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**A supersymmetric model for lattice fermions**

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# Chapter 8

## Discussion and outlook

### 8.1 Quantum criticality

We have seen that for a variety of one dimensional lattices, the model shows quantum critical behavior. For the chain this is in fact an exact result [22]. Here the model is integrable and in the continuum limit one can derive the thermodynamic Bethe Ansatz equations. From this it was found that the low energy spectrum in the continuum limit is described by the massless free boson at the  $\mathcal{N} = 2$  supersymmetric point. In chapter 4 we investigate the continuum limit in great detail. Combining numerical and analytic results, we relate operators and sectors in the continuum theory, on the one hand, to lattice operators and chains with different boundary conditions on the other hand. Furthermore, the chain serves as a good model to test and develop the spectral flow analysis [34]. By relating a boundary twist in the lattice model to a spectral flow in the continuum theory, we can extract important parameters in the continuum theory from the numerics. This technique has several important pros. First of all, it tells us on a qualitative level that if the energy of a state has a parabolic dependence on the twist parameter, this is clear evidence of quantum criticality. Second of all, the analysis can be carried out in a single system, that is, no scaling of the size of the system is necessary. This is very useful in particular for systems with oscillations as a function of the number of sites (for the chain these oscillations have a period of only 3 sites, but for example for the 3 leg ladder discussed in chapter 7 this is 18 sites!). Finally, we find that the quantitative results extracted from the spectral flow analysis are quite accurate already in very small systems, at least for the lowest energy state. The only true con of the spectral flow analysis, is that it only gives ratios, consequently, it cannot be used to extract the value of the central charge.

In chapter 7 we apply the spectral flow analysis on various ladder models and find convincing evidence for critical modes in these systems. The ubiquity of criticality in the supersymmetric model on one dimensional lattices may suggest a deep relation with the imposed supersymmetry. Since quantum criticality usually occurs in an isolated point in the parameter space of interactions, it typically requires fine tuning. Here it seems that imposing supersymmetry, which indeed leads to fine tuning of the interactions, is a sufficient condition for criticality. In [90] this question is addressed for a specific lattice. It is argued that by tuning to the supersymmetric point, one lands on a multi-critical point, representing a phase transition between at least three different types of order. Although the reported results are very suggestive, further investigations are needed to confirm this hypothesis. Finally, there are also a few counter examples, where the supersymmetric model on a one dimensional lattice is gapped (the zig-zag ladder at quarter filling for example).

## 8.2 Ground states and tilings

The numerous results on the number of zero-energy ground states of the supersymmetric model, discussed in this thesis, suggest that there is a profound connection with tilings. For the square lattice there is an explicit relation between zero energy ground states of the supersymmetric model and tilings. However, this relation is not exclusive to the square lattice, but seems to be a more generic feature. In chapter 5, we discussed the work of Jonsson [39] which relates homology elements of  $Q$  to tilings for the triangular and hexagonal lattice. In chapter 7, we encountered the kagomé ladder as yet another example. Moreover, there is compelling evidence from the zig-zag ladder that the tilings not only count, but indeed seem to dominate the actual ground states. In this section we wish to exploit this relation and discuss various possible implications for two dimensional systems.

### 8.2.1 Edge modes in the square lattice

For the square lattice we can exploit the effective geometric description of the space of ground states by comparing periodic and free boundary conditions [34]. This comparison suggests the existence of critical edge modes in the system. For the square lattice wrapped around the torus with  $\vec{u} = (m, -m)$  and  $\vec{v} = (n, n)$  the number of ground states grows exponentially with the linear system size, i.e. as  $2^{2(n+m)/3}$  [38]. On the cylinder, however, if one cuts the torus open along the  $(m, -m)$ -direction only  $2^{2n/3}$  ground states remain. Finally, if one also cuts the cylinder open along the  $(n, n)$ -direction one is left with a unique ground state on the plane.

What happens to this vast number of ground states that disappear upon changing the boundary conditions? Consider the picture in fig. 8.1. If one identifies the dotted, zig-zagged boundaries both in the horizontal as well as in the vertical direction, one finds that both the tiling with the drawn lines as well as the one with the dashed lines represent ground states. However, if one only identifies the left and right boundaries, then the tiling with the drawn lines no longer represents a ground state. Instead it has two defects at the edges, which can propagate along the edge. The only available scale for the energy of the edge mode is one over the length of the edge, which suggests that the edge modes are gapless.

Further evidence for the existence of these gapless edge modes comes from the spectral flow analysis of the ladder realizations of the square lattice. In all three cases compelling evidence for critical modes was found. The fact that the ladders are rather confined in one direction, suggests that they essentially probe the edge modes and we have thus confirmed that they are gapless. Whether there are also gapless modes in the bulk remains unclear from this analysis.

While the physical understanding of the quantum phase on the square lattice remains far from complete, we speculate on the following picture. The ground state corresponding to a given tiling has fermions that are confined to the area set by an individual tile, but quantum fluctuating within that space. Explicit support for this picture comes from the ground states of the zig-zag ladder at quarter filling (see section 7.4.1). The tiling based physical picture of the ground state wavefunctions is reminiscent of electrons in a filled magnetic Landau level, each of them effectively occupying an area set by the strength of the magnetic field. Critical edge modes naturally fit into a picture of this sort. In the

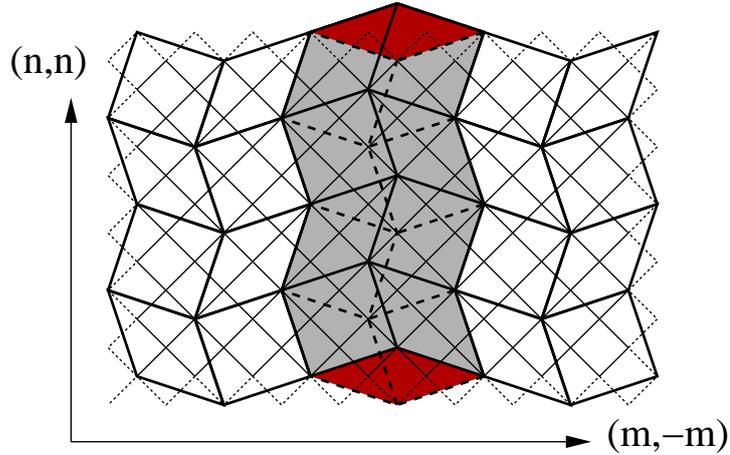


Figure 8.1: Edge modes.

next section, we briefly discuss the octagon-square lattice, which we propose as a suitable model to test these ideas.

### 8.2.2 Octagon-square lattice

As discussed in section 5.2.2, the octagon-square lattice is another 2D lattice where the supersymmetric model displays sub-extensive ground state entropy. The growth behavior of the numbers of ground states on the plane, cylinder and torus is similar to that of the square lattice. A big difference, however, is that here all ground states reside at  $1/4$  filling. This hugely simplifies the computation of the degeneracies. For the plane we find that the ground state is unique. For the cylinder with  $M \times L$  square plaquettes, where  $M$  is the number of square plaquettes along the periodic, horizontal direction and  $L$  along the open, vertical direction, the number of ground states is  $2^L$ . Finally, for the torus the number of ground states is  $2^M + 2^L - 1$  [21].

There is again a relatively simple physical picture which we propose as a basis for further analysis of physical properties [85]. This picture reflects the systematics uncovered by the analysis of the associated cohomology problem as well as results for small system sizes. The basic building block of the many-body ground states is the 1-fermion ground state on an isolated square plaquette. The unique many-body ground state on the plane essentially has individual fermions occupying this lowest 1-plaquette orbital, again allowing the analogy with a filled Landau level. Closing boundaries leads to the possibility that electrons on horizontal or vertical rows of plaquettes ‘shift’ into a second 1-fermion state, this way building up the total of  $2^M + 2^L - 1$  ground states. This picture can be further substantiated by allowing defects, which can be brought in by adding diagonal links in individual plaquettes. The diagonal link in a plaquette could be interpreted as an additional magnetic flux through the plaquette. Indeed the effect of the defect on the number of ground states seems reminiscent of the effect of adding holes in a filled Landau level. In particular, the cohomology picture of these ground states suggests that these defects may have non-trivial braid properties.

At this point, these ideas are highly speculative and it is unclear whether the cohomology elements can truly reveal so much of the physics of the model. Among the key issues that are presently on the agenda for further study are: the existence of energy gaps, the presence of bulk or edge critical modes, and interactions and braiding properties of defects. Currently, we have joined forces with J. Vala and N. Moran to address some of these questions.

### 8.2.3 Bulk criticality

On a rather speculative note one may argue that the results for the square lattice suggest the existence of bulk critical modes for the triangular and hexagonal lattice. The argument goes as follows. For the square lattice the properties of the tilings tell us that to go from one tiling to another, one has to change (at least) all the tiles along a line that wraps around one direction of the torus. This statement is equivalent to the observation that the number of tilings grows exponentially with the linear size of the system. In contrast, for the triangular and hexagonal lattice there is compelling evidence that the number of ground states grows exponentially with the (two dimensional) volume of the system. Furthermore, also here, there is a relation between the ground states and tilings. If indeed all ground states can be represented by tilings, one would find that the different tilings are related via local moves of the tiles in order to account for the extensive ground state entropy. In analogy with the square lattice, we can then speculate that a local defect can drastically reduce the number of ground states, leading to a bulk critical mode. Whether this picture is correct, is obviously an extremely difficult question.

## 8.3 Extensions of the supersymmetric model

The models discussed in this thesis are all described by the same supersymmetric hamiltonian for hardcore spinless fermions. By considering different underlying lattices a whole class of models is generated, exhibiting a rich variety of properties, ranging from broken to unbroken supersymmetry to an exponential ground state degeneracy and from quantum critical to gapped to possibly topological phases.

However, there are many other ways to adjust the supersymmetric model, some of which we would like to mention here. Possibly the simplest generalization of the model is to allow for site-dependent coefficients in the definition of the supercharges:

$$Q = \sum_i \alpha_i c_i^\dagger \prod_{j \text{ next to } i} (1 - n_j) \quad Q^\dagger = \sum_i \alpha_i c_i \prod_{j \text{ next to } i} (1 - n_j), \quad (8.1)$$

with the  $\alpha_i$  real numbers. It is easily verified that these generalized supercharges are still nilpotent. Also the hamiltonian constructed from these operators still consists of the same local terms. The parameters of the kinetic and potential terms, however, now depend on the coefficients  $\alpha_i$ . The hopping term from site  $i$  to site  $j$  next to  $i$  is parametrized by  $t_{ij} = \alpha_i \alpha_j$ , the chemical potential becomes site dependent,  $\mu_i = \alpha_i^2$  and also the two-body repulsive terms (and three- or four-body terms for  $D > 1$  dimensional lattices) acquire a similar site dependence. An interesting choice for the  $\alpha_i$  is inspired by the chemical potential: choosing  $\alpha_i = \alpha$  for  $i \bmod p = 0$  for some value of  $p$  and setting  $\alpha_i$  to unity for all other  $i$ , creates a lattice potential with a periodic staggering for the fermions.

For the one dimensional chain, for example, it is interesting to choose  $p = 3$ , since the ground states then take a very simple form in the limits  $\alpha \rightarrow 0$  and  $\alpha \rightarrow \infty$ . Exploring such staggered versions of the supersymmetric model holds the promise of interesting new insights, especially if they are combined with our understanding of the ground states in terms of cohomology elements.

Since the supersymmetric model is motivated by the wish to understand phases of strongly correlated electrons, a natural generalization of the model is to incorporate spin. This was first considered in [91]. Insisting on an  $SU(2)$  spin symmetry leads to an algebraic structure with  $\mathcal{N} = 4$  rather than  $\mathcal{N} = 2$  supersymmetry. They show that an  $\mathcal{N} = 4$  supersymmetric model for itinerant spin-1/2 fermions in  $D=1$  dimension can be constructed and the results indicate that this system describes a hole doped antiferromagnet. Unfortunately, the construction is rather involved and an extension to  $D>1$  dimensions is still an open challenge. An intermediate step between spinless fermions and spin-1/2 fermions with a full  $SU(2)$  symmetry, is to consider the supersymmetric model with two types of particles, suggestively called ”+” and ”-”. This model is studied in [90] both on the  $D=1$  dimensional chain<sup>1</sup> and on the  $D=2$  dimensional copper-oxide lattice, which is the square lattice with one additional site on each link. Various (non-supersymmetric) limits of the model are investigated, leading to a very intriguing conjectured phase diagram.

Finally, we mention another  $\mathcal{N} = 2$  supersymmetric model constructed from supercharges quite different from the ones considered in this thesis [92]. In this one dimensional model the lattice particles and their interactions are such that the excitations are cooper pairs with zero energy and free particles obeying exclusion statistics. This integrable model is closely related to the Haldane-Shastry chain [93], which is a variant of the Heisenberg model with long-range interactions. The degrees of freedom are two types of spinless free fermions with a non-trivial exchange statistics between the two types. Each type of fermion is restricted to move on a chain, but the two chains are coupled to form a zig-zag ladder. Now the supercharges change a particle of type 1 into a particle of type 2 taking into account their mutual statistics. The model is analyzed in detail via Bethe ansatz, however, a number of the derived properties can be easily understood exploiting the supersymmetry. This leads to the idea that a supersymmetry preserving deformation of the model can be used to gap out the exclusions, while preserving the cooper pairs, resulting in a truly superconducting system.

Clearly, there are numerous other generalizations one may think of and it is reasonable to expect that the study of supersymmetric models in their various guises will lead to further interesting developments.

## 8.4 Superfrustration and black holes

The focus of the research presented in this thesis has been on the square lattice. We have found that this system only shows a sub-extensive ground state entropy. Nevertheless, we believe that the findings presented here may also have implications for the truly extensive cases. It is likely that we were able to obtain especially the exact result in  $D = 2$ , precisely because the square lattice is special and therefore simpler.

Recent developments in a seemingly unrelated field are the reason that we wish to make

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<sup>1</sup>The ”+/-” model on the chain is equivalent to the spinless model on the square ladder, see section 7.5.

some closing remarks on the superfrustrated systems. As was mentioned before an extensive ground state entropy, is in contradiction with the third law of thermodynamics, which says that as the temperature goes to zero, the entropy vanishes. Experimentally, indications for a developing zero temperature entropy have recently been observed in highly pure  $\text{Sr}_3\text{Ru}_2\text{O}_7$  single crystals at magnetic field strengths for which the compound is believed to have a zero temperature quantum critical point [79] (see also [80] for a nice perspective). In [94], J. Zaanen relates this feature to observations in theoretical studies of condensed matter systems using the AdS/CFT-correspondence [95].

The AdS/CFT correspondence [96] is a conjectured -but generally accepted- mapping between a D dimensional theory with (quantum) gravity and a D-1 dimensional quantum field theory without gravity. The power of the correspondence is that it constitutes a duality between a strongly coupled theory on one side of the correspondence and a weakly coupled theory on the other side, thus providing a way to access the strong coupling regime of a quantum field theory. The rapidly growing field of AdS/CMT, which applies the AdS/CFT correspondence to condensed matter systems, has seen some recent successes. In particular, there is substantial evidence that the correspondence can be employed to describe a condensed matter system in the Fermi liquid phase [97, 98].

Interestingly, in trying to get AdS/CMT to work, physicists face the obstacle that the best understood examples of the AdS/CFT correspondence have supersymmetric large N gauge theories on the boundary, which are rather different from the theories one usually encounters in condensed matter theory. Furthermore, one typically finds an extremal black hole on the gravity side of the correspondence, which has the property that their mass and charge are equal. Additionally, these extremal black holes have a finite zero-temperature entropy, which translates into an extensive ground state entropy for the (supersymmetric) quantum field theory. The observed superfrustration in the supersymmetric model for lattice fermions indicates that the continuum theory of these systems should also exhibit this feature. It would be very interesting to investigate if the continuum theory of the supersymmetric lattice model could be a quantum field theory for which a weakly coupled gravitational dual exists. Clearly, if such a connection can be made and the supersymmetric lattice model can serve as a toy model to explore AdS/CMT, this would be quite spectacular.