Analysis of dependence metrics for queueing processes
Es-Saghouani, A.

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Chapter 4

Correlation structure of Lévy-driven queues

In Chapters 2 and 3 we have studied a single queue fed by Gaussian sources. The present chapter and Chapter 5 deal with another class of input processes, namely, Lévy processes. In this chapter we specialize to the class of spectrally-positive Lévy processes, for which we prove a number of structural properties of the correlation function $r(t)$ of the stationary workload process. Furthermore, we study the asymptotics of $r(t)$ for $t \to \infty$ for light and heavy-tailed spectrally-positive Lévy inputs.

4.1 Introduction

Consider a queueing system, and, more particularly, its workload process $\{Q(t) : t \geq 0\}$. Where one usually focuses on the characterization of the (transient or steady-state) workload, another interesting problem relates to the identification of the correlation function $r(t)$, cf. (1.10). For several queueing systems this correlation function has been explicitly computed; Morse [91], for instance, analyzes the number of customers in the M/M/1 queue. Often explicit formulae were hard to obtain, but the analysis simplified greatly when looking at the Laplace transform

$$
\rho(\vartheta) := \int_0^{\infty} r(t) e^{-\vartheta t} dt.
$$

Beneš [18] managed to compute $\rho(\cdot)$ for the workload in the M/G/1 queue; relying on the concept of complete monotonicity, Ott [95] elegantly proved that, in this case, $r(\cdot)$ is positive, decreasing and convex. We further mention the survey by Reynolds [102], and interesting results by Abate and Whitt [2].

The primary aim of this chapter is to extend the results mentioned above to the class of single-server queues fed by Lévy processes. Notice that the M/G/1 queue is contained in this class (then the Lévy process under consideration is a compound Poisson process with drift). One could expect that such an extension is possible, as the classical Pollaczek-Khintchine result for the M/G/1 queue, carries over to queues with general Lévy input, see Zolotarev [113] for an early reference; we refer also to Bingham [22], and references therein, and the book by Kyprianou [74], for an extensive account of fluctuation theory for Lévy processes. The only condition one
usually needs to impose in order to obtain explicit results, is that no negative jumps are allowed.

In more detail, the setting we consider is the following. We define a ‘net input process’ \( \{X(t) : t \geq 0\} \), which is assumed to be a Lévy process with no negative jumps. Then the workload process \( \{Q(t) : t \geq 0\} \) is defined as the reflected process of \( \{X(t) : t \geq 0\} \) at 0. Because of the lack of explicit formulae for the probability distributions of the processes considered, we will work most of the time with their Laplace transforms; in our analysis the Laplace exponent \( \varphi(\cdot) \) of the process \( \{X(t) : t \geq 0\} \), as well as its inverse \( \psi(\cdot) \), play an important role.

We first obtain an explicit expression, in terms of \( \varphi(\cdot) \) and \( \psi(\cdot) \), of the transform \( \rho(\cdot) \) of the correlation function. Using the concept of complete monotonicity, we use this transform to establish a series of structural properties of the correlation function, viz. we prove that \( r(\cdot) \) is positive, decreasing, and convex. These results indeed generalize those obtained by Ott [95] and Abate and Whitt [2] for the M/G/1 queue. We then consider the asymptotic behavior of \( r(t) \) for \( t \) large. For light-tailed Lévy input these asymptotics are essentially exponential; for the M/G/1 case they resemble those of the busy period. For heavy-tailed input we can use results for regularly varying functions, e.g. Karamata’s Tauberian theorem, to obtain the asymptotics of \( r(\cdot) \).

The remainder of this chapter is organized as follows. In Section 4.2 we obtain the Laplace transform of the correlation function, where in Section 4.3 its structural properties are studied. The cases of light-tailed and heavy-tailed input are treated in Sections 4.4 and 4.5, respectively. Concluding remarks are found in Section 4.6.

### 4.2 Laplace transform of the correlation function

In this section we find an expression for the transform \( \rho(\cdot) \) of the correlation function. We start this section, however, with a formal introduction of our queueing system.

**Lévy processes.** Let \( \{X(t) : t \geq 0\} \) be a Lévy process without negative jumps, with drift \( E_X(1) < 0 \). Its Laplace exponent is given by the function \( \varphi(\cdot) : [0, \infty) \to [0, \infty) \), i.e., \( \varphi(\alpha) := \log E e^{-\alpha X(1)} \). It is known that \( \varphi(\cdot) \) is increasing and convex on \( [0, \infty) \), with slope \( \varphi'(0) = -E_X(1) \) in the origin. Therefore the inverse \( \psi(\cdot) \) of \( \varphi(\cdot) \) is well-defined on \( [0, \infty) \). In the sequel we also require that \( X(t) \) is not a subordinator, i.e., a monotone process; thus \( X(1) \) has probability mass on the positive half-line, which implies that \( \lim_{\alpha \to -\infty} \varphi(\alpha) = \infty \).

Important examples of such Lévy processes are the following.

1. **Brownian motion with drift.** We write \( X \in \mathcal{Bm}(-\delta, \sigma^2) \) when \( \varphi(\alpha) = \alpha \delta + \frac{1}{2} \alpha^2 \sigma^2 \).
4.2. Laplace transform of the correlation function

(2) Compound Poisson with drift. Jobs arrive according to a Poisson process of rate \( \lambda \); the jobs \( J_1, J_2, \ldots \) are i.i.d. samples from a distribution with Laplace transform \( \beta(\alpha) := \mathbb{E} e^{-\alpha J} \); the storage system is continuously depleted at a rate \(-1\).

We write \( X \in \text{CP}(\lambda, \beta(\cdot)) \); it can be verified that \( \varphi(\alpha) = \alpha - \lambda + \lambda \beta(\alpha) \).

Reflected Lévy processes; queues. We consider the reflection of \( \{X(t) : t \geq 0\} \) at 0, which we denote by \( \{Q(t) : t \geq 0\} \). It is formally introduced as follows, see for instance [11, Ch. IX]. Define the increasing process \( \{L(t) : t \geq 0\} \) by

\[
L(t) = -\inf_{0 \leq s \leq t} X(s).
\]

Then the reflected process (or: workload process, queueing process) \( \{Q(t) : t \geq 0\} \) is given through

\[
Q(t) := X(t) + \max\{L(t), Q(0)\};
\]

observe that \( Q(t) \geq 0 \) for all \( t \geq 0 \). Then the steady-state distribution of \( Q_e := \lim_{t \to \infty} Q(t) \) is characterized by [113]:

\[
\kappa(\alpha) := \mathbb{E} e^{-\alpha Q_e} = \frac{\alpha \varphi'(0)}{\varphi(\alpha)};
\]

(4.1)

for the special case of CP input this is the celebrated Pollaczek-Khintchine formula. This relation reveals all moments of the steady-state queue \( Q_e \), and in particular its mean and variance:

\[
\mu := \mathbb{E} Q_e = -\frac{d}{d\alpha} \frac{\alpha \varphi'(0)}{\varphi(\alpha)} \bigg|_{\alpha=0} = \frac{\varphi''(0)}{2\varphi'(0)},
\]

(4.2)

and similarly

\[
v := \text{Var} Q_e = \frac{1}{4} \left( \varphi''(0) \right)^2 - \frac{1}{3} \varphi'''(0) \varphi'(0),
\]

(4.3)

which from now on are assumed to be finite.

Correlation structure of the queue. In this chapter we are interested in the correlation structure of the queue process \( \{Q(t) : t \geq 0\} \). Our analysis relies on the following useful relation, see e.g. [11, Section IX.3] and [65]:

\[
\mathbb{E} \left( e^{-\alpha Q(\tau)} \mid Q(0) = q \right) = \frac{\vartheta}{\vartheta - \varphi(\alpha)} \left( e^{-\alpha q} - \alpha \cdot \frac{e^{-\varphi(\vartheta)} q}{\vartheta(\vartheta)} \right),
\]

(4.4)

where \( \tau \) is exponentially distributed with mean \( \vartheta^{-1} \), independently of the Lévy process. (As an aside we mention that (4.4) implies Pollaczek-Khintchine in at least two ways: (a) let \( \vartheta \downarrow 0 \), so that \( \tau \) corresponds with some epoch infinitely far away, and
use elementary calculus (‘L’Hôpital’); (b) find $\mathbb{E}e^{-\alpha Q(t)}$ by deconditioning, use that in stationarity $\mathbb{E}e^{-\alpha Q(t)}$ should coincide with $\mathbb{E}e^{-\alpha Q(0)}$, and then solve $\mathbb{E}e^{-\alpha Q(0)}$.

Formula (4.4) enables us to find explicitly the Laplace transform $\rho(\cdot)$ of

$$r(t) := \text{Corr}(Q(0), Q(t)) = \frac{\mathbb{Cov}(Q(0), Q(t))}{\sqrt{\mathbb{Var}Q(0) \cdot \mathbb{Var}Q(t)}} = \frac{EQ(0)Q(t) - (EQ_t)^2}{\mathbb{Var}Q_t},$$

as we show now. Here it is assumed that the system is in steady-state at time 0, that is, $Q_t$ obeys the ‘generalized Pollaczek-Khintchine’ formula (4.1). First realize that

$$\int_0^\infty \vartheta e^{-\vartheta t} \mathbb{E}\left(e^{-\alpha Q(t)} \mid Q(0) = q\right) dt = -\frac{\vartheta \mu(Q_0)}{\vartheta^2} + q + \frac{e^{-\psi(q)q}}{\psi(q)}.$$

Concentrate on the Laplace transform $\gamma(\vartheta)$ of $R(t)$, Straightforward calculus reveals that

$$\gamma(\vartheta) := \int_0^\infty \mathbb{Cov}(Q(0), Q(t)) e^{-\vartheta t} dt = \int_0^\infty \left(\mathbb{E}Q(0)Q(t) - \mu^2\right) e^{-\vartheta t} dt\]

$$= \int_0^\infty \int_0^\infty q \cdot \mathbb{E}(Q(t) \mid Q(0) = q) \cdot e^{-\vartheta t} dP(Q(0) \leq q) dt - \frac{\mu^2}{\vartheta};$$

it is assumed that the queue is in stationarity at time 0 (and hence it is in stationarity at time $t$ as well). By invoking (4.5) we find that the expression in the previous display equals

$$\int_0^\infty q \cdot \frac{\vartheta' e^{-\vartheta} q}{\vartheta^2} + q + \frac{e^{-\psi(q)q}}{\psi(q)} dP(Q(0) \leq q) - \frac{\mu^2}{\vartheta} = -\frac{\mu^2}{\vartheta} \int_0^\infty \vartheta \mathbb{E}(Q(0) e^{-\psi(q)Q(0)}).$$

From the generalized Pollaczek-Khintchine formula (4.1) we obtain by differentiating

$$\mathbb{E}(Q(0) e^{-\alpha Q(0)}) = \varphi'(0) \left( -\frac{1}{\varphi(\alpha)} + \alpha \frac{\varphi'(\alpha)}{(\varphi(\alpha))^2} \right).$$

Inserting this relation, in addition to (4.2), into (4.6) we obtain the Laplace transform of $\mathbb{Cov}(Q(0), Q(t))$:

$$\gamma(\vartheta) = -\frac{\vartheta''(0)}{2 \vartheta^2} + \frac{\vartheta'(0)}{\vartheta^2} \left( \frac{1}{\psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} \right).$$

This trivially also provides us with the Laplace transform of the correlation function $r(t) = \text{Corr}(Q(0), Q(t))$, as stated in the following theorem. When specializing to CP input, we retrieve Equation (6.2) of Beneš [18].
4.3. Structural properties of the correlation function

Theorem 4.2.1. For any \( \vartheta \geq 0 \), and \( \nu \) as in (4.3),
\[
\rho(\vartheta) := \int_0^\infty r(t) e^{-\vartheta t} dt = \frac{\gamma(\vartheta)}{\nu} = \frac{1}{\vartheta} - \frac{\varphi''(0)}{2\nu\vartheta^2} + \frac{\varphi'(0)}{\nu\vartheta^2} \left[ \frac{1}{\vartheta \psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} \right].
\]
(4.7)

Remark 4.2.2. Using the generalized Pollaczek-Khintchine formula (4.1), it is readily verified that the result in Theorem 4.2.1 can be simplified to
\[
\rho(\vartheta) = \frac{1}{\vartheta} - \frac{1}{\nu} \left( \frac{\varphi''(0)}{2\vartheta^2} + \frac{\kappa'(\psi(\vartheta))}{\vartheta \psi(\vartheta)} \right).
\]

Example 4.2.3. Consider the situation that \( \{X(t) : t \geq 0\} \) corresponds to standard Brownian motion decreased by a linear drift (say of rate 1, so \( X \in \mathcal{Bm}(-1, 1) \)). In other words: the Laplace exponent of the Lévy process is given by \( \varphi(\alpha) = \alpha + \frac{1}{2} \alpha^2 \), and its inverse is \( \psi(\vartheta) = -1 + \sqrt{1 + 2\vartheta} \). Now consider the workload process \( \{Q(t) : t \geq 0\} \) and its correlation function. The above theory yields that the Laplace transform of \( r(\cdot) \) is given by
\[
\rho(\vartheta) = \frac{1}{\vartheta} - \frac{2}{\vartheta^2} + \frac{2}{\vartheta^3} \left( \sqrt{1 + 2\vartheta} - 1 \right).
\]
(4.8)
It turns out to be possible to explicitly invert \( \rho(\cdot) \):
\[
r(t) = 2(1 - 2t - t^2) \left( 1 - \Phi_N(\sqrt{t}) \right) + 2\sqrt{t}(1 + t)\phi_N(\sqrt{t}),
\]
with \( \Phi_N(\cdot) \) (resp. \( \phi_N(\cdot) \)) the standard Normal distribution (resp. density). Equation (4.8) is in agreement with the results in [1] and [80, Section 12.1].

4.3 Structural properties of the correlation function

This section concentrates on the derivation of a number of key structural properties of the correlation function \( r(\cdot) \). More specifically, relying on the concept of completely monotone functions [19, 95], we prove in Theorem 4.3.6 that \( r(\cdot) \) is a positive, decreasing, and convex function. To this end, we first establish a number of auxiliary results; a key result is Proposition 4.3.1.

Proposition 4.3.1. Define \( \xi(\vartheta) \) by
\[
\xi(\vartheta) := \frac{1}{\mu} \left( \frac{1}{\vartheta \psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} \right);
\]
(4.9)
then \( \xi(\vartheta) \) is the Laplace transform of a (non-negative) random variable \( Z \).
Remark 4.3.2. The Laplace transform of the stationary-excess distribution $Z_e$ associated with $Z$ is given by [2]

$$\xi_e(\vartheta) = \frac{\xi(\vartheta) - 1}{\vartheta \xi'(0)} = \frac{\varphi''(0)}{2\vartheta} \left(1 - \xi(\vartheta)\right).$$

(4.10)

Hence, the first moment of $Z$ is $2\nu/\varphi''(0)$.

To prove Proposition 4.3.1, we need a number of lemmas. These are stated and proved now. They extensively use the concept of complete monotonicity [19, 54]. The class $\mathcal{C}$ of completely monotone functions is defined in the Appendix, where also a series of standard properties is given.

Lemma 4.3.3. $\psi'(\vartheta) \in \mathcal{C}$.

Proof. Consider for $x \geq 0$,

$$H(x) := \inf\{t \geq 0 : X(t) = -x\};$$

then $H(x)$ is a Lévy process with Laplace exponent $-\psi(\vartheta)$, see e.g. [106, Theorem 46.3]. More specifically, $H(x)$ is a subordinator. Now apply Lemma 4.A.4.

Lemma 4.3.4. If $f(\alpha) \in \mathcal{C}$, then so does

$$\frac{f(0) - f(\alpha) + \alpha f'(\alpha)}{\alpha^2}.$$

Proof. The result is a consequence of subsequently applying Lemma 4.A.3.(4) and 4.A.3.(5).

Lemma 4.3.5. For $\sigma^2 > 0$ and measure $\Pi_\varphi(\cdot)$ such that

$$\int_{(0,\infty)} \min\{1, x^2\} \Pi_\varphi(dx) < \infty,$$

$$\frac{\alpha \varphi'(\alpha) - \varphi(\alpha)}{\alpha^2} = \frac{1}{2} \sigma^2 + \frac{\alpha}{\alpha^2} \int_{(0,\infty)} (1 - e^{-\alpha x} - \alpha x e^{-\alpha x}) \Pi_\varphi(dx) \in \mathcal{C}.$$  

(4.11)

Proof. The Laplace exponent $\varphi(\alpha)$ can be written as, with $\sigma^2 > 0$ and measure $\Pi_\varphi(\cdot)$ such that

$$\int_{(0,\infty)} \min\{1, x^2\} \Pi_\varphi(dx) < \infty,$$

$$\varphi(\alpha) = \alpha \delta + \frac{1}{2} \alpha^2 \sigma^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1)}) \Pi_\varphi(dx),$$

which immediately yields the equality in (4.11). The claim that this function is in $\mathcal{C}$ follows from the fact that any positive constant is in $\mathcal{C}$, Lemma 4.3.4, and Lemma 4.A.3.(1).
Proof of Proposition 4.3.1. We first decompose
\[ \frac{1}{\vartheta \psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} = \eta_1(\vartheta) \eta_2(\vartheta), \]
with
\[ \eta_1(\vartheta) := \frac{\psi(\vartheta)}{\vartheta}, \quad \eta_2(\vartheta) := \frac{1}{\psi(\vartheta)\psi'(\vartheta)} - \frac{\vartheta}{(\psi(\vartheta))^2}. \]

Because of (4.1), we have that \( \alpha/\phi(\alpha) \in \mathcal{C} \); now applying Lemma 4.A.3.(3), in conjunction with Lemma 4.3.3, we obtain that \( \eta_1(\vartheta) \in \mathcal{C} \).

To prove that also \( \eta_2(\vartheta) \in \mathcal{C} \), we first recall from Lemma 4.3.5 that
\[ \frac{\alpha \phi'(\alpha)}{\alpha^2} - \frac{\phi(\alpha)}{\alpha^3} \in \mathcal{C}. \]
Again applying Lemma 4.A.3.(3), in conjunction with Lemma 4.3.3, it follows that \( \eta_2(\vartheta) \in \mathcal{C} \).

As both \( \eta_1(\vartheta) \) and \( \eta_2(\vartheta) \) are in \( \mathcal{C} \), Lemma 4.A.3.(2) yields that \( \xi(\vartheta) \in \mathcal{C} \). Applying 'L'Hôpital' twice, and using that \( \psi''(0)(\phi'(0))^3 = -\phi''(0) \), it is readily verified that
\[ \xi(0) = \lim_{\vartheta \downarrow 0} \xi(\vartheta) = 1. \]

Now Theorem 4.A.2 yields the stated. \( \square \)

Let \( \rho^{(1)}(\vartheta) \) and \( \rho^{(2)}(\vartheta) \) be the Laplace transforms of, respectively, \( r'(t) \) and \( r''(t) \). Their expressions are given respectively as follows
\[ \rho^{(1)}(\vartheta) := \int_0^\infty r'(t) e^{-\vartheta t} dt = -\frac{\phi''(0)}{2v} \left( 1 - \xi(\vartheta) \right) = -\xi(\vartheta); \tag{4.12} \]
\[ \rho^{(2)}(\vartheta) := \int_0^\infty r''(t) e^{-\vartheta t} dt = \frac{\phi''(0)}{2v} \xi(\vartheta), \tag{4.13} \]
for \( \vartheta \geq 0 \). Here the properties that \( r(0) = 1 \) and
\[ r'(0) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}Q(0)Q(\varepsilon) - \mu^2}{\varepsilon v} = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}Q(0)X(\varepsilon)}{\varepsilon v} = -\frac{\phi''(0)}{2v}, \]
in conjunction with integration by parts, are used.

**Theorem 4.3.6.** \( r(t) \) is positive, decreasing and convex. Moreover, \( r(t) \) can be written as the tail of the stationary-excess distribution function associated with \( Z \). More specifically, \( r(t) = \mathbb{P}(Z_\varepsilon > t) \). Furthermore, if \( Z \) has a finite second moment, then \( r(t) \) is integrable and
\[ \int_0^\infty r(t) dt = \frac{1}{8v} \frac{\varphi^{(4)}(0)}{\varphi'(0)^2} - \frac{5}{12v} \frac{\varphi''(0)\varphi'(0)}{\varphi'(0)^3} + \frac{1}{4v} \frac{\varphi''(0)^3}{\varphi'(0)^3}. \tag{4.14} \]
Convexity follows from the expression for \( \rho^{(2)}(\vartheta) \) in \((4.13)\); it is concluded from Proposition 4.3.1 that \( \rho^{(2)}(\vartheta) \in \mathcal{C} \), thus \( r''(t) \) is non-negative (for \( t \geq 0 \)). The monotonicity follows from the expression for \( \rho^{(1)}(\vartheta) \) in Equation \((4.12)\), by applying Lemma 4.A.3.(4) to \( \rho^{(2)}(\vartheta) \in \mathcal{C} \); we find that \( -\rho^{(1)}(\vartheta) \) is in \( \mathcal{C} \), implying that \( r'(t) \leq 0 \) (for \( t \geq 0 \)). Then it is easily verified that applying Lemma 4.A.3.(4) to \( -\rho^{(1)}(\vartheta) \in \mathcal{C} \), in conjunction with Equation \((4.7)\), implies \( \rho(\vartheta) \in \mathcal{C} \), and hence \( r(t) \geq 0 \) (for \( t \geq 0 \)).

Observe that combining Equations \((4.7)\) and \((4.10)\) yields
\[
\rho(\vartheta) = 1 - \xi(\vartheta) e^{\vartheta}.
\]

It is straightforward to verify that the right-hand side of the previous display is just the Laplace transform of \( P(Z_e > t) \). It is concluded that \( r(t) = P(Z_e > t) \) by the uniqueness of the Laplace transform. Equation \((4.14)\) is found, after considerable calculus (i.e., application of ‘L’Hôpital’ several times, and various series expansions), by evaluating
\[
\int_0^\infty r(t) dt = \rho(0) = \lim_{\alpha \downarrow 0} \rho(\alpha);
\]

it is noted that \( \varphi^{(4)}(0) \) exists if the second moment of \( Z \) is finite.

Remark 4.3.7. For the GI/G/1 queue, Borovkov et al. [27, Theorem 7.3] obtained the following expression for \( \int_0^\infty R(t) dt \):
\[
\int_0^\infty R(t) dt = \frac{1}{2} \left( \frac{a_A}{a_I} \sigma_I \right)^2 + \sigma_A^2 - 2 \frac{a_A}{a_I} \sigma_I \sigma_A c(A, l)
\]
where \( A \) is the area swept under the workload process \( Q(t) \) during the busy period and \( l \) is the length of the busy cycle. Furthermore, \( a_A, \sigma_A^2 \) and \( a_I, \sigma_I^2 \) denote the mean and the variance of \( A \) and \( l \), respectively, whereas \( c(A, l) \) is the correlation between \( A \) and \( l \). It can be checked that this formula coincides with ours in the M/G/1 queue case.

4.4 Correlation asymptotics for light-tailed input

When \( \varphi(\cdot) \) has an analytic continuation for \( \alpha < 0 \), we are in the regime of light tails, as \( a \) fortiori then all moments \((-1)^n \varphi^{(n)}(0)\) of \( X(1) \) exist. When \( \{X(t) : t \geq 0\} \) does not correspond to a decreasing subordinator, we also have that \( \lim_{\alpha \to -\infty} \varphi(\alpha) = \infty \). Bearing in mind the fact that \( \varphi(\cdot) \) has a positive slope at the origin, and that convexity of \( \varphi(\cdot) \) implies continuity, there is a unique minimizer \( \zeta < 0 \) such that \( \varphi(\zeta) < 0 \), \( \varphi'(\zeta) = 0 \) and \( \varphi''(\zeta) > 0 \).
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In this situation, also $\psi(\cdot)$ is well-defined for negative arguments; more precisely: for all $\vartheta \geq \varphi(\zeta)$ the inverse $\psi(\vartheta)$ has a meaningful interpretation. In fact, $\vartheta^* := \varphi(\zeta)$ can be regarded as branching point. We thus see that Theorem 4.2.1 does not only apply for $\vartheta \geq 0$, but also for $\vartheta \in [\vartheta^*, 0)$. Around $\zeta$, we can write $\varphi(\cdot)$ as

$$\varphi(\alpha) = \varphi(\zeta) + \frac{1}{2} (\alpha - \zeta)^2 \varphi''(\zeta) + O((\alpha - \zeta)^3),$$

and hence for $\theta \downarrow \vartheta^*$

$$(\psi(\vartheta) - \zeta) \sim \frac{2}{\varphi''(\zeta)} \sqrt{\vartheta - \varphi(\zeta)} = \sqrt{\vartheta - \vartheta^*}$$

(as before $\sim$' indicates that the ratio of the left-hand side and right-hand side tends to 1). Routine calculations reveal that, for $\theta \downarrow \vartheta^*$, we have that $\rho(\vartheta)$ looks like

$$\rho(\vartheta) = -\frac{1}{v} \left( -\frac{\varphi''(0)}{2(\vartheta^*)^2} + \frac{1}{4\vartheta^*} \left( \frac{\varphi''(0)}{\varphi''(0)} \right)^2 - \frac{1}{3\vartheta^*} \frac{\varphi''(0)}{\varphi''(0)} - \frac{1}{(\vartheta^*)^2} \frac{\varphi'(0)}{\psi(\vartheta)} + \frac{1}{(\vartheta^*)^3} \frac{\varphi'(0)}{\psi'(\vartheta)} \right),$$

or, more precisely,

$$\rho(\vartheta) \sim \frac{\sqrt{2} \varphi'(0)}{\sqrt{\varphi''(\zeta)} v(\vartheta^*)^2} \left( \frac{1}{\xi^2} + \frac{\varphi''(\zeta)}{\vartheta^*} \right) \sqrt{\vartheta - \vartheta^*}.$$

We now relate the behavior of a transform $\int_0^\infty e^{-\vartheta t} f(t) dt$ (around a branching point $\vartheta^* < 0$) to the behavior of the ‘transformed’ function $f(t)$ (for $t$ large). We heuristically obtain the following result, cf. for instance the ‘Heaviside approach’ of [3, Equations (3.21)–(3.23)]; see also [33, pp. 153-154]: Suppose $\varphi(\alpha) < \infty$ for some $\alpha < 0$. Then

$$r(t) \sim \ell \cdot \frac{e^{\vartheta^* t}}{t^{1/2}} \quad \text{as } t \to \infty, \quad (4.16)$$

where

$$\ell := -\frac{\varphi'(0)}{\sqrt{2\pi \varphi''(\zeta) v(\vartheta^*)^2}} \left( \frac{1}{\xi^2} + \frac{\varphi''(\zeta)}{\vartheta^*} \right). \quad (4.17)$$

**Remark 4.4.1.** $\ell$, as given in (4.17), is positive, as is seen as follows. From (4.11) we know that

$$f(\alpha) := 2 \frac{\alpha \varphi'(\alpha) - \varphi(\alpha)}{\alpha^2}.$$
is a Laplace transform, and hence also \(-f'(\alpha)/f'(0)\), so that for all \(\alpha\) holds that \(f'(\alpha) < 0\), or
\[
\alpha^3 \varphi''(\alpha) - 2\alpha^2 \varphi'(\alpha) + 2\alpha \varphi(\alpha) < 0.
\]
Now insert \(\alpha := \zeta < 0\); using \(\varphi'(\zeta) = 0\) and \(\zeta < 0\), we obtain \(\zeta^2 \varphi''(\zeta) + 2\varphi(\zeta) > 0\), which implies
\[
\frac{\varphi''(\zeta)}{\varphi(\zeta)} + \frac{2}{\zeta^2} < 0,
\]
(use \(\varphi(\zeta) < 0\)), and hence also
\[
-\frac{\varphi''(\zeta)}{\vartheta^*} > \frac{2}{\zeta^2} > \frac{1}{\zeta^2},
\]
thus implying \(\ell > 0\).

Example 4.4.2. It can be checked that for Brownian motion with drift, i.e., \(X \in \mathbb{B}m(-1, 1)\) as in the setting of Example 4.2.3,
\[
r(t) \sim 8 \sqrt{\frac{2}{\pi}} \frac{e^{-t/2}}{t^{1/4}};
\]
this could be found directly from (4.8) as well, cf. again [1] and [80, Section 12.1].

Example 4.4.3. For the compound Poisson model with exponential jobs (i.e., \(M/M/1\) queue), it can be checked that
\[
\psi(\vartheta) = \frac{1}{2} \left( \lambda - \mu + \vartheta + \sqrt{(\lambda - \mu + \vartheta)^2 + 4\vartheta \mu} \right),
\]
so that the branching point is \(\vartheta^* = -\left(\sqrt{\mu} - \sqrt{\lambda}\right)^2\). Also, \(\zeta = -\mu + \sqrt{\lambda}\mu\). Equation (4.16) now yields an explicit expression for the correlation asymptotics:
\[
r(t) \sim \frac{1}{2\rho \sqrt{\pi}} \left( \frac{1 - \sqrt{\rho}}{1 + \sqrt{\rho}} \right)^3 \exp\left(\frac{-1}{\rho} \sqrt{\rho} \sqrt{\mu t}\right) \frac{\sqrt{\mu t}}{(\sqrt{\mu t})^{3/2}} \text{ as } t \to \infty.
\]

Remark 4.4.4. For compound Poisson input, that is, \(X \in \mathbb{C}P(\lambda, \beta(\cdot))\), the tail asymptotics of the correlation function are proportional to those of the busy period, at least in this light-tailed regime (where light-tailedness here means that we should require that \(\beta(\alpha) < \infty\) for some \(\alpha < 0\)). This can be seen as follows.

First recall that the Laplace exponent is \(\varphi(\alpha) = \alpha - \lambda + \lambda \beta(\alpha)\). With \(\pi(\cdot)\) the Laplace transform of the busy period, it is known that it satisfies \(\pi(\vartheta) = \beta(\vartheta + \lambda - \lambda \pi(\vartheta))\). Therefore
\[
0 = \beta(\vartheta + \lambda - \lambda \pi(\vartheta)) - \pi(\vartheta) = \frac{1}{\lambda} \varphi(\vartheta + \lambda - \lambda \pi(\vartheta)) - \frac{\vartheta}{\lambda},
\]
4.5. Correlation asymptotics for heavy-tailed input

and hence $\varphi(\vartheta + \lambda - \lambda \pi(\vartheta)) = \vartheta$. Apply $\psi(\cdot)$ to both sides, and we obtain

$$
\pi(\vartheta) = \frac{\lambda + \vartheta}{\lambda} - \frac{1}{\lambda} \psi(\vartheta).
$$

Considering the tail asymptotics of the busy period, first observe that $\pi(\cdot)$ also has a branching point at $\vartheta^* < 0$ (i.e., it has the same branching point as $\rho(\vartheta)$), such that, for $\vartheta \downarrow \vartheta^*$,

$$
\pi(\vartheta) \sim \frac{\lambda - \vartheta}{\lambda} - \frac{1}{\lambda} \cdot \left( \zeta + \sqrt{\frac{2}{\varphi''(\zeta)} \cdot \sqrt{\vartheta - \vartheta^*}} \right).
$$

Applying ‘Heaviside’ now yields, with $P$ the busy period,

$$
\frac{d}{dt} P(P \leq t) \sim \frac{1}{\lambda} \sqrt{\frac{2}{\varphi''(\zeta)}} \cdot \frac{1}{2\sqrt{\pi} \cdot t \sqrt{t}} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{\beta''(\zeta) \cdot \lambda t \sqrt{t}}}.
$$

in line with the results of Cox and Smith [33, Section 5.6]. These asymptotics are indeed proportional to those of Equation (4.16). Similarly, applying the relation

$$
\mathbb{E} e^{-P} = 1 - \vartheta \int_0^\infty P(P > t) dt,
$$

we obtain

$$
P(P > t) \sim -\sqrt{\frac{2}{\varphi''(\zeta)}} \cdot \frac{1}{2\sqrt{\pi}} \cdot \frac{e^{\vartheta^* t}}{\vartheta^* \lambda t \sqrt{t}} = -\frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{\beta''(\zeta) \cdot \vartheta^* \lambda t \sqrt{t}}}.
$$

4.5 Correlation asymptotics for heavy-tailed input

Where the previous section focused on light-tailed Lévy input, we now consider the heavy-tailed case. We extensively use the concept of slowly (and regularly) varying functions. Proposition 4.5.4 is the main result of this section; in Corollary 4.5.5 it is applied to the situation of a queue with CP input with regularly varying jobs.

The following class of functions plays a crucial role in our analysis.

**Definition 4.5.1.** We say that $f(x) \in \mathcal{R}_\delta(n, \sigma)$, with $n \in \mathbb{N}$, $\sigma \in \mathbb{R}$, and $\delta \in (n, n + 1)$, for $x \downarrow 0$ if

$$
f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i + \sigma x^\delta l(1/x) \quad (x \downarrow 0),
$$

for a slowly varying function $l(\cdot)$ (i.e., $l(x)/l(tx) \to 1$ for $x \to \infty$, for any $t > 0$).
Lemma 4.5.2. Suppose $\varphi'(\alpha) \in \mathcal{R}_{\delta-1}(n-1, \sigma)$. Then

$$
\varphi(\alpha) \in \mathcal{R}_{\delta}(n, \sigma/\delta);
$$

$$
\psi(\vartheta) \in \mathcal{R}_{\delta}(n, \tau), \quad \text{with} \quad \tau := -\frac{\sigma}{\delta(\varphi'(0))^{\delta+1}};
$$

$$
\psi'(\vartheta) \in \mathcal{R}_{\delta-1}(n-1, 1, \tau \delta),
$$

for $\alpha \downarrow 0$, resp. $\vartheta \downarrow 0$.

Proof. The first statement is an immediate consequence of Karamata’s theorem; the second statement follows from $\psi(\varphi(\alpha)) = \alpha$; the third statement follows in an elementary way by using $\psi'(\vartheta) = 1/\varphi'(\psi(\vartheta))$.

The following lemma presents the behavior of $\xi_e(\vartheta)$ as $\vartheta \downarrow 0$. We need this type of results, as Karamata’s Tauberian theorem then enables us to translate the behavior of transforms around 0 into the behavior of $r(t)$ for $t$ large.

Lemma 4.5.3. If $\varphi'(\alpha) \in \mathcal{R}_{\delta-1}(n-1, \sigma)$, with $n \in \{3, 4, \ldots\}$ and $\delta \in (n, n+1)$, then

$$
\xi_e(\vartheta) = 1 - \vartheta \rho(\vartheta) \in \mathcal{R}_{\delta-3}(n-3, \omega), \quad \text{with}
$$

$$
\omega := \frac{\vartheta(\delta - 1)}{\vartheta(\varphi'(0))^{\delta-2}}.
$$

Proof. Recall Equations (4.9) and (4.10). The crucial step is to verify that

$$
\frac{\vartheta}{\psi(\vartheta)} \in \mathcal{R}_{\delta-1}(n-1, -\tau(\varphi'(0))^2) \quad \text{and} \quad \frac{1}{\psi'(\vartheta)} \in \mathcal{R}_{\delta-1}(n-1, -\tau \delta(\varphi'(0))^{\delta-2});
$$

use Lemma 4.5.2. Verification of the claim is now straightforward (though tedious).

The Tauberian theorem in Bingham, Goldie, and Teugels [24, Theorem 8.1.6] now yields the following result; see also [23].

Proposition 4.5.4. If $\varphi'(\alpha) \in \mathcal{R}_{\delta-1}(n-1, \sigma)$, with $n \in \{3, 4, \ldots\}$ and $\delta \in (n, n+1)$, then

$$
r(t) \sim \frac{\omega \cdot (-1)^{n+1}}{\Gamma(4-\delta)} t^{3-\delta} l(t) \quad \text{as} \quad t \to \infty.
$$

Proof. First recall that $r(t) = \mathbb{P}(Z_e > t)$, and that $Z_e$ has transform $\xi_e(\vartheta)$. Lemma 4.5.3 and Theorem 8.1.6 of [24] yield the stated.
Remark 4.5.5. Interestingly, we can now also find a criterion for long-range dependence, cf. the remarks in the introduction of [95].

Suppose \( \varphi'(\alpha) \in R_{n-1}(n-1, \sigma) \), with \( n \in \{3, 4, \ldots \} \) and \( \delta \in (n, n+1) \). Then the queueing process is long-range dependent if \( n = 3 \), as in this case \( \int_0^\infty r(t)dt = \infty \). Consider for instance the case that \( X \in \mathbb{C}P(\lambda, \beta(\cdot)) \), with \( \mathbb{P}(B > t) \sim t^{-\nu} \), for \( \nu \in (3, 4) \). Then the first three moments of \( B \) exist, and hence also the first two moments of the steady-state queue length, as well as the covariance \( R(t) \). The tail of \( J \), however, is so heavy that \( R(t) \) decays roughly as \( t^{3-\nu} \), giving rise to a long-range dependent queueing process.

Likewise, it follows that the queueing process is short-range dependent if \( n \in \{4, 5, \ldots \} \), for instance when considering \( \mathbb{C}P \) input with \( \mathbb{P}(B > t) \sim t^{-\nu} \), for \( \nu \in (4, \infty) \).

4.6 Concluding remarks

In this chapter we studied the correlation function of the workload process of a single-queue fed by a Lévy process (that is, a Lévy process reflected at 0). Relying on the concept of complete monotonicity we have been able to derive a set of structural properties of the correlation function, viz. that it is a positive, decreasing, and convex function. Importantly, we have shown how to represent the correlation function \( r(\cdot) \) as the complementary distribution function of a specific random variable. This representation, as well as an explicit characterization of the Laplace transform of \( r(\cdot) \), enabled the analysis of the asymptotic behavior of \( r(t) \) for \( t \) large; both the light-tailed and heavy-tailed cases were studied.

An alternative way to conclude that the correlation function is positive, decreasing, and convex, may be the following. The Laplace exponent of any Lévy process can be approximated arbitrarily closely by that of an appropriately chosen \( \mathbb{C}P \) process, see e.g. [54, Theorem XVII.1]. As the claim has been proved for \( \mathbb{C}P \) input [95], a limit argument may lead to an alternative proof of Theorem 4.3.6. Exploration of this approach is a subject for further research.

Restricting ourselves to the case of \( \mathbb{C}P \) input, one could say that Section 4 covers the case in which the jobs have a finite moment generating function in a neighborhood of the origin: \( \beta(\alpha) < \infty \) for some \( \alpha < 0 \), and hence all moments are finite. On the other hand, Section 5 addresses the situation in which just a finite set of moments are finite. In between, however, there is a third class of distributions: those for which all moments are finite, but without an analytic continuation for \( \alpha < 0 \) (that is \( \beta(\alpha) = \infty \) for all \( \alpha < 0 \)). Examples of distributions in this class are the Weibull and LogNormal distributions. A subject for further research would be the analysis of the correlation asymptotics for this class of distributions.
As we lack, in most cases, an explicit formula for \( r(t) \), one may attempt to estimate it through simulation. This is particularly challenging, as \( r(t) \) can be extremely small for large \( t \), and is the difference of two (potentially large) numbers. A way to circumvent this problem is to use importance sampling [12, Section V.1], that is, sampling under an alternative measure and correcting the simulation output by likelihood ratios (that keep track of the relative likelihood of the realization under the actual measure, relative to the alternative measure). The resemblance with the busy period asymptotics suggests that, for light-tailed input, the (exponentially-twisted) change of measure proposed in [87] may work well; it is noted that the analysis of [87] indicates that the twisting of the work present at time 0 should be handled with care. An other option could be to rely on the representation of the correlation function \( r(\cdot) \) as the complementary distribution function of the random variable \( Z_e \), see Theorem 4.3.6.

A potential application area of our results is the following. Suppose that no measurements of the queue’s input process can be made, and hence estimation of the probabilistic law of the input process has to be performed in an alternative manner. One approach could be to measure the queue’s workload (for instance periodically), and to infer the input characteristics from the resulting measurements. Insight into the correlation between subsequent measurements, as obtained in the present chapter, may be useful when devising such a procedure. Work along these lines for queues with Gaussian input was done in [84] (in a somewhat more experimental context), and for M/G/\( \infty \) systems in [25] (building on the results presented in [102]); see also [60].

**Appendix**

**4.A Complete monotonicity**

The concept of complete monotonicity is summarized in the following definition.

**Definition 4.A.1.** A function \( f(\alpha) \) on \([0, \infty)\) is completely monotone if for all \( n \in \mathbb{N}, \alpha \geq 0,\) 
\[
(-1)^n \frac{d^n}{d\alpha^n} f(\alpha) \geq 0.
\]

We write \( f(\alpha) \in \mathcal{C} \).

The following deep and powerful result is due to Bernstein [19]. It says that there is equivalence between \( f(\alpha) \) being completely monotone, and the possibility of
writing \( f(\alpha) \) as a Laplace transform. For more background on completely monotone functions, see [54, pp. 439-442].

**Theorem 4.A.2.** A function \( f(\alpha) \) on \([0, \infty)\) is the Laplace transform of a non-negative random variable if and only if (i) \( f(\alpha) \in \mathcal{C} \), and (ii) \( f(0) = 1 \).

The concept of complete monotonicity is easy to work with, as one can use a set of practical properties.

**Lemma 4.A.3.** The following properties apply:

1. \( \mathcal{C} \) is closed under addition: if \( f(\alpha) \in \mathcal{C} \) and \( g(\alpha) \in \mathcal{C} \), then \( f(\alpha) + g(\alpha) \in \mathcal{C} \). This extends to: if \( f_x(\alpha) \in \mathcal{C} \) for \( x \in \Xi \), then \( \int_{x \in \Xi} f_x(\alpha) \mu(dx) \in \mathcal{C} \) for any measure \( \mu(\cdot) \).

2. \( \mathcal{C} \) is closed under multiplication: if \( f(\alpha) \in \mathcal{C} \) and \( g(\alpha) \in \mathcal{C} \), then so does \( f(\alpha)g(\alpha) \in \mathcal{C} \).

3. Properties of composite \( \mathcal{C} \) functions: if \( f(\alpha) \in \mathcal{C} \) and \( g(\alpha) \geq 0 \) with \( g'(\alpha) \in \mathcal{C} \), then \( f(g(\alpha)) \in \mathcal{C} \).

4. Let \( U(\alpha) \) non-decreasing on \([0, \infty)\), and \( U(0) = 0 \), \( u := \lim_{\alpha \to \infty} U(\alpha) < \infty \), and
   \[
   f(\alpha) := \int_{[0, \infty)} e^{-\alpha x} dU(x);
   \]
   clearly \( f(\alpha) \in \mathcal{C} \) and \( u = f(0) \). Then also
   \[
   g(\alpha) := \frac{f(0) - f(\alpha)}{\alpha} \in \mathcal{C}.
   \]

5. \( \mathcal{C} \) closed under differentiation: if \( f(\alpha) \in \mathcal{C} \), then \( -f'(\alpha) \in \mathcal{C} \).

**Proof.** (1) is trivial from the definition. (2) follows from [54, Criterion 1], and (3) from [54, Criterion 2]. Property (4) can be found in for instance [95, Equation (4.2)]. (5) is trivial.

**Lemma 4.A.4.** Let \( \{Y(t) : t \geq 0\} \) be an increasing subordinator Lévy process, with Laplace exponent \( \xi(\alpha) \), then \( -\xi'(\alpha) \in \mathcal{C} \).

**Proof.** According to Bertoin [20, Ch. III, Equation (3)], we can write
   \[
   \xi(\alpha) = -d\alpha + \int_{(0, \infty)} (e^{-\alpha x} - 1) \Pi_\xi(dx),
   \]
with \( d \geq 0 \), and measure \( \Pi_\xi(\cdot) \) such that \( \int_{(0, \infty)} \min\{1, x^2\} \Pi_\xi(dx) < \infty \). This implies that
   \[
   -\xi'(\alpha) = d + \int_{(0, \infty)} xe^{-\alpha x} \Pi_\xi(dx),
   \]
so that \( -\xi'(\alpha) \in \mathcal{C} \); use Lemma 4.A.3.(1).