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# On a $q$ -extension of Mehta's eigenvectors of the finite Fourier transform for $q$ a root of unity

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## Abstract

It is shown that the continuous  $q$ -Hermite polynomials for  $q$  a root of unity have simple transformation properties with respect to the classical Fourier transform. This result is then used to construct  $q$ -extended eigenvectors of the finite Fourier transform in terms of these polynomials.

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## 1 Introduction

The *finite Fourier transform* [26, Ch. 7] (also called *discrete Fourier transform*) is defined as the Fourier transform associated with the *finite abelian group*  $\mathbb{Z}_N := \mathbb{Z}/(N\mathbb{Z})$  of integers modulo  $N$  [25, 19], just as the classical integral Fourier transform is the Fourier transform associated with  $\mathbb{R}$ . In concrete terms, it is a linear transformation  $\Phi^{(N)}$  of the space of functions on  $\mathbb{Z}$  with period  $N$  defined by

$$(\Phi^{(N)}f)(r) := \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N} rs\right) f(s), \quad r \in \mathbb{Z}. \quad (1.1)$$

Equivalently, if we identify the  $N$ -periodic function  $f$  on  $\mathbb{Z}$  with the vector

$$(f(0), f(1), \dots, f(N-1))$$

in  $\mathbb{C}^N$ , then  $\Phi^{(N)}$  is a unitary operator on  $\mathbb{C}^N$  with matrix elements

$$\Phi_{rs}^{(N)} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} rs\right), \quad 0 \leq r, s \leq N-1. \quad (1.2)$$

We are interested in a suitable basis of  $N$  eigenvectors  $f_n^{(N)}$  ( $n = 0, 1, \dots, N-1$ ) of  $\Phi^{(N)}$ , which thus should satisfy the eigenvalue equation

$$\sum_{s=0}^{N-1} \Phi_{rs}^{(N)} f_n^{(N)}(s) = \lambda_n f_n^{(N)}(r) \quad (1.3)$$

for suitable eigenvalues  $\lambda_n$ . Since the fourth power of  $\Phi^{(N)}$  is the identity operator (or matrix), the  $\lambda_n$ 's can only be equal to  $\pm 1$  or  $\pm i$ .

The finite Fourier transform has deep roots in classical pure mathematics and it is also extremely useful in applications. See [12], [27] for mathematical and historical details and [13], [14] for the relation with Gauss sums.

Mehta studied in [22] the eigenvalue problem (1.3) and found analytically a set of eigenvectors  $F_n^{(N)}$  of the finite Fourier transform  $\Phi^{(N)}$  of the form

$$F_n^{(N)}(r) := \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} H_n\left(\sqrt{\frac{2\pi}{N}}(kN+r)\right), \quad r = 0, 1, \dots, N-1, \quad (1.4)$$

where  $H_n(x)$  is the Hermite polynomial of degree  $n$  in  $x$ . These eigenvectors  $F_n^{(N)}$  correspond to the eigenvalues  $\lambda_n = i^n$ , that is,

$$\sum_{s=0}^{N-1} \Phi_{rs}^{(N)} F_n^{(N)}(s) = i^n F_n^{(N)}(r), \quad r = 0, \dots, N-1. \quad (1.5)$$

They can be considered as a discrete analogue of the well-known continuous case where the Hermite functions  $e^{-x^2/2} H_n(x)$  are constant multiples of their own Fourier transforms:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy-x^2/2} H_n(x) dx = i^n e^{-y^2/2} H_n(y). \quad (1.6)$$

It was conjectured by Mehta [22] that the  $F_n^{(N)}$  ( $n = 0, 1, \dots, N-1$  if  $N$  is odd and  $n = 0, 1, \dots, N-2, N$  if  $N$  is even) are linearly independent, but this problem has not been any further considered until now. As shown by Ruzzi [24], these systems are in general not orthogonal.

In later work it was shown that many  $q$ -extensions of the classical orthogonal polynomials satisfy simple transformation properties under the Fourier transform (see [7] and references therein). Thus it was natural to repeat Mehta's construction of eigenfunctions of the finite Fourier transform in that context. This was done in [9], [8] and [11]. In particular, it was shown there that the finite Fourier transform provides a link between continuous  $q$ -Hermite and  $q^{-1}$ -Hermite polynomials of Rogers, as well as between families of Rogers-Szegő and Stieltjes-Wigert polynomials. It turned out that the same form of connection exists also between discrete  $q$ -Hermite polynomials of types I and II, see [3].

Zhedanov [28] considered continuous  $q$ -Hermite polynomials for  $q$  a root of unity and he obtained a discrete orthogonality on finitely many points with complex weights for a finite system of such polynomials. In the present paper we consider the transformation properties of these polynomials under the integral and finite Fourier transforms. Analytically, this

turns out to be a straightforward extension of the earlier results for the continuous  $q$ -Hermite polynomials with  $0 < q < 1$ . However, the resulting formulas, a little different from the case that  $q$  is real, are interesting enough to be displayed.

Another feature of the present paper, compared with [9], [8] and [11], is that we emphasize a more conceptual approach by using Theorem 4.1 due to Dahlquist [16] and Matveev [21] and the (trivial) Lemma 5.1, rather than repeating a technical argument in each special situation.

The contents of the paper are as follows. In section 2 we recall some properties of the continuous  $q$ -Hermite polynomials for general complex  $q$ , and in particular for  $q$  a root of unity. In section 3 we consider the behaviour of these polynomials times a Gaussian under the integral Fourier transform. This result, together with the Dahlquist-Matveev Theorem 4.1, then gives a construction of functions behaving nicely under the finite Fourier transform. In section 5, using Lemma 5.1 we obtain from these functions and their Fourier images eigenfunctions of the finite Fourier transform. Finally, section 6 concludes the paper with a brief discussion of some further research directions of interest.

Throughout our exposition we employ standard notations of the theory of special functions (see, for example, [2] and [17]).

## 2 $q$ -Hermite polynomials for $q$ a root of unity

The continuous  $q$ -Hermite polynomials of Rogers (see [23], [1], [6]), denoted by  $H_n(x|q)$ , can be generated for any  $q \in \mathbb{C}$  by the three-term recurrence relation

$$2x H_n(x|q) = H_{n+1}(x|q) + (1 - q^n) H_{n-1}(x|q) \quad (2.1)$$

with initial condition  $H_0(x|q) = 1$ . Their explicit form as a finite Fourier series in  $\theta$  ( $x = \cos \theta$ ) is given by

$$H_n(\cos \theta|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}, \quad (2.2)$$

where the symbol  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  stands for the  $q$ -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \quad (2.3)$$

Here  $(a; q)_n$  is the  $q$ -shifted factorial, see [17, (1.2.15)]. The right-hand sides of (2.3) and (2.2) are well-defined for all  $q \in \mathbb{C}$  because the  $q$ -binomial coefficients are polynomials in  $q$ .

Since  $\sin \theta = \cos(\frac{1}{2}\pi - \theta)$  we can rewrite (2.2) as

$$H_n(\sin \theta|q) = i^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k e^{i(2k-n)\theta}. \quad (2.4)$$

The polynomials  $H_n(x|q)$  are orthogonal polynomials for  $0 < q < 1$ . For  $q > 1$  they are orthogonal polynomials in  $ix$ : the  $q^{-1}$ -Hermite polynomials (see [5]) denoted by

$$h_n(x|q) := i^{-n} H_n(ix|q^{-1}). \quad (2.5)$$

Another case of orthogonality, but not with positive weights, was considered for  $q$  a root of unity, see [28]. For  $M$  a positive integer put

$$q_{j,M} := \exp(2\pi i j/M), \quad j \in \{1, 2, \dots, M-1\}. \quad (2.6)$$

For such  $q = q_{j,M}$  (2.1) and (2.2) remain valid. In particular, for  $q = q_{j,M}$  with  $j$  and  $M$  co-prime and for  $n = M$ , the only non-vanishing terms in (2.2) occur for  $k = 0$  and  $M$ . Hence, for  $j$  and  $M$  co-prime, we have

$$H_M(\cos \theta | q_{j,M}) = 2 \cos M\theta =: 2 T_M(\cos \theta), \quad (2.7)$$

where  $T_M(x)$  is a Chebyshev polynomial of the first kind (see, for example, [2, Remark 2.5.3]).

As pointed out in [28], the polynomials  $H_n(x|q_{j,M})$  ( $n = 0, 1, \dots, M-1$ ) satisfy a discrete orthogonality with possibly complex weights on the  $M$  zeros of  $T_M(x)$  if  $j$  and  $M$  are co-prime. It is for functions suitably defined in terms of these polynomials that we will discuss their integral and finite Fourier transforms.

Finally observe that, by induction with respect to  $m$  and  $n$ , we derive from (2.1) and (2.7) that, for  $j$  and  $M$  co-prime,

$$H_{mM+n}(x|q_{j,M}) = (2T_M(x))^m H_n(x|q_{j,M}), \quad n = 0, 1, \dots, M-1, \quad m = 0, 1, \dots \quad (2.8)$$

### 3 Integral Fourier transform

There are  $q$ -extensions of the eigenfunction result (1.6) for the integral Fourier transform. These interrelate certain  $q$ -polynomial families (see [7] and references therein). For the continuous  $q$ -Hermite polynomials we obtain:

**Lemma 3.1.** *The Fourier transform of the functions  $e^{-x^2/2} H_n(\sin(\lambda x)|q)$  is given by:*

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy - x^2/2} H_n(\sin(\lambda x)|q) dx &= i^n e^{-n^2\lambda^2/2} e^{-y^2/2} \\ &\times \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \left( q^{-1} e^{-2\lambda^2} \right)^{k(k-n)} (-1)^k e^{-(2k-n)\lambda y}, \quad \lambda, q \in \mathbb{C}, \quad q \neq 0, 1. \end{aligned} \quad (3.1)$$

In particular, if  $q = e^{-2\lambda^2}$  then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy - x^2/2} H_n(\sin(\lambda x)|q) dx = q^{n^2/4} e^{-y^2/2} H_n(\sin(i\lambda y)|q^{-1}). \quad (3.2)$$

**Proof** Substitute (2.4) on the left-hand side, take termwise Fourier transforms (which turns down to the Fourier transform of the Gaussian), and use that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

If  $q = e^{-2\lambda^2}$  then we see by (2.4) that (3.1) simplifies to (3.2).  $\square$

We consider two cases of equation (3.2). First let  $0 < q < 1$  and define  $\kappa$  by

$$q = \exp(-2\kappa^2), \quad 0 < \kappa < \infty. \quad (3.3)$$

Then, by (2.5), we obtain (see [10], [7]):

**Proposition 3.2.** *The Fourier transform of the functions  $e^{-x^2/2} H_n(\sin \kappa x|q)$  is given by:*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy - x^2/2} H_n(\sin \kappa x|q) dx = i^n q^{n^2/4} e^{-y^2/2} h_n(\sinh \kappa y|q). \quad (3.4)$$

Second, let  $q := q_{j,M}$  as in (2.6) and put  $\lambda := \alpha_{j,M}$ , where

$$\alpha_{j,M} := \sqrt{\frac{\pi j}{M}} e^{-\pi i/4}, \quad \text{hence } e^{-2\alpha_{j,M}^2} = q_{j,M} \quad \text{and} \quad i\alpha_{j,M} = \overline{\alpha_{j,M}}. \quad (3.5)$$

**Proposition 3.3.** *The Fourier transform of the functions  $e^{-x^2/2} H_n(\sin(\alpha_{j,M} x)|q_{j,M})$  is given by:*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy - x^2/2} H_n(\sin(\alpha_{j,M} x)|q_{j,M}) dx = q_{j,M}^{n^2/4} e^{-y^2/2} H_n(\sin(\overline{\alpha_{j,M}} y)|q_{j,M}^{-1}). \quad (3.6)$$

The Fourier inversion formula of (3.6) is just the result of taking complex conjugates on both sides of (3.6). We will mostly work with (3.6) for  $n = 0, 1, \dots, M-1$ , but this formula remains valid for all nonnegative integer values of  $n$ . In particular, for  $n = mM$  ( $m = 0, 1, \dots$ ), formula (3.6) takes by substitution of (2.8) the form

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos^m \left( M\pi/2 - \sqrt{\pi j M} e^{-i\pi/4} x \right) e^{ixy - x^2/2} dx \\ = i^{m^2 j M} e^{-y^2/2} \cos^m \left( M\pi/2 - \sqrt{\pi j M} e^{i\pi/4} y \right). \end{aligned} \quad (3.7)$$

Depending on the value of  $M \pmod{4}$  this may be further simplified.

**Remark 3.4.** Formulas (3.4) and (3.6) can be considered as  $q$ -analogues of formula (1.6), in the sense that (1.6) can be obtained as the limit for  $q \uparrow 1$  of (3.4) and as the limit for  $M \rightarrow \infty$  of (3.6) with  $j$  fixed. Indeed, from [18, (5.26.1)] we have

$$\lim_{q \rightarrow 1} \left( \sqrt{\frac{1}{2}(1-q)} \right)^{-n} H_n \left( x \sqrt{\frac{1}{2}(1-q)} \mid q \right) = H_n(x). \quad (3.8)$$

Since  $\kappa \sim ((1-q)/2)^{\frac{1}{2}}$  as  $q \uparrow 1$ , it follows from (3.8) that

$$\begin{aligned} \lim_{q \uparrow 1} \kappa^{-n} H_n(\sin(\kappa x) \mid q) &= H_n(x), \\ \lim_{q \uparrow 1} (i\kappa)^{-n} H_n(\sin(i\kappa y) \mid q^{-1}) &= H_n(y). \end{aligned}$$

Hence, in view of (2.5), equation (3.4) with both sides multiplied by  $\kappa^{-n}$  tends to (1.6) as  $q \uparrow 1$ .

As for (3.6) with  $j$  fixed, we have  $\alpha_{j,M} \sim ((1-q_{j,M})/2)^{\frac{1}{2}}$  as  $M \rightarrow \infty$ . Hence it follows from (3.8) that

$$\begin{aligned} \lim_{M \rightarrow \infty} \alpha_{j,M}^{-n} H_n(\sin(\alpha_{j,M} x) \mid q_{j,M}) &= H_n(x), \\ \lim_{M \rightarrow \infty} \alpha_{j,M}^{-n} H_n(\sin(\overline{\alpha_{j,M}} y) \mid q_{j,M}^{-1}) &= H_n(y). \end{aligned}$$

Then (3.6) with  $j$  fixed and with both sides multiplied by  $\alpha_{j,M}^{-n}$ , tends to (1.6) as  $q \uparrow 1$ .

## 4 Finite Fourier transform

It was observed by Dahlquist [16, Theorem 1] and Matveev [21, Theorem 8.1] that Mehta's result (1.5) is a special case of the following more general relationship between integral Fourier transform and finite Fourier transform, which can be obtained as an immediate consequence of the Poisson summation formula. This result will hold for  $f$  in a wide class of functions on  $\mathbb{R}$ , but for convenience we only formulate it for  $f \in \mathcal{S}$ , the space of Schwartz functions on  $\mathbb{R}$  (see [26, Ch. 5, §1.3]).

**Theorem 4.1.** *Define a linear map  $\mathcal{M}^{(N)}$  from  $\mathcal{S}$  to the space of  $N$ -periodic functions on  $\mathbb{Z}$  by*

$$(\mathcal{M}^{(N)} f)(r) := \sum_{k \in \mathbb{Z}} f \left( \sqrt{\frac{2\pi}{N}} (kN + r) \right), \quad f \in \mathcal{S}, r \in \mathbb{Z}. \quad (4.1)$$

*If  $f, g \in \mathcal{S}$  are related by the integral Fourier transform*

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(x) dx, \quad (4.2)$$

and if  $F := \mathcal{M}^{(N)}f$ ,  $G := \mathcal{M}^{(N)}g$  then  $F$  and  $G$  are related by the finite Fourier transform (1.1):

$$G(r) = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N} rs\right) F(s). \quad (4.3)$$

In particular, in the case when  $g(x) = \lambda f(x)$ ,  $\lambda = \pm 1, \pm i$ , one has  $G(r) = \lambda F(r)$ ; so Mehta's eigenvectors (1.4) are a particular case of the general statement.

Let us now apply Theorem 4.1 to the case where  $f$  and  $g$  are implied by (3.2), i.e.,

$$f(x) := e^{-x^2/2} H_n(\sin(\lambda x)|q), \quad g(y) := e^{-y^2/2} H_n(\sin(i\lambda y)|q^{-1}),$$

where  $\lambda, q \in \mathbb{C}$ ,  $q \neq 0, 1$ , and  $q = e^{-2\lambda^2}$ . Put

$$f_n^{(N)}(r|q) := (\mathcal{M}^{(N)}f)(r) = \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} H_n\left(\sin\left(\lambda\sqrt{\frac{2\pi}{N}}(kN+r)\right)\middle|q\right), \quad (4.4)$$

$$\begin{aligned} g_n^{(N)}(r|q) &:= q^{-n^2/4} (\mathcal{M}^{(N)}g)(r) \\ &= \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} H_n\left(\sin\left(i\lambda\sqrt{\frac{2\pi}{N}}(kN+r)\right)\middle|q^{-1}\right). \end{aligned} \quad (4.5)$$

Then we obtain by Theorem 4.1:

**Proposition 4.2.** *The finite Fourier transform of the functions  $f_n^{(N)}$  is given by:*

$$\Phi^{(N)}(f_n^{(N)}(\cdot|q))(r) = \sum_{s=0}^{N-1} \exp\left(\frac{2\pi i}{N} rs\right) f_n^{(N)}(s|q) = q^{n^2/4} g_n^{(N)}(r|q). \quad (4.6)$$

Just as we did in §3 for (3.2), we consider two special cases of (4.6). First, let  $0 < q < 1$  and take  $\lambda = \kappa$  as in (3.3). Then (4.6) holds (see [9] and [11]) with

$$g_n^{(N)}(r|q) = i^n \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} h_n\left(\sinh\left(\kappa\sqrt{\frac{2\pi}{N}}(kN+r)\right)\middle|q\right). \quad (4.7)$$

Second, consider (4.6) with  $q = q_{j,M}$  and  $\lambda = \alpha_{j,M}$  as in (3.5). Then (4.6) holds with

$$\begin{aligned} f_n^{(N)}(r|q_{j,M}) &= \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} H_n\left(\sin\left(\alpha_{j,M}\sqrt{\frac{2\pi}{N}}(kN+r)\right)\middle|q_{j,M}\right), \quad (4.8) \\ g_n^{(N)}(r|q_{j,M}) &= \sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{N}(kN+r)^2} H_n\left(\sin\left(\overline{\alpha_{j,M}}\sqrt{\frac{2\pi}{N}}(kN+r)\right)\middle|q_{j,M}^{-1}\right) \\ &= \overline{f_n^{(N)}(r|q_{j,M})}. \end{aligned} \quad (4.9)$$



By similar reasoning as in Remark 3.4 we see that (1.4) is a limit of (4.6) in these two cases (as  $q \uparrow 1$  in the first case and as  $M \rightarrow \infty$  in the second case). These limits are formal because we have to take termwise limits for  $f_n^{(N)}(s|q)$  and  $g_n^{(N)}(r|q)$ . Just as for the functions  $F_n^{(N)}$  given by (1.4), we do not know if the functions  $f_n^{(N)}$  defined by (4.4) are linearly independent.

Let  $\epsilon = \pm 1$ . From (4.1) we see that, if  $f(-r) = \epsilon f(r)$  for all  $r$ , then also  $(\mathcal{M}^{(N)}f)(-r) = \epsilon(\mathcal{M}^{(N)}f)(r)$  for all  $r$ . Also note from (2.1) that  $H_n(-x|q) = (-1)^n H_n(x|q)$ . Hence, by (4.4) and (4.5),

$$f_n^{(N)}(-r|q) = (-1)^n f_n^{(N)}(r|q), \quad g_n^{(N)}(-r|q) = (-1)^n g_n^{(N)}(r|q).$$

Thus consider (4.6) with  $f_n^{(N)}$  and  $g_n^{(N)}$  as in (4.8), (4.9). Then take complex conjugates on both sides and use the simple facts just mentioned above. Then we obtain:

$$\Phi^{(N)}(f_n^{(N)}(\cdot|q_{j,M}))(r) = q_{j,M}^{n^2/4} \overline{f_n^{(N)}(r|q_{j,M})}, \quad (4.10)$$

$$\Phi^{(N)}(\overline{f_n^{(N)}(\cdot|q_{j,M})})(r) = (-1)^n q_{j,M}^{-n^2/4} f_n^{(N)}(r|q_{j,M}). \quad (4.11)$$

**Remark 4.3.** It is tempting to consider the case  $q = q_{j,M}$  and  $\lambda = \alpha_{j,M}$  of (4.6) with  $M = N$ , even if one loses then the possibility to take the limit to (1.4) for  $M \rightarrow \infty$ . One might hope to arrive at some transform acting on the polynomials  $H_n(x|q_{j,N})$  ( $n = 0, 1, \dots, N-1$ ;  $j$  and  $N$  co-prime) but only involving their values at the zeros of  $T_N(x)$ , i.e., at the points involved in Zhedanov's [28] orthogonality relations for these polynomials. At the moment we have no idea how to proceed here.

## 5 Finite Fourier $q$ -extended eigenvectors

We now show that relations (4.10), (4.11) enable us to construct  $q$ -extensions of Mehta's eigenvectors (1.4) of the finite Fourier transform when the deformation parameter  $q$  is a root of unity. For this we need the following trivial observation.

**Lemma 5.1.** *Let  $V$  be a complex linear space. Let  $f, g \in V$  and  $a, b \in \mathbb{C}$  such that  $\Phi f = a^2 g$  and  $\Phi g = b^2 f$ . Then*

$$\Phi(bf \pm ag) = \pm ab(bf \pm ag).$$

Combination of this lemma with (4.10) and (4.11) shows that the functions

$$i^n q_{j,N}^{-n^2/8} f_n^{(N)}(\cdot|q_{j,M}) \pm q_{j,M}^{n^2/8} \overline{f_n^{(N)}(\cdot|q_{j,M})}$$

are eigenfunctions of  $\Phi^{(N)}$  with eigenvalues  $\pm i^n$ . So put

$$F_n^{(N)}(r|q_{j,M}) := \Re \left( e^{i\pi n/8} q_{j,M}^{-n^2/8} f_n^{(N)}(r|q_{j,M}) \right),$$

$$G_n^{(N)}(r|q_{j,M}) := \Im \left( e^{i\pi n/8} q_{j,M}^{-n^2/8} f_n^{(N)}(r|q_{j,M}) \right).$$

Then

$$\Phi^{(N)}(F_n^{(N)}(\cdot|q_{j,M}))(r) = i^n F_n^{(N)}(r|q_{j,M}), \quad (5.1)$$

$$\Phi^{(N)}(G_n^{(N)}(\cdot|q_{j,M}))(r) = -i^n G_n^{(N)}(r|q_{j,M}). \quad (5.2)$$

## 6 Concluding comments and outlook

We have demonstrated that the continuous  $q$ -Hermite polynomials for  $q$  a root of unity are interrelated by the classical Fourier transform, see (3.6). Then the technique of constructing eigenvectors of the finite Fourier transform, pioneered by Mehta [22] and formulated in a more systematic way by Dahlquist [16] and Matveev [21], has been employed in order to construct  $q$ -extended eigenvectors of the finite Fourier transform.

Quite naturally, it would be of considerable interest to find out whether there are other families of  $q$ -polynomials for  $q$  a root of unity, which also possess such simple transformations properties with respect to the Fourier transform and, consequently, give rise to other  $q$ -variants of Mehta's eigenfunctions (1.4) of the finite Fourier transform. The point is that the continuous  $q$ -Hermite polynomials, considered in the present paper, belong to the lowest level in the Askey hierarchy of basic hypergeometric polynomials [18]. Therefore it will be natural to attempt to apply the same technique to the study of other  $q$ -families, which depend on some additional parameters (and therefore occupy the higher levels in the Askey  $q$ -scheme).

Finally, another direction for further study is connected with  $q$ -extensions of the harmonic oscillator in quantum mechanics [4], [20], [15]. We remind the reader that for proving the fundamental formula (1.5) for Mehta's eigenvectors  $F^{(n)}$  of the finite Fourier transform operator (1.2) it is vital to use the simple transformation property (1.6) of the Hermite functions  $H_n(x) \exp(-x^2/2)$  with respect to the Fourier transform. Moreover, these eigenvectors  $F^{(n)}$  are actually built in terms of these Hermite functions taken at the infinite set of discrete points  $x_r^{(k)} := \sqrt{\frac{2\pi}{N}}(kN + r)$ ,  $0 \leq r \leq N - 1$ ,  $k \in \mathbb{Z}$  (cf. (4.1)). In other words, Mehta's technique of proving (1.5) is based on introducing a discrete analogue of the quantum harmonic oscillator. It seems that  $q$ -extended eigenvectors of the finite Fourier transform, constructed in the foregoing sections, can be similarly viewed as discrete analogues of the  $q$ -harmonic oscillator of Macfarlane and Biedenharn for  $q$  a root of unity.

We plan to continue our studies in both of these directions. But a focus of our attention will be on constructing an explicit form of a finite-difference equation for suitable eigenvectors (1.3) of the finite Fourier transform operator  $\Phi^{(N)}$ . From the group-theoretic point of view this amounts to finding an adequate finite representation, associated with the Heisenberg–Weyl group [25, 19]. Purely analytically this is reduced to the construction of an explicit finite-difference operator  $L$  which commutes with  $\Phi^{(N)}$ , so that the eigenspaces of  $\Phi^{(N)}$  can be decomposed by spectral decomposition of  $L$ .

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