Topological strings and quantum curves
Hollands, L.

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Chapter 3

I-brane Perspective on Vafa-Witten Theory and WZW Models

In the last decades enormous progress has been made in analyzing 4-dimensional supersymmetric gauge theories. Partition functions and correlation functions of some theories have been computed, spectra of BPS operators have been discovered and many other structures have been revealed. Most fascinating to us is that many exact results can be related to two-dimensional geometries.

Since 4-dimensional supersymmetric gauge theories appear in several contexts in string theory, much of this progress is strongly influenced by string theory. Often, string theory tools can be used to compute important quantities in supersymmetric gauge theories. Moreover, in many cases string theory provides a key understanding of new results. For example, when auxiliary structures in the gauge theory can be realized geometrically in string theory and when symmetries in the gauge theory can be understood as stringy dualities.

In this chapter we study a remarkable correspondence between 4-dimensional gauge theories and 2-dimensional conformal field theories. This correspondence connects a “twisted” version of supersymmetric Yang-Mills theory to a so-called Wess-Zumino-Witten model. In particular, generating functions of $SU(N)$ gauge instantons on the 4-manifold $\mathbb{C}^2/\mathbb{Z}_k$ are related to characters of the affine Kac-Moody algebra $\widehat{su}(k)$ at level $N$. This connection was originally discovered by Nakajima [39], and further analyzed by Vafa and Witten [40]. The goal of this chapter is to make it more transparent. Once again, we find that string theory offers the right perspective.
We have strived to make this chapter self-contained by starting in Section 3.1 with a short introduction in gauge and string theory. We review how supersymmetric gauge theories show up as low energy world-volume theories on D-branes and how they naturally get twisted. Twisting emphasizes the role of topological contributions to the theory. Furthermore, we introduce fundamental string dualities as T-duality and S-duality.

In Section 3.2 we introduce Vafa-Witten theory as an example of a twisted 4-dimensional gauge theory, and study it on non-compact 4-manifolds that are asymptotically Euclidean. In Section 3.3 we show that Vafa-Witten theory on such a 4-manifold is embedded in string theory as a D4-D6 brane intersection over a torus $T^2$. We refer to the intersecting brane wrapping $T^2$ as the I-brane. Since the open 4-6 strings introduce chiral fermions on the I-brane, we find a duality between Vafa-Witten theory and a CFT of free fermions on $T^2$.

In Section 3.4 we show that the full I-brane partition function is simply given by a fermionic character, and reduces to the Nakajima-Vafa-Witten results after taking a decoupling limit. The I-brane thus elucidates the Nakajima-Vafa-Witten correspondence from a string theoretic perspective. Moreover, we gain more insights in level-rank duality and the McKay correspondence from this stringy point of view.

3.1 Instantons and branes

A four-dimensional gauge theory with gauge group $G$ on a Euclidean 4-manifold $M$ is mathematically formulated in terms of a $G$-bundle $E \to M$. A gauge field $A$ is part of a local connection $D = d + A$ of this bundle, whose curvature is the electro-magnetic field strength

$$F = dA + A \wedge A.$$ 

If we denote the electro-magnetic gauge coupling by $e$ and call $\ast$ the Hodge star operator in four dimensions, the Yang-Mills path integral is

$$Z = \int_{A/G} DA \exp \left( -\frac{1}{e^2} \int_M d^4x \text{Tr} F \wedge *F \right).$$

This path integral, over the moduli space of connections $A$ modulo gauge invariance, defines quantum corrections to the classical Yang-Mills equation $D * F = 0$. When the gauge group is abelian, $G = U(1)$, the equation of motion plus Bianchi identity combine into the familiar Maxwell equations

$$d * F = 0, \quad dF = 0.$$
Topological terms

Topologically non-trivial configurations of the gauge field are measured by characteristic classes. If $G$ is connected and simply-connected the gauge bundle $E$ is characterized topologically by the instanton charge

$$ch_2(F) = \text{Tr} \left[ \frac{F \wedge F}{8\pi^2} \right] \in H^4(M, \mathbb{Z}).$$

(3.1)

Instanton configuration are included in the Yang-Mills formalism by adding a topological term to the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{e^2} F \wedge *F + \frac{i\theta}{8\pi^2} F \wedge F$$

(3.2)

Note that this doesn’t change the equations of motion. The path integral is invariant under $\theta \rightarrow \theta + 2\pi$, and the parameter $\theta$ is therefore called the $\theta$-angle. The total Yang-Mills Lagrangian can be rewritten as

$$\mathcal{L} = \frac{i\tau}{4\pi} F_+ \wedge F_+ + \frac{i\bar{\tau}}{4\pi} F_- \wedge F_-,$$

(3.3)

where $F_\pm = \frac{1}{2} (F \pm *F)$ are the (anti-)selfdual field strengths while

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

(3.4)

is the complexified gauge coupling constant. When $G$ is not simply-connected magnetic fluxes on 2-cycles in $M$ are detected by the first Chern class

$$c_1(F) = \text{Tr} \left[ \frac{F}{2\pi} \right] \in H^2(M, \mathbb{Z}).$$

(3.5)

Electro-magnetic duality

The Maxwell equations are clearly invariant under the transformation $F \leftrightarrow *F$ that exchanges the electric and the magnetic field. To see that this is even a symmetry at the quantum level, we introduce a Lagrange multiplier field $A_D$ in the $U(1)$ Yang-Mills path integral that explicitly imposes $dF = 0$:

$$\int DADA_D \exp \int_M \left( \frac{i\tau}{4\pi} F_+ \wedge F_+ + \frac{i\bar{\tau}}{4\pi} F_- \wedge F_- + \frac{1}{2\pi} F \wedge *dA_D \right).$$

Integrating out $A$ yields the dual path integral

$$\int DA_D \exp \int_M \left( \frac{i}{4\pi \tau} F^D_+ \wedge F^D_+ + \frac{i}{4\pi \bar{\tau}} F^D_- \wedge F^D_- \right).$$
So electric-magnetic duality is a strong-weak coupling duality, that sends the complexified gauge coupling $\tau \mapsto -1/\tau$. Moreover, this argument suggests an important role for the modular group $Sl(2,\mathbb{Z})$. This group acts on $\tau$ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z}).$$

and is generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Hence, $S$ is the generator of electro-magnetic duality (later also called S-duality) and $T$ the generator of shifts in the $\theta$-angle. The gauge coupling $\tau$ is thus part of the fundamental domain of $Sl(2,\mathbb{Z})$ in the upper-half plane, as shown in Fig. 3.1.

**Figure 3.1:** The fundamental domain of the modular group $Sl(2,\mathbb{Z})$ in the upper half plane.

Montonen and Olive [41] where pioneers in conjecturing that electro-magnetic duality is an exact non-abelian symmetry, that exchanges the opposite roles of electric and magnetic particles in 4-dimensional gauge theories. This involves replacing the gauge group $G$ by the dual group $\hat{G}$ (whose weight lattice is dual to that of $G$). The first important tests of S-duality have been performed in supersymmetric gauge theories.

For $U(1)$ theories the partition function $Z^{U(1)}$ can be explicitly computed [42, 43]. The classical contribution to the partition function is given by integral fluxes $p \in H^2(M,\mathbb{Z})$, as in equation (3.5), to the Langrangian (3.3). Decomposing the flux $p$ into a self-dual and anti-selfdual contribution yields the generalized theta-function

$$\theta_{\Gamma}(q, \bar{q}) = \sum_{(p_+, p_-) \in \Gamma} q^{\frac{1}{2}p^2_+} \bar{q}^{\frac{1}{2}p^2_-}$$  \hspace{1cm} (3.6)
with \( q = \exp(2\pi i \tau) \), whereas \( \Gamma = H^2(M, \mathbb{Z}) \) is the intersection lattice of \( M \) and \( p^2 = \int_M p \wedge p \). The total \( U(1) \) partition function is found by adding quantum corrections to the above result, which are captured by some determinants [42]. Instead of transforming as a modular invariant, \( Z^{U(1)} \) transforms as a modular form under \( SL(2, \mathbb{Z}) \)-transformation of \( \tau \)

\[
Z^{U(1)} \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^u/(c\overline{\tau} + d)^v/2 Z^{U(1)}(\tau), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}).
\]

We will come back on this in Section 3.2.

### 3.1.1 Supersymmetry

In *supersymmetric* theories quantum corrections are much better under control, so that much more can be learned about non-perturbative properties of the theory. We will soon discuss such elegant results, but let us first introduce supersymmetric gauge theories.

The field content of the simplest supersymmetric gauge theories just consists of a bosonic gauge field \( A \) and a fermionic gaugino field \( \lambda \). Supersymmetry relates the gauge field \( A \) to its superpartner \( \chi \). In any supersymmetric theory the number of physical bosonic degrees of freedom must be the same as the number of physical fermionic degrees of freedom. This constraints supersymmetric Yang-Mills theories to dimension \( d \leq 10 \).

The Lagrangian of a minimal supersymmetric gauge theory is

\[
\mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2} \bar{\chi} \Gamma_{\mu} D_{\mu} \chi,
\]

and supersymmetry variations of the fields \( A \) and \( \chi \) are generated by a spinor \( \epsilon \)

\[
\delta A_{\mu} = \frac{i}{2} \epsilon \Gamma_{\mu} \chi, \quad \delta \chi = \frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon.
\] (3.7)

The number of supersymmetries equals the number of components of \( \epsilon \).

Dimensionally reducing the above minimal \( \mathcal{N} = 1 \) susy gauge theories to lower-dimensional space-times yield \( \mathcal{N} = 2, 4 \) and possibly \( \mathcal{N} = 8 \) susy gauge theories. Their supersymmetry variations are determined by an extended supersymmetry algebra. In four dimensions this is a unique extension of the Poincaré algebra generated by the supercharges \( Q^A_{\alpha} \) and \( Q_{A\dot{\alpha}} \), with \( A \in \{1, \ldots, \mathcal{N}\} \) and \( \alpha, \dot{\alpha} \in \{1, 2\} \) are indices in the 4-dimensional spin group \( su(2)_L \times su(2)_R \). Non-
vanishing anti-commutation relations are given by
\[
\{ Q^A_\alpha, \overline{Q}^\beta_{B\dot{\beta}} \} = 2 (\sigma^\mu)_{\alpha\dot{\beta}} P^\mu \delta^A_B \\
\{ Q^A_\alpha, Q^B_\beta \} = \epsilon_{\alpha\beta} Z^{AB} \\
\{ \overline{Q}_{A\dot{\alpha}}, \overline{Q}^\beta_{B\dot{\beta}} \} = \epsilon_{A\dot{\alpha}} \dot{Z}^{\dagger AB}
\]
where $Z^{AB}$ and its Hermitean conjugate are the central charges. The automorphism group of this algebra, that acts on the supercharges, is known as the R-symmetry group.

**BPS states**

A special role in extended supersymmetric theories is played by supersymmetric BPS states \[44\]. They are annihilated by a some of the supersymmetry generators, e.g. quarter BPS states satisfy
\[
Q^A_\alpha |\text{BPS}\rangle = 0,
\]
for $1/4 \mathcal{N}$ indices $A \in \{1, \ldots, \mathcal{N}\}$. BPS states saturate the bound $M^2 \leq |Z|^2$ and form “small” representations of the above supersymmetry algebra. This implies that supersymmetry protects them against quantum corrections: a small deformation won’t just change the dimension of the representation.

**Twisting**

Supersymmetric Yang-Mills requires a covariantly constant spinor $\epsilon$ in the rigid supersymmetry variations (3.7). Since these are impossible to find on a generic manifold $M$, the concept of *twisting* has been invented. Twisting makes use of the fact that supersymmetric gauge theories are invariant under a non-trivial internal symmetry, the R-symmetry group. By choosing a homomorphism from the space-time symmetry group into this internal global symmetry group, the spinor representations change and often contain a representation that transforms as a scalar under the new Lorentz group.

Such an odd scalar $Q_T$ can be argued to obey $Q_T^2 = 0$. It is a topological supercharge that turns the theory into a cohomological quantum field theory. Observables $\mathcal{O}$ can be identified with the cohomology generated by $Q_T$, and correlation functions are independent of continuous deformations of the metric
\[
\frac{\partial}{\partial g_{\mu\nu}} \langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = 0
\]
This yields techniques to study the dynamics of these theories non-perturbatively.
For these topological theories it is sometimes possible to compute the partition function and other correlators. Witten initiated twisting in the context of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills \cite{Witten1988}. He showed that correlators in the so-called Donaldson-Witten twist compute the famous Donaldson invariants.

A general theme in cohomological field theories is localization. Unlike in general physical theories, in these topological theories the saddle point approximation is actually exact. The path integral only receives contributions from fixed point locus \( \mathcal{M} \) of the scalar supercharge \( \mathcal{Q} \). Since the kinetic part of the action (that contains all metric-dependent terms) is \( \mathcal{Q} \)-exact, the only non-trivial contribution to the path integral is given by topological terms:

\[
Z_{\text{cohTFT}} = \int D\mathcal{X} \exp \left( -\frac{1}{\ell^2} S_{\text{kin}}(\mathcal{X}) + S_{\text{top}}(\mathcal{X}) \right) \rightarrow \int_{\mathcal{M}} D\mathcal{X} \exp (S_{\text{top}}(\mathcal{X})).
\]

Here \( \mathcal{X} \) represents a general field content. An elegant example in this respect is 2-dimensional gauge theory \cite{Witten1989}. Extensive reviews of localization are \cite{Vafa1994, Gukov2003}. We will encounter localization on quite a few occasions, starting with Vafa-Witten theory in Section 3.2.

### 3.1.2 Extended objects

Whereas Yang-Mills theory is formulated in terms of a single gauge potential \( A \), string theory is equipped with a whole set of higher-form gauge fields. Instead of coupling to electro-magnetic particles they couple to extended objects, such as D-branes. This is analogous to the coupling of a particle of electric charge \( q \) to the Maxwell gauge field \( A \)

\[
q \int_{\mathcal{W}} A = q \int_{\mathcal{W}} A_\mu \frac{\partial x^\mu}{\partial t} dt,
\]

where \( \mathcal{W} \) is the worldline of the particle. Notice that we need to pull-back the space-time gauge field \( A \) in the first term before integrating it over the worldline. We often don’t write down the pull-back explicitly to simplify notation. D-branes and other extended objects appear all over this thesis. Let us therefore give a very brief account of the properties that are relevant for us.

**Couplings and branes in type II**

Gauge potentials in type II theory either belong to the so-called RR or the NS-NS sector. The RR potentials couple to D-branes, whereas the NS-NS potential couples to the fundamental string (which is often denoted by F1) and the NS5-brane. Let us discuss these sectors in a little more detail.

The only NS-NS gauge field is the 2-form \( B \). The \( B \)-field plays a crucial role in Chapter 5. Aside from the \( B \)-field the NS-NS sector contains the dilaton field \( \phi \).
and the space-time metric $g_{mn}$. Together these NS-NS fields combine into the sigma model action

$$S_{\sigma\text{-model}} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} g_{mn} dx^m \wedge \ast dx^n + iB_{mn} dx^m \wedge dx^n + \alpha' \phi R.$$  \hspace{1cm} (3.9)

This action describes a string that wraps the Riemann surface $\Sigma$ and is embedded in a space-time with coordinates $x^m$. In particular, it follows that the $B$-field couples to a (fundamental) string $F1$.

Remember that the 1-form $dx^m$ refers to the pull-back $\partial_{\alpha} x^m d\sigma^\alpha$ to the world-sheet $\Sigma$ with coordinates $\sigma^\alpha$. Furthermore, the symbol $\ast$ stands for the 2-dimensional worldsheet Hodge star operator and $R$ is the worldsheet curvature 2-form.

This formula requires some more explanation though. The symbol $\sqrt{\alpha'} = l_s$ sets the string length, since $\alpha'$ is inversely proportional to its tension. Since the Ricci scalar of a Riemann surface equals its Euler number, the last term in the action contributes $2g_s - 2$ powers of $g_s = e^\phi$ to a stringy $g$-loop diagram; $g_s$ is therefore called the string coupling constant.

The extended object to which the B-field couples magnetically is called the NS5-brane. It can wrap any 6-dimensional geometry in the full 10-dimensional string background, but its presence will deform the transverse geometry. In the transverse directions the dilaton field $\phi$ is non-constant, and there is a flux $H = dB$ of the $B$-field through the boundary of the transverse 4-dimensional space. The tension of a NS5-brane is proportional to $1/g_s^2$ so that it is a very heavy object when $g_s \to 0$. Moreover, unlike for D-branes open strings cannot end on it. This makes it quite a mysterious object.

For the RR-sector it makes a difference whether we are in type IIA or in type IIB theory: type IIA contains all odd-degree RR forms and type IIB the even ones. In particular, the gauge potentials $C_1$ and $C_4$ are known as the graviphoton fields for type IIA and type IIB, respectively, and the RR potential $C_0$ is the axion field. Any RR potential $C_{p+1}$ couples electrically to a $Dp$-brane. This is a $p$-dimensional extended object that sweeps out a $(p+1)$-dimensional world-volume $\Sigma_{p+1}$. Type IIA thus contains $Dp$-branes with $p$ even, whereas in type IIB $p$ is odd.

The electric $Dp$-brane coupling to $C_{p+1}$ introduces the term

$$T_p \int_{\Sigma_{p+1}} C_{p+1}$$  \hspace{1cm} (3.10)

in the 10-dimensional string theory action, where $1/T_p = (2\pi)^p \sqrt{\alpha'}^{p+1} g_s$ is the inverse tension of the $Dp$-brane. Magnetically, the RR-potential $C_{p+1}$ couples to
3.1. Instantons and branes

a D$(6 - p)$-brane that wraps a $(7 - p)$-dimensional submanifold $\Sigma_{7-p}$.

**Calibrated cycles**

D-branes are solitonic states as their tension $T_p$ is proportional to $1/g_s$. To be stable against decay the brane needs to wrap a submanifold that preserves some supersymmetries. Geometrically, such configurations are defined by a *calibration* \cite{49}. A calibration form is a closed form $\Phi$ such that $\Phi \leq \text{vol}$ at any point of the background. A submanifold $\Sigma$ that is calibrated satisfies

$$\int_{\Sigma} \Phi = \int_{\Sigma} \text{vol},$$

and minimizes the volume in its holomogy class. On a Kähler manifold a calibration is given by the Kähler form $t$, and the calibrated submanifolds are complex submanifolds. On a Calabi-Yau threefold the holomorphic threeform $\Omega$ provides a calibration, whose calibrated submanifolds are special Lagrangians. Calibrated submanifolds support covariantly constant spinors, and therefore preserve some supersymmetry. D-branes wrapping them are supersymmetric BPS states.

**Worldvolume theory**

D-branes have a perturbative description in terms of open strings that end on them. The massless modes of these open strings recombine in a Yang-Mills gauge field $A$. When the D-brane worldvolume is flat the corresponding field theory on the $p$-dimensional brane is a reduction of $\mathcal{N} = 1$ susy Yang-Mills from 10 dimensions to $p + 1$. The $9 - p$ scalar fields in this theory correspond to the transverse D-brane excitations. When $N$ D-branes coincide the wordvolume theory is a $U(N)$ supersymmetric Yang-Mills theory.

For more general calibrated submanifolds the low energy gauge theory is a twisted topological gauge theory \cite{50}, which we introduced in Section 3.1.1. Which particular twist is realized, can be argued by determining the normal bundle to the submanifold. Sections of the normal bundle fix the transverse bosonic excitations of the gauge theory, and should correspond to the bosonic field content of the twisted theory.

**I-branes and bound states**

Branes can intersect each others such that they preserve some amount of supersymmetry. This is called an *I-brane* configuration. In such a set-up there are more degrees of freedom than the ones (we described above) that reside on the individual branes. These extra degrees of freedom are given by the modes of open strings that stretch between the branes. In stringy constructions of the standard model on a set of branes they often provide the chiral fermions.
Chiral fermions are intimately connected with quantum anomalies, and brane intersections likewise. To cure all possible anomalies in an I-brane system, a topological Chern-Simons term has to be added to the string action

\[ S_{CS} = T_p \int_{\Sigma_{p+1}} \text{Tr} \exp \left( \frac{F}{2\pi} \right) \wedge \sum_i C_i \wedge \sqrt{\hat{A}(R)}. \]  

(3.11)

This term is derived through a so-called anomaly inflow analysis [51, 52]. The last piece contains the $A$-roof genus for the 10-dimensional curvature 2-form $R$ pulled back to $\Sigma_{p+1}$. It may be expanded as

\[ \hat{A}(R) = 1 - \frac{p_1(R)}{24} + \frac{7p_1(R)^2 - 4p_2(R)}{5760} + \ldots, \]

where $p_k(R)$ is a Pontryagin class. For example, the Chern-Simons term (3.11) includes a factor

\[ T_p \int_{\Sigma_{p+1}} \text{Tr} \left( \frac{F}{2\pi} \right) \wedge C_{p-1} \]

when a gauge field $F$ on the worldvolume $\Sigma_{p+1}$ is turned on. It describes an induced D$(p-2)$-brane wrapping the Poincare dual of $[F/2\pi]$ in $\Sigma_{p+1}$.

Vice versa, a bound state of a D$(p-2)$-brane on a D$p$-brane may be interpreted as turning on a field strength $F$ on the D$p$-brane. Analogously, instantons (3.1) in a 4-dimensional gauge theory, say of rank zero and second Chern class $n$, have an interpretation in type IIA theory bound states of $n$ D0-branes on a D4-brane. More generally, topological excitations in the worldvolume theory of a brane are often caused by other extended objects that end on it [53].

### 3.1.3 String dualities

The different appearances of string theory, type I, II, heterotic and M-theory, are connected through a zoo of dualities. Let us briefly introduce T-duality and S-duality in type II. There are many more dualities, some of which we will meet on our way.

**T-duality**

T-duality originates in the worldsheet description of type II theory in terms of open and closed strings. T-duality on an $S^1$ in the background interchanges the Dirichlet and Neumann boundary conditions of the open strings on that circle, and thereby maps branes that wrap this $S^1$ into branes that don’t wrap it (and vice versa). It thus interchanges type IIA and type IIB theory.

Similar to electro-magnetic duality (see Section 3.1), T-duality follows from a path integral argument [54]. The sigma model action for a fundamental string
is based on the term
\[- \frac{1}{2\pi\alpha'} \int_{\Sigma} g_{mn} dx^m \wedge * dx^n,\]
in equation (3.9) when we forget the $B$-field for simplicity. Let us suppose that the metric is diagonal in the coordinate $x$ that parametrizes the $T$-duality circle $S^1$. Then we can add a Lagrange multiplier field $dy$ to the relevant part of the action
\[
\int DxDy \exp \int \left( -\frac{1}{2\pi\alpha'} dx \wedge * dx + \frac{i}{\pi} dx \wedge * dy \right). 
\]
On the one hand the Lagrange multiplier field $dy$ forces $d(dx) = 0$, which locally says that $dx$ is exact. On the other hand integrating out $dx$ yields
\[
\int Dy \exp \int \left( -\frac{\alpha'}{2\pi} dy \wedge * dy \right) 
\]
So $T$-duality exchanges
\[\alpha' \leftrightarrow \frac{1}{\alpha'}\]
and is therefore a strong-weak coupling on the worldsheet. More precisely, one should also take into account the $B$-field coupling (3.9), which is related to non-diagonal terms is the space-time metric. This leads to the well-known Buscher rules [55].

Since the differential $dx$ may be identified with a component of the gauge field $A$ on the brane, and $dy$ with a normal 1-form to the brane, the reduction of supersymmetric Yang-Mills from ten to $10 - d$ dimensions can be understood as applying $T$-duality $d$ times.

**S-duality**

Since $\mathcal{N} = 4$ supersymmetric Yang-Mills is realized as the low-energy effective theory on a D3-brane wrapping $\mathbb{R}^4$. Since electro-magnetic duality in this theory is an exact symmetry, it should have a string theoretic realization in type IIB theory as well. Indeed it does, and this symmetry is known as S-duality. In type IIB theory S-duality is a (space-time) strong-weak coupling duality that maps $g_s \leftrightarrow 1/g_s$. Analogous to Yang-Mills theory the complete symmetry group is $SL(2, \mathbb{Z})$, where the complex coupling constant $\tau$ (3.4) is realized as
\[\tau = C_0 + ie^{-\phi}.\]
Since the ratio of the tensions of the fundamental string $F_1$ and the D1-brane is equal to $g_s$, S-duality exchanges these objects as well as the $B$-field and the
$C_2$-field they couple to. Likewise, it exchanges the NS5-brane with the D5-brane.

**M-theory**

Type IIA is not invariant under S-duality. Instead, in the strong coupling limit another dimension of size $g_s l_s$ opens up. This eleven dimensional theory is called M-theory. The field content of M-theory contains a 3-form gauge field $C_3$ with 4-form flux $G_4 = dC_3$ that couples to an M2-brane. Its magnetic dual is an M5-brane. Other extended objects include the KK-modes and Taub-NUT space. We will introduce them in more detail later. For now, let us just note that a reduction over the M-theory circle consistently reproduces all the fields and objects of type IIA theory. (An extensive review can for instance be found in [56].)

### 3.2 Vafa-Witten twist on ALE spaces

The maximal amount of supersymmetry in 4-dimensional gauge theories is $\mathcal{N} = 4$ supersymmetry. This gauge theory preserves so many supercharges that it has a few very special properties. Its beta function is argued to vanish non-perturbatively, making the theory exactly finite and conformally invariant [57]. It is also, not unrelated, the only 4-dimensional gauge theory where electromagnetic and $Sl(2, \mathbb{Z})$ duality are conjectured to hold at all energy scales.

In this section we apply the techniques of the previous section to study a twisted version of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. It is called the Vafa-Witten twist. We study Vafa-Witten theory on ALE (asymptotically locally Euclidean) spaces, which are defined as hyper-Kähler resolutions of the singularity $\mathbb{C}^2/\Gamma$, where $\Gamma$ is a finite subgroup of $SU(2)$. ALE spaces are intimately connected to ADE Lie algebras.

The Vafa-Witten twist is an example of a topological gauge theory. It localizes on anti-selfdual instantons that are defined by the vanishing of the selfdual component $F_+$ of the field strength. The Vafa-Witten partition function is therefore a generating function that counts anti-selfdual instantons. On an ALE space this partition function turns out to compute the character of an affine ADE algebra.

This section starts off with the Vafa-Witten twist and ALE spaces. We discuss Vafa-Witten theory on ALE spaces, and embed the gauge theory into string theory as a worldvolume theory on top of D4-branes. This is the first step in finding a deeper explanation for the duality between $\mathcal{N} = 4$ supersymmetric gauge theories and 2-dimensional conformal field theories.
3.2. Vafa-Witten twist on ALE spaces

3.2.1 Vafa-Witten twist

C. Vafa and E. Witten have performed an important S-duality check of \( \mathcal{N} = 4 \) supersymmetric gauge theory by computing a twisted partition function on certain 4-manifolds \( M \).\(^{[40]} \)

In total three inequivalent twists of \( \mathcal{N} = 4 \) Yang-Mills theory are possible. These are characterized by an embedding of the rotation group \( SO(4) \cong SU(2)_L \times SU(2)_R \) in the R-symmetry group \( SU(4)_R \) of the supersymmetric gauge theory. The Vafa-Witten twist considers the branching

\[
SU(4)_R \rightarrow SU(2)_A \times SU(2)_B
\]

and twists either \( SU(2)_L \) or \( SU(2)_R \) with \( SU(2)_A \). Both twists are related by changing the orientation of the 4-fold \( M \) and at the same time changing \( \tau \) with \( \bar{\tau} \). Let us choose the left-twist here. This results in a bosonic field content consisting of a gauge field \( A \), an anti-selfdual 2-form and three scalars.

The twisted theory is a cohomological gauge theory with \( \mathcal{N}_T = 2 \) equivariant topological supercharges \( Q_{\pm} \), whose Lagrangian can be written in the form

\[
\mathcal{L} = \frac{i\tau}{4\pi} F \wedge F - \frac{2}{e^2} F_- \wedge *F_- + \ldots
\]

\[
= \frac{i\tau}{4\pi} F \wedge F + Q_+ Q_- \mathcal{F}, \tag{3.12}
\]

where \( \mathcal{F} \) is called the action potential \([58]\). In the spirit of our discussion in Section 3.1.1 this implies that the path integral localizes onto the critical points of the potential \( \mathcal{F} \) modulo gauge equivalence. On Kähler manifolds this critical locus is characterized by the vanishing of the anti-selfdual 2-form and the three scalars, whereas the gauge field obeys

\[
F_- = 0.
\]

The Vafa-Witten twist thus localizes to the instanton moduli space

\[
\mathcal{M} = \mathcal{W}/G, \quad \mathcal{W} = \{ A : F_-(A) = 0 \}.
\]

of selfdual connections.\(^1\) The moduli space \( \mathcal{M} \) naturally decomposes in con-

\(^1\)Alternatively, the localization to \( F_- \) follows from the field equations. Since the field content of the right twist involves only anti-selfdual (instead of selfdual) fields, setting the fermion variations to zero forces \( F_- = 0 \). Likewise, performing the left twist corresponds to changing \( \tau \) with \( \bar{\tau} \) as well changing \( F_- \) with \( F_+ \) in the Lagrangian (3.12). Ultimately, our conventions in equation (3.2) and (3.3) imply that the selfdual instantons receive contributions in \( \tau \). This choice is non-standard in the Vafa-Witten literature, but it fits better with the content of this thesis.
Chapter 3. I-brane Perspective on Vafa-Witten Theory and WZW Models

Connected components $M_n$ that are labeled by the instanton number

$$n = \int_M \text{Tr} \left[ \frac{F \wedge F}{8\pi^2} \right].$$

The Vafa-Witten partition function computes the Euler characteristic of these components (without $\pm$ signs). Up to possible holomorphic anomalies it is a holomorphic function of $\tau$ with a Fourier-expansion of the type

$$Z^G(\tau) \sim \sum_n d(n)q^n,$$

where the numbers $d(n)$ represent the Euler characteristic of $M_n$, whereas $q^n = \exp(2\pi in\tau)$ denotes the contribution of the instantons to the topological term in the Lagrangian (3.12). The numbers $d(n)$ are integers when $G$ is connected and simply connected.

S-duality and modular forms

The Vafa-Witten partition function only transforms nicely under S-duality once local curvature corrections in the Euler characteristic $\chi$ and the signature $\sigma$ of $M$ have been added to the action [40]. Notice that this is justified since they do not change the untwisted theory on $\mathbb{R}^4$. In particular, these additional terms introduce an extra exponent in (3.13)

$$Z^G(\tau) = q^{-c/24} \sum_n d(n)q^n,$$

where $c$ is a number depending on $\chi$ and $\sigma$. The resulting Vafa-Witten partition function conjecturally transforms as a modular form of weight $w = -\chi(M)$ that exchanges $G$ with its dual $\hat{G}$

$$Z^G(-1/\tau) \sim \tau^{w/2} Z^\hat{G}(\tau).$$

Since $\hat{G}$ is often not simply connected (for example the dual of $G = SU(2)$ is $\hat{G} = SO(3) = SU(2)/\mathbb{Z}_2$) one has to take into account magnetic fluxes $v \in H^2(M, \pi_1(\hat{G}))$. The components of the partition function $Z_v$ mix under the S-duality transformation $\tau \rightarrow -1/\tau$. Furthermore, since for such $\hat{G}$ instantons numbers are not integer, the vector valued partition function $Z_v$ will only be covariant under a subgroup of $Sl(2,\mathbb{Z})$.

Characters of affine Lie algebras are examples of such vector valued modular forms. We will soon introduce them and see that they indeed appear as Vafa-Witten partition functions.
Unitary gauge group and Jacobi forms

In the following we will be especially interested in Vafa-Witten theory with gauge group $U(N)$. This gauge group is not simply-connected, since it contains an abelian subgroup $U(1) \subset U(N)$. Instanton bundles are therefore not only characterized topologically by their second Chern class $\text{ch}_2$, but also carry abelian fluxes measured by the first Chern class $c_1$.

The $U(N)$ Yang-Mills partition function gets extra contributions from these magnetic fluxes. The path integral can be performed by first taking care of the $U(1)$ part of the field strength. This gives a contribution in the form of a Siegel theta function, precisely as explained in Section 3.1.

We can make these fluxes more explicit by introducing a topological coupling $v \in H^2(M, \mathbb{Z})$ in the original Yang-Mills Lagrangian:

$$\mathcal{L} = \frac{i\tau}{4\pi} \text{Tr} F_+ \wedge F_+ + v \wedge \text{Tr} F_+ + \text{c.c.} \quad (3.14)$$

Here we define complex conjugation c.c. not only to change $\tau$ and $v$ into their anti-holomorphic conjugates, but also to map the selfdual part $F_+$ of the field strength to the anti-selfdual part $F_-$. The $v$-dependence of the partition function is entirely captured by the $U(1)$ factor of the field strength. It results in a Siegel theta-function of signature $(b_+^2, b_-^2)$

$$\theta_G(\tau, \bar{\tau}; v, \bar{v}) = \sum_{p \in \Gamma} e^{i\pi (\tau p_+^2 - \bar{\tau} p_-^2)} e^{2\pi i (v \cdot p_+ - \bar{v} \cdot p_-)}. \quad (3.15)$$

Here $p$ and $v$ are elements of $\Gamma = H^2(M, \mathbb{Z})$, so that $v \cdot p$ (and likewise $p^2$) refers to the intersection product $\int_M v \wedge p$.

In this chapter we focus on non-compact hyper-Kähler manifolds whose Betti number $b_2^+ = 0$. We change their orientation to find a non-trivial Vafa-Witten partition function. The $U(1)$ contribution to their partition function is then purely holomorphic.

Because of S-duality the total $U(N)$ partition function $Z(v, \tau)$ is expected to be given by a Jacobi form determined by the geometry $M$. That is, for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \text{and} \quad n, m \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}^{b_2},$$

it should have the transformation properties

$$Z\left(\frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{w/2} e^{\frac{2\pi i n c v^2}{c\tau + d}} Z(v, \tau)$$

$$Z(v + n \tau + m, \tau) = e^{-2\pi i n (\tau + 2n \cdot v)} Z(v, \tau),$$
where \( w \) is the weight and \( \kappa \) is the index of the Jacobi form. Using the localization to instantons, the partition function has a Fourier expansion of the form

\[
Z(v, \tau) = \sum_{m \in H^2(M), n \geq 0} d(m, n) y^m q^{n - c/24},
\]

where \( y = e^{2\pi iv}, q = e^{2\pi i\tau} \) and \( c = N\chi \). The coefficients \( d(n, m) \) are roughly computed as the Euler number of the moduli space of \( U(N) \) instantons on \( M \) with total instanton numbers \( c_1 = m \) and \( ch_2 = n \).

### 3.2.2 M5-brane interpretation

In string theory \( U(N) \) Vafa-Witten theory is embedded as the topological subsector of \( \mathcal{N} = 4 \) super Yang-Mills theory on \( N \) D4-branes that wrap a holomorphic 4-cycle \( M \subset X \) in the IIA background

\[
\text{IIA} : \quad X \times \mathbb{R}^4.
\]  

Topological excitations in the gauge theory amount to bound states of D0 and D2-branes on the D4-brane.

Let us now consider the 5-dimensional gauge theory on a Euclidean D4-brane wrapping \( M \times S^1 \). The partition function of this theory is given by a trace over its Hilbert space, whose components are labeled by the number \( n = ch_2(F) \) of D0-branes and the number \( m = c_1(F) \) of D2-branes. The coefficients \( d(n, m) \) in the Fourier expansion (3.16) thus have a direct interpretation as computing BPS invariants: the number of bound states of \( n \) D0-branes and \( m \) D2-branes on the D4-brane. For this reason they are believed to be integers in general. We can compute \( d(m, n) \) as the index

\[
d(m, n) = \text{Tr}(-1)^F \in \mathbb{Z},
\]

in the subsector of field configurations on \( M \) of given instanton numbers \( m, n \).

From the string theory point of view the modular invariance of \( Z \) is explained naturally by lifting the D4-brane to M-theory, where it becomes an M5-brane on the product manifold

\[
M \times T^2.
\]

The world-volume theory of the M5-brane is (in the low-energy limit) the rather mysterious 6-dimensional \( U(N) \) conformal field theory with \( (0, 2) \) supersymmetry. The complexified gauge coupling \( \tau \) can now be interpreted as the modulus of the elliptic curve \( T^2 \), while the Wilson loops of the 3-form potential \( C_3 \) along

\footnote{Here and in the subsequent sections we assume that the two fermion zero modes associated to the center of mass movements of the D4-brane have been absorbed.}
this curve are related to the couplings $v$, as we explain in more detail in Section 3.3.3. With this interpretation the action of modular group $SL(2,\mathbb{Z})$ on $v$ and $\tau$ is the obvious geometric one.

Instead of compactifying over $T^2$, we can also consider a compactification over $M$. We then find a 2-dimensional $(0,8)$ CFT on the two-torus, whose moduli space consists of the solutions to the Vafa-Witten field equations on $M$. This duality motivates the appearance of CFT characters in Vafa-Witten theory. In Section 3.3 we will reach a deeper understanding.

**$\mathbb{R}^4$ – Example**

The simplest example is $U(1)$ Vafa-Witten theory on $\mathbb{R}^4$ corresponding to a single D4-brane on $\mathbb{R}^4$. Point-like instantons in this theory correspond to bound states with D0-branes and yield a non-trivial partition function

$$Z(\tau, v) = \frac{\theta_3(v, \tau)}{\eta(\tau)} = \sum_{p \in \mathbb{Z}} e^{\pi i v p^2} q^{3/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The Dedekind eta function $\eta(\tau)$ can be rewritten as a generating function

$$\frac{1}{\eta(\tau)} = q^{-1/24} \frac{1}{\prod_{n>0} (1 - q^n)} = q^{-1/24} \sum_{n \geq 0} p(n) q^n,$$

of the number $p(n)$ of partitions $(n_1, \ldots, n_k)$ of $n$. We can identify each such a partition with a bosonic state

$$\alpha_{-n_1} \cdots \alpha_{-n_k} |p\rangle$$

in the Fock space $\mathcal{H}_p$ of a chiral boson $\phi(x)$ with mode expansion

$$\partial \phi(x) = \sum_{n \in \mathbb{Z}} \alpha_n x^{-n-1}.$$

The state $|p\rangle$ is the vacuum whose Fermi level is raised by $p$ units. The partition function $Z(\tau, v)$ is exactly reproduced by the $\widehat{u}(1)$ character

$$Z(\tau, v) = \text{Tr}_{\mathcal{H}_p} \left( y^{J_0} q^{L_0 - c/24} \right) = \chi_{\widehat{u}(1)}(\tau, v),$$

where $L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n>0} \alpha_{-n} \alpha_n$ measures the energy of the states and $J_0 = \alpha_0$ the $U(1)$ charge. The instanton zero point energy $c = 1$ now corresponds to the central charge for a single free boson.
3.2.3 Vafa-Witten theory on ALE spaces

So far we motivated that the Vafa-Witten partition function transforms under S-duality in a modular way. Furthermore, we have seen a simple example with $M = \mathbb{R}^4$ where the partition function equals a CFT-character. In this section we will see that this relation is more generally true for ALE spaces.

In the forthcoming sections we introduce quite a few notions from the theory of affine Lie algebras $\hat{g}$ and their appearance in WZW (Wess-Zumino-Witten) models. The classic reference for this subject is [59].

ALE spaces and geometric McKay correspondence

An ALE space $M_\Gamma$ is a non-compact hyper-Kähler surface. It is obtained by resolving the singularity at the origin of $\mathbb{C}^2/\Gamma$,

$$M_\Gamma \to \mathbb{C}^2/\Gamma,$$

where $\Gamma$ is a finite subgroup of $SU(2)$ that acts linearly on $\mathbb{C}^2$. These Kleinian singularities are classified into three families: the cyclic groups $A_k$, the dihedral groups $D_k$ and the symmetries of the platonic solids $E_k$. For example, an $A_{k-1}$ singularity is generated by the element

$$(z, w) \mapsto (e^{2\pi i/k}z, e^{-2\pi i/k}w).$$

of the cyclic subgroup $\Gamma = \mathbb{Z}_k$.

![Diagram](image)

**Figure 3.2:** The left picture (1) illustrates an $A_4$-singularity, which is a hyper-Kähler resolution of $\mathbb{C}^2/\mathbb{Z}_5$. Its homology is generated by 4 independent 2-cycles. They have self-intersection number $-2$ and intersect once with their neighbours. This Kleinian singularity is therefore dual to the Dynkin diagram of the Lie algebra $su(5)$, which is illustrated on the right in picture (2). The dotted lines complete this diagram into the Dynkin diagram of the extended Lie algebra $\hat{su}(5)$. The labels are the dual Dynkin indices of the simple roots.

A hyper-Kähler resolution replaces the singularity at the origin with a number of
2-spheres. The (oriented) intersection product

$$(S_i^2, S_j^2) \mapsto S_i^2 \cup S_j^2$$

puts a lattice structure on the second homology. This turns out to be determined by the Cartan matrix of the corresponding ADE Lie algebra $g$, so that there is a bijection between a basis of 2-cycles and a choice of simple roots. $A_{k-1}$ singularities correspond to the Lie group $SU(k)$, $D_k$ singularities lead to $SO(2k)$ and $E_k$ ones are related to one of the exceptional Lie groups $E_6$, $E_7$ or $E_8$.

The Dynkin diagram of each Lie algebra is thus realized geometrically in terms of intersections of 2-cycles in the resolution of the corresponding Kleinian singularity. This is the famous geometric McKay correspondence [60, 61]. We will encounter its string theoretic interpretation in the next chapter.

In this thesis we mainly consider $A_{k-1}$ surface singularities, for which $\Gamma = \mathbb{Z}_k$. Let us work out this example in some more detail. A resolved $A_{k-1}$ singularity $M_k$ is defined by an equation of the form:

$$W_k = \prod_{i=1}^{k} (z - a_i) + u^2 + v^2 = 0,$$

for $z, u, v \in \mathbb{C}$.

More precisely, this equation defines a family of $A_{k-1}$ spaces that are parametrized by $k$ complex numbers $a_i$. For any configuration with $a_1 \neq \ldots \neq a_k$ the surface $M_k$ is smooth.

The 4-manifold $M_k$ can be thought of $S^1 \times \mathbb{R}$-fibration over the complex plane $\mathbb{C}$, where the fiber is defined by the equation $u^2 + v^2 = \mu = -\prod_{i=1}^{k} (z - a_i)$ over a point $z \in \mathbb{C}$. Notice, however, that the size $\mu$ of the circle $S^1$ becomes infinite when $z \to \infty$.

Over each of the points $z = a_i$ the fiber circle vanishes. Hence, non-trivial 2-cycles $C_{ij}$ in the 4-manifold can be constructed as $S^1$-fibrations over some line segment $[a_i, a_j]$ in the $z$-plane. In fact, the second homology of the 4-manifold $M_k$ is spanned by $k-1$ of these two-spheres, say $C_{i(i+1)}$ for $1 \leq i \leq k-1$. This is illustrated in in Fig. 3.2.

Since the $(k-1) \times (k-1)$ intersection form on $H_2(M_k)$ in this basis

$$
\begin{pmatrix}
-2 & 1 & 0 & \cdots \\
1 & -2 & 1 & \cdots \\
0 & 1 & -2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

coincides with the Cartan form of the Lie algebra $su(k)$ (up to an overall minus-sign), the 2-cycles $C_{i(i+1)}$ generate the root lattice of $A_{k-1}$.
McKay-Nakajima correspondence

Let us now study Vafa-Witten theory on these ALE spaces. First of all we have to address the fact that the 4-manifold $M_{\Gamma}$ is non-compact, so that we have to fix boundary conditions for the gauge field. The boundary at infinity is given by the Lens space $S^3/\Gamma$ and here the $U(N)$ gauge field should approach a flat connection. Up to gauge equivalence this flat connection is labeled by an $N$-dimensional representation of the quotient group $\Gamma$, that is, an element

$$ \rho \in \text{Hom}(\Gamma, U(N)). $$

If $\rho_i$ label the irreducible representations of $\Gamma$ (with $\rho_0$ the trivial representation), then $\rho$ can be decomposed as

$$ \rho = \bigoplus_{i=1}^{r} N_i \rho_i, $$

where the multiplicities $N_i$ are non-negative integers satisfying the restriction

$$ \sum_{i=1}^{r} N_i d_i = N, \quad d_i = \text{dim} \rho_i. $$

and $r$ is the rank of the gauge group. Now the classic McKay correspondence (without the adjective “geometric”) relates the irreducible representations $\rho_i$ of the finite subgroup $\Gamma$ to the nodes of the Dynkin diagram of the corresponding affine extension $\hat{g}$, such that the dimensions $d_i$ of these irreps can be identified with the dual Dynkin indices (see Fig. 3.2).

Furthermore, the non-negative integers $N_i$ label a dominant weight of the affine algebra $\hat{g}$ whose level is equal to $N$. Through the McKay correspondence each $N$-dimensional representation $\rho$ of $\Gamma$ thus determines an integrable highest-weight representation of $\hat{g}_N$ at level $N$. We will denote this (infinite-dimensional) Lie algebra representation as $V_{\rho}$. For $\Gamma = \mathbb{Z}_k$, which is the case that we will mostly concentrate on, flat connections on $S^3/\mathbb{Z}_k$ get identified with integrable representations of $\hat{su}(k)_N$. In this particular case all Dynkin indices satisfy $d_i = 1$.

With $\rho$ labeling the boundary conditions of the gauge field at infinity, we will get a vector-valued partition function $Z_{\rho}(v, \tau)$. Formally the $U(N)$ gauge theory partition function on the ALE manifold again has an expansion

$$ Z_{\rho}(v, \tau) = \sum_{n,m} d(m, n) y^m q^{h_{\rho} + n - c/24}, $$

where $c = Nk$ with $k$ the regularized Euler number of the $A_{k-1}$ manifold [40]. The usual instanton numbers given by the second Chern class $n = ch_2$ in the
exponent are now shifted by a rational number $h_\rho$, which is related to the Chern-Simons invariant of the flat connection $\rho$. As we explain in Section 3.4.2, $h_\rho$ gets mapped to the conformal dimension of the corresponding integrable weight in the affine Lie algebra $\hat{g}$ related to $\Gamma$ by the McKay correspondence. S-duality will act non-trivially on the boundary conditions $\rho$, and therefore $Z_\rho(v, \tau)$ will be a vector-valued Jacobi form \cite{40}.

For these ALE spaces the instanton computations can be explicitly performed, because there exists a generalized ADHM construction in which the instanton moduli space is represented as a quiver variety. The physical intuition underlying this formalism has been justified by the beautiful mathematical work of H. Nakajima \cite{39, 62}, who has proved that on the middle dimensional cohomology of the instanton moduli space one can actually realize the action of the affine Kac-Moody algebra $\hat{g}_N$ in terms of geometric operations. In fact, this work leads to the identification

$$Z_\rho(v, \tau) = \text{Tr}_{V_\rho} \left( y^{J_0} q^{L_0 - c/24} \right) = \chi_\rho(v, \tau),$$

with $V_\rho$ the integrable highest-weight representation of $\hat{g}_N$ and $\chi_\rho$ its affine character. Here $c$ is the appropriate central charge of the corresponding WZW model. A remarkable fact is that, in the case of a $U(N)$ gauge theory on a $\mathbb{Z}_k$ singularity, we find an action of $\hat{su}(k)_N$ and not of the gauge group $SU(N)$. This is a important example of level-rank duality of affine Lie algebras. This setup has been studied from various perspectives in for instance \cite{63, 64, 65}.

Interestingly, I. Frenkel has suggested \cite{66} that, if one works equivariantly for the action of the gauge group $SU(N)$ at infinity (we ignore the $U(1)$ part for the moment), there would similarly be an action of the $\hat{su}(N)_k$ affine Lie algebra.

Physically this means “ungauging” the $SU(N)$ at infinity. In other words, we consider making the $SU(N)$ into a global symmetry instead of a gauge symmetry at the boundary. This suggestion has recently been confirmed in \cite{67}. So, depending on how we deal with the theory at infinity, there are reasons to expect both affine symmetry structures to appear and have a combined action of the Lie algebra

$$\hat{su}(N)_K \times \hat{su}(k)_N.$$

We will now turn to a dual string theory realization, where this structure indeed becomes transparent.

### 3.3 Free fermion realization

In this section we discover a string theoretic set-up to study the correspondence between Vafa-Witten theory on ALE spaces and the holomorphic part of a WZW
model. We find that Vafa-Witten theory is dual to a system of intersecting D4 and D6-branes on a torus $T^2$.

### 3.3.1 Taub-NUT geometry

To study Vafa-Witten theory on ALE spaces within string theory, we use a trick that proved to be very effectively in relating 4d and 5d black holes [68, 69, 70, 71, 72] and is in line with the duality between ALE spaces and 5-brane geometries [73]. We will replace the local $A_{k-1}$ singularity with a Taub-NUT geometry. This can be best understood as an $S^1$ compactification of the singularity. The $TN_k$ geometry is a hyper-Kähler manifold with metric [74, 75],

$$ds^2_{TN} = R^2 \left[ \frac{1}{V} d\chi^2 + \Omega^2 + V d\vec{x}^2 \right],$$

with $\chi \in S^1$ (with period $4\pi$) and $\vec{x} \in \mathbb{R}^3$. Here the function $V$ and 1-form $\alpha$ are determined as

$$V(\vec{x}) = 1 + \sum_{a=1}^{k} \frac{1}{|\vec{x} - \vec{x}_a|}, \quad d\alpha = \ast_3 dV.$$

Just like a local $A_{k-1}$ singularity, the Taub-NUT manifold may be seen as a circle fibration

$$S^1_{TN} \rightarrow TN_k \quad \downarrow \quad \mathbb{R}^3$$

where the size of the $S^1_{TN}$ shrinks at the points $\vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^3$, whose positions are the hyperkähler moduli of the space. The main difference with the (resolved) $A_{k-1}$ singularity is that the Taub-NUT fiber stays of finite size $R$ at infinity.

The total Taub-NUT manifold is perfectly smooth. At infinity it approximates the cylinder $\mathbb{R}^3 \times S^1_{TN}$, but is non-trivially fibered over the $S^2$ at infinity as a monopole bundle of charge (first Chern class) $k$

$$\int_{S^2} d\alpha = 2\pi k.$$

In the core, where we can ignore the constant 1 that appears in the expression for the potential $V(\vec{x})$, the Taub-NUT geometry can be approximated by the (resolved) $A_{k-1}$ singularity.

The manifold $TN_k$ has non-trivial 2-cycles $C_{a,b} \cong S^2$ that are fibered over the line segments joining the locations $\vec{x}_a$ and $\vec{x}_b$ in $\mathbb{R}^3$. Only $k - 1$ of these cycles are homologically independent. As a basis we can pick the cycles

$$C_a := C_{a,a+1}, \quad a = 1, \ldots, k - 1.$$
The intersection matrix of these 2-cycles gives the Cartan matrix of $A_{k-1}$.

From a dual perspective, there are $k$ independent normalizable harmonic 2-forms $\omega_a$ on $TN_k$, that can be chosen to be localized around the centers or NUTs $\vec{x}_a$. With

$$V_a = \frac{1}{|\vec{x} - \vec{x}_a|}, \quad d\alpha_a = \ast dV_a,$$

they are given as

$$\omega_a = d\eta_a, \quad \eta_a = \alpha_a - \frac{V_a}{V}(d\chi + \alpha).$$

Furthermore, these 2-forms satisfy

$$\int_{TN} \omega_a \wedge \omega_b = 16\pi^2 \delta_{ab},$$

and are dual to the cycles $C_{a,b}$

$$\int_{C_{a,b}} \omega_c = 4\pi (\delta_{ac} - \delta_{bc}).$$

A special role is played by the sum of these harmonic 2-forms

$$\omega_{TN} = \sum_a \omega_a. \quad (3.18)$$

This is the unique normalizable harmonic 2-form that is invariant under the triholomorphic $U(1)$ isometry of $TN$. The form $\omega_{TN}$ has zero pairings with all the cycles $C_{ab}$. In the “decompactification limit”, where $TN_k$ gets replaced by $A_{k-1}$, the linear combination $\omega_{TN}$ becomes non-normalizable, while the $k-1$ two-forms orthogonal to it survive.

We will make convenient use of the following elegant interpretation of the two-form $\omega_{TN}$. Consider the $U(1)$ action on the $TN_k$ manifold that rotates the $S^1_{TN}$ fiber. It is generated by a Killing vector field $\xi$. Let $\eta_{TN}$ be the corresponding dual one-form given as $(\eta_{TN})_\mu = g_{\mu\nu}\xi^\nu$, where we used the $TN$-metric to convert the vector field to a one-form. Up to an overall rescaling this gives

$$\eta_{TN} = \frac{1}{V}(d\chi + \alpha). \quad (3.19)$$

In terms of this one-form, $\omega_{TN}$ is given by $\omega_{TN} = d\eta_{TN}$.

### 3.3.2 The D4-D6 system

Our strategy will be that, since we consider the twisted partition function of the topological field theory, the answer will be formally independent of the radius
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$R$ of the Taub-NUT geometry. So we can take both the limit $R \to \infty$, where we recover the result for the ALE space $\mathbb{C}^2/\mathbb{Z}_k$, and the limit $R \to 0$, where the problem becomes essentially 3-dimensional.

Now, there are some subtleties with this argument, since a priori the partition function of the gauge theory on the $TN$ manifold is not identical to that of the ALE space. In particular there are new topological configurations of the gauge field that can contribute. These can be thought of as monopoles going around the $S^1$ at infinity. We will come back to this subtle point later.$^{3}$

In type IIA string theory, the partition function of the $\mathcal{N} = 4$ SYM theory on the $TN_k$ manifold can be obtained by considering a compactification of the form

\[(\text{IIA}) \quad TN \times S^1 \times \mathbb{R}^5,\]

and wrapping $N$ D4-branes on $TN \times S^1$. This is a special case of the situation presented in the box on the right-hand side in Fig. 1.6 with $\Gamma = S^1$, $B_3 = S^1 \times \mathbb{R}^2$, and $S^1$ decompactified. In the decoupling limit the partition function of this set of D-branes will reproduce the Vafa-Witten partition function on $TN_k$. This partition function can be also written as an index

\[Z(v, \tau) = \text{Tr} \left( (-1)^F e^{-\beta H} e^{i n \theta} e^{2 \pi i m v} \right)\]

where $\beta = 2\pi R_9$ is the circumference of the “9th dimension” $S^1$, and $m = c_1$, $n = ch_2$ are the Chern characters of the gauge bundle on the $TN_k$ space. Here we can think of the theta angle $\theta$ as the Wilson loop for the graviphoton field $C_1$ along the $S^1$. Similarly $v$ is the Wilson loop for $C_3$. The gauge coupling of the 4d gauge theory is now identified as

\[\frac{1}{g^2} = \frac{\beta}{gs \ell_s}.\]

Because only BPS configurations contribute in this index, again only the holomorphic combination $\tau$ (3.4) will appear.

We can now further lift this configuration to M-theory with an additional $S^1$ of size $R_{11} = gs \ell_s$, where we obtain the compactification

\[(\text{M}) \quad TN \times T^2 \times \mathbb{R}^5,\]

now with $N$ M5-branes wrapping the 6-manifold $TN_k \times T^2$. This corresponds to the top box in Fig. 1.6 with $\Sigma = T^2$. As we remarked earlier, after this lift the coupling constant $\tau$ is interpreted as the geometric modulus of the elliptic curve

$^{3}$Recently, instantons on Taub-NUT spaces have been studied extensively in [76, 77, 78]. In particular, [79] gives a closely related description of the duality between $\mathcal{N} = 4$ supersymmetric gauge theory on Taub-NUT space and WZW models from the perspective of an M5-brane wrapping $\mathbb{R} \times S^1 \times TN$. It is called a geometric Langlands duality for surfaces.
3.3. Free fermion realization

In particular its imaginary part is given by the ratio $R_9/R_{11}$. Dimensionally reducing the 6-dimensional $U(N)$ theory on the M5-brane world-volume over the Taub-NUT space gives a 2-dimensional $(0,8)$ superconformal field theory, in which the gauge theory partition function is computed as the elliptic genus

$$Z = \text{Tr} \left( (-1)^F y^{J_0} q^{L_0 - c/24} \right).$$

In order to further analyze this system we switch to yet another duality frame by compactifying back to Type IIA theory, but now along the $S^1$ fiber in the Taub-NUT geometry. This is the familiar 9-11 exchange. In this fashion we end up with a IIA compactification on

$$\text{(IIA)} \quad \mathbb{R}^3 \times T^2 \times \mathbb{R}^5,$$

with $N$ D4-branes wrapping $\mathbb{R}^3 \times T^2$. However, because the circle fibration of the $TN$ space has singular points, we have to include D6-branes as well. In fact, there will be $k$ D6-branes that wrap $T^2 \times \mathbb{R}^5$ and are localized at the points $\vec{x}_1, \ldots, \vec{x}_k$ in the $\mathbb{R}^3$. This situation is represented in the box on the left-hand side in Fig. 1.6.

Summarizing, we get a system of $N$ D4-branes and $k$ D6-branes intersecting along the $T^2$. This intersection locus is called the I-brane. It is pictured in Fig. 3.3.2. We will now study this I-brane system in greater detail.

Figure 3.3: Configuration of intersecting D4 and D6-branes with one of the 4-6 open strings that gives rise to a chiral fermion localized on the I-brane.
3.3.3 Free fermions

A collection of D4-branes and D6-branes that intersect along two (flat) dimensions is a supersymmetric configuration. One way to see this is that after some T-dualities, it can be related to a D0-D8 or D1-D9 system. The supersymmetry in this case is of type $(0, 8)$. The massless modes of the 4-6 open strings stretching between the D4 and D6 branes reside entirely in the Ramond sector. All modes in the NS sector are massive. These massless modes are well-known to be chiral fermions on the 2-dimensional I-brane \[51, 80, 81\]. If we have \( N \) D4-branes and \( k \) D6-branes, the chiral fermions

\[
\psi_{i, a}(z), \quad \psi_{\bar{i}, a}^\dagger(z), \quad i = 1, \ldots, N, \quad a = 1, \ldots, k
\]

transform in the bifundamental representations \((N, k)\) and \((\bar{N}, k)\) of \(U(N) \times U(k)\). Since we are computing an index, we can take the \(\alpha' \to 0\) limit, in which all massive modes decouple. In this limit we are just left with the chiral fermions. Their action is necessarily free and given by

\[
S = \int d^2z \, \psi^\dagger \tilde{\partial}_{A+\bar{A}} \psi,
\]

where \(A\) and \(\tilde{A}\) are the restrictions to the I-brane \(T^2\) of the \(U(N)\) and \(U(k)\) gauge fields, that live on the worldvolumes of the D4-branes and the D6-branes respectively. (Here we have absorbed the overall coupling constant).

Under the two \(U(1)\)'s the fermions have charge \((+1, -1)\). Therefore the overall (diagonal) \(U(1)\) decouples and the fermions effectively couple to the gauge group

\[
U(1) \times SU(N) \times SU(k),
\]

where the remaining \(U(1)\) is the anti-diagonal. At this point we ignore certain discrete identifications under the \(\mathbb{Z}_N\) and \(\mathbb{Z}_k\) centers, that we will return to later.

Zero modes

A special role is played by the zero-modes of the D-brane gauge fields. In the supersymmetric configuration we can have both a non-trivial flat \(U(N)\) and \(U(k)\) gauge field turned on along the \(T^2\). We will denote these moduli as \(u_i\) and \(v_a\) respectively. The partition function of the chiral fermions on the I-brane will be a function \(Z(u, v, \tau)\) of both the flat connections \(u, v\) and the modulus \(\tau\). It will transform as a (generalized) Jacobi-form under the action of \(SL(2, \mathbb{Z})\) on the two-torus.

The couplings \(u\) and \(v\) have straightforward identifications in the \(N = 4\) gauge theory on the \(TN\) space. First of all, the parameters \(u_i\) are Wilson loops along the circle of the \(D4\) compactified on \(TN \times S^1\), and so in the 4-dimensional theory
they just describe the values of the scalar fields on the Higgs moduli space. That is, they parametrize the positions $u_i$ of the $N$ D4-branes along the $S^1$. Clearly, we are not interested in describing these kind of configurations where the gauge group $U(N)$ gets broken to $U(1)^N$ (or some intermediate case). Therefore we will in general put $u = 0$.

The parameters $v_a$ are the Wilson lines on the D6-branes and are directly related to fluxes along the non-trivial two-cycles of $TN_k$ and (in the limiting case) on the $A_{k-1}$ geometry. To see this, let us briefly review how the world-volume fields of the D6-branes are related to the $TN$ geometry in the M-theory compactification.

The positions of the NUTs $\vec{x}_a$ of the $TN$ manifold are given by the vev’s of the three scalar Higgs fields of the 6+1 dimensional gauge theory on the D6-brane. In a similar fashion the $U(1)$ gauge fields $\tilde{A}_a$ on the D6-branes are obtained from the 3-form $C_3$ field in M-theory. More precisely, if $\omega_a$ are the $k$ harmonic two-forms on $TN_k$ introduced in Section 3.3.1, we have a decomposition

$$C_3 = \sum_a \omega_a \wedge \tilde{A}_a.$$  \hspace{1cm} (3.20)

We recall that the forms $\omega_a$ are localized around the centers $\vec{x}_a$ of the $TN$ geometry (the fixed points of the circle action). So in this fashion the bulk $C_3$ field gets replaced by $k$ $U(1)$ brane fields $\tilde{A}_a$. This relation also holds for a single D6-brane, because the two-form $\omega$ is normalizable in the $TN_1$ geometry. Relation (3.20) holds in particular for a flat connection, in which case we get the M-theory background

$$C_3 = \sum_a v_a \omega_a \wedge dz + c.c.$$  

Reducing this 3-form down to the type IIA configuration on $TN \times S^1$ gives a mixture of NS $B$ fields and RR $C_3$ fields on the Taub-NUT geometry. Finally, in the $\mathcal{N} = 4$ gauge theory this translates (for an instanton background) into a topological coupling

$$\int v \wedge \text{Tr} F_+ + \bar{v} \wedge \text{Tr} F_-,$$

with $v$ the harmonic two-form

$$v = \sum_a v_a \omega_a.$$  

The existence of this coupling can also be seen by recalling that the M5-brane action contains the term $\int H \wedge C_3$. On the manifold $M \times T^2$ the tensor field strength $H$ reduces as $H = F_+ \wedge d\bar{z} + F_- \wedge dz$ and similarly one has $C_3 = v \wedge dz + \bar{v} \wedge d\bar{z}$, which gives the above result. If one thinks of the gauge theory in terms of a D3-brane, the couplings $v, \bar{v}$ are the fluxes of the complexified 2-form combination $B_{RR} + \tau B_{NS}$. 
Chiral anomaly

We should address another point: the chiral fermions on the I-brane are obviously anomalous. Under a gauge transformation of, say, the $U(N)$ gauge field

$$\delta A = D \xi,$$

the effective action of the fermions transforms as

$$k \int_{T^2} \text{Tr}(\xi F_A).$$

A similar story holds for the $U(k)$ gauge symmetry. Nonetheless, the overall theory including both the chiral fermions on the I-brane and the gauge fields in the bulk of the D-branes is consistent, due to the coupling between both systems. The consistency is ensured by Chern-Simons terms \[3.11\] in the D-brane actions, which cancel the anomaly through the process of anomaly inflow \[51, 82\]. For example, on the D4-brane there is a term coupling to the RR 2-form (graviphoton) field strength $G_2$:

$$I_{CS} = \frac{1}{2\pi} \int_{T^2 \times \mathbb{R}^3} G_2 \wedge CS(A), \quad \text{(3.21)}$$

with Chern-Simons term

$$CS(A) = \text{Tr}(A dA + \frac{2}{3} A \wedge A \wedge A).$$

Because of the presence of the D6-branes, the 2-form $G_2$ is no longer closed, but satisfies instead

$$dG_2 = 2\pi k \cdot \delta T^2.$$

Therefore under a gauge transformation $\delta A = D \xi$ the D4-brane action gives the required compensating term

$$\delta I_{CS} = \frac{1}{2\pi} \int G_2 \wedge d \text{Tr}(\xi F_A) = -k \int_{T^2} \text{Tr}(\xi F_A),$$

which makes the whole system gauge invariant.

3.4 Nakajima-Vafa-Witten correspondence

So far we have obtained a configuration of $N$ D4-branes and $k$ D6-branes that intersect transversely along a 2-torus. Moreover, the massless modes of the 4-6 open strings combine into $Nk$ free fermions on this 2-torus. This already relates $SU(N)$ Vafa-Witten theory on an $A_k$-singularity to a 2-dimensional conformal
field theory of free fermions. However, the I-brane system contains more information than just the Vafa-Witten partition function. In this section we analyze the I-brane system and extract the Nakajima-Vafa-Witten correspondence from the I-brane partition function.

### 3.4.1 Conformal embeddings and level-rank duality

The system of intersecting branes gives an elegant realization of the level-rank duality

\[ \widehat{su}(N)_k \leftrightarrow \widehat{su}(k)_N \]

that is well-known in CFT and 3d topological field theory. The analysis has been conducted in [82] for a system of D5-D5 branes, which is of course T-dual to the D4-D6 system that we consider here. Hence we can follow this analysis to a large extent.

The system of \( Nk \) free fermions has central charge \( c = Nk \) and gives a realization of the \( \widehat{u}(Nk)_1 \) affine symmetry at level one. In terms of affine Kac-Moody Lie groups we have the embedding

\[
\widehat{u}(1)_{Nk} \times \widehat{su}(N)_k \times \widehat{su}(k)_N \subset \widehat{u}(Nk)_1.
\]

This is a conformal embedding, in the sense that the central charges of the WZW models on both sides are equal. Indeed, using that the central charge of \( \widehat{su}(N)_k \) is

\[
c_{N,k} = \frac{k(N^2 - 1)}{k + N},
\]

it is easily checked that

\[
1 + c_{N,k} + c_{k,N} = Nk.
\]

The generators for these commuting subalgebras are bilinears constructed out of the fermions \( \psi_{i,a} \) and their conjugates \( \psi_{i,a}^\dagger \). In terms of these fields one can define the currents of the \( \widehat{u}(N)_k \) and \( \widehat{u}(k)_N \) subalgebras as respectively

\[
J_{j \overline{k}}(z) = \sum_a \psi_j^a \psi_{k,a}^\dagger,
\]

and

\[
J_{\overline{m} \overline{b}}(z) = \sum_j \psi_{\overline{m}, \overline{b}} \psi_{j}^\dagger.
\]

Now it is exactly the conformal embedding (3.22) that gives the most elegant explanation of level-rank duality. This correspondence should be considered as the affine version of the well-known Schur-Weyl duality for finite-dimensional Lie groups. Let us recall that the latter is obtained by considering the (commuting)
actions of the unitary group and symmetric group

$$U(N) \times S_k \subset U(Nk)$$
on the vector space $\mathbb{C}^{Nk}$, regarded as the $k$-th tensor product of the fundamental representation $\mathbb{C}^N$. Schur-Weyl duality is the statement that the corresponding group algebras are maximally commuting in $\text{End}((\mathbb{C}^N)^{\otimes k})$, in the sense that the two algebras are each other’s commutants. Under these actions one obtains the decomposition

$$\mathbb{C}^{Nk} = \bigoplus_{\rho} V_{\rho} \otimes \tilde{V}_{\rho},$$

with $V_{\rho}$ and $\tilde{V}_{\rho}$ irreducible representations of $u(N)$ and $S_k$ respectively. Here $\rho$ runs over all partitions of $k$ with at most $N$ parts. This duality gives the famous pairing between the representation theory of the unitary group and the symmetric group.

In the affine case we have a similar situation, where we now take the $k$th tensor product of the $N$ free fermion Fock spaces, viewed as the fundamental representation of $\hat{u}(N)_1$. The symmetric group $S_k$ gets replaced by $\hat{s}(k)_N$ (which reminds one of constructions in D-branes and matrix string theory, where the symmetry group appears as the Weyl group of a non-Abelian symmetry). The affine Lie algebras

$$\hat{u}(1)_{Nk} \times \hat{su}(N)_k \times \hat{su}(k)_N$$

again have the property that they form maximally commuting subalgebras within $\hat{u}(Nk)_1$. The total Fock space $\mathcal{F}^{\otimes Nk}$ of $Nk$ free fermions now decomposes under the embedding (3.22) as

$$\mathcal{F}^{\otimes Nk} = \bigoplus_{\rho} U_{\|\rho\|} \otimes V_{\rho} \otimes \tilde{V}_{\rho}.$$

(3.23)

Here $U_{\|\rho\|}$, $V_{\rho}$ and $\tilde{V}_{\rho}$ denote irreducible integrable representations of $\hat{u}(1)_{Nk}$, $\hat{su}(N)_k$, and $\hat{su}(k)_N$ respectively.

The precise formula for the decomposition (3.23) is a bit complicated, in particular due to the role of the overall $U(1)$ symmetry, and is given in detail in Appendix A. But roughly it can be understood as follows: the irreducible representations of $\hat{u}(N)_k$ are given by Young diagrams that fit into a box of size $N \times k$. Similarly, the representations of $\hat{u}(k)_N$ fit in a reflected box of size $k \times N$. In this fashion level-rank duality relates a representation $V_{\rho}$ of $\hat{u}(N)_k$ to the representation $\tilde{V}_{\rho}$ of $\hat{u}(k)_N$ labeled by the transposed Young diagram. If we factor out the $\hat{u}(1)_{Nk}$ action, we get a representation of charge $\|\rho\|$, which is related to the total number of boxes $|\rho|$ in $\rho$ (or equivalently $\tilde{\rho}$).

At the level of the partition function we have a similar decomposition into char-
acters. To write this in more generality it is useful to add the Cartan generators. That is, we consider the characters for $\widehat{u}(N)_k$ that are given by
\[ \chi_{\rho}^{\widehat{u}(N)_k}(u, \tau) = \text{Tr}_{V_{\rho}} \left( e^{2\pi i u_j J_j^0} q^{L_0 - c_{N,k}/24} \right), \]
and similarly for $\widehat{u}(k)_N$ we have
\[ \chi_{\rho}^{\widehat{u}(k)_N}(v, \tau) = \text{Tr}_{V_{\rho}} \left( e^{2\pi i v_a J_a^0} q^{L_0 - c_{k,N}/24} \right). \]
Here the diagonal currents
\[ J_j^0 = \oint \frac{dz}{2\pi i} J_{jj}(z), \quad J_a^0 = \oint \frac{dz}{2\pi i} J_{aa}(z) \]
generate the Cartan tori $U(1)^N \subset U(N)$ and $U(1)^k \subset U(k)$. Including the Wilson lines $u$ and $v$ for the $U(N)$ and $U(k)$ gauge fields, the partition function of the I-brane system is given by the character of the fermion Fock space
\[ Z_I(u, v, \tau) = \text{Tr}_F \left( e^{2\pi i (u_j J_j^0 + v_a J_a^0)} q^{L_0 - \frac{Nk}{24}} \right) \]
\[ = q^{-\frac{Nk}{24}} \prod_{j=1}^{N} \prod_{a=1}^{k} \left( 1 + e^{2\pi i (u_j + v_a)} q^{n+1/2} \right) \left( 1 + e^{-2\pi i (u_j + v_a)} q^{n+1/2} \right). \]
Writing the decomposition (3.23) in terms of characters gives
\[ Z_I(u, v, \tau) \]
\[ = \sum_{[\rho] \in \mathcal{Y}_{N-1,k}} \sum_{j=0}^{N-1} \sum_{k=0}^{k-1} \chi_{[\rho]+jk+aN}(N|u| + k|v|, \tau) \chi_{\sigma_k^{\rho}(\rho)}(\bar{u}, \bar{v}) \chi_{\sigma_n^{\rho}(\rho)}(\bar{u}, \bar{v}), \]
where the Young diagrams $\rho \in \mathcal{Y}_{N-1,k}$ of size $(N-1) \times k$ represent $\widehat{su}(N)_k$ integrable representations and $\sigma$ denote generators of the outer automorphism groups $\mathbb{Z}_N$ and $\mathbb{Z}_k$ that connect the centers of $SU(N)$ and $SU(k)$ to the $U(1)$ factor (see again Appendix A for notation and more details).

### 3.4.2 Deriving the McKay-Nakajima correspondence

In the intersecting D-brane configuration both the D4-branes and the D6-branes are non-compact. So, we can choose both the $U(N)$ and $U(k)$ gauge groups to be non-dynamical and freeze the background gauge fields $A$ and $\tilde{A}$. In fact, this set-up is entirely symmetric between the two gauge systems, which makes level-rank duality transparent.
However, in order to make contact with the $\mathcal{N} = 4$ gauge theory computation, we will have to break this symmetry. Clearly, we want the $U(N)$ gauge field to be dynamical — our starting point was to compute the partition function of the $U(N)$ Yang-Mills theory. The $U(k)$ symmetry should however not be dynamical, since we want to freeze the geometry of the Taub-NUT manifold. So, to derive the gauge theory result, we will have to integrate out the $U(N)$ gauge field $A$ on the I-brane. Particular attention has to be payed to the $U(1)$ factor in the CFT on the I-brane. We will argue that in this string theory set-up we should not take that to be dynamical.

Therefore we are dealing with a partially gauged CFT or coset theory

$$\hat{u}(Nk)_1/\hat{su}(N)_k.$$ 

In particular the $\hat{su}(N)_k$ WZW model will be replaced by the corresponding $G/G$ model. Gauging the model will reduce the characters. (Note that this only makes sense if the Coulomb parameters $u$ are set to zero. If not, we can only gauge the residual gauge symmetry, which leads to fractionalization and a product structure.) In the gauged WZW model, which is a topological field theory, only the ground state remains in each irreducible integrable representation. So we have a reduction

$$\chi^{{\hat{su}(N)}_k}(\pi, \tau) \to q^{h_\rho - c/24},$$

with $h_\rho$ the conformal dimension of the ground state representation $\rho$. Note that the choice of $\rho$ corresponds exactly to the boundary condition for the gauge theory on the $A_{k-1}$ manifold. We will explain this fact, that is crucial to the McKay correspondence, in a moment.

Gauging the full I-brane theory and restricting to the sector $\rho$ finally gives

$$Z_I(u, v, \tau) \to Z^N_{\rho} (v, \tau) = q^{h_\rho - c/24} \sum_{a=0}^{k-1} \chi^{(1)N}_{|\rho|+aN}(k|v|, \tau) \chi^{\hat{su}(k)_N} (\nu, \tau).$$

Up to the $\chi^{(1)N}_{k}$ factor, this reproduces the results presented in [40, 39] for ALE spaces, which involve just $\hat{su}(k)_N$ characters. This extra factor is due to additional monopoles mentioned in Section 3.3.2. They are related to the finite radius $S^1$ at infinity of the Taub-NUT space and are absent in case of ALE geometries.

In fact, the extra $U(1)$ factor can already be seen at the classical level, because the extra normalizable harmonic two-form $\omega$ in (3.18) disappears in the decompactification limit where $TN_k$ degenerates into $A_{k-1}$. The lattice $H^2(TN_k, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^k$ with the standard inner product and contains the root lattice $A_{k-1}$ as a sublattice given by $\sum_I n_I = 0$. Note also that the lattice $\mathbb{Z}^k$ is not even, which explains why the I-brane partition function has a fermionic charac-
ter and only transforms under a subgroup of $SL(2, \mathbb{Z})$ that leaves invariant the spin structure on $T^2$.

**Relating the boundary conditions**

By relating the original 4-dimensional gauge theory to the intersecting brane picture one can in fact derive the McKay correspondence directly. Moreover we can understand the appearance of characters of the WZW models (for both the $SU(N)$ and the $SU(k)$ symmetry) in a more natural way in this set-up. Recall that the $SU(N)$ gauge theory on the $A_k$ singularity or $T N_k$ manifold is specified by a boundary state. This state is given by picking a flat connection on the boundary that is topologically $S^3/\mathbb{Z}_k$. If we think of this system in radial quantization near the boundary, where we consider a wave function for the time evolution along

$$S^3/\mathbb{Z}_k \times \mathbb{R},$$

we have a Hilbert space with one state $|\rho\rangle$ for each $N$-dimensional representation

$$\rho : \mathbb{Z}_k \to U(N).$$

After the duality to the I-brane system, we are dealing with a 5-dimensional $SU(N)$ gauge theory on $\mathbb{R}^3 \times T^2$, with $k$ D6-branes intersecting it along $\{p\} \times T^2$ where $p$ is (say) the origin of $\mathbb{R}^3$. Here the boundary of the D4-brane system is $S^2 \times T^2$. In other words, near the boundary the space-time geometry looks like $\mathbb{R} \times S^2 \times T^2$. We now ask ourselves what specifies the boundary states for this theory. Since we need a finite energy condition, this is equivalent to considering the IR limit of the theory. In M-theory the $S^1$-bundle over $S^2$ carries a first Chern class $k$, which translates into the flux of the graviphoton field strength

$$\int_{S^2} G_2 = 2\pi k.$$

Therefore the term

$$\int_{S^2 \times T^2 \times \mathbb{R}} G_2 \wedge CS(A),$$

living on the D4 brane, leads upon reduction on $S^2$ (as is done in [82]) to the term

$$I_{CS} = 2\pi k \int_{T^2 \times \mathbb{R}} CS(A).$$

Hence we have learned that the boundary condition for the D4-brane requires specifying a state of the $SU(N)$ Chern-Simons theory at level $k$ living on $T^2$. The Hilbert space for Chern-Simons theory on $T^2$ is well-known to have a state for each integrable representation of the $\hat{u}(N)_k$ WZW model, which up to the level-rank duality described in the previous section, gives the McKay correspondence.
In fact, the full level-rank duality can be brought to life. Just as we discussed for the \( N \) D4-branes, a \( SU(k) \) gauge theory lives on the \( k \) D6-branes on \( T^2 \times \mathbb{R}^5 \). The boundary of the space is \( S^4 \times T^2 \). Furthermore, taking into account that the \( N \) D4-branes source the \( G_4 \) RR flux through \( S^4 \), we get, as in the above, a \( SU(k) \) Chern-Simons theory at level \( N \) living on \( T^2 \times \mathbb{R} \). Therefore the boundary condition should be specified by a state in the Hilbert space of the \( SU(k) \) Chern-Simons theory on \( T^2 \). So we see three distinct ways to specify the boundary conditions: as a representation of \( \mathbb{Z}_k \) in \( SU(N) \), as a character of \( SU(N) \) at level \( k \), and as a character of \( SU(k) \) at level \( N \). Thus we have learned that, quite independently of the fermionic realization, there should be an equivalence between these objects.

To make the map more clearly we could try to show that the choice of the flat connection of the \( SU(N) \) theory on \( S^3/\mathbb{Z}_k \) gets mapped to the characters that we have discussed in the dual intersecting brane picture. To accomplish this, recall that the original \( SU(N) \) action on the \( A_{k-1} \) space leads to a boundary term (modulo an integer multiple of \( 2\pi i \tau \)) given by the Chern-Simons invariant

\[
\frac{\tau}{4\pi i} \int_{A_{k-1}} \text{Tr} F \wedge F = \frac{\tau}{4\pi i} \int_{S^3/\mathbb{Z}_k} \text{CS}(A).
\]

Restricting to a particular flat connection on \( S^3/\mathbb{Z}_k \) yields the value of the classical Chern-Simons action.

If we show that

\[
S(\rho) = \frac{1}{8\pi^2} \int_{S^3/\mathbb{Z}_k} \text{CS}(A)
\]

for the flat connection \( \rho \) on \( S^3/\mathbb{Z}_k \) gets mapped to the conformal dimension \( h_\rho \) of the corresponding state of the quantum Chern-Simons theory on \( T^2 \), we would have completed a direct check of the map, because the gauge coupling constant \( \tau \) above is nothing but the modulus of the torus in the dual description.

To see how this works, let us first consider the abelian case of \( N = 1 \). In that case the flat connection \( \rho \) is given by a phase \( e^{2\pi in/k} \) with \( n \in \mathbb{Z}/k\mathbb{Z} \). The corresponding CS term gives

\[
S^{U(1)}(\rho) = \frac{n^2}{2k}.
\]

This is the conformal dimension of a primary state of the \( U(1) \) WZW model at level \( k \).

A general \( U(N) \) connection can always be diagonalized to \( U(1)^N \), which therefore gives integers \( n_1, \ldots, n_N \in \mathbb{Z}/k\mathbb{Z} \). The Chern-Simons action is therefore
given by
\[ S^{U(N)}(\rho) = \sum_{i=1}^{N} \frac{n_i^2}{2k}. \]

On the other hand, a conformal dimension of a primary state in the corresponding WZW model is given by
\[ h_\rho = \frac{C_2(\rho)}{2(k+N)}, \]
where \( \rho \) is an irreducible integrable \( \hat{u}(N)_k \) weight. Such a weight can be encoded in a Young diagram with at most \( N \) rows of lengths \( R_i \). There is a natural change of basis \( n_i = R_i + \rho_\text{Weyl} \) where we shift by the Weyl vector \( \rho_\text{Weyl} \). If we decompose \( U(N) \) into \( SU(N) \) and \( U(1) \), the basis \( n_i \) cannot be longer than \( k \), which relates to the condition \( n_i \in \mathbb{Z}_k \) on the Chern-Simons side. In this basis the second Casimir \( C_2 \) takes a simple form. Therefore the conformal dimension becomes
\[ h_\rho = -\frac{N(N^2-1)}{24(k+N)} + \frac{1}{2(k+N)} \sum_{i=1}^{N} n_i^2. \]
The constant term combines nicely with the central charge contribution \(-c_{N,k}/24\) to give an overall constant \((N^2-1)/24\). Apart from this term we see that \( h_\rho \) indeed matches the expression for \( S^{U(N)}(\rho) \) given above, up to the usual quantum shift \( k \to k+N \).

According to the McKay correspondence one might expect to find a relation between representations of \( \mathbb{Z}_k \) and \( \hat{u}(k)_N \) integrable weights. Instead, we have just shown how \( \hat{u}(N)_k \) weights \( \rho \) arise. Nonetheless, one can relate integrable weights of those algebras by a transposition of the corresponding Young diagrams. Then the conformal dimensions of \( \hat{u}(k)_N \) weights \( \tilde{\rho} \) are determined by the relation
\[ h_\rho + h_{\tilde{\rho}} = \frac{|\rho|}{2} - \frac{|\rho|^2}{2Nk}, \]
which is a consequence of the level-rank duality described in Appendix A. The above chain of arguments connects \( \mathbb{Z}_k \) representations and \( \hat{u}(k)_N \) integrable weights, thereby realizing the McKay correspondence.

### 3.4.3 Orientifolds and \( SO/Sp \) gauge groups

In this chapter we have considered a system of \( N \) D4-branes and \( k \) D6-branes intersecting along a torus, whose low energy theory is described by \( U(N) \) and \( U(k) \) gauge theories on each stack of branes, together with bifundamental fermions. We can reduce this system to orthogonal or symplectic gauge groups in a standard way by adding an orientifold plane. This construction can also be lifted to
M-theory. Let us recall that D6-branes in our system originated from a Taub-NUT solution in M-theory. The O6-orientifold can also be understood from M-theory perspective, and it corresponds to the Atiyah-Hitchin space [84]. Combining both ingredients, it is possible to construct the M-theory background for a collection of D6-branes with an O6-plane. The details of this construction are explained in [84].

Let us see what are the consequences of introducing the orientifold into our I-brane system. We start with a stack of \(k\) D6-branes. To get orthogonal or symplectic gauge groups one should add an orientifold O6-plane parallel to D6-branes [85], which induces an orientifold projection \(\Omega\) which acts on the Chan-Paton factors via a matrix \(\gamma_{\Omega}\). Let us recall there are in fact two species O6\(^\pm\) of such an orientifold. As the \(\Omega\) must square to identity, this requires

\[
\gamma_{\Omega}^T = \pm \gamma_{\Omega},
\]

with the \(\pm\) sign corresponding to O6\(^\pm\)-plane, which gives respectively \(SO(k)\) and \(Sp(2k)\) gauge group. In the former case \(k\) can be even or odd; \(k\) odd requires having half-branes, fixed to the orientifold plane (as explained e.g. in [86]).

Let us add now \(N\) D4-branes intersecting D6 along two directions. The presence of O6\(^\pm\)-plane induces appropriate reduction of the D4 gauge group as well. The easiest way to argue what gauge group arises is as follows. We can perform a T-duality along three directions to get a system of D1-D9-branes, now with a spacetime-filling O9-plane. This is analogous to the D5-D9-09 system in [85], in which case the gauge groups on both stacks of branes must be different (either orthogonal on D5-branes and symplectic on D9-branes, or the other way round); the derivation of this fact is a consequence of having 4 possible mixed Neumann-Dirichlet boundary conditions for open strings stretched between branes. On the contrary, for D1-D9-O9 system there is twice as many possible mixed boundary conditions, which in consequence leads to the same gauge group on both stack of branes. By T-duality we also expect to get the same gauge groups in D4-D6 system under orientifold projection.

Let us explain now that the appearance of the same type of gauge groups is consistent with character decompositions resulting from consistent conformal embeddings or the existence of the so-called dual pairs of affine Lie algebras related to systems of free fermions. We have already come across one such consistent embedding in (3.22) for \(\hat{u}(Nk)_1\). A dual pair of affine algebras in this case is \((\hat{su}(N)_k, \hat{su}(k)_N)\). These two algebras are related by the level-rank duality discussed in Appendix A. As proved in [87, 88], all other consistent dual pairs are necessarily of one of the following forms

\[
(\hat{sp}(2N)_k, \hat{sp}(2k)_N),
\]
\[ (\widetilde{so}(2N + 1)_{2k+1}, \widetilde{so}(2k + 1)_{2N+1}), \]
\[ (\widetilde{so}(2N)_{2k+1}, \widetilde{so}(2k + 1)_{2N}), \]
\[ (\widetilde{so}(2N)_{2k}, \widetilde{so}(2k)_{2N}). \]

Corresponding expressions in terms of characters, analogous to (A.5), are also given in [88]. The crucial point is that both elements of those pair involve algebras of the same type, which confirms and agrees with the string theoretic orientifold analysis above.

Finally we wish to stress that the appearance of $U$, $Sp$ and $SO$ gauge groups which we considered so far in this paper is related to the fact that their respective affine Lie algebras can be realized in terms of free fermions, which arise on the I-brane from our perspective. It turns out there are other Lie groups $G$ whose affine algebras have free fermion realization. There is a finite number of them, and fermionic realizations can be found only if there exists a symmetric space of the form $G'/G$ for some other group $G'$ [89]. It is an interesting question whether I-brane configurations can be engineered in string theory that support fermions realizing all these affine algebras.

From a geometric point of view we can remark the following. For ALE singularities of $A$-type and $D$-type a non-compact dimension can be compactified on a $S^1$ to give Taub-NUT geometries. For exceptional groups such manifolds do not exist. But one can compactify two directions on a $T^2$ to give an elliptic fibration. In this setting exotic singularities can appear as well. Such construction have a direct analogue in type IIB string theory where they correspond to a collection of $(p,q)$ 7-branes [90, 91]. The I-brane is now generalized to the intersection of $N$ D3-branes with this non-abelian 7-brane configuration [92]. However, there is in general no regime where all the 7-branes are weakly coupled, so it is not straightforward to write down the I-brane system.