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# Utility Maximization under Solvency Constraints and Unhedgeable Risks

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## Abstract

We consider the utility maximization problem for an investor who faces a solvency or risk constraint in addition to a budget constraint. The investor wishes to maximize her expected utility from terminal wealth subject to a bound on her expected solvency at maturity. We measure solvency using a solvency function applied to the terminal wealth. The motivation for our analysis is an optimal investment problem where the investor faces a random and non-hedgeable liability at maturity.

Keywords: Utility maximisation, Risk constraint, Value-at-Risk, Merton problem, Expected shortfall, Tail-Value-at-Risk

## 1 Introduction

Since the influential work by Merton (1969) on optimal portfolio selection a number of authors has studied extensions to his original model. In this article we study the optimal portfolio selection problem in a situation where the investor's decisions are subject to hedgeable and unhedgeable risks and solvency constraints. Instead of considering a particular risk measure we introduce a solvency function which is applied to the terminal wealth of the investor. The expectation of the solvency function applied to the terminal

wealth will then serve as our risk measure. This allows us to generalize our framework to incorporate situations in which the investor faces both hedgeable and unhedgeable risks.

The use of risk measures for portfolio optimization subject to risk constraints has recently attracted some interest. Basak & Shapiro (2001) studied the optimal portfolio choice for an investor who faces a risk constraint on her terminal wealth. Rogers (2008) considers a similar problem and applies his results to the problem of designing an optimal contract between an principal and an agent. Danielsson, Jorgensen, de Vries & Yang (2008) show that in a complete market with several financial instruments the optimal portfolio allocation becomes a computational complex problem and how financial innovations can help to reduce this complexity.

Our ambition in this paper is to join two branches of literature which have not been linked before. On the one hand we are inspired by results obtained by Basak & Shapiro (2001). These authors study the optimal terminal wealth of an investor who's decisions are subject to a risk constraint. In particular, Basak & Shapiro (2001) consider a Value-at-Risk constraint, that is, the investor's terminal wealth has to be above a certain value with a certain probability. This lower bound of the terminal wealth can be interpreted as a liability that the investor has to meet with a certain probability. They also consider a Tail-VaR constraint for a given VaR-level.

On the other hand, we are inspired by the results on portfolio optimization in an incomplete market setting. There has been a lot of research interest in this area recently. We refer to the papers by Henderson & Hobson (2004), Musiela & Zariphopoulou (2004), Young (2004) and Young & Zariphopoulou (2002). These papers consider optimal investment problems where the agent is trying to hedge as good as possible a liability which is not (fully) spanned by the available financial instruments. However, up to now these papers have not explicitly considered the presence of solvency constraints imposed externally upon the agent.

Our motivation is to generalize these results to situations where the investor's constraint is such that the investor has to meet a liability whose value is not known yet. We therefore model the liability as a random variable and assume that this random payoff can not be (fully) replicated by a self-financing strategy. In that sense we consider an incomplete financial market since this random liability is non-hedgeable. Typical examples include insurance companies and pension funds who have to pay benefits to policyholders which might depend on future mortality rates or future inflation.

Therefore, our approach to managing the risk associated with this random liability is to invest such that with a certain probability there are sufficient

funds available to pay for it rather than trying to hedge it directly. This approach is based on considering a solvency function rather than a particular risk measure. However, we show how the choice of the solvency function is related to the choice of risk-measures. The type of optimization problem we consider in our paper belongs to the class of so-called non-convex optimization problems. Such problems, in the setting of optimal investment problems in complete markets, have been considered previously by Boyle & Tian (2007).

The main contribution of this paper is to consider non-hedgeable liabilities that the investor has to meet at maturity. This leads us to consider a general solvency function instead of a particular risk measure.

The paper is organized as follows. We set the scene in sections 2 and 3 by introducing the financial market and the optimal investment problem. In section 4 we introduce the pointwise optimization problem and show that its solution is also a solution to original problem faced by the investor. We continue in section 5 with studying the properties of the optimal terminal wealth. Numerical illustrations are provided in section 6.

## 2 The Financial Market

The financial market is modelled by a probability space  $(\Omega, \mathcal{F}, P)$ . We define a standard Brownian motion  $W = \{W(t), t \in [0, T]\}$  on  $(\Omega, \mathcal{F}, P)$  and consider the filtration  $\{\mathcal{F}_t : t \in [0, T]\}$  generated by  $W$ , that is  $\mathcal{F}_t = \sigma\{W(s), s \leq t\}$ .

There are two investment opportunities, a bond with price process  $B$  and a risky asset with price process  $S$ . The bond price is the solution of

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1 \quad (1)$$

where the short rate  $r$  is a non-random function on  $[0, T]$ . The price  $S$  of the risky asset follows

$$dS(t) = S(t)\{\mu(t)dt + \sigma(t)dW(t)\}$$

where  $\mu$  and  $\sigma > 0$  are non-random functions on  $[0, T]$ .

It is well-known that the financial market is arbitrage-free and complete and there exists a unique state price density (deflator)  $\xi$ , given as the solution of

$$d\xi(t) = -\xi(t) \left\{ r(t)dt + \frac{\mu(t) - r(t)}{\sigma(t)} dW(t) \right\}. \quad (2)$$

Since the financial market is arbitrage-free, the price of any payoff  $A(T)$  ( $A(T)$  is an  $\mathcal{F}_T$ -measurable random variable) at time  $T$  is given by

$$A(0) = \frac{1}{\xi(0)} \mathbb{E}[\xi(T)A(T)].$$

Furthermore, the completeness of the market ensures that any payoff  $A(T)$  can be replicated by a self-financing trading strategy. Therefore, we only consider the optimal asset value of an investor at time  $T$  and not the portfolio strategy that generates the desired payoff.

### 3 The Optimization Problem

We consider an investor with initial wealth  $A(0)$  at time 0 who wishes to maximize her expected utility at time  $T$  from trading in the financial market. The maximization problem faced by the investor is the well known Merton-problem:

$$\max \mathbb{E}u(A(T)) \tag{3}$$

subject to the initial wealth restriction

$$\mathbb{E}[\xi(T)A(T)] = \xi(0)A(0) \tag{4}$$

where  $u : \mathbb{R}_+ \mapsto \mathbb{R}$  is a utility function. We assume that

**(A1)**  $u$  is a strictly increasing, concave function on  $\mathbb{R}_+$  with  $\lim_{x \rightarrow 0} u'(x) = \infty$  and  $\lim_{x \rightarrow \infty} u'(x) = 0$ .

It is well known that the solution to this problem is

$$A(T, \lambda) = I(\lambda\xi(T))$$

where  $I$  is the inverse of  $u'$  and  $\lambda$  is chosen such that

$$\mathbb{E}[\xi(T)A(T, \lambda)] = \mathbb{E}[\xi(T)I(\lambda\xi(T))] = \xi(0)A(0). \tag{5}$$

We now assume that the investor also faces some risk-management constraints. To this end we assume that there is a function  $R : \mathbb{R}_+ \mapsto \mathbb{R}$  that is used to measure the solvency of an investor. We choose this function such that a high value of  $R$  indicates a bad solvency level. If the value of  $R$  is too high the investor will be insolvent. We call  $R$  the solvency (or risk-management) function of the investor and assume

**(A2)**  $R$  is non-increasing

**(A3)**  $R(a)/a \rightarrow 0$  for  $a \rightarrow \infty$  .

We apply  $R$  to the final wealth  $A(T)$  of the investor and, therefore, Assumption (A2) has the economical interpretation: the larger the final wealth  $A(T)$  the more solvent is the investor. (A3) states that the marginal solvency improvement tends to 0 for high values of the terminal wealth.

The optimization problem now faced by the investor is still (3) but it is now subject to (4) and the additional solvency constraint

$$\mathbb{E}[R(A(T))] \leq \varepsilon \quad (6)$$

where  $\varepsilon$  is a positive non-random constant. This means, the expected shortfall weighted according to the solvency function is limited. This concept of using a solvency function directly applied to the terminal wealth allows us to cover a number risk-management problems as illustrated in the following examples:

**Example 1** Value-at-Risk with constant level  $L$ : *Such a constraint was considered by Basak & Shapiro (2001). Here the investor chooses to invest such that her final wealth is larger than a certain level  $L \in \mathbb{R}_+$  with some probability  $1 - \varepsilon$ . In this case the solvency function  $R$  has the form*

$$R(a) = \mathbb{1}_{[0,L]}(a). \quad (7)$$

The solvency constraint then becomes

$$\mathbb{E}[R(A(T))] = \mathbb{P}[A(T) \leq L] \leq \varepsilon.$$

**Example 2** Value-at-Risk with random level  $L$ : *Such a constraint was considered by Boyle & Tian (2007) for the case where  $L$  is hedgeable (i.e.  $L$  is spanned by the financial assets). Our formulation of the optimization problem allows us to consider the more general case where  $L$  is non-hedgeable. This is of particular interest to life-insurance companies or pension funds where the wealth  $A(T)$  has to be managed in a way that allows the company to meet liabilities in the future which depend on non-hedgeable risks like inflation or mortality. This particular example is studied in detail in Sections 5 and 6. Let  $L$  be a random variable with values in  $\mathbb{R}$ . We then obtain for the shortfall probability*

$$\mathbb{P}[A(T) \leq L] = \mathbb{E} \left[ \mathbb{P} \left[ A(T) \leq L \mid A(T) \right] \right] \leq \varepsilon$$

Using the notation  $F_{X|Y=y}$  for the conditional distribution function of a random variable  $X$  given the realisation  $y$  of another random variable  $Y$ , we obtain that the solvency function  $R$  is given by

$$R(a) = 1 - F_{L|A(T)=a}(a) \text{ and } \varepsilon = \alpha.$$

**Example 3** Expected Loss with random level  $L$ : We could also consider a risk measure based on Expected loss by choosing  $R$  as follows.

$$R(A(T)) = \mathbb{E} \left[ (L - A(T)) \mathbf{1}_{\{A(T) \leq L\}} \mid A(T) \right] .$$

**Example 4** Tail-Value-at-Risk with random level  $L$ : In this case we define the function  $R$  by

$$R(A(T), K) = K + (1 - \beta)^{-1} \mathbb{E} \left[ ((L - A(T)) - K) \mathbf{1}_{\{L - A(T) \geq K\}} \mid A(T) \right] ,$$

for given constants  $K$  and  $\beta$ . This example is studied in more detail in section 6.

The above examples are not exclusive. The solvency function could have different forms reflecting the needs of the investor or the regulatory environment the investor is operating in.

## 4 The Point-Wise Optimization Problem

To solve the optimisation problem (3) subject to (4) and (6) and find the optimal terminal wealth for the investor we introduce a point-wise optimization problem first. We define the function

$$G : \mathbb{R}_+^4 \mapsto \mathbb{R}, \quad G(x, a, \lambda_1, \lambda_2) = u(a) - \lambda_1 x a - \lambda_2 R(a) \quad (8)$$

and consider the optimization problem

$$\max_{a \in \mathbb{R}_+} G(x, a, \lambda_1, \lambda_2) .$$

The following definition explains what we mean by a solution to the point-wise optimization problem.

**Definition 1** For any fixed  $\lambda_1^*, \lambda_2^* \in \mathbb{R}$ , we call the function

$$A_{max}(\cdot, \lambda_1^*, \lambda_2^*) : \mathbb{R}_+ \mapsto \mathbb{R}_+$$

a solution to the point-wise optimization problem if

- for all  $x \in \mathbb{R}_+$ :

$$G(x, A_{max}(x, \lambda_1^*, \lambda_2^*), \lambda_1^*, \lambda_2^*) = \max_{a \in \mathbb{R}_+} G(x, a, \lambda_1^*, \lambda_2^*)$$

and

- the following system of equations is solved

$$\mathbb{E}[\xi(T)A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)] = \xi(0)A(0) \quad (9)$$

$$\mathbb{E}[R(A_{max}(\xi(T), \lambda_1^*, \lambda_2^*))] = \varepsilon \quad (10)$$

where  $\xi(T)$  denotes the state price deflator defined in (2).

We are now going to study the existence of a solution to the point-wise problem and the relationship between the point-wise problem and the original optimization problem (3) subject to (4) and (6).

It is well-known that a point-wise solution to the Merton-problem, (3) subject to (4) exists. However, when we include the additional constraint (6) a point-wise solution as defined above might not exist anymore.

The reason is that the investor's initial wealth  $A(0)$  might not be sufficient to satisfy the solvency constraint (6) with a terminal wealth that is a function of the state price deflator  $\xi$ . This means, for all  $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$  with

$$\mathbb{E}[\xi(T)f(\xi(T))] = \xi(0)A(0)$$

holds

$$\mathbb{E}\left[R\left(f(\xi(T))\right)\right] > \varepsilon.$$

To avoid this situation we assume for the remainder of the paper

**(A4)**  $R(B(T)A(0)) \leq \varepsilon$ , where  $B(T)$  is the price of the risk-free bond at time  $T$  as defined in (1).

This assumption ensures that the investor has sufficient initial capital to meet the solvency constraint by investing the entire initial wealth in the risk-free asset. Under this assumption we immediately obtain for the constant function  $f(x) \equiv B(T)A(0)$  that  $A(T) = f(\xi(T)) = B(T)A(0)$  is a feasible solution that satisfies (4) and (6) and, therefore, an optimal point-wise solution exists.

On the other hand, a point-wise solution might exist but the solvency constraint (6) is not binding. This situation is characterized by the following lemma.

**Lemma 1** *If a point-wise solution  $A_{max}(\cdot, \lambda_1^*, \lambda_2^*)$  exists with  $\lambda_2^* \leq 0$  then the solvency constraint (6) is not binding and the optimal terminal wealth is the solution to the Merton-problem, that is, the optimal final payoff (solution to (3) subject to (4) and (6)) is*

$$A^*(T) = I(\lambda\xi(T))$$

where  $I$  is the inverse of  $u'$  and  $\lambda$  is chosen such that (5) is fulfilled.

**Proof of lemma 1:** Let  $A^* = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)$  and  $A$  be the solution to (3) subject to (4) only, that is,  $A$  is the solution to the Merton-problem. We therefore have that  $\mathbb{E}[u(A)] \geq \mathbb{E}[u(A^*)]$  and  $\mathbb{E}[\xi(T)A] = \mathbb{E}[\xi(T)A^*] = \xi(0)A(0)$ . We only have to show that  $A$  fulfills the solvency constraint (6). From definition 1 we know that

$$G(\xi(T), A^*, \lambda_2^*, \lambda_2^*) \geq G(\xi(T), A, \lambda_2^*, \lambda_2^*)$$

and, since  $\mathbb{E}[\xi(T)A^*] = \mathbb{E}[\xi(T)A] = \xi(0)A(0)$ , we obtain

$$\mathbb{E}[u(A^*)] - \lambda_2^* \mathbb{E}[R(A^*)] \geq \mathbb{E}[u(A)] - \lambda_2^* \mathbb{E}[R(A)]$$

Since  $\lambda_2^* < 0$ , it follows, that

$$\mathbb{E}[R(A)] \leq \mathbb{E}[R(A^*)] + \frac{1}{\lambda_2^*} [\mathbb{E}[u(A)] - \mathbb{E}[u(A^*)]] \leq \mathbb{E}[R(A^*)] = \varepsilon$$

□

We now consider the case in which a solution to the point-wise problem exists and the solvency constraint is binding.

**Lemma 2** *If a point-wise solution  $A_{max}(\cdot, \lambda_1^*, \lambda_2^*)$  exists with  $\lambda_1^*, \lambda_2^* > 0$ , then*

$$A^*(T) = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)$$

*is a solution to (3) subject to (4) and (6).*

**Proof of lemma 2:** Let  $A^* = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)$  and  $A$  be any  $\mathcal{F}_T$ -measurable random variable that fulfills (4) and (6). From the definition of the point-wise solution we obtain that

$$\mathbb{P}\left[G(\xi(T), A^*, \lambda_1^*, \lambda_2^*) \geq G(\xi(T), A, \lambda_1^*, \lambda_2^*)\right] = 1$$

Since  $\lambda_1^*$  and  $\lambda_2^*$  are positive we obtain

$$\begin{aligned} \mathbb{E}[u(A^*)] - \mathbb{E}[u(A)] &= \mathbb{E}[u(A^*)] - \mathbb{E}[u(A)] - \lambda_1^* \xi(0)A(0) + \lambda_1^* \xi(0)A(0) \\ &\quad - \lambda_2^* \varepsilon + \lambda_2^* \varepsilon \\ &\geq \mathbb{E}[u(A^*)] - \mathbb{E}[u(A)] - \lambda_1^* \mathbb{E}[\xi(T)A^*] + \lambda_1^* \mathbb{E}[\xi(T)A] \\ &\quad - \lambda_2^* \mathbb{E}[R(A^*)] + \lambda_2^* \mathbb{E}[R(A)] \\ &= \mathbb{E}\left[u(A^*) - \lambda_1^* \xi(T)A^* - \lambda_2^* R(A^*)\right] \\ &\quad - \mathbb{E}\left[u(A) - \lambda_1^* \xi(T)A - \lambda_2^* R(A)\right] \\ &= \mathbb{E}\left[G(\xi(T), A^*, \lambda_1^*, \lambda_2^*)\right] - \mathbb{E}\left[G(\xi(T), A, \lambda_1^*, \lambda_2^*)\right] \\ &\geq 0. \end{aligned}$$

□

A theorem about the existence of an optimal terminal wealth can now be obtained by combining Lemma 1 and Lemma 2.

**Theorem 1** *If assumptions (A1)-(A4) are satisfied then there exists at least one solution to (3) subject to (4) and (6). This solution is either the solution to the Merton-problem,*

$$A^*(T) = I(\lambda\xi(T)) \quad \text{with} \quad \mathbb{E}[\xi(T)I(\lambda\xi(T))] = \xi(0)A(0)$$

or is given by

$$A^*(T) = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)$$

where  $A_{max}(\cdot, \lambda_1^*, \lambda_2^*)$  is a solution to the point-wise problem and  $\lambda_1^* > 0$  and  $\lambda_2^* > 0$ .

**Proof of theorem 1:** We prove the result in three steps.

- 1) A solution to the Merton-problem, (3) subject to (4) exists.
- 2) It follows from lemma 1 that if there is a solution to the point-wise problem with  $\lambda_2^* < 0$  then the solution to (3) subject to (4) and (6) is  $A^* = I(\lambda\xi(T))$ .
- 3) It follows from assumptions (A1) and (A3) that  $G(x, a, \lambda_1, \lambda_2)$  has for all  $x$  at least one maximum if  $\lambda_1, \lambda_2 > 0$ . Therefore, it follows from (A4) that a point-wise solution exists. It now follows from lemma 2 that this point-wise solution applied to  $\xi(T)$  is a solution to (3) subject to (4) and (6). □

We do not discuss the uniqueness of the optimal terminal wealth here since an investor would be indifferent between two solutions to the utility optimization problem if both solutions maximize the expected utility and fulfill the constraints. However, the following section contains some remarks about the uniqueness of the point-wise solution under particular additional assumptions.

## 5 Properties of the Optimal Terminal Wealth

We are now going to study the properties of the optimal terminal wealth. Dybvig (1988) has shown that “any cheapest way to achieve a lottery assigns the outcomes of the lottery to the states in reverse order of the state-price density...”. The following lemma states this result, in the sense that our optimal terminal wealth (the “lottery”) is a non-increasing function of the state price deflator. Although, this is just the result already obtained by Dybvig (1988), we provide the proof in the context of this paper.

**Lemma 3** *If assumptions (A1) - (A4) are satisfied then the optimal terminal wealth  $A^*(T) = A(\xi(T))$  is a non-increasing function of  $\xi(T)$ .*

**Proof of lemma 3:** If the solvency constraint is not binding, the assertion follows immediately, since the optimal terminal wealth is the solution to the Merton-problem.

We now prove the assertion for a binding solvency constraint ( $\lambda_2 > 0$ ). We use the notation  $G(x, a) = G(x, a, \lambda_1, \lambda_2)$  and  $A_{max}(x) = A_{max}(x, \lambda_1, \lambda_2)$ . We fix any  $x_1 < x_2$  and define  $a_i := A_{max}(x_i)$  for  $i = 1, 2$ . Assume now that  $a_2 > a_1$ . We then obtain for any  $\lambda_1 > 0$  and  $\lambda_2 > 0$ :

$$\begin{aligned} G(x_1, a_2) &= u(a_2) - \lambda_1 x_1 a_2 - \lambda_2 R(a_2) + \lambda_1 x_2 a_2 - \lambda_1 x_2 a_2 \\ &= G(x_2, a_2) + \lambda_1 a_2 (x_2 - x_1) \\ &> G(x_2, a_2) + \lambda_1 a_1 (x_2 - x_1) & (11) \\ &\geq G(x_2, a_1) + \lambda_1 a_1 (x_2 - x_1) & (12) \\ &= u(a_1) - \lambda_1 x_2 a_1 - \lambda_2 R(a_1) + \lambda_1 x_2 a_1 - \lambda_1 x_1 a_1 \\ &= G(x_1, a_1) \end{aligned}$$

where (11) follows from our assumption  $a_2 > a_1$  and  $x_2 > x_1$ , and (12) follows from the definition of  $a_2$  as the global maximum of  $G(x_2, a)$ . However,  $G(x_1, a_2) > G(x_1, a_1)$  is a contradiction to  $a_1 := A_{max}(x_1)$ .  $\square$

To get some further insight into the structure of the optimal terminal wealth we make the additional assumption

**(A5)**  $u$  and  $R$  are three times continuously differentiable

and define the function

$$\Xi(a, \lambda_1, \lambda_2) := \frac{1}{\lambda_1} \left[ u'(a) - \lambda_2 R'(a) \right] \quad (13)$$

We now obtain the following lemma.

**Lemma 4** *If assumptions (A1) - (A5) are satisfied then*

1.  $A^*(T) = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*)$  is a strictly decreasing function of  $\xi(T)$ .
2. If  $R''(a) \geq 0$  for all  $a \in \mathbb{R}_+$  then

$$A^*(T) = A_{max}(\xi(T), \lambda_1^*, \lambda_2^*) = \Xi^{-1}(\xi(T), \lambda_1^*, \lambda_2^*)$$

where  $\Xi^{-1}(\cdot, \lambda_1, \lambda_2)$  is the inverse function of  $\Xi(\cdot, \lambda_1, \lambda_2)$ , and  $\lambda_1^*$  and  $\lambda_2^*$  solve (9) and (10).

**Proof :** We assume in the following that  $\lambda_1$  and  $\lambda_2$  are positive constants, and we use the notations  $G(x, a) = G(x, a, \lambda_1, \lambda_2)$  and  $\Xi(a) = \Xi(a, \lambda_1, \lambda_2)$ .

We first prove the second part of the lemma. From assumptions (A5) follows that  $G(x, a)$  is twice differentiable with respect to  $a$  and

$$G'(x, a) := \frac{\partial}{\partial a} G(x, a) = u'(a) - \lambda_1 x - \lambda_2 R'(a) \quad (14)$$

$$G''(x, a) := \frac{\partial^2}{\partial a^2} G(x, a) = u''(a) - \lambda_2 R''(a) = \lambda_1 \Xi'(a). \quad (15)$$

It follows from (14) that

$$G'(x, a) = 0 \iff x = \Xi(a) \quad (16)$$

This means that for every  $x$  there is exactly one extreme value of  $G(x, \cdot)$  at  $a = \Xi^{-1}(x)$ . Since  $u$  is concave and  $R'' \geq 0$  it follows from (15) that  $G(x, \cdot)$  has a maximum at  $\Xi^{-1}(x)$ .

To prove the second part, note that it follows from lemma 3 that we only have to show that there is no  $x$  such that  $A_{max}(x)$  is constant in a vicinity of  $x$ . With the same arguments as above we obtain that  $G(x, \cdot)$  has a local maximum in  $a$  if and only if  $x = \Xi(a)$  and  $\Xi'(a) < 0$ . Therefore, there exists a  $\delta > 0$  such that  $\Xi$  is strictly decreasing on  $[a - \delta, a + \delta]$  if and only if  $G(x, \cdot)$  has a local maximum at  $a$ . It follows that  $\Xi$  is invertible on  $[a - \delta, a + \delta]$  and the inverse function is strictly decreasing. Since any global maximum of  $G(x, \cdot)$  is also a local maximum the assertion follows.  $\square$

Let us mention here that if  $\Xi$  is invertible the structure of the solution is similar to the structure of the solution to the Merton-problem without additional constrains.

We will now consider the more interesting case in which  $\Xi$  might not be invertible. Instead of considering this situation for general solvency functions  $R$  fulfilling a number of technical conditions we make the following assumption.

**(A6)** the utility function  $u(a)$  and the solvency function  $R(a)$  fulfill (A1) - (A5). Furthermore,  $u(a)$  and  $R(a)$  satisfy the property that the function  $u'(a) - \lambda R'(a)$  has at most one local maximum  $\forall \lambda > 0$  in  $\mathbb{R}_+$ , and  $\lim_{a \rightarrow 0} |R'(a)| < \infty$ .

Since assumption (A6) holds for all  $\lambda > 0$  and the derivative  $u'$  of the utility function is strictly decreasing, this assumption implies that the function  $-R'(a)$  has at most one local maximum.

**Example 5** An example of a utility function  $u$  and a solvency function  $R$  that satisfy assumption (A6) is a power utility function  $u(a) = a^{1-\gamma}/(1-\gamma)$ , and  $R(a) = 1 - \Phi_{\mu,\sigma}(a)$  where  $\Phi_{\mu,\sigma}$  is the distribution function of a normal distribution with mean  $\mu$  and variance  $\sigma^2$ .

Using the notation  $\phi = \Phi'_{\mu,\sigma}$ , that is,  $\phi$  is the density of a normal distribution, and omitting the dependence of  $\Xi$  and its derivatives on  $\lambda_1$  and  $\lambda_2$  in the notation, we obtain for any  $\lambda_1$  and  $\lambda_2$

$$\Xi(a) = \frac{1}{\lambda_1} [u'(a) - \lambda_2 R'(a)] = \frac{1}{\lambda_1} [a^{-\gamma} + \lambda_2 \phi(a)], \quad (17)$$

$$\Xi'(a) = \frac{1}{\lambda_1} \left\{ -\gamma a^{-(\gamma+1)} + \lambda_2 \frac{\phi(a)}{\sigma^2} (\mu - a) \right\}, \quad (18)$$

$$\Xi''(a) = \frac{1}{\lambda_1} \left\{ \gamma(\gamma+1) a^{-(\gamma+2)} + \lambda_2 \frac{\phi(a)}{\sigma^2} \left[ \left( \frac{\mu - a}{\sigma} \right)^2 - 1 \right] \right\}. \quad (19)$$

To see that  $\Xi$  has at most one local maximum, let  $a^*$  be the smallest value in  $\mathbb{R}_+$  at which  $\Xi$  has a local maximum. It follows from (18) that  $a^* < \mu$  since  $\Xi'(a^*) = 0$ . We now choose any positive  $\varepsilon < \mu - a^*$  and find that

$$\lambda_1 \Xi''(a^* + \varepsilon) < \lambda_1 \Xi''(a^*) < 0. \quad (20)$$

Since  $\Xi'(a^*) = 0$ , it follows from (20) that  $\Xi'(a) < 0$  for all  $a \in (a^*, \mu]$ . Therefore,  $\Xi$  does not have a local maximum or minimum in  $(a^*, \mu]$ . Since  $\Xi$  also does not have a local maximum or minimum in  $(\mu, \infty)$ , the only point at which there is a local maximum is  $a^*$ .

This example is studied in more detail in section 6.

Given assumption (A6), we immediately obtain from theorem 1 that a solution to (3) subject to (4) and (6) exists. Furthermore, we obtain from lemma 4 that the optimal terminal wealth is a strictly decreasing function of the state price deflator  $\xi(T)$ .

We will now study the solution to this problem in a bit more detail. Let us remark that the arguments presented in the following can be extended to other solvency functions  $R$  for which  $R'$  has more than one local maxima.

**Lemma 5** Assume that (A6) is fulfilled. For all  $\lambda_1 > 0$  and  $\lambda_2 > 0$  we obtain that either

1.  $\Xi(\cdot, \lambda_1, \lambda_2)$  is strictly decreasing on  $\mathbb{R}_+$  and, therefore, invertible or

2. there exists a unique pair of positive constants  $a_1 = a_1(\lambda_1, \lambda_2)$  and  $a_2 = a_2(\lambda_1, \lambda_2)$  with  $a_1 < a_2$ ,

$$\Xi(a_1, \lambda_1, \lambda_2) = \Xi(a_2, \lambda_1, \lambda_2) \quad (21)$$

$$G\left(\Xi(a_1, \lambda_1, \lambda_2), a_1, \lambda_1, \lambda_2\right) = G\left(\Xi(a_2, \lambda_1, \lambda_2), a_2, \lambda_1, \lambda_2\right) \quad (22)$$

and  $\Xi(\cdot, \lambda_1, \lambda_2)$  is invertible on  $(0, a_1] \cup (a_2, \infty)$ .

**Proof :** We fix  $\lambda_1 > 0$  and  $\lambda_2 > 0$ . We will omit the dependence of  $G$ ,  $\Xi$  and their derivatives on  $\lambda_1$  and  $\lambda_2$  in the notation used for this proof.

Recall that  $\Xi(a) = 1/\lambda_1[u'(a) - \lambda_2 R'(a)]$ , where  $u'(a)$  is a strictly decreasing function. From assumptions (A1) and (A6), it follows that the function  $\Xi(a)$  is strictly decreasing in a vicinity of 0. Furthermore, it follows from (A3), (A5) and (A6) that  $\Xi(a)$  is strictly decreasing for  $a \rightarrow \infty$ . We now consider the two cases mentioned in the lemma.

If  $\Xi$  does not have a local maximum, then  $\Xi$  is strictly decreasing on  $\mathbb{R}_+$  and, therefore, invertible on  $\mathbb{R}_+$ . The proof for that situation is therefore finished.

We now consider the case of a local maximum. This implies that the function  $-R'(a)$  has exactly one local maximum. Denote the argument for which this local maximum is attained by  $\mu$ . This case is illustrated in Figure 1 below.

By assumption (A6) the function  $\Xi$  has exactly one local maximum. We will denote the argument for which the maximum is attained by  $a^*$ . Note that since  $u'(a)$  is strictly decreasing, we have that  $a^* < \mu$ .

Furthermore, since  $\Xi$  is continuous and strictly decreasing in a vicinity of 0, it must have a local minimum at a point  $a_*$  with  $0 < a_* < a^*$ .

From  $\lim_{a \rightarrow 0} \Xi(a) = \infty$  and  $\lim_{a \rightarrow \infty} \Xi(a) = 0$ , combined with the continuity of  $\Xi$ , follows that there exist  $a_{**}, a^{**}$  such that

$$a^{**} < a_* < a^* < a_{**}, \quad \Xi(a_{**}) = \Xi(a_*) \text{ and } \Xi(a^{**}) = \Xi(a^*).$$

We define  $x_* = \Xi(a_*)$  and  $x^* = \Xi(a^*)$ . We now obtain from (14) – (16) that

$$G'(x_*, a_*) = G''(x_*, a_*) = G'(x^*, a^*) = G''(x^*, a^*) = 0$$

and

$$G'(x_*, a_{**}) = G'(x^*, a^{**}) = 0, \quad G''(x_*, a_{**}) < 0 \text{ and } G''(x^*, a^{**}) < 0.$$

That means that  $G(x_*, \cdot)$  and  $G(x^*, \cdot)$  obtain their local maxima at  $a_{**}$  and  $a^{**}$  respectively. It follows that

$$G(x_*, a_*) < G(x_*, a_{**}) \text{ and } G(x^*, a^*) < G(x^*, a^{**}).$$

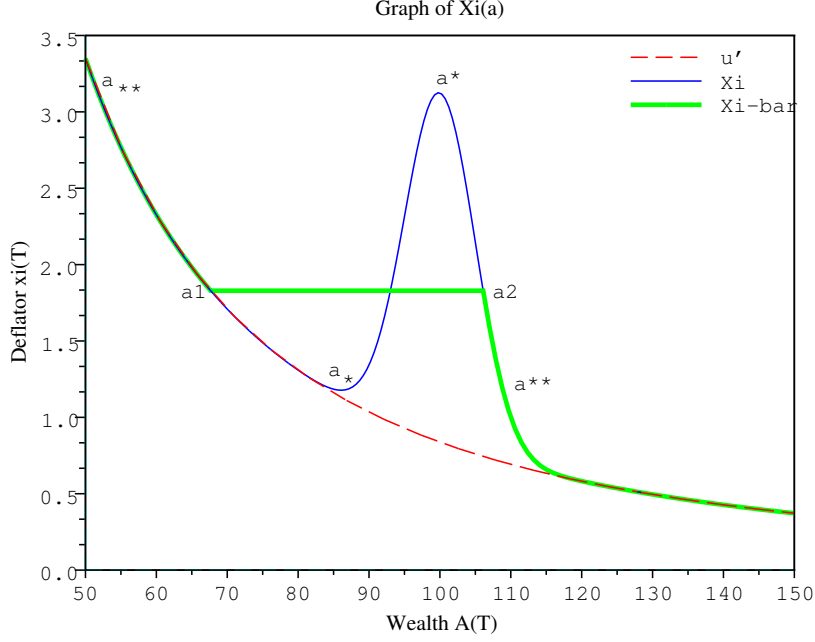


Figure 1: Construction of invertible function  $\Xi(a, \lambda_1, \lambda_2)$ .

The existence of  $a_1 \in (a^{**}, a_*)$  and  $a_2 \in (a^*, a^{**})$  which solve (21) and (22) follows from the continuity of  $G$  and  $\Xi$ , and an application of the intermediate value theorem. Since  $a_1 < a_* < a^* < a_2$  the function  $\Xi$  is invertible on  $(0, a_1] \cup (a_2, \infty)$ .

We still have to prove the uniqueness of  $a_1$  and  $a_2$ . Let us assume that we have found the smallest  $a_1$  for which there exists an  $a_2 > a_1$  such that (21) and (22) are fulfilled, and  $\Xi$  is invertible on  $(0, a_1) \cup [a_2, \infty)$ .

We now choose any  $b_1 > a_1$  such that  $\Xi$  is invertible on  $(0, b_1)$ . It follows that  $b_1 \in (a_1, a_*]$ . Let  $b_2 > b_1$  be such that (21) is fulfilled for  $b_1$  and  $b_2$  instead of  $a_1$  and  $a_2$ , and  $\Xi$  is invertible on  $(0, b_1) \cup [b_2, \infty)$ . It follows that  $b_2 > a_2$ . In summary, we have  $a_1 < b_1 \leq a_* < a^* < a_2 < b_2$ .

We now obtain from

$$\frac{\partial}{\partial a} G(\Xi(a), a) = -\lambda_1 \Xi'(a) a$$

and

$$\Xi'(a) < 0 \quad \forall a \in [a_1, b_1) \cup [a_2, b_2)$$

that

$$\begin{aligned}
G(\Xi(b_2), b_2) &= G(\Xi(a_2), a_2) + \int_{a_2}^{b_2} -\lambda_1 \Xi'(a) da \\
&> G(\Xi(a_2), a_2) + a_2 \int_{a_2}^{b_2} -\lambda_1 \Xi'(a) da \\
&= G(\Xi(a_2), a_2) - a_2 \lambda_1 [\Xi(b_2) - \Xi(a_2)] \\
&= G(\Xi(a_1), a_1) - a_2 \lambda_1 [\Xi(b_1) - \Xi(a_1)] \\
&= G(\Xi(a_1), a_1) + a_2 \int_{a_1}^{b_1} -\lambda_1 \Xi'(a) da \\
&> G(\Xi(a_1), a_1) + \int_{a_1}^{b_1} -\lambda_1 \Xi'(a) da \\
&= G(\Xi(b_1), b_1)
\end{aligned}$$

□

In the following we use the notation  $\bar{\Xi}(\cdot, \lambda_1, \lambda_2)$  for the restriction of  $\Xi(\cdot, \lambda_1, \lambda_2)$  to  $[0, a_1] \cup [a_2, \infty)$ , that is,

$$\bar{\Xi}(\cdot, \lambda_1, \lambda_2) : [0, a_1] \cup [a_2, \infty) \mapsto \mathbb{R}_+ \text{ with } \bar{\Xi}(a, \lambda_1, \lambda_2) = \Xi(a, \lambda_1, \lambda_2).$$

Note that  $a_1$  and  $a_2$  depend on  $\lambda_1$  and  $\lambda_2$ .

The following corollary about the structure of the optimal terminal wealth is a direct consequence of Lemma 5.

**Corollary 1** *The solution to (3) subject to (4) and (6) is given by*

$$A^*(T) = \bar{\Xi}^{-1}(\xi(T), \lambda_1^*, \lambda_2^*)$$

where  $\bar{\Xi}$  is the restriction of  $\Xi$  to  $(0, a_1] \cup (a_2, \infty)$  and  $\bar{\Xi}^{-1}$  denotes its inverse function. The constants  $a_1$  and  $a_2$  are chosen such that (21) and (22) are fulfilled and  $\bar{\Xi}(\cdot, \lambda_1^*, \lambda_2^*)$  is invertible on  $(0, a_1] \cup (a_2, \infty)$ .

To obtain an economic interpretation for the jump of  $A^*(\xi(T))$  at  $\bar{\xi} = \bar{\Xi}(a_1) = \bar{\Xi}(a_2)$  we take  $\lambda_1$  and  $\lambda_2$  as fixed positive values and obtain

$$\lambda_1 \bar{\xi} = u'(a_1) + \lambda_2 R'(a_1) \tag{23}$$

$$= u'(a_2) + \lambda_2 R'(a_2) \tag{24}$$

$$= \frac{u(a_2) - u(a_1)}{a_2 - a_1} + \lambda_2 \frac{R(a_2) - R(a_1)}{a_2 - a_1} \tag{25}$$

This means that in terms of the Lagrangian function  $G$  which measures the combined benefit to the investor of increasing utility and solvency, the investor is indifferent between decreasing  $a_1$  marginally and decreasing  $a_1$  to  $a_2$ .

## 6 Numerical Illustrations

In this section we are going to study some particular numerical examples which might be of interest to the insurance and pensions industry.

In the example we will consider below, we will use a very simple Black-Scholes economy where  $r, \mu$  and  $\sigma$  are constants. For such a Black-Scholes economy, we can explicitly express the state price density (deflator) as

$$\xi(T) = \exp \left\{ \left( -r - \frac{1}{2} \left( \frac{\mu-r}{\sigma} \right)^2 \right) T - \left( \frac{\mu-r}{\sigma} \right) W(T) \right\},$$

where we have set  $\xi(0) = 1$ . Note that  $W(T)$  is the same Brownian Motion that drives the risky asset  $S(T)$ . Hence, we can express the deflator  $\xi(T)$  explicitly in terms of the return  $S(T)/S(0)$  of the risky asset:

$$\xi(T) = C \left( \frac{S(T)}{S(0)} \right)^{-\frac{\mu-r}{\sigma^2}},$$

where  $C = \exp \left\{ \left( -r - \frac{1}{2} \left( \frac{\mu-r}{\sigma} \right)^2 \right) T + \frac{\mu-r}{\sigma^2} (\mu - \frac{1}{2} \sigma^2) T \right\}$ . We will use this relation in the examples we will discuss in this section to express the optimal wealth  $A^*(T)$  directly in terms of the risky asset, which makes the interpretation of the results easier.

For our numerical calculations we will take  $r = 0, \mu = 0.03$  and  $\sigma = 0.15$ , and a time-horizon  $T = 1$ . Note, that for ease of exposition we assume that the risk-free rate is equal to zero (or, equivalently stated, that all monetary quantities are expressed in units of the  $T$ -bond).

Furthermore we will assume that we have liabilities  $L$  that are uncorrelated to the risky asset. The liabilities follow a normal distribution with mean  $\mu_L = 100$  and standard deviation  $\sigma_L = 5$ .

We also assume that the agent's preferences are given by a power-utility function  $u(x) = x^{1-\gamma}/(1-\gamma)$  with  $\gamma = 2$ .

Finally, we assume that we have an initial wealth  $A(0) = 110$ . Note, that we explicitly allow for some surplus between the "best estimate" of the liabilities ( $\mu_L = 100$ ) and the initial wealth  $A(0)$ . This creates a buffer that the company can use to find an optimal investment policy, while at the same time ensuring that the solvency constraints can be met.

### 6.1 Merton solution with no solvency constraint

To start our example, we will first consider an investor that does not care about any solvency constraints. Therefore, this investor simply solves the Merton-problem (3) subject to the wealth constraint (4). Given the parameters we have chosen, the optimal asset allocation is given by  $A^*(T) =$

$I(\lambda_1 \xi(T))$ . A value of  $\lambda_1 = 8.18 \cdot 10^{-5}$  ensures that the optimal wealth allocation  $A^*(T)$  satisfies the wealth restriction  $\mathbb{E}[A^*(T)\xi(T)] = 110$ .

In Figure 2 we have plotted the optimal wealth allocation in two panels. The dashed line (with the label “NoSolv”) denotes the Merton solution in both panels. In the left panel we have plotted the “inverse” function  $\xi(A) = 1/\lambda_1 u'(A)$ . This is the first order condition from differentiating the Lagrangean with respect to  $A$ . In the right panel we have plotted the optimal wealth  $A^*(T)$  as a function of the return  $S(T)/S(0)$  of the risky asset (i.e. the market-index). From the right panel we observe that the optimal wealth allocation is a monotonically increasing function of the risk asset. As is well known, the steepness of this curve depends on the product of the “risk tolerance”  $1/\gamma$  and the market price of risk  $(\frac{\mu-r}{\sigma})$ .

For this particular optimal asset allocation  $A^*(T)$  we can calculate (via numerical integration over the joint probability distribution of  $A^*$  and  $L$ ) the shortfall probability  $\mathbb{P}[A^*(T) < L] = 0.159$ . Hence, the optimal Merton asset-allocation has a shortfall probability of almost 16%.

In the sub-sections that follow, we want to investigate what happens if we impose additional solvency constraints upon the investor.

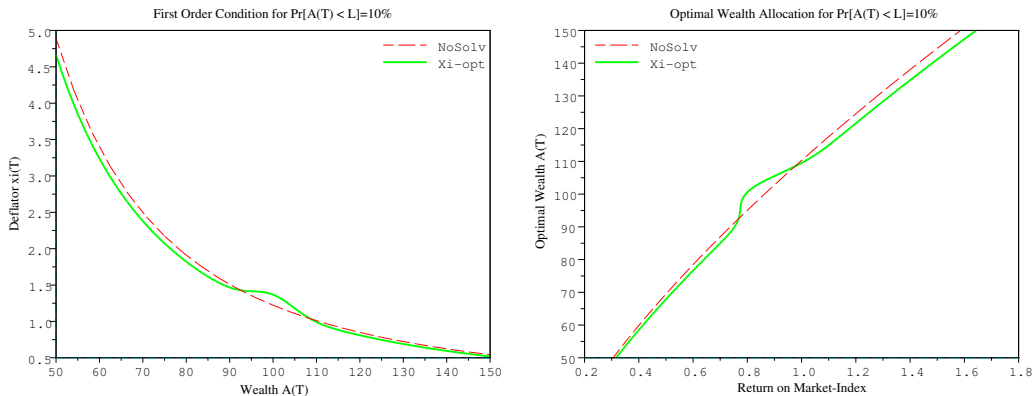


Figure 2: Optimal wealth allocation for  $\mathbb{P}[A^*(T) < L] = 0.10$ .

## 6.2 Monotonically declining function $\Xi$

In our first example we consider a case where we impose a “mild” solvency constraint upon our investor. In particular, we impose that  $\mathbb{P}[A(T) < L] = 0.10$ . This Value-at-Risk restriction translates into the solvency function  $R$  given by  $R(A) = 1 - \Phi((A - \mu_L)/\sigma_L)$ , where  $\Phi(\cdot)$  denotes the cumulative standard normal density function.

In the left panel of Figure 2 we have plotted the first order condition given by the function  $\Xi(A) = 1/\lambda_1 [u'(A) - \lambda_2 R'(A)]$  defined in (13). The values for the Lagrange multipliers are given by  $\lambda_1 = 8.58 \cdot 10^{-5}$  and  $\lambda_2 = 2.17 \cdot 10^{-4}$ . This is depicted by the solid line labelled “Xi-opt”.

If we compare the first order condition for the Merton problem to our problem we see immediately that the solvency constraints translates into a hump around the level  $\mu_L = 100$ . Because we have a random liability which is not hedgeable, the hump is dispersed over an interval. An economic explanation for this effect is that even at asset levels above  $\mu_L$  there is a chance that due to a bad outcome of the liability  $L$  the company may be pushed (unexpectedly) into insolvency. Conversely, for asset levels below  $\mu_L$  there is also a chance that the company may escape insolvency due to a low realisation of the liabilities. Hence, this uncertainty about the true outcome of the liabilities is explicitly incorporated into the optimal strategy.

As it turns out, the level of  $\lambda_2$  is still so low, that the function  $\Xi$  is still monotonically decreasing, and the optimal wealth allocation at time  $T$  is given by

$$A^*(T) = A_{max}(\xi(T)) = \Xi^{-1}(\xi(T))$$

where  $\Xi^{-1}$  is the inverse function of  $\Xi$ . The difference between the solution to the Merton-problem without VaR-constraint and our solution is illustrated in the right panel of Figure 2, where we have expressed the optimal wealth not in terms of  $\xi(T)$ , but in terms of the return  $S(T)/S(0)$  of the risky asset, i.e. the market-index.

If we compare the two optimal asset allocations in the right panel, we see that “Xi-opt” solution lies below the Merton “NoSolv” solution, except for wealth values around  $\mu_L = 100$ . This is, of course, due to the budget constraint. Both solutions cost a total amount of initial wealth equal to  $A(0) = 110$ . Therefore the only way to finance the hump around  $\mu_L$  is to reduce the overall exposure to the market-index.

### 6.3 Non-monotone function $\Xi$

We will now consider the case where we have a Value-at-Risk constraint  $\mathbb{P}[A^*(T) < L] = 0.025$ , which is much more severe than the case in the previous sub-section. In this case the function  $\Xi$  as defined in (13) is not invertible.

This is illustrated in Figure 3. In the left panel we have plotted three curves. The dashed curve (“NoSolv”) is again the Merton solution, which we have included here for reference. The next curve is the thin solid line “Xi” which shows the first order condition  $\Xi(A)$ . This curve is obviously not

invertible, as the “hump” around  $\mu_L$  is very high.

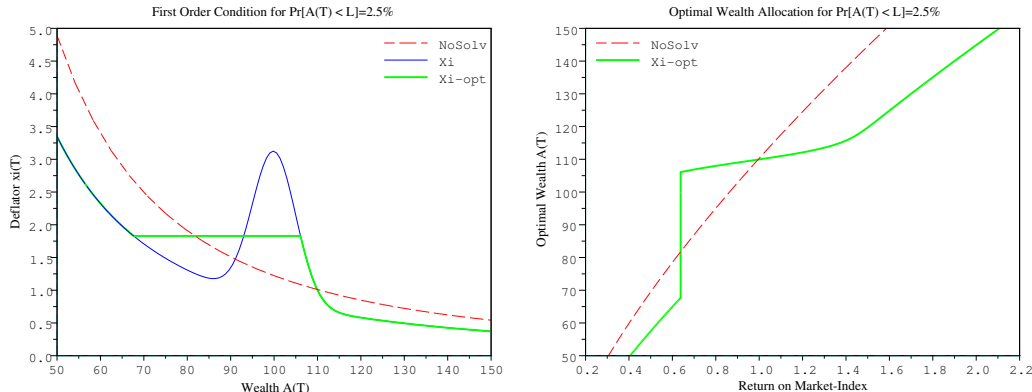


Figure 3: Optimal wealth allocation for  $\mathbb{P}[A^*(T) < L] = 0.025$ .

The solution “Xi-opt” is constructed by restricting the domain of  $\Xi$  to  $(0, a_1] \cup (a_2, \infty)$  where  $a_1 < a_2$  and the following system of equations is satisfied:

$$\begin{aligned} \Xi(a_1) &= \Xi(a_2) \\ G(\Xi(a_1), a_1) &= G(\Xi(a_2), a_2) \end{aligned}$$

To solve our problem numerically we have to work at two levels. In the “inner level” we solve for  $a_1$  and  $a_2$  for given values of the Lagrange multipliers  $\lambda_1$  and  $\lambda_2$ . The wealth and solvency constraints can then be evaluated by numerically integrating over the restricted domain  $(0, a_1] \cup (a_2, \infty)$ . Then on the “outer level” we search (numerically) for the two values of  $\lambda_1$  and  $\lambda_2$  that let the optimal wealth allocation satisfy the wealth and solvency constraints.

For this particular example we find that the values for the Lagrange multipliers are given by  $\lambda_1 = 1.19 \cdot 10^{-4}$  and  $\lambda_2 = 3.42 \cdot 10^{-3}$ . The restricted domain of  $\Xi$  that correspond to these  $\lambda$ 's is given by  $a_1 = 67.7$  and  $a_2 = 106.1$ .

The domain-restricted function “Xi-opt” is now invertible and satisfies all constraints. The optimal wealth allocation that results is depicted in the right panel of Figure 3.

The optimal wealth allocation we obtain in this way represents a non-trivial investment and hedging strategy for the insurance company. For very high levels of the wealth, we basically follow a “Merton-type” optimal strategy, but we behave as if we have a much lower wealth. (The curve of “Xi-opt” lies considerably lower than “NoSolv”.) When the wealth drops below a level of roughly 115, we begin to invest much more conservatively. In essence, accumulating put-options to ensure that enough wealth is retained to meet the

(uncertain) liabilities with sufficient probability. Then at the wealth level of 106.1 the optimal strategy makes a discrete jump back down to the wealth level of 67.7, and resumes the “Merton strategy”.

If we compare our optimal strategy to a Basak-Shapiro strategy, we see several interesting differences. First, our optimal strategy gradually begins to “anticipate” on the VaR constraint at wealth levels much higher than  $\mu_L$ . This is due to the random nature of the liabilities. Second, the “down-jump” in wealth occurs at a wealth which is also higher than  $\mu_L$  since  $a_2 > \mu_L$ .

## 6.4 Convergence to Basak-Shapiro solution

Our solution to finding an optimal investment strategy under a solvency constraint for a random liability includes the Basak & Shapiro (2001) result as a special case. Basak-Shapiro treat the case where the liability is deterministic. Our setup reduces to the Basak-Shapiro setting when  $\sigma_L = 0$ .

We conclude this example by demonstrating how our optimal solution converges to the Basak-Shapiro solution for small values of  $\sigma_L$ . When the uncertainty in the liability  $L$  converges to zero, the “hump” in the first order condition  $\Xi(A)$  approaches a Dirac delta-function. In the optimal wealth allocation this will have the effect that the “safe investment region” flattens until it becomes perfectly horizontal at  $\mu_L$ , which is exactly the Basak-Shapiro solution.

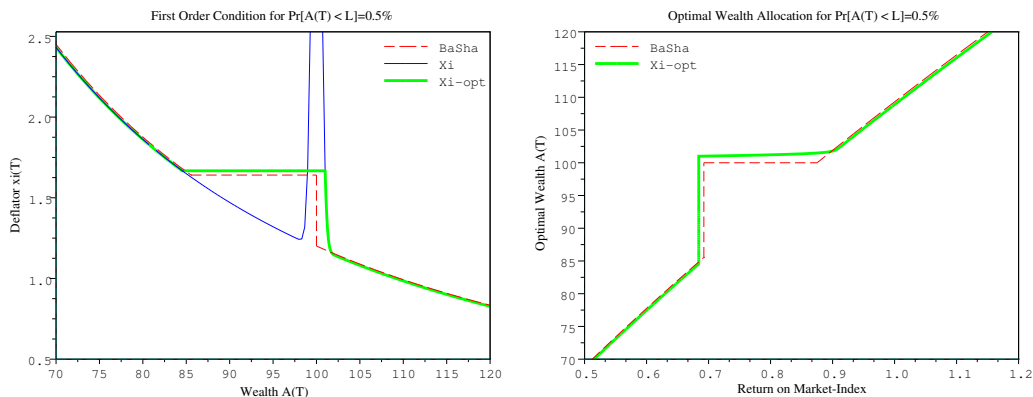


Figure 4: Optimal wealth allocation for small  $\sigma_L$ .

We have illustrated this behaviour in Figure 4. We have set  $\sigma_L = 0.5$ , and we have imposed the solvency constraint  $\mathbb{P}[A^*(T) < L] = 0.005$ . For these parameter settings we have plotted both the Basak-Shapiro solution (“BaSha”) and our solution (“Xi” and “Xi-opt”).

In the left panel we see that the first order condition (“Xi”) has become very spiked. Nevertheless, the domain-restricted optimal solution (“Xi-opt”) is already close to the limiting Basak-Shapiro solution. The optimal Lagrange multipliers for our problem that satisfy the constraints are  $\lambda_1 = 8.39 \cdot 10^{-5}$  and  $\lambda_2 = 3.84 \cdot 10^{-4}$ , yielding  $a_1 = 84.55, a_2 = 100.10$ . The optimal Lagrange multiplier for the Basak-Shapiro problem is  $\lambda = 8.33 \cdot 10^{-5}$  and a “jump-point”  $\bar{\xi} = 1.64$ , leading to  $a_1 = 85.55$  and  $a_2 = \mu_L = 100$ .

## 6.5 Tail-VaR solvency constraint

We want to conclude this section by showing an example where we use a Tail-VaR solvency constraint, instead of “simple” VaR. As discussed in Example 3, we can model an Expected Loss as

$$R(A(T)) = \mathbb{E} [(L - A(T)) \mathbf{1}_{\{A(T) \leq L\}} | A(T)] .$$

However, if we want to model a Tail-VaR the situation is slightly more complicated. The Tail-VaR is defined as the expected loss *given* that the losses exceed a given VaR level. In our setup, we cannot control the Value-at-Risk directly, but we obtain the VaR (and consequently the Tail-VaR) indirectly by choosing an optimal wealth  $A^*(T)$  strategy that satisfies certain solvency constraints.

In the case of a Tail-VaR solvency constraint, we can recover the Tail-VaR from the Expected Loss as follows. We use here well-known results from Rockafellar & Uryasev (2000) on optimisation under a Tail-VaR (or CVaR) constraint. First, we define the solvency function

$$R(A(T), K) = K + (1 - \beta)^{-1} \mathbb{E} [((L - A(T)) - K) \mathbf{1}_{\{L - A(T) \geq K\}} | A(T)] ,$$

where  $\beta$  denotes the probability threshold of the VaR. As shown by Rockafellar & Uryasev (2000), the Tail-VaR is then given by

$$TV_\beta = \min_K \mathbb{E}[R(A(T), K)].$$

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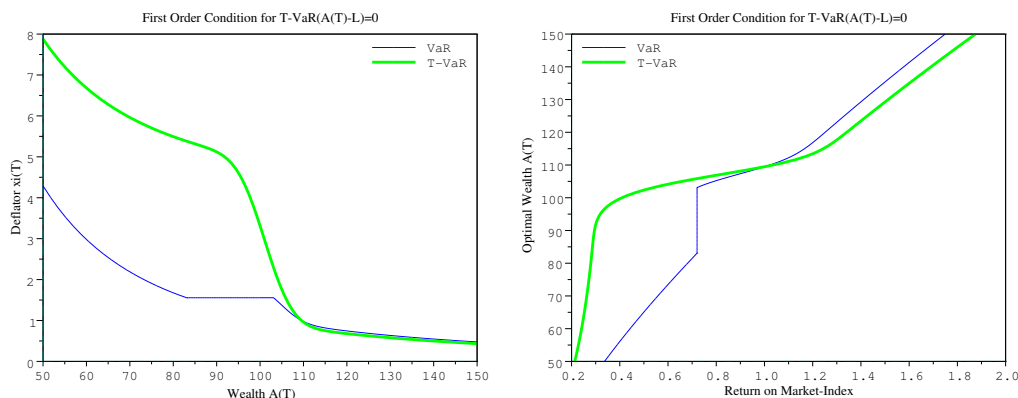


Figure 5: Optimal wealth allocation under Tail-VaR constraint.

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