Analysis of dependence metrics for queueing processes
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Chapter 7

Exact multivariate workload asymptotics

In this chapter we consider a discrete-time queue fed by a general process assuming only stationarity of the increments. The main contribution of this chapter concerns the derivation of the exact asymptotics of the joint probability $P(Q(0) > p, Q(T) > q)$ under the many sources scaling.

7.1 Introduction

As already established in the preceding chapters, a way to measure the degree of dependence between the workloads $Q(0)$ at time 0 and $Q(T)$ at time $T$, is to consider the measure $r(T|p, q)$ defined in (1.13) for (given) positive $p$ and $q$. While for various input models, considerable insight has been gained into the steady-state distribution $Q_e$, less is known about $P(Q(0) > p, Q(T) > q)$, the joint probability of the workload exceeding the threshold $p$ at time 0 and exceeding the threshold $q$ at time $T$. This joint probability gives an insight in the way the events $\{Q(0) > p\}$ and $\{Q(T) > q\}$ are dependent. Clearly, the analysis of $P(Q(0) > p, Q(T) > q)$ is of practical and theoretical interest.

The objective of this chapter is to analyze the joint probability given above, in case a large number $n$ of i.i.d. sources feed into the queue, with the queueing resources (buffer and service speed) scaled by $n$ as well. In this many-sources framework, considered already in Chapter 2, we find exact asymptotics (as $n \to \infty$) of the probability of interest. Our approach relies on ideas developed by Likhanov and Mazumdar [79] to obtain the exact asymptotics of the steady-state distribution of the workload under the many-sources scaling, in conjunction with results by Chaganty and Sethuraman [30] for the large deviations of sample means of multivariate random variables. As in [79], we consider a slotted-time model, i.e., a discrete-time model.

The remainder of this chapter is organized as follows. In Section 7.2 we introduce the model, and determine the tail probabilities of bivariate sample means. These are used in Section 7.3 to determine the exact asymptotics of the probability of interest. We also include a number of remarks, and indicate how to extend the results to the setting where one would consider more than two time epochs.


7.2 Model, objective, and preliminaries

Traffic model. In this chapter we consider a queueing resource fed by \( n \) i.i.d. sources. Let \( A_i(s,t) \), with \( s, t \in \mathbb{Z} \) such that \( s < t \), be the amount of traffic generated by the \( i \)-th source in timeslots \( s + 1, \ldots, t \). The \( A_i(s,t), i = 1, \ldots, n, \) are distributed as the (generic) stochastic process \( A(s,t) \). It is assumed that this process has stationary increments: \( A(s,t) \) has the same distribution as \( A(s+u,t+u) \) for all \( u \in \mathbb{Z} \).

We define the cumulant function of \( A(s,t) \) by \( \Lambda_t := \log \mathbb{E} \exp ( \vartheta A(s,t) ) \), which we assume to exist for some positive \( \vartheta \); the corresponding Legendre-Fenchel transform is given by \( I(x|s) := \sup_{\vartheta} (\vartheta x - \Lambda_t(s,t)) \). We also need the two-dimensional counterparts of these objects:

\[
\Lambda^T(\vartheta, \eta|s,t) := \log \mathbb{E} \exp ( \vartheta A(s-0) + \eta A(T-t,T) ),
\]

\[
I^T(x,y|s,t) := \sup_{\vartheta, \eta} (\vartheta x + \eta y - \Lambda^T(\vartheta, \eta|s,t) ).
\]

Let \( \vartheta(x|s) \) and \( (\vartheta(x, y|s,t), \eta(x, y|s,t)) \) be the optimizing arguments in the definitions of \( I(x|s) \) and \( I^T(x, y|s,t) \); they may be found from the obvious first order conditions.

We finally define, suppressing the arguments of \( \Lambda \) and \( \Lambda^T \),

\[
\sigma^2(x|s) := \frac{d^2\Lambda}{d\vartheta^2} \bigg|_{\vartheta = \vartheta(x|s)},
\]

\[
\sigma^2_\vartheta(x, y|s,t) := \det \left( \begin{array}{cc} \frac{\partial^2\Lambda^T}{\partial\vartheta^2} & \frac{\partial^2\Lambda^T}{\partial\vartheta\partial\eta} \\ \frac{\partial^2\Lambda^T}{\partial\vartheta\partial\eta} & \frac{\partial^2\Lambda^T}{\partial\eta^2} \end{array} \right) \bigg|_{\vartheta = \vartheta(x, y|s,t), \eta = \eta(x, y|s,t)}.
\]

Queueing model, objective, reduction property. The \( n \) sources fed into a buffered resource that is drained at a constant rate \( nc \). To ensure stability, it is assumed that \( \mathbb{E} A(0,1) < c \). Let \( Q^n(t) \) denote the workload in the system at time \( t \in \mathbb{Z} \). It is well-known that the stationary queue obeys

\[
Q^n(t) = \sup_{s \in \mathbb{N}} \left( \sum_{i=1}^{n} A_i(t-s,t) - ncs \right), \quad (7.1)
\]

where \( A(t,t) \) is to be understood as 0. The goal of this chapter is to find the exact asymptotics, as \( n \to \infty \), of the probability \( \pi^n_0(p,q) := \mathbb{P}(Q^n(0) \geq np, Q^n(T) \geq nq) \), for given positive \( p \) and \( q \).

We will now rewrite \( \pi^n_0(p,q) \) as follows. With \( p_s := p + cs \) and \( q_t := q + ct \), let \( E^n_T(p, q|s, t) \) denote the event

\[
E^n_T(p, q|s, t) := \left\{ \sum_{i=1}^{n} A_i(-s,0) > np_s, \sum_{i=1}^{n} A_i(T-t,T) > nq_t \right\}.
\]
Then
\[ \pi_n^T(p, q) = \mathbb{P} \left( \exists (s, t) \in \mathbb{N}^2 : E_n^T(p, q|s, t) \right) = \mathbb{P} \left( \exists (s, t) \in \mathcal{E} : E_n^T(p, q|s, t) \right), \]  
(7.2)
where \( \mathcal{E} \) is the set of elements \((s, t)\) such that \( s \in \mathbb{N} \) and \( t \in \{0, \ldots, T\} \cup \{T + s\} \).

The first equality in (7.2) is directly from (7.1), whereas the second follows from a reduction property, established in [38], see also Chapter 3: we can reduce \( \mathbb{N}^2 \) in (7.2) to \( \mathcal{E} \) (this is essentially due to the fact that the busy period in which \( T \) is contained can start (i) either after time 0, (ii) or at the same time as the start of the busy period in which 0 is contained).

The logarithmic asymptotics of \( \mathbb{P}(Q_n^e > np) \), with \( Q_n^e \) denoting the stationary workload under the many-sources scaling, were found before [28]:
\[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Q_n^e > np) = - \inf_{s \in \mathbb{N}} I(p_s|s); \]  
(7.3)
for exact asymptotics, see [79]. We denote by \( \bar{s} \) an optimizing argument in the right hand side of (7.3), which is not necessarily unique; also, let \( \bar{t} \) be a \( t \) for which \( I(q_t|t) \) is minimal.

**Exact two-dimensional sample-mean asymptotics.** Now consider
\[ \pi_n^T(p, q|s, t) := \mathbb{P}(E_n^T(p, q|s, t)). \]
From (7.2) we find
\[ \pi_n^T(p, q|s, t) \leq \pi_n^T(p, q) \leq \sum_{(s,t) \in \mathcal{E}} \pi_n^T(p, q|s, t). \]  
(7.4)
As our goal is to derive the exact asymptotics of \( \pi_n^T(p, q) \), we first consider in this section those of \( \pi_n^T(p, q|s, t) \).

As a first remark, we note that \( \check{I}(x, y|s, t) \geq \max \{I(x|s), I(y|t)\} \), as follows from
\[ \sup_{(\vartheta, \eta) \in \mathbb{R}^2} (\vartheta x + \eta y - \check{\Lambda}_T(\vartheta, \eta|s, t)) \geq \sup_{(\vartheta, \eta) \in \mathbb{R} \times \{0\}} (\vartheta x + \eta y - \check{\Lambda}_T(\vartheta, \eta|s, t)) = I(p_s|s). \]
The bivariate version of ‘Cramér’ states
\[ \lim_{n \to \infty} \frac{1}{n} \log \pi_n^T(p, q|s, t) = - \inf_{x \geq p_s, \, y \geq q_t} \check{I}(x, y|s, t). \]  
(7.5)
Realizing (A) that \( \mathbb{E}A\{-s, 0\} < p_s \) and \( \mathbb{E}A\{T - t, T\} < q_t \), and (B) that the contour lines of \( \check{I}(\cdot, \cdot|s, t) \) are convex, there are three possibilities for the optimizer \( p^*, q^* \) in the right-hand side of (7.5): (i) \( p^* = p_s \) and \( q^* > q_t \); (ii) \( p^* > p_s \) and \( q^* = q_t \); (iii) \( p^* = p_s \) and \( q^* = q_t \). We refer to Figure 7.1 for a pictorial illustration; the left panel depicts Case (i), the right panel Case (iii).
Let us first consider Case (i). Write \( \pi_n^T(p, q|s, t) \) as
\[
\pi_n^T(p, q|s, t) = \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} A_i\{-s, 0\} > p_s \right) - \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} A_i\{-s, 0\} > p_s, \frac{1}{n} \sum_{i=1}^{n} A_i\{T - t, T\} \leq q_t \right). \tag{7.6}
\]

Using the Bahadur-Rao estimate [15], we have for the first probability in (7.6) that, as \( n \to \infty \),
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} A_i\{-s, 0\} > p_s \right) \sim e^{-nI(p_s|s)} \frac{e^{-nI(p_s|s)}}{\vartheta(p_s|s) \sqrt{2\pi n\sigma^2(p_s|s)}}.
\]
where \( g(n) \sim f(n) \) as \( n \to \infty \) denotes \( f(n)/g(n) \to 1 \). On the other hand, the second probability in (7.6) decays faster than the first probability, and is therefore asymptotically negligible: bearing in mind the left panel of Figure 7.1, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} A_i\{-s, 0\} > p_s, \frac{1}{n} \sum_{i=1}^{n} A_i\{T - t, T\} \leq q_t \right) = -\hat{I}(p_s, q_t|s, t) < -I(p_s|s).
\]

As Case (ii) can be dealt with similarly, we have arrived at the following result.

**Proposition 7.2.1.** If \( p^* = p_s \) and \( q^* > q_t \), then as \( n \to \infty \)
\[
\pi_n^T(p, q|s, t) \sim e^{-nI(p_s|s)} \frac{e^{-nI(p_s|s)}}{\vartheta(p_s|s) \sqrt{2\pi n\sigma^2(p_s|s)}}.
\]

If \( p^* > p_s \) and \( q^* = q_t \), then as \( n \to \infty \)
\[
\pi_n^T(p, q|s, t) \sim e^{-nI(q_t|t)} \frac{e^{-nI(q_t|t)}}{\vartheta(q_t|t) \sqrt{2\pi n\sigma^2(q_t|t)}}.
\]
Now we consider Case (iii). The decay rate of the probability of interest can alternatively be found through the Lagrangian $\hat{I}_T(x, y|s, t) - \lambda(x - p_s) - \mu(y - q_t)$. As we know that the optimum is attained at $p^* = p_s$ and $q^* = q_t$, we know that at the stationary point $\lambda^* > 0$ and $\mu^* > 0$ (complementary slackness). It is standard from convex analysis that
$$
\frac{\partial}{\partial x} \hat{I}_T(x, y|s, t) = \vartheta(x, y|s, t), \quad \frac{\partial}{\partial y} \hat{I}_T(x, y|s, t) = \eta(x, y|s, t),
$$
but at the same time these partial derivatives are, at $(p^*, q^*)$, equal to $\lambda^*$ and $\mu^*$, respectively, and hence they are strictly positive. We conclude that Condition (3.4) of Chaganty and Sethuraman [30] is fulfilled, so that we can use their Theorem 3.4.

We have the following result.

**Proposition 7.2.2.** If $p^* = p_s$ and $q^* = q_t$, then
$$
\pi_n T(p, q|s, t) \sim e^{-n \hat{I}_T(p_s, q_t|s, t)} \vartheta(p_s, q_t|s, t) \cdot \eta(p_s, q_t|s, t) \cdot 2\pi n \sqrt{\hat{\sigma}_T^2(p_s, q_t|s, t)}.
$$

### 7.3 Exact workload asymptotics

In this section we use the estimates for exact bivariate sample-mean large deviations, as derived in the previous section, to determine the exact asymptotics of $\pi_n T(p, q)$. As will become clear, the main idea is that these asymptotics are, under mild assumptions, fully determined by the contribution of busy periods starting at a single time epoch $(s^*, t^*)$, cf. [79].

Let $\kappa^{(i)}(s, t)$ be 1 if Case (i) applies for $s$ and $t$ (that is $p^* = p_s$ and $q^* > q_t$) and 0 otherwise; $\kappa^{(ii)}(s, t)$ and $\kappa^{(iii)}(s, t)$ are defined likewise. Then we introduce, in self-evident notation,
$$
K_T(s, t) := I(p_s|s) \cdot \kappa^{(i)}(s, t) + I(q_t|t) \cdot \kappa^{(ii)}(s, t) + \hat{I}_T(p_s, q_t|s, t) \cdot \kappa^{(iii)}(s, t),
$$
so that Props. 7.2.1-7.2.2 entail that $n^{-1} \log \pi_n T(p, q|s, t) \to -K_T(s, t)$ as $n \to \infty$.

We now impose the following two assumptions, in line with those needed to find the exact asymptotics of the stationary workload $Q^*_n$, see [79].

**Assumption 7.3.1.** $(s^*, t^*) := \arg \min_{(s, t) \in E} K_T(s, t)$ is unique.

**Assumption 7.3.2.** $\liminf_{s \to \infty} I(p_s|s) / \log s > 0$.

It will turn out that Assumption 7.3.1 entails that the event of overflow over level $np$ at time 0 and over level $nq$ at time $T$ is essentially exclusively caused by the event $E^*_T(p^*, q^*|s^*, t^*)$. Assumption 7.3.2 will be needed to make sure that contributions
of $E_T^n(p_s, q_t | s, t)$ for large $s$ and $t$ do not contribute significantly; below we will comment on what happens if the uniqueness assumption is not fulfilled. We are now ready to prove our main result.

**Theorem 7.3.3.** As $n \to \infty$,

$$\frac{\pi_T^n(p, q)}{\pi_T^n(p, q|s^*, t^*)} \to 1.$$  

**Proof.** The lower bound is evident due to (7.4), so let us focus on the upper bound. First observe that by applying (7.4), for any finite $M$,

$$\pi_T^n(p, q) \leq \sum_{(s,t) \in E} \pi_T^n(p, q|s, t) \leq \sum_{s=0}^{M} \left( \sum_{t=0}^{T+s} \pi_n(p, q|s, t) \right) \sum_{s=M+1}^{\infty} \left( \sum_{t=0}^{T+s} \pi_n(p, q|s, t) \right). \quad (7.7)$$

For $(s, t) \neq (s^*, t^*)$, because of Assumption 7.3.1,

$$\lim_{n \to \infty} \frac{1}{n} \log \pi_T^n(p, q|s, t) = -K_T(s, t) < -K_T(s^*, t^*) = \lim_{n \to \infty} \frac{1}{n} \log \pi_T^n(p, q|s^*, t^*),$$

so that for these $(s, t)$ it holds that $\pi_T^n(p, q|s, t) = o(\pi_T^n(p, q|s^*, t^*))$. Choosing $M$ large enough such that $s^* \in \{0, \ldots, M\}$ and $t^* \in \{0, \ldots, T + s^*\}$, it follows that

$$\sum_{s=0}^{M} \left( \sum_{t=0}^{T+s} \pi_T^n(p, q|s, t) \right) \sim \pi_T^n(p, q|s^*, t^*).$$

Now consider the second sum in the right-hand side of (7.7). Trivially,

$$\sum_{s=M+1}^{\infty} \left( \sum_{t=0}^{T+s} \pi_T^n(p, q|s, t) \right) \leq \sum_{s=M+1}^{\infty} \left( \sum_{t=0}^{T+s} p \left( \frac{1}{n} \sum_{i=1}^{n} A_i(-s, 0) > p_s \right) \right) \leq \sum_{s=M+1}^{\infty} (T + s + 1)P \left( \frac{1}{n} \sum_{i=1}^{n} A_i(-s, 0) > p_s \right). \quad (7.8)$$

Now applying the Chernoff bound in conjunction with the fact that there is an $\alpha > 0$ such that $I(p_s | s) > \alpha \log s$ for $s$ sufficiently large (due to Assumption 7.3.2), we have

$$P \left( \frac{1}{n} \sum_{i=1}^{n} A_i(-s, 0) > p_s \right) \leq e^{-nI(p_s | s)} \leq s^{-n\alpha}.$$  

Consequently, (7.8) is further bounded by, taking $n > 2/\alpha$,

$$\sum_{s=M+1}^{\infty} (T + s + 1)s^{-n\alpha} \leq \int_{M}^{\infty} (T + s + 1)s^{-n\alpha} ds = (T + 1)\frac{M^{-n\alpha+1}}{n\alpha - 1} + \frac{M^{-n\alpha+2}}{n\alpha - 2}.$$
which is \( o(\pi^n_T(p,q|s^*,t^*)) \) as \( n \to \infty \), by picking \( M \) sufficiently large (that is, \( M \) should be chosen such that \( \alpha \log M > K_T(s^*,t^*) \)). □

The following corollary is an immediate consequence of Props. 7.2.1-7.2.2 and Theorem 7.3.3.

**Corollary 7.3.4.** If \( \kappa^{(i)}(s^*,t^*) = 1 \), then

\[
\pi^n_T(p,q) \sim \frac{e^{-nI(p_\ast|s^*)}}{\theta(p_\ast|s^*) \sqrt{2 \pi n \sigma^2(p_\ast|s^*)}}.
\]

If \( \kappa^{(ii)}(s^*,t^*) = 1 \), then

\[
\pi^n_T(p,q) \sim \frac{e^{-nI(q_\ast|t^*)}}{\theta(q_\ast|t^*) \sqrt{2 \pi n \sigma^2(q_\ast|t^*)}}.
\]

If \( \kappa^{(iii)}(s^*,t^*) = 1 \), then

\[
\pi^n_T(p,q) \sim \frac{e^{-nI(p_\ast|q_\ast|s^*,t^*)}}{\theta(p_\ast,q_\ast|s^*,t^*) \cdot \eta(p_\ast,q_\ast|s^*,t^*) \cdot 2 \pi n \sqrt{\sigma^2_T(p_\ast,q_\ast|s^*,t^*)}}.
\]

**Remark 7.3.5.** Non-unique optimizers \( s^* \) and \( t^* \). Suppose \( K(s,t) \) is minimal at two \((s,t)\)-pairs, viz. \((s_1^*,t_1^*)\) and \((s_2^*,t_2^*)\). If \( \kappa^{(i)}(s_1^*,t_1^*) = 1 \) and \( \kappa^{(ii)}(s_2^*,t_2^*) = 1 \), then we are essentially in case (i) of the above corollary (as the 1/n factor is negligible compared to the 1/\( \sqrt{n} \) factor). The same line of reasoning applies if \( \kappa^{(i)}(s_1^*,t_1^*) = 1 \) and \( \kappa^{(ii)}(s_2^*,t_2^*) = 1 \).

The other cases are harder to deal with. If \( \kappa^{(iii)}(s_1^*,t_1^*) = 1 \) and \( \kappa^{(iii)}(s_2^*,t_2^*) = 1 \), then the asymptotics look like \( \gamma \exp(-nK(s_1^*,t_1^*)/n) \), but now the constant \( \gamma > 0 \) cannot be determined explicitly. A similar property applies if

\[
\sum_{k=1}^2 \kappa^{(i)}(s_k^*,t_k^*) + \kappa^{(ii)}(s_k^*,t_k^*) = 2;
\]

then the asymptotics look like \( \delta \exp(-nK(s_1^*,t_1^*)/\sqrt{n}) \), with a constant \( \delta > 0 \) that cannot be determined explicitly. ♣

**Remark 7.3.6.** The optimizing \( s^* \) and \( t^* \) can be interpreted as follows [38]. Given overflow over level \( np \) at time \( 0 \) and over level \( nq \) at time \( T \), the busy period in which \( 0 \) is contained started with overwhelming probability at time \( -s^* \), whereas the busy period in which \( T \) is contained started at time \( T - t^* \). This means that if \( t^* + s^* \), epochs \( 0 \) and \( T \) lie in the same busy period. It is expected that for large \( T \) this is typically not the case: then it is more likely that \( 0 \) and \( T \) are contained in separate busy periods. We now determine a \( T^- \) such that for \( T > T^- \) we have that \( t^* \in \{0, \ldots T\} \).
We first observe that, due to Assumption 7.3.2, for some \( \alpha > 0 \),
\[
\inf_{s \in \mathbb{N}} \inf_{x \geq p_s, y \geq q_{T+s}} \hat{I}(x, y|s, T + s) \geq \inf_{s \in \mathbb{N}} \inf_{y \geq q_{T+s}} I(y|T + s) \geq \inf_{s \in \mathbb{N}} \alpha \log(T + s) \geq \alpha \log T.
\]

We impose the condition of positive input correlation:
\[
\hat{I}(x, y|s, t) \leq I(x|s) + I(y|t),
\]
with \( E^- \) denoting \( \mathbb{N} \times \{1, \ldots, T\} \),
\[
\inf_{(s, t) \in E^-} \inf_{x \geq p_s, y \geq q_{t}} \hat{I}(x, y|s, t) \leq \inf_{s \in \mathbb{N}} \inf_{x \geq p_s} I(x|s) + \inf_{t \in \{1, \ldots, T\}} \inf_{y \geq q_{t}} I(y|t),
\]
which equals \( I(p_{\bar{s}}|\bar{s}) + I(q_{\bar{t}}|\bar{t}) \) for \( T \geq \bar{t} \) (recall that \( \bar{s} \) and \( \bar{t} \) were defined in Section 2); note that \( I(p_{\bar{s}}|\bar{s}) + I(q_{\bar{t}}|\bar{t}) \) would be the decay rate if \( Q_n(0) \) and \( Q_n(T) \) would be independent. We conclude that if
\[
T > T^- := \max \left\{ \exp \left( \frac{I(p_{\bar{s}}|\bar{s}) + I(q_{\bar{t}}|\bar{t})}{\alpha} \right), \bar{t} \right\},
\]
we can restrict ourselves to \( (s, t) \in E^- \): the decay rate of interest equals
\[
\lim_{n \to \infty} \frac{1}{n} \log \pi_n(p, q) = - \inf_{(s, t) \in E^-} \inf_{x \geq p_s, y \geq q_t} \hat{I}(x, y|s, t).
\]

Intuitively, for \( T \) larger than \( T^- \) the time epochs 0 and \( T \) lie in separate busy periods with overwhelming probability. ♠

Remark 7.3.7. The bivariate results presented above can be easily extended to dimensions \( d \in \{3, 4, \ldots\} \). Then the probability \( \mathbb{P}(Q_n(T_i) \geq np_i, i = 1, \ldots, d) \) is studied, for time epochs \( 0 = T_1 \leq \ldots \leq T_d \) and positive numbers \( p_1, \ldots, p_d \). Again, under mild assumptions, the corresponding asymptotics are fully determined by the contribution of busy periods starting at a single time epoch \( (s^*_1, \ldots, s^*_d) \). The result from [30] can be used again, to obtain that the asymptotics look like \( \gamma_d n^{-d^*/2} \exp(-nI_d) \), for some positive \( \gamma_d \) and \( I_d \); where \( d^* \in \{1, \ldots, d\} \) denotes the number of constraints that are tightly met in the \( d \)-dimensional counterpart of (7.5) evaluated at the point \( (s^*_1, \ldots, s^*_d) \). ♠