems, on the one hand, from the fact that Jonsson's result for the Witten index holds for general periodicities and on the other hand, from numerical and analytic results for small systems (see chapter 7).
Clearly, as an immediate consequence of this theorem, we obtain for the Euler characteristic, or equivalently, the Witten index

$$\chi \equiv \sum_n \left[(-1)^n \mathrm{dim} H_Q^{(n)}\right] = \sum_n (-1)^n (t_n + \Delta_n).$$

which is precisely the result obtained by Jonsson for the hard squares at activity $z = -1$ [37].
Another direct consequence follows from the area of the tiles. The diamonds have area 4, and thus a tiling with solely diamonds will contain $L/4$ tiles. This corresponds to an element in the $L/4$-th cohomology and a ground state with $L/4$ particles. Conversely, a tiling consisting of squares only corresponds to an element in the $L/5$-th cohomology and a ground state with $L/5$ particles. Continuing this argument for general tilings with the diamonds and squares, we find on the infinite plane that for all rational numbers $r \in [\frac{1}{5}, \frac{1}{4}] \cap \mathbb{Q}$ the cohomology at grade rL is non vanishing, or, equivalently, there exists a zero energy ground state with rL particles. Clearly, this nicely agrees with the bounds on the size of the homology of $Q^\dagger$ obtained by Jonsson using the cross-cycles. However, for the cross-cycles it was an open question whether they are independent, i.e. in different homology classes, and whether they constitute a basis. A comparison with our result, theorem 2, suggests that the cross-cycles are indeed independent and span the full homology with the exception of Δ_n elements at the n-th grade.
Finally, the theorem provides insight in the growth behavior of the number of ground states, since this is now directly related to the growth behavior of the number of tilings. In [38] various results on the number of tilings on the doubly periodic square lattice are reported (see also section 6.6). Here we mention two of these results for the case that $\vec{u} = (m, -m)$ and $\vec{v} = (k, k)$.

1 For m and k such that $\gcd(m, k) = 1$, there are no rhombus tilings that satisfy the periodicities given by $\vec{u}$ and $\vec{v}$.

2 For $m = 3\mu q$ and $k = 3\lambda q$, with μ and λ positive integers and q large, the total number of rhombus tilings t grows as

$$t \equiv \sum_n t_n \sim \frac{9}{2} \frac{4^{\mu q + \lambda q}}{\pi q \sqrt{\mu \lambda}}. \tag{5.4}$$

In the first case, it follows that the number of ground states with n particles is given by Δ_n, which is non-zero only for $n = [2m/3]p$ given that $2k = 3p$. In the second case, the number of ground states shows the same growth behavior as the number of tilings.

This number turns out to be dominated entirely by the number of tilings with $2L/9$ tiles. Furthermore, it is noteworthy that the number of tilings grows exponentially with the linear dimensions, instead of the area, of the system. It follows that, even though the system is highly frustrated, this leads only to a sub-extensive ground state entropy. This is in contrast with results discussed in the previous sections for the triangular, hexagonal and martini lattices, for which the ground state entropy was found to be extensive.
We finish this chapter by stating the results for the square lattice on the plane and on the cylinder. For open boundary conditions in either one or both of the diagonal directions along the square lattice ($(m,-m)$ and (n,n)) the number of ground states reduces dramatically [37, 75, 34]. One finds that it is either one or zero, except for the cylindrical case periodic in the $(m,-m)$-direction with $m = 3p$ and $n = 3q+2$ or $n = 3q+3$. In that case the number of ground states is $4^{(q+1)}$. These results are fairly easily obtained using the 'tic-tac-toe' lemma. The proofs are given in section 6.4. Finally, there is a result for the Witten index for the square lattice of $M \times N$ sites with periodic boundary conditions along the $(N,0)$-direction and N odd [84]. It was proven that for odd N

$$W = \begin{cases} 2 & \text{if 3 divides } \gcd(M-1,N) \\ -1 & \text{otherwise.} \end{cases}$$