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Airiau, S.; Endriss, U.; Halpern, J.Y.

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Solution to Exchanges 8.1 Puzzle: Identifying the Champion

STÉPHANE AIRIAU and ULLE ENDRISS
ILLC, University of Amsterdam
and
JOSEPH Y. HALPERN
Cornell University

The Editor's Puzzle published in SIGeom Exchanges 8.1 was based on an ancient Japanese prophecy [Conitzer 2009]. It was concerned with the complications associated with determining the champion of all warriors. What follows is a synthesis of two correct solutions that were received.

1. TERMINOLOGY AND ASSUMPTIONS

First, some terminology, notation, and assumptions: We call n_i the *skill level* of warrior i. Let n_c be the skill level of the champion (we shall assume that $n_c > 0$ and that this is common knowledge). Define $n_v = \max\{n_i \mid i \text{ is a warrior and } n_c > n_i\}$, the skill level of (one of) the vice champion(s). Let m be the number of warriors.

2. WARRIORS AND MUDDY CHILDREN

Suppose that there are $\ell \leq m$ warriors in a room, all of whom were qualified to enter this room, and that ℓ is common knowledge. Also suppose that k of these ℓ warriors perform satisfactorily, and thus are qualified to enter the next room.

If $\ell = k = 1$, then the single warrior in the room will be able to infer that he must be the one (or rather: the One), and will leave by the end of the day.

Otherwise (that is, if $\ell > 1$), we can check how long it will take the qualified warriors to realise that they are indeed qualified using the well-known *muddy children argument*, familiar from epistemic logic.[1] If $k = 1$, then the warrior concerned will be able to observe that everybody else's performance is not satisfactory, infer that he must be the One, and leave by the end of the day. If $k = 2$, then nobody will have left by the end of the first day, by which time each of the two qualified warriors will know that the one qualified warrior they observe cannot be the only one (as he would have left otherwise); so by the end of the second day they will both leave. We can iterate this kind of reasoning (and so can the warriors), which means that if there are k qualified warriors in a room, it takes until the end of the kth day for all of them to realise that they are qualified to move on and leave.

[1] This puzzle is a variant of the well-known muddy children puzzle [Fagin et al. 1995]. The muddy children puzzle has a long history. It is itself a variant of the "unfaithful wives" puzzle discussed by Littlewood [1953] and Gamow and Stern [1958]. Gardner [1984] also presents a variant of the puzzle; a number of other variants of the puzzle are discussed by Moses et al. [1986].

Authors' email addresses: s.airiau@uva.nl, u.endriss@uva.nl, halpern@cs.cornell.edu

Hence, by induction on k we are able to show that for each room it takes as many days for the warriors qualified for the next room to move on as there are warriors that are qualified to do so. Now, by induction on j, it follows that, for any $j \leq n_v$, the number of days it takes for all the warriors of skill level j to move on to room $j+1$ is $\sum_{r=1}^{j} \#\{i \mid n_i \geq r\}$.

3. SOLUTION TO THE PUZZLE

We are now in a position to answer the various questions posed:

—*Will the champion be identified?* Yes.

—*How long does the process take?* The process stops as soon as the champion and the vice champion(s) reach a room where only the champion can perform at a satisfactory level. To be precise, on that day, the champion will come to know that he is the One (but the vice champion(s) do(es) not yet know that they aren't). As the champion doesn't talk to his fellow warriors, we have to wait until the end of that day for him to announce himself by moving on to the next room. To summarise, the process will end at midnight on day $D = 1+\sum_{r=1}^{n_v} \#\{i \mid n_i \geq r\}$.

In the worst case, every non-champion is a vice champion, in which case the process will take $n_v \times m + 1$ days. In the best case, only the champion will perform satisfactorily in the first room and the process takes just one day.

There is another, perhaps more intuitive, description of D. Let $\vec{n}$ be the vector describing the skill levels of each player, and let $\vec{n}'$ be the result of replacing n_c by $n_v + 1$ in $\vec{n}$. It follows from the discussion above that the process takes just as long if the qualification levels of the players are described by $\vec{n}'$ as if they are described by $\vec{n}$. It is not hard to show that $D = \sum_{i=1}^{m} n'_i$. Intuitively, if we consider a sequence of m vertical rectangles, where the ith rectangle has base 1 and height n'_i, then $\sum_{i=1}^{m} n'_i$ and $1 + \sum_{r=1}^{n_v} \#\{i \mid n_i \geq r\}$ describe two ways of computing the total area of the rectangles, either "vertically" or "horizontally". (A formal proof proceeds by induction on $\sum_{i=1}^{j} n'_i$.)

—*Where does each warrior end up?* All but the champion end up in the room just beyond their skill level ($n_i + 1$ for warrior i). The champion ends up in the room next to the room occupied by the vice champion(s) ($n_v + 2$).

—*Why is the prophecy Japanese?* This we do not know, but we certainly are delighted to finally understand where people derived their inspiration from when they came up with the Japanese auction protocol.

REFERENCES

CONITZER, V. 2009. Editor's puzzle: Identifying the champion. *ACM SIGecom Exchanges 8,* 1.

FAGIN, R., HALPERN, J. Y., MOSES, Y., AND VARDI, M. Y. 1995. *Reasoning About Knowledge.* MIT Press, Cambridge, Mass.

GAMOW, G. AND STERN, M. 1958. *Puzzle Math.* Viking Press, New York.

GARDNER, M. 1984. *Puzzles From Other Worlds.* Viking Press, New York.

LITTLEWOOD, J. E. 1953. *A Mathematician's Miscellany.* Methuen and Co., London.

MOSES, Y., DOLEV, D., AND HALPERN, J. Y. 1986. Cheating husbands and other stories: A case study of knowledge, action, and communication. *Distributed Computing 1,* 3, 167–176.