Topological strings and quantum curves
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Chapter 6

Quantum Curves in Matrix Models and Gauge Theory

This chapter illustrates the $D$-module formalism of Chapter 5 from a string theory perspective, with examples from the theory of random matrices, minimal (non-critical) string theory, supersymmetric gauge theory and topological strings. As a result we connect these familiar ingredients in a common framework centered around $D$-modules. String theory provides solutions to integrable hierarchies of the KP type. This was first noted in the context of non-critical ($c \neq 26$) bosonic string theory, which has a dual formulation in terms of Hermitean random matrices. The matrix model partition function

$$Z_{\text{mm}}(\lambda) = \frac{1}{\text{vol}(U(N))} \int DM e^{-\frac{1}{\lambda} \text{Tr} W(M)}$$  \hspace{1cm} (6.1)$$

is known to be a tau-function of the KP integrable system. Although an algebraic curve $\Sigma$ emerges in the limit that the size $N$ of the square matrix $M$ tends to infinity, these matrix model solutions do not correspond to geometric Krichever solutions. In particular, the relevant Fock space state $|\mathcal{W}\rangle$ does not have a purely geometric interpretation as being swept out by regular free fermions living on the matrix model spectral curve $\Sigma$.

The matrix model partition function admits a formal expansion

$$Z_{\text{mm}}(\lambda) = \exp \sum_{g} \lambda^{2g-2} \mathcal{F}_g$$

in the string coupling constant $\lambda$. In the ’t Hooft limit $N$ is sent to infinity while the product of $N$ with $\lambda$ is held fixed, so that the geometric curve $\Sigma$ equivalently emerges in the classical limit $\lambda \to 0$. This suggests that $\lambda$ should be
interpreted as some form of non-commutative deformation of the underlying algebraic curve. In fact, there have been many indications that this is the right point of view.

In the simplest matrix models $\Sigma$ appears as an affine rational curve given by a relation of the form

$$H(x, y) = 0$$

in the complex two-plane $\mathbb{C}^2$, with a (local) parametrization $x = p(z)$ and $y = q(z)$, with $p, q$ polynomials. Of course, $p$ and $q$ commute: $[p, q] = 0$. However, the string-type solutions with $\lambda \neq 0$ are characterized by quantities $P$ and $Q$ that no longer commute but instead satisfy the canonical commutation relation

$$[P, Q] = \lambda.$$

In this case clearly $P, Q$ cannot be polynomials, but are represented as differential operators, i.e. polynomials in $z$ and $\partial_z$.

As we will point out in Section 6.1 these solutions are naturally captured by a $\mathcal{D}$-module. Instead of classical curve in the $(x, y)$-plane, we should think of a quantum curve as its analogue in the non-commutative plane $[x, y] = \lambda$. If we interpret

$$y = -\lambda \frac{\partial}{\partial x},$$

one can identify such a quantum curve as a holonomic $\mathcal{D}$-module $\mathcal{W}$ for the algebra $\mathcal{D}$ of differential operators in $x$. Roughly speaking, $\mathcal{W}$ can be considered as the space of local sections that can be continued as sections of a (non-commutative) $\mathcal{D}$-module, instead of sections of a line bundle over a curve.

One other important instance of integrable hierarchies in string theory is in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. In Chapter 4 we have seen that the low energy effective description of Seiberg-Witten theories is determined by a twice-punctured algebraic curve $\Sigma_{SW}$, defined by an equation of the form

$$H(t, v) = 0$$

with $t \in \mathbb{C}^*$ and $v \in \mathbb{C}$, that appears as a spectral curve of a Hitchin integrable system. In Chapter 4 we geometrically engineered Seiberg-Witten theory as a Calabi-Yau compactification and encountered additional gravitational corrections $\mathcal{F}_g$ to the effective action.

Like in the matrix model setting these $\mathcal{F}_g$-terms are multiplied by some power of the string coupling constant $\lambda$, suggesting that the full genus free energy $\mathcal{F} = \sum_g \lambda^{2g-2} \mathcal{F}_g$ has an interpretation in terms of a quantum Seiberg-Witten curve. This motivates us to quantize the Seiberg-Witten surface $\Sigma_{SW}$ in Section 6.3.
Again, we see that a $\mathcal{D}$-module underlies the structure of the total partition function.

### 6.1 Matrix model geometries

Hermitian one-matrix models with an algebraic potential

$$W(M) = \sum_{j=0}^{d+1} u_j M^j$$

are defined through the matrix integral (6.1). In the large $N$ limit the distribution of the eigenvalues $\lambda_i$ of $M$ on the real axis becomes continuous and defines a hyperelliptic curve

$$\Sigma_{\text{mm}}: \quad y^2 - W'(x)^2 + f(x) = 0,$$

(6.2)

called the (matrix model) spectral curve. The polynomial $f(x) = 4\mu \sum_{j=0}^{d-1} b_j x^j$ is determined as

$$f(x) = \frac{4\mu}{N} \sum_{i=1}^{N} \frac{W'(x) - W'((\lambda_i)}{x - \lambda_i},$$

with $\mu = N\lambda$. The potential $W(x)$ determines the positions of the cuts of the hyperelliptic curve, and contains the non-normalizable moduli. On the other hand, the size of the cuts is determined by the polynomial $f(x)$, that comprises the normalizable moduli $b_0, \ldots, b_{d-2}$ and the log-normalizable modulus $b_{d-1}$.

R. Dijkgraaf and C. Vafa discovered that this matrix model has a dual description in string theory. In the 't Hooft limit $N \to \infty$ (with $\mu$ fixed) it is equivalent to the topological B-model on a Calabi-Yau geometry $X_{\Sigma}$ modeled on the matrix model spectral curve $\Sigma_{\text{mm}}$ [123, 124, 125]. A good review is [172]. This duality may be generalized by starting with multi-matrix models, whose spectral curve is a generic (in contrast to hyperelliptic) algebraic curve in the variables $x$ and $y$.

The I-brane picture suggests that the full B-model partition function on these Calabi-Yau geometries can be understood in terms of $\mathcal{D}$-modules. Even better, we will find that finite $N$ matrix models are determined by an underlying $\mathcal{D}$-module structure.

In the past, as well as recently, Hermitean matrix models have been studied in great detail in many contexts. Already in [173, 174] an attempt has been made to understand the string equation $[P, Q] = \lambda$ in terms of a quantum curve
in terms of the expansion in the parameter $\lambda$. In Moore’s approach this surface seemed to emerge from an interpretation of the string equation as isomonodromy equations.

More recently, quantum curves have appeared in the description of branes in a dual string model. In topological string theory as well as non-critical string theory a dominant role is played by holomorphic branes: either topological B-branes [128] or FZZT branes [175, 176, 177, 171]. Their moduli space equals the spectral curve, whereas the branes themselves may be interpreted as fermions on the quantized spectral curve. As was reviewed in Chapter 4, in these theories it is possible to compute correlation functions using a $W_{1+\infty}$-algebra that implements complex symplectomorphisms of the complex plane $B$ – as in (5.34) – in quantum theory as Ward identities [128, 178, 179].

These advances strongly hint at a fundamental appearance of $\mathcal{D}$-modules in the theory of matrix models. Indeed, this section unifies recent developments in matrix models in the framework of Chapter 5. Firstly, after a self-contained introduction in double scaled models we uncover the $\mathcal{D}$-module underlying the double-scaled $(p, 1)$-models. In the second part of this section we shift our focus to general Hermitian multi-matrix models, and unravel their $\mathcal{D}$-module structure.

### 6.1.1 Double scaled matrix models and the KdV hierarchy

Our first goal is to find the $\mathcal{D}$-modules that explain the quantum structure of double scaled Hermitian matrix models. This double scaling limit is a large $N$ limit in which one also fine-tunes the parameters to find the right critical behaviour of the multi-matrix model potential. Geometrically the double scaling limit zooms in on some branch points of the spectral curve that move close together. Spectral curves of double scaled matrix models are therefore of genus zero and parametrized as

$$\Sigma_{p,q} : \quad y^p + x^q + \ldots = 0.$$  

The one-matrix model only generates hyperelliptic spectral curves, whereas the two-matrix model includes all possible combinations of $p$ and $q$. These double scaled multi-matrix models are known to describe non-critical ($c < 1$) bosonic string theory based on the $(p, q)$ minimal model coupled to two-dimensional gravity [180, 181, 182, 183, 184, 185] (reviewed extensively in e.g. [186, 187]). This field is called minimal string theory.

Zooming in on a single branch point yields the geometry

$$\Sigma_{p,1} : \quad y^p = x,$$
corresponding to the \((p, 1)\) topological minimal model. This model is strictly not a well-defined conformal field theory, but does make sense as 2d topological field theory. For \(p = 2\) it is known as topological gravity [188, 189, 190, 165].

All \((p, q)\) minimal models turn out to be governed by two differential operators

\[
P = (\lambda \partial_x)^p + u_{p-2}(x)(\lambda \partial_x)^{p-2} + \ldots + u_0(x),
\]

\[
Q = (\lambda \partial_x)^q + v_{q-2}(x)(\lambda \partial_x)^{q-2} + \ldots + v_0(x),
\]

of degree \(p\) and \(q\) respectively, which obey the string (or Douglas) equation

\[
[P, Q] = \lambda.
\]

\(P\) and \(Q\) depend on an infinite set of times \(t = (t_1, t_2, t_3, \ldots)\), which are closed string couplings in minimal string theory, and evolve in these times as

\[
\frac{\lambda}{\partial t_j} P = [(P^{j/p})_+, P],
\]

\[
\frac{\lambda}{\partial t_j} Q = [(P^{j/p})_+, Q],
\]

The fractional powers of \(P\) define a basis of commuting Hamiltonians. This integrable system defines the \(p\)-th KdV hierarchy and the above evolution equations are the KdV flows.

The differential operator \(Q\) is completely determined as a function of fractional powers of the Lax operator \(P\) and the times \(t\)

\[
Q = - \sum_{j \geq 1, j \neq 0 \mod p} \left(1 + \frac{j}{p}\right) t_{j+p} P^{j/p}.
\]

This implies that when we turn off all the KdV times except for \(t_1 = x\) and fix \(t_{p+1}\) to be constant we find \(Q = \lambda \partial_x\). This defines the \((p, 1)\)-models

\[
P = (\lambda \partial_x)^p - x, \quad Q = \lambda \partial_x. \tag{6.3}
\]

One can reach any other \((p, q)\) model by flowing in the times \(t\).

The partition function of the \(p\)-th KdV hierarchy is a tau-function as in equation (5.31). The associated subspace \(\mathcal{W} \in Gr\) may be found by studying the eigenfunctions \(\psi(t, z)\) of the Lax operator \(P\)

\[
P \psi(t, z) = z^p \psi(t, z).
\]

The Baker function \(\psi_\lambda(t, z)\) represents the fermionic field that sweeps out the subspace \(\mathcal{W}\) in the times \(t\).
If we restrict to the \((p, 1)\)-models the Baker function \(\psi(x, z)\) can be expanded in a Taylor series

\[
\psi(x, z) = \sum_{k=0}^{\infty} v_k(z) \frac{x^k}{k!}.
\]

Since \(\psi(x, z)\) is an element of \(\mathcal{W}\) for all times, this defines a basis \(\{v_k(z)\}_{k \geq 0}\) of the subspace \(\mathcal{W}\). In fact, it is not hard to see that the \((p, 1)\) Baker function is given by the generalized Airy function

\[
\psi(x, z) = e^{pzp + 1} \sqrt{zp} \int dw e^{\frac{(z+w)^p + 1}{zp + 1} + \frac{z^p + 1}{zp + 1}},
\]

which is normalized such that its Taylor components \(v_k(z)\) can be expanded as

\[
v_k(z) = z^k(1 + O(\lambda/z^{p+1})).
\]

The \((p, 1)\) model thus determines the fermionic state

\[
|W\rangle = v_0 \wedge v_1 \wedge v_2 \wedge \ldots,
\]

where the \(v_k(z)\) can be written explicitly in terms of Airy-like integrals (see \[165\] for a nice review). The invariance under

\[
z^p \cdot \mathcal{W} \subset \mathcal{W}
\]

characterizes this state as coming from a \(p\)-th KdV hierarchy. In the other direction, the state \(|W\rangle\) determines the Baker function (and thus the Lax operator) as the one-point function

\[
\psi(t, z) = \langle t|\psi(z)|W\rangle.
\]

In the dispersionless limit \(\lambda \to 0\) the derivative \(\lambda \partial_x\) is replaced by a variable \(d\), and the Dirac commutators by Poisson brackets in \(x\) and \(d\). The leading order contribution to the string equation is given by the Poisson bracket

\[
\{P_0, Q_0\} = 1,
\]

where \(P_0\) and \(Q_0\) equal \(P\) and \(Q\) at \(\lambda = 0\). The solution to this equation is

\[
P_0(d; t) = x
\]

\[
Q_0(d; t) = y(x; t)
\]

and recovers the genus zero spectral curve \(\Sigma_{p,q}\) of the double scaled matrix model, parametrized by \(d\). The KdV flows deform this surface such a way that
6.1. Matrix model geometries

its singularities are preserved. (See the appendix of [171] for a detailed discussion.)

Note that $\Sigma_{p,q}$ is not a spectral curve for the Krichever map. The Krichever curve is instead found as the space of simultaneous eigenvalues of the differential operators

$$[P, Q] = 0,$$

that is preserved by the KdV flow as a straight-line flow along its Jacobian. In fact, there is no such Krichever spectral curve corresponding to the doubled scaled matrix model solutions.

Wrapping an I-brane around $\Sigma_{p,q}$ quantizes the semi-classical fermions on the spectral curve $\Sigma_{p,q}$. The only point at infinity on $\Sigma_{p,q}$ is given by $x \to \infty$. The KdV tau-function should thus be the fermionic determinant of the quantum state $|\mathcal{W}\rangle$ that corresponds to this $D$-module. In the next subsection we write down the $D$-module describing the $(p, 1)$ model and show precisely how this reproduces the tau-function using the formalism developed in Chapter 5.

6.1.2 $D$-module for topological gravity

We are ready to reconstruct the $D$-module that yields the fermionic state $|\mathcal{W}\rangle$ in equation (6.5). For simplicity we study the $(2, 1)$-model, associated to an I-brane wrapping the curve

$$\Sigma_{(2,1)} : \quad y^2 = x \quad \text{with } x, y \in \mathbb{C}.$$

Notice that this is an $2 : 1$ cover over the $x$-plane. It contains just one asymptotic region, where $x \to \infty$. Fermions on this cover will therefore sweep out a subspace $\mathcal{W}$ in the Hilbert space

$$\mathcal{W} \subset \mathcal{H}(S^1) = \mathbb{C}((y^{-1})),$$

the space of formal Laurent series in $y^{-1}$. The fermionic vacuum $|0\rangle \subset \mathcal{H}(S^1)$ corresponds to the subspace

$$|0\rangle = y^{1/2} \wedge y^{3/2} \wedge y^{5/2} \wedge \ldots,$$

which encodes the algebra of functions on the disk parametrized by $y$ and with boundary at $y = \infty$. Exponentials in $y^{-1}$ represent non-trivial behaviour near the origin and therefore act non-trivially on the vacuum state. In contrast, exponentials in $y$ are holomorphic on the disk and thus act trivially on the vacuum. The $B$-field $B = \frac{1}{\chi} dx \wedge dy$ quantizes the algebra of functions on $\mathbb{C}^2$ into the
differential algebra

\[ D_\lambda = \langle x, \lambda \partial_x \rangle. \]

Furthermore, it introduces a holomorphic connection 1-form \( A = \frac{1}{\lambda} y dx \) on \( \Sigma_{(2,1)} \), which pushes forward to the rank two \( \lambda \)-connection

\[ \nabla_A = \lambda \partial_x - \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \] \tag{6.6}

on the base \( \mathbb{C} \), parametrized by \( x \). We claim that the corresponding \( D_\lambda \)-module \( \mathcal{M} \), generated by

\[ P = (\lambda \partial_x)^2 - x, \]

describes the \((2,1)\) model. Let us verify this.

Trivializing the \( \lambda \)-connection \( \nabla_A \) in [6.6] implies finding a rank two matrix \( g(x) \) such that

\[ \nabla_A = \lambda \partial_x - g'(x) \circ g^{-1}(x). \]

The columns of \( g \) define a basis of solutions \( \Psi(x) \) to the differential equation \( \nabla_A \Psi(x) = 0 \). They are meromorphic flat sections for \( \nabla_A \) that determine a trivialization of the bundle near \( x = \infty \). As the connection \( \nabla_A \) is pushed forward from the cover, \( \Psi(x) \) is of the form

\[ \Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}. \]

Independent solutions have different asymptotics in the semi-classical regime where \( x \to \infty \). In the \((2,1)\)-model the two independent solutions \( \psi_\pm(x) \) solve the differential equation

\[ P \psi_\pm(x) = ((\lambda \partial_x)^2 - x) \psi_\pm(x) = 0. \]

Hence these are the functions \( \psi_+(x) = \text{Ai}(x) \) and \( \psi_-(x) = \text{Bi}(x) \), that correspond semi-classically to the two saddles

\[ w_\pm = \pm \sqrt{x/\lambda^{1/3}} \]

of the Airy integral

\[ \psi(x) = \frac{1}{2\pi i} \int dw \ e^{-\frac{w^3}{3\sqrt{\pi}} + \frac{w^3}{3}}. \]
The $\mathcal{D}$-module $\mathcal{M}$ can be quantized into a fermionic state for any choice of boundary conditions. Depending on this choice we find an $\mathcal{O}(x)$-module $\mathcal{W}_\pm$ spanned by linear combinations of $\psi_\pm(x)$ and of $\psi'_\pm(x)$. The fermionic state is generated by asymptotic expansions in the parameter $\lambda$ of these elements.

The saddle-point approximation around the saddle $w_\pm = \pm \sqrt{x}/\lambda^{1/3}$ yields

$$
\psi_\pm(x) \sim y^{-1/2} e^{\pm \frac{2y^3}{3\lambda}} \left( 1 + \sum_{n \geq 1} c_n \lambda^n (\pm y)^{-3n} \right)
\sim y^{-1/2} e^{\pm \frac{2y^3}{3\lambda}} v_0(\pm y).
$$

To see the last step just recall the definition of $v_0(z)$ as being equal to the Baker function $\psi(x, z)$ evaluated at $x = 0$.

A similar expansion can be made for $\psi'(x)$ with the result

$$
\psi'_\pm(x) \sim y^{1/2} e^{\pm \frac{2y^3}{3\lambda}} v_1(\pm y).
$$

Note that both expansions in $\lambda$ are functions in the coordinate $y$ on the cover. They contain a classical term (the exponential in $1/\lambda$), a 1-loop piece and a quantum expansion in $\lambda y^{-3}$. When we restrict to the saddle $w = \sqrt{x}/\lambda^{1/3}$, these series blend the into the fermionic state

$$
|\mathcal{W}_+\rangle = \psi_+(y) \wedge \psi'_+(y) \wedge y^2 \psi_+(y) \wedge y^2 \psi'_+(y) \wedge \ldots.
$$

Does this agree with the well-known result (6.5)?

First of all, notice that the basis vectors $x^k \psi(x)$ and $x^k \psi'(x)$, with $k > 0$, contain in their expansions the function $v_k(y)$ plus a sum of lower order terms in $v_l(y)$ (with $l < k$). The wedge product obviously eliminates all these lower order terms. Secondly, the extra factor $y^{-1/2}$ factors just reminds us that we have written down a fermionic state.

Furthermore, the WKB exponentials are exponentials in $y$ and thus elements of $\Gamma_+$, whereas the expansions $v_k(y)/y^l$ are part of $\Gamma_-$. Up to normal ordering ambiguities this shows that the WKB part gives a trivial contribution. In fact, the tau-function even cancels these ambiguities.

This shows that

$$
|\mathcal{W}_+\rangle = v_0(y) \wedge v_1(y) \wedge v_2(y) \wedge \ldots,
$$

which is indeed the same as in equation (6.5), when we change variables from $z$ to $y$ in that equation. Of course, this doesn’t change the tau-function.

\footnote{Remark that $x$ and $z^2$ appear equivalently in $\psi(x, z)$ in equation (6.4), while $\psi(x)$ and $\psi(x, z)$ only differ in the normalization term in $z$.}
So our conclusion is that the $\mathcal{D}$-module underlying topological gravity is the canonical $\mathcal{D}$-module

$$\mathcal{M} = \frac{\mathcal{D}_\lambda}{\mathcal{D}_\lambda((\lambda \partial_x)^2 - x)}.$$  

This $\mathcal{D}$-module gives the definition of the quantum curve corresponding to the $(2, 1)$ model and defines its quantum partition function in an expansion around $\lambda$. Exactly the same reasoning holds for the $(p, 1)$-model, where we find a canonical rank $p$ connection on the base. It would be good to be able to write down a $\mathcal{D}$-module for general $(p, q)$-models as well.

### 6.1.3 $\mathcal{D}$-module for Hermitean matrix models

$\mathcal{D}$-modules continue to play an important role in any Hermitean matrix model. In this subsection we are guided by [191] and [192, 193] of Bertola, Eynard and Harnad.

We first summarize how the partition function for a 1-matrix model defines a tau-function for the KP hierarchy. As we saw before, such a tau-function corresponds to a fermionic state $|\mathcal{W}\rangle$, whose basis elements we will write down. Following [191] we discover a rank two differential structure in this basis, whose determinant reduces to the spectral curve in the semi-classical limit. This $\mathcal{D}$-module structure is somewhat more complicated then the $\mathcal{D}$-module we just found describing double scaled matrix models.

We continue with 2-matrix models, based on [193]. Instead of one differential equation, these models determine a group of four differential equations, that characterize the $\mathcal{D}$-module in the local coordinates $z$ and $w$ at infinity. The matrix model partition function may of course be computed in either frame.

### 1-matrix model

Let us start with the 1-matrix model partition function

$$Z_N = \frac{1}{\text{vol}(U(N))} \int DM \ e^{-\frac{1}{\alpha} \text{Tr} W(M)}.$$  \hfill (6.7)

By diagonalizing the matrix $M$ the matrix integral may be reduced to an integral over the eigenvalues $\lambda_i$

$$Z_N = \frac{1}{N!} \int \prod_{i=1}^N \frac{d\lambda_i}{2\pi} \Delta(\lambda)^2 e^{-\frac{1}{\alpha} \sum_i W(\lambda_i)},$$

with the Vandermonde determinant $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = \det(\lambda_i^{j-1})$. The method of orthogonal polynomials solves this integral by introducing an infinite
set of polynomials \( p_k(x) \), defined by the properties

\[
p_k(x) = x^k (1 + \mathcal{O}(x^{-1})),
\]

\[
\int dx \ p_k(x) \ p_l(x) \ e^{-\frac{1}{2\lambda} W(x)} = 2\pi h_k \delta_{k,l}.
\]

The normalization of their leading term determines the coefficients \( h_n \in \mathbb{C} \). Since the Vandermonde determinant \( \Delta(x) \) is not sensitive to exchanging its entries \( x_i^{j-1} \) for \( p_j-1(x_i) \), substituting \( \Delta(x) = \det(p_j-1(x_i)) \) turns the partition function into a product of coefficients

\[
Z_N = \prod_{k=0}^{N-1} h_k.
\]

With the help of orthogonal polynomials the large \( N \) behaviour of \( Z_N \) may be studied, while keeping track of \( 1/N \) corrections.

The orthogonal polynomials are crucial since they build up a basis for the fermionic KP state. In an appendix of [191] it is shown that one should start at \( t = 0 \) with a state \( |W_0\rangle \) generated by the polynomials \( p_k(x) \) for \( k \geq N \)

\[
|W_0\rangle = p_N(x) \wedge p_{N+1}(x) \wedge p_{N+2}(x) \wedge \ldots.
\]

Notice that the vector \( p_N(x) \) thus corresponds to the Fermi level and defines the Baker function in the double scaling limit. Acting on them with the commuting flow generated by

\[
\Gamma_+ = \left\{ g(t) = e^{\sum_{n \geq 1} \frac{1}{n} t_n x^n} \right\}
\]

defines a state \( |W_t\rangle = |g(t)W_0\rangle \) at time \( t \), which allows to compute a tau-function at time \( t \). If the coefficients \( u_j \) in the potential \( W(x) \) are taken to be \( u_j = u_j^{(0)} + t_j \), this \( \tau \)-function equals the ratio of the matrix model partition function \( Z_N \) at time \( t \) divided by that at \( t = 0 \).

Multiplying the orthogonal polynomials by \( \exp\left(-\frac{1}{2\lambda} W(x)\right) \) doesn’t change the fermionic state \( W = W_0 \) in a relevant way, since this factor is an element of \( \Gamma_+ \). To find the right \( \mathcal{D} \)-module structure, it is necessary to proceed with the quasi-polynomials

\[
\psi_k(x) = \frac{1}{\sqrt{h_k}} p_k e^{-\frac{1}{2\lambda} W(x)},
\]

which form an orthonormal basis with respect to the bilinear form

\[
(\psi_k, \psi_l) = \int dx \ \psi_k \psi_l. \tag{6.8}
\]
It is possible to express both multiplication by $x$ and differentiation with respect to $x$ in terms of the basis of $\psi_m$'s. The Weyl algebra $\langle x, \lambda \partial_x \rangle$ acts on these (quasi)-polynomials by two matrices $Q$ and $P$

$$
 x \psi_k(x) = \sum_{l=0}^{\infty} Q_{kl} \psi_l \\
 \lambda \partial_x \psi_k(x) = \sum_{l=0}^{\infty} P_{kl} \psi_l(x),
$$

and the space of quasi-polynomials $\psi_k$ is thus a $D_\lambda$-module.

Notice that we anticipate that the $D$-module possesses a rank two structure, since we started with a flat connection $A = \frac{1}{\lambda} y dx$ on an I-brane wrapped on a hyperelliptic curve.

Now, the matrices $Q$ and $P$ only contain non-zero entries in a finite band around the diagonal. The action of $\lambda \partial_x$ on the semi-infinite set of $\psi_k(x)$'s can therefore indeed be summarized in a rank two differential system ([191] and references therein)

$$
\lambda \partial_x \left[ \begin{array}{c} \psi_N(x) \\ \psi_{N-1}(x) \end{array} \right] = A_N(x) \left[ \begin{array}{c} \psi_N(x) \\ \psi_{N-1}(x) \end{array} \right],
$$

(6.9)

where $A_N(x)$ is a rather complicated $2 \times 2$-matrix involving the derivative $W'$ of the potential and the infinite matrix $Q$:

$$
A_N(x) = \frac{1}{2} W'(x) \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] + \gamma_N \left[ \begin{array}{cc} -\tilde{W}'(Q,x)_{N,N-1} & \tilde{W}'(Q,x)_{N,N} \\ -\tilde{W}'(Q,x)_{N-1,N-1} & \tilde{W}'(Q,x)_{N-1,N} \end{array} \right],
$$

with

$$
\tilde{W}'(Q,x) = \left( \frac{W'(Q) - W'(x)}{Q - x} \right) \quad \text{and} \quad \gamma_N = \sqrt{\frac{h_N}{h_{N-1}}}.
$$

Equation (6.9) is thus the rank two $\lambda$-connection defining the $D_\lambda$-module structure on $\mathcal{W}$ that we were searching for! As a check, the determinant of this connection reduces to the spectral curve in the semiclassical, or dispersionless, limit [191]:

$$
\Sigma_N : \quad 0 = \det (y_{12 \times 2} - A_N(x))
$$

$$
= y^2 - W'(x)^2 + 4 \lambda \sum_{j=0}^{N-1} \left( \frac{W'(Q) - W'(x)}{Q - x} \right)_{jj}
$$

(To make the coefficients in the above equation agree with (6.2), we rescaled
y \mapsto y/2.) In conclusion we found the D-module structure underlying Hermitean 1-matrix models.

Remark that in the $N \to \infty$ limit we expect that the hyperelliptic curve defining the B-model Calabi-Yau (6.2) emerges from $\Sigma_N$. Indeed, in the 't Hooft limit $Q$ corresponds classically to the coordinate $x$ on the curve, whereas quantum-mechanically it is an operator whose spectrum is described by the eigenvalues $\lambda_i$ of the infinite matrix $M$. In the large $N$ limit we can therefore replace the matrix $Q_{ij}$ in the definition for $\Sigma_N$ by $\lambda_i \delta_{ij}$.

We can rewrite the rank two connection for the vector $(\psi_N, \psi'_N)^t$ as

$$\lambda \partial_x \begin{bmatrix} \psi_N(x) \\ \psi'_N(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\det(A_N(x)) + \lambda Y & \lambda Z \end{bmatrix} \begin{bmatrix} \psi_N(x) \\ \psi'_N(x) \end{bmatrix},$$

at least when $\tr(A_N(x)) = 0$, with $Y$ and $Z$ some derivatives of entries of $A_N(x)$. This brings the $\lambda$-connection in the familiar form of Chapter 5. In the next subsection we clarify the differential structure in a simple example.

### 2-matrix model

Let us first say a few words on the $D$-module structure underlying multi-matrix models, which capture spectral curves of any degree in $x$ and $y$ \[192, 193\]. The partition function for a two-matrix model, with two rank $N$ matrices $M_1$ and $M_2$, is

$$Z_N = \frac{1}{\Vol(U(N))^2} \int D M_1 D M_2 \ e^{-\frac{1}{\lambda} \Tr(W_1(M_1) + W_2(M_2) - M_1 M_2)},$$

where $W_1$ and $W_2$ are two potentials of degree $d_1 + 1$ and $d_2 + 1$. Choosing $W_2$ to be Gaussian reduces the 2-matrix model to a 1-matrix model. The 2-matrix model is solved by introducing two sets of orthogonal polynomials $\pi_k(x)$ and $\sigma_k(y)$. Again it is convenient to turn them into quasi-polynomials

$$\psi_k(x) = \pi_k(x) e^{-\frac{1}{\lambda} W_1(x)}, \quad \phi_k(y) = \sigma_k(y) e^{-\frac{1}{\lambda} W_2(y)}.$$

obeying the orthogonality relations

$$\int dx dy \ \psi_k(x) \phi_l(y) e^{\frac{x y}{\lambda}} = h_k \delta_{kl}. \quad (6.10)$$

Multiplying with or taking a derivative with respect to either $x$ or $y$ yields (just) two operators $Q$ and $P$ (and their transposes because of (6.10)), that form a representation of string equation $[P, Q] = 0$. Since $Q$ is only non-zero in a band around the diagonal of size $d_2 + 1$ and $P$ of size $d_1 + 1$, the quasi-polynomials
may be folded into the vectors 
\[
\vec{\psi} = [\psi_N, \ldots, \psi_{N-d_2}]^t, \quad \vec{\phi} = [\phi_N, \ldots, \phi_{N-d_1}]^t.
\]

Any other quasi-polynomial can be expressed as a sum of entrees of these vectors, with coefficients in the polynomials in \( x \) and \( y \). These vectors are called windows. The differential operators \( \lambda \partial_x \) and \( \lambda \partial_y \) respect them, so that their action is summarized in a rank \( d_2 + 1 \) resp. rank \( d_1 + 1 \) \( \lambda \)-connection

\[
\lambda \partial_x \vec{\psi}(x) = A_1(x) \vec{\psi}(x), \quad \lambda \partial_y \vec{\phi}(y) = A_2(y) \vec{\phi}(x).
\]

This we interpret as two representations of the \( D_\lambda \)-module underlying 2-matrix models. Indeed, [192] proves that the determinant of both differential systems equals the same spectral curve \( \Sigma \), in the limit \( \lambda \to 0 \) when we replace \( \lambda \partial_x \to y \) and \( \lambda \partial_y \to x \). The defining equation of \( \Sigma \) is of degree \( d_1 + 1 \) in \( x \) and of degree \( d_2 + 1 \) in \( y \).

In fact, it is useful to introduce two more semi-infinite sets of quasi-polynomials \( \psi_k(x) \) and \( \phi_k(y) \), as the Fourier transforms of \( \psi_k(x) \) and \( \phi_k(y) \) respectively. The action of the Weyl algebra on them may be encoded as the transpose of the above linear systems. The full system can therefore be summarized by (compare to (6.22))

\[
x\text{-axis} : \quad \{ \psi_k(x), \phi_k(x) \}, \quad \nabla_\lambda = \lambda \partial_x - A_1(x),
\]

\[
y\text{-axis} : \quad \{ \phi_k(y), \psi_k(y) \}, \quad \nabla_\lambda = \lambda \partial_y - A_2(y).
\]

Moreover, the matrix model partition function can be rewritten as a fermionic correlator in either local coordinate

\[
Z_N \propto \frac{1}{N!} \int \prod_i d\lambda_1^i d\lambda_2^i \Delta(\lambda^1) \Delta(\lambda^2) e^{-\frac{1}{\lambda} \sum_i W_1(\lambda_1^i) + W_2(\lambda_2^i) - \lambda_1^i \lambda_2^i} = \prod_{k=0}^{N-1} \langle \psi_k(x) | \phi_k(x) \rangle = \prod_{k=0}^{N-1} \langle \phi_k(y) | \psi_k(y) \rangle
\]

with respect to the bilinear form in (6.8).

Furthermore, Bertola, Eynard and Harnad study the dependence on the parameters \( u_j^{(1)} \) and \( u_j^{(2)} \) appearing in the potentials \( W_1 \) and \( W_2 \). Deformations in these parameters leave the two sets of quasi-polynomials invariant as well. On \( \vec{\psi} \) and \( \vec{\phi} \) they act as matrices \( U_j^{(1)} \) and \( U_j^{(2)} \). This yields the 2-Toda system

\[
\partial_{u_j^{(1)}} Q = -[Q, U_j^{(1)}] \quad \partial_{u_j^{(2)}} P = -[P, U_j^{(2)}] \quad \partial_{u_j^{(1)}} P = [P, U_j^{(1)}].
\]
In [192] it is proved that the linear differential systems (6.11) are compatible with these deformations, so that the parameters $u_j^{(1)}$ and $u_j^{(2)}$ in fact generate isomonodromic deformations. This shows precisely how the non-normalizable parameters in the potential respect the central role of the $D_\lambda$-module (6.11) in the 2-matrix model.

### 6.1.4 Gaussian example

Let us consider the Gaussian 1-matrix model with quadratic potential

$$W(x) = \frac{x^2}{2}, \quad (6.12)$$

that is associated to the spectral curve

$$y^2 = x^2 - 4\mu \quad (6.13)$$

in the large $N$ limit. In the Dijkgraaf-Vafa correspondence this matrix model is thus dual to the topological B-model on the deformed conifold geometry (see Fig. 4.5).

The Hermite functions

$$\psi^\lambda_k(x) = \frac{1}{\sqrt{h_k}} e^{-\frac{x^2}{4\lambda}} H^\lambda_k(x), \quad \text{with}$$

$$H^\lambda_k(x) = \lambda^{k/2} H_k \left( \frac{x}{\sqrt{\lambda}} \right) = x^k \left( 1 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right),$$

form an orthogonal basis for this model. Their inner product is given by

$$\int \frac{dx}{2\pi} \psi^\lambda_k(x) \psi^\lambda_l(x) = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}} \delta_{kl} \implies h_k = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}}.$$

The partition function of the Gaussian matrix model can be computed as a product of the normalization constants $h_k$. Using the asymptotic expansion of the Barnes function $G_2(z)$, that is defined by $G_2(z + 1) = \Gamma(z)G_2(z)$, the free energy can be expanded in powers of $\lambda$

$$\mathcal{F}_N = \log \prod_{k=1}^{N-1} h_k = \log \left( G_2(N + 1) \frac{\lambda^{N^2/2}}{(2\pi)^{N/2}} \right) \quad (6.14)$$

$$= \frac{1}{2} \left( \frac{\mu}{\lambda} \right)^2 \left( \log \mu - \frac{3}{2} \right) + \frac{1}{12} \log \mu + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g - 2)} \left( \frac{\lambda}{\mu} \right)^{2g-2},$$

where $B_{2g}$ are the Bernoulli numbers and $\mu = N\lambda$. 
The derivatives of the Hermite functions are related as
\[ \lambda \frac{d}{dx} \begin{bmatrix} \psi^k_\lambda(x) \\ \psi^{-1}_\lambda(x) \end{bmatrix} = \begin{bmatrix} -x/2 & \sqrt{k \lambda} \\ \sqrt{k \lambda} & x/2 \end{bmatrix} \begin{bmatrix} \psi^k_\lambda(x) \\ \psi^{-1}_\lambda(x) \end{bmatrix}. \]

So, according to the previous discussion, the \( D_\lambda \)-module connection is given by
\[ \lambda \frac{d}{dx} - A_N(x) = \lambda \frac{d}{dx} + \begin{bmatrix} x/2 & -\sqrt{N \lambda} \\ \sqrt{N \lambda} & -x/2 \end{bmatrix}. \] (6.15)

Here we choose \( \vec{\psi} = [\psi_N, \psi_{N-1}]^t \) as window. In the large \( N \) limit the determinant of this rank two differential system indeed yields the spectral curve (6.13) with \( \mu = N \lambda \).

Instead of using \( \psi^k_\lambda \) and \( \psi^{-1}_\lambda \) as a basis, we can also write down the differential system for \( \psi^k_\lambda \) and its derivative \( \psi'_\lambda(x) = \lambda \partial_x \psi^k_\lambda(x) \). Since this derivative is a linear combination of \( \psi^{-1}_\lambda \) and \( x \psi^k_\lambda(x) \) (as we saw above), it is equivalent to use this basis to generate the fermionic state \( \mathcal{W} \). We compute that
\[ \lambda \frac{d}{dx} \begin{bmatrix} \psi^k_\lambda(x) \\ \psi'_\lambda(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ x^2 - \lambda N - \lambda/2 & 0 \end{bmatrix} \begin{bmatrix} \psi^k_\lambda(x) \\ \psi'_\lambda(x) \end{bmatrix}. \]

The spectral curve in the large \( N \) limit hasn’t changed. Notice that in this form it is clear that the rank 2 connection is the push-forward of the connection \( A = \frac{1}{\lambda} ydx \) on the spectral curve \( y^2 = x^2 - 4 \mu \) to the \( \mathbb{C} \)-plane, up to some \( \lambda \)-corrections.

In the double scaling limit the limits \( N \to \infty \) and \( \lambda \to 0 \) are not independent as in the ’t Hooft limit, but correlated, such that the higher genus contributions to the partition function are taken into account. In terms of the Gaussian spectral curve this limit implies that one zooms in onto one of the endpoints of the cuts. The orthogonal function \( \psi^k_\lambda(x) \) turns into the Baker function \( \psi(x) \) of the double scaled state \( \mathcal{W} \).

In the Gaussian matrix model this is implemented by letting \( x \to \sqrt{\mu} + \epsilon x \), where \( \epsilon \) is a small parameter. So the double scaled spectral curve reads
\[ y^2 = x, \]
while the differential system reduces to
\[ \lambda \frac{d}{dx} \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}. \]

This is indeed the \( D \)-module corresponding to the \((2,1)\)-model.
### 6.2 Conifold and $c = 1$ string

The free energy (6.14) of the Gaussian matrix model pops up in the theory of bosonic $c = 1$ strings. This $c = 1$ string theory is formulated in terms of a single bosonic coordinate $X$, that is compactified on a circle of radius $r$ in the Euclidean theory. A critical bosonic string theory (with $c = 26$) is obtained by coupling the above CFT to a Liouville field $\phi$. The Liouville field corresponds to the non-decoupled conformal mode of the worldsheet metric. The local worldsheet action reads

$$
\frac{1}{4\pi} \int d^2 \sigma \left( \frac{1}{2} (\partial X)^2 + (\partial \phi)^2 + \mu e^{\sqrt{2} \phi} + \sqrt{2} \phi R \right),
$$

where the coupling $\mu$ is seen as the worldsheet cosmological constant. In the Euclidean model there are only two sets of operators, that describe the winding and momenta modes of the field $X$. These vertex and vortex operators can be added to the action as marginal deformations with coefficients $t_n$ and $\tilde{t}_n$.

Just like in $c < 1$ minimal string theories (the $(p,q)$-models of last section), the partition function of the $c = 1$ string is first computed using a dual matrix model description [194]. At the self-dual radius $r = 1$ it agrees with the Gaussian matrix model partition function in equation (6.14), where $\lambda$ now plays the role of the $c = 1$ string coupling constant.

The matrix model dual to the $c = 1$ string is called matrix quantum mechanics. This duality is reviewed in much detail in e.g. [195, 196, 197]. Matrix quantum mechanics is described by a gauge field $A$ and a scalar field $M$ that are both $N \times N$ Hermitean matrices. The momentum modes of the $c = 1$ string correspond to excitations of $M$, whereas the winding modes are excitations of $A$. If we focus on the momentum modes, the (double scaled) matrix model is governed by the Hamiltonian

$$
H = \frac{1}{2} \text{Tr} \left( -\lambda^2 \frac{\partial^2}{\partial M^2} - M^2 \right).
$$

Let us focus on solutions that depend purely on the eigenvalues $\lambda_i$ of $M$. The Hamiltonian may be rewritten in terms of the eigenvalues as

$$
H = \frac{1}{2} \Delta^{-1}(\lambda) \sum_i \left( -\lambda^2 \frac{\partial^2}{\partial \lambda_i^2} - \lambda_i^2 \right) \Delta(\lambda),
$$

where $\Delta(\lambda)$ is Vandermonde determinant. It is convenient to absorb the factor $\Delta$ in the wavefunction solutions, making them anti-symmetric. Hence, the singlet sector of matrix quantum mechanics describes a system of $N$ free fermions in an upside-down Gaussian potential.
To describe the partition function of the \( c = 1 \) model it is convenient to move over to light-cone coordinates \( \lambda_{\pm} = \lambda \pm p \), so that elementary excitations of the \( c = 1 \) model are represented as collective excitations of free fermions near the Fermi level
\[
\lambda_+ \lambda_- = \mu. \quad (6.16)
\]
When we restrict to \( \lambda_{\pm} > 0 \), scattering amplitudes can be computed by preparing asymptotic free fermionic states \( \langle \tilde{t} | \) and \( | t \rangle \) at the regions where one of \( \lambda_{\pm} \) becomes very large.

In this picture the generating function of scattering amplitudes has a particularly simple form. It can be formulated as a fermionic correlator \[198\]
\[
Z = \langle t | S | \tilde{t} \rangle, \quad (6.17)
\]
where the fermionic scattering matrix \( S \in GL(\infty, \mathbb{C}) \) was first computed in \[199\]. Moreover, in \[200\] (see also Chapter V of \[197\]) and later in \[128\] it is noticed that \( S \) just equals the Fourier transformation
\[
(S \psi)(\lambda_-) = \int d\lambda_+ e^{\frac{1}{2} \lambda_- \lambda_+} \psi(\lambda_+). \quad (6.18)
\]
In the next section we show that this follows naturally from the perspective of \( \mathcal{D} \)-modules.

The result \((6.17)\) shows that \( c = 1 \) string theory is an integrable system, just like the \((p,q)\)-models in the last section. Since it depends on two sets of times this integrable system is not a KP system. Instead, the above expression defines a tau function of a 2-Toda hierarchy.

Notice that the Fermi level \((6.16)\) is a real cycle on the complex curve
\[
\Sigma : \quad zw = \mu, \quad (6.19)
\]
which is a different parametrization of the spectral curve \( y^2 = x^2 - \mu \) of the Gaussian 1-matrix model. In the revival of this subject a few years ago, a number of other matrix model interpretations have been found. This includes a duality with the Hermitean 2-matrix model, which makes the 2-Toda structure manifest \[201\], a Kontsevich-type model \[202, 203\] at the self-dual radius, and a so-called normal matrix model \[204, 205\], that parametrizes the dual real cycle on the complex curve \( \Sigma \). Let us also mention that the well-known duality of the \( c = 1 \) string with the topological B-model on the deformed conifold \[206\], that follows, with a detour, from the more general Dijkgraaf-Vafa correspondence.
6.2. Conifold and \( c = 1 \) string

This paragraph reproduces the \( c = 1 \) partition function (6.17) from a \( \mathcal{D} \)-module point of view. The discussion continues the line of thought in Section 5.5 of [128].

As we have just seen, the \( c = 1 \) string is geometrically characterized by the presence of a holomorphic curve in \( \mathbb{C} \times \mathbb{C} \) defined by

\[ \Sigma_{c=1} : zw = \mu. \]

Let us consider an I-brane wrapping the curve \( \Sigma_{c=1} \). When we assume \( z \) as local coordinate the curve quantizes into the differential operator

\[ P = -\lambda z \partial_z - \mu. \quad (6.20) \]

It is amusing that the differential operator \( P \) appears as a canonical example in the theory of \( \mathcal{D} \)-modules (see e.g. [159]) in the same way as the \( c = 1 \) string is an elementary example of a string theory.

We recognize this example from Chapter 5 where a \( \mathcal{D} \)-module was associated to the differential operator \( P \). However, now it is important not to forget that there are two asymptotic points \( z_\infty \) and \( w_\infty \). Let us call their local neighbourhoods \( U_z \) and \( U_w \), as local coordinates are \( z \) and \( w \) respectively. At both asymptotic points the I-brane fermions will sweep out an asymptotic state. The quantum partition function should therefore be constructed from two quantum states.

Before constructing these states for general \( \lambda \), let us first consider the semi-classical limit \( \lambda \to 0 \). In this limit the I-brane degrees of freedom are just conventional chiral fermions on \( \Sigma_{c=1} \). The genus 1 part \( F_1 \) of the free energy is obtained as the partition function of these semi-classical fermions. It can be computed by assigning the Dirac vacuum

\[ |0\rangle_z = z^{1/2} \wedge z^{3/2} \wedge z^{5/2} \wedge \ldots \]

to \( U_z \) and likewise the conjugate state

\[ |0\rangle_w = w^{1/2} \wedge w^{3/2} \wedge w^{5/2} \wedge \ldots \]

to \( U_w \). To compare these states, we need an operator \( S \) that relates \( z \) to \( 1/z \).

The semi-classical partition can then be computed as a fermionic correlator \( \langle 0|S|0\rangle_z \), with the result that

\[ e^{F_1} = \langle 0|S|0\rangle_z = \prod_{k \geq 0} \mu^{k+1/2}. \quad (6.21) \]

Using \( \zeta \)-function regularization we find that this expression yields the familiar
answer $\mathcal{F}_1 = -\frac{1}{12} \log \mu$.

In order to go beyond 1-loop, we should think in terms of $\mathcal{D}$-modules. Let us for a moment not represent their elements in terms of differential operators yet. In both asymptotic regions we then find the $\mathcal{D}$-modules

\begin{align*}
U_z : \quad & \mathcal{M} = \mathcal{D}/\mathcal{D} P, \quad \text{with} \quad P = \hat{z}\hat{w} - \mu, \\
U_w : \quad & \mathcal{M} = \mathcal{D}/\mathcal{D} P, \quad \text{with} \quad P = \hat{w}\hat{z} - \mu + \lambda.
\end{align*}

Notice that the Weyl algebra $\mathcal{D} = \langle \hat{z}, \hat{w} \rangle$, with the relation $[\hat{z}, \hat{w}] = \lambda$, acts on monomials $z^k$ and $w^k$ in the module $\mathcal{M}$ as

\begin{align*}
\hat{z}(z^k) &= z^{k+1}, & \hat{z}(w^k) &= \left(\lambda \partial_{\hat{w}} + \frac{\mu - \lambda}{w}\right) w^k \\
\hat{w}(z^k) &= \left(-\lambda \partial_{\hat{z}} + \frac{\mu}{z}\right) z^k, & \hat{w}(w^k) &= w^{k+1}.
\end{align*}

Here, we just used the relation $\mathcal{D} P \equiv 0$ and wrote the elements in the basis $\{z^k, w^k \mid k \in \mathbb{Z}\}$ of $\mathcal{M}$. A basis of a representation of $\mathcal{M}$ on which $\hat{z}$ and $\hat{w}$ just act by multiplication by $z$ resp. differentiation with respect to $z$ is given by

\begin{align*}
v_z^k(z) &= z^k \cdot z^{-\mu/\lambda}, \\
v_w^k(z) &= \int dw \ e^{\frac{z w}{\lambda}} w^{k-1} \cdot w^{\mu/\lambda}.
\end{align*}

Indeed, differentiation with respect to $z$ clearly gives the same result as applying $\hat{w}$. Moreover, multiplying $v_w^k$ by $z$ gives

\begin{align*}
z \cdot v_w^k(z) &= \lambda \int dw \ e^{\frac{z w}{\lambda}} \frac{\partial}{\partial w} \left(w^{k-1+\mu/\lambda}\right) = (\mu + \lambda (k - 1)) v_w^{k-1}.
\end{align*}

Similarly, in the module $\mathcal{M}$ one can verify that

\begin{align*}
\hat{w}(w^k) &= w^{k+1}, & \hat{w}(z^k) &= \left(-\lambda \partial_{\hat{w}} + \frac{\mu}{w}\right) w^k \\
\hat{z}(w^m) &= \left(\lambda \partial_{\hat{z}} + \frac{\mu - \lambda}{z}\right) z^k, & \hat{z}(z^k) &= z^{k+1}.
\end{align*}

Hence in the representation of $\mathcal{M}$ defined by

\begin{align*}
v_k^w(w) &= w^{k-1} \cdot w^{\mu/\lambda}, \\
v_k^z(w) &= \int dz \ e^{\frac{z w}{\lambda}} z^k \cdot z^{-\mu/\lambda},
\end{align*}

$w$ and $\partial_w$ act in the usual way.
Since we moved over to representations of the $D$-module where the differential operator acts as we are used to, the $S$ transformation, that connects the $U_z$ and the $U_w$ patch and thereby exchanges $\hat{z}$ and $\hat{w}$, must be a Fourier transformation. This is clear from the expressions for the basis elements $w$ and $\tilde{w}$: $S$ interchanges $v^z_k(z)$ with $v^z_k(w)$, and $v^w_k(z)$ with $v^w_k(w)$. In total we thus find the $D$-module elements

$$U_z : \ v^z_k, v^w_k$$

$$U_w : \ v^w_k, v^z_k$$

(6.22)

Representing the $D$-module in terms of differential operators of course gives the same result. A fundamental solution of $P\Psi(z) = 0$ is $\Psi(z) = z^{-\mu/\lambda}$, so that acting with $D = \langle z, \partial_z \rangle$ on $\Psi(z)$ gives the elements $v^z_k$ in $M$. Likewise, we reconstruct the elements $v^w_k$ from the fundamental solution of $\tilde{P}\Psi(w) = 0$. Since $D = \langle z, \partial_z \rangle$ and $\tilde{D} = \langle w, \partial_w \rangle$ are related by a Fourier transform, an element $v_k$ of the $D$-module in one asymptotic region is represented by its Fourier transform in the opposite region. This reproduces all elements in (6.22).

A $\lambda$-expansion of the $D$-module element $v^z_k$, using for example the stationary phase approximation, yields as zeroth order contribution

$$e^{\mu/\lambda} \left( \frac{\mu}{w} \right)^{k-\mu/\lambda},$$

while the subdominant contribution is given by

$$\sqrt{-\frac{2\pi \lambda \mu}{w^2}}.$$

So in total we find that

$$v^z_k(w) = \sqrt{-2\pi \lambda} \left( \frac{\mu}{w} \right)^{-\mu/\lambda} w^{\mu/\lambda} \mu^{k+1/2} w^{-k-1} \psi_{qu} \left( \frac{\mu}{w} \right).$$

This summarizes the contributions that we found before: the genus zero $w^{\mu/\lambda}$ and genus one $\mu^{k+1/2} w^{-k-1}$ results, plus the higher order contributions that are collected in $\psi_{qu}$.

The all-genus partition function $Z$ of this I-brane system can be easily computed exactly. Schematically it equals the correlation function

$$Z_{c=1} = \langle W_w | S_{\mu} | W_z \rangle,$$

where the $S$-matrix implements the Fourier transform between the two asymptotic patches. Similar to the arguments in (the appendices of) [200] and [128].

\footnote{The argument presented in the appendix of [128] is not fully correct. The proper argument (as}
we find that the result reproduces the perturbative expansion of the free energy as in equation (6.14). For completeness let us review the argument by comparing \( \varphi^z_k(w) \) with \( \varphi^w_k(w) \).

Notice that \( \varphi^z_k(w) \) almost equals the gamma-function \( \Gamma(z) = \int_0^\infty dt \ e^{-t} t^{z-1} \).

Indeed, let us take the integration contour from \(-i\infty\) to \(i\infty\) and choose the cut of the logarithm to run from 0 to \(\infty\). Then

\[
\varphi^z_k(w) = \left( \frac{i\lambda}{w} \right)^{k+1-\frac{\mu}{\lambda}} \Gamma \left( k + 1 - \frac{\mu}{\lambda} \right).
\]

which is the same as the theory of type II result in the appendix of [200]. Ignoring the exponential factor (which will only play a role non-perturbatively), we find that the free energy \( \mathcal{F} \) equals the sum

\[
\mathcal{F}(\lambda, \mu) = \sum_{k \geq 0} \left( k + 1 - \frac{\mu}{\lambda} \right) \log \lambda + \log \Gamma \left( k + 1 - \frac{\mu}{\lambda} \right).
\]

It obeys the recursion relation

\[
\mathcal{F} \left( \lambda, \mu + \frac{\lambda}{2} \right) - \mathcal{F} \left( \lambda, \mu - \frac{\lambda}{2} \right) = \left( \frac{1}{2} - \frac{\mu}{\lambda} \right) \log \lambda + \log \Gamma \left( \frac{1}{2} - \frac{\mu}{\lambda} \right).
\]

which is known to be fulfilled by the \( c = 1 \) string (see for example Appendix A in [147]), up to a term \(-\frac{1}{2} \log(2\pi\lambda)\) that can be taken care of by normalizing the functions \( \varphi_k \). The same result is found when analyzing the function \( \varphi_k \).

This concludes our discussion of the \( c = 1 \) string. It is the first \( D \)-module example where we see how to handle curves with two punctures. The physical interpretation of the I-brane set-up furthermore provides a check of our formalism. Moreover, this example agrees with the claim that the \( D \)-module partition function should be invariant under different parametrizations. Both the repre-
sentation as \( c = 1 \) curve, \( \Sigma_{c=1} : zw = \mu \), and that as a Gaussian matrix model spectral curve, \( \Sigma_{mm} : y^2 = x^2 + \mu \), yield the same partition function.

### 6.3 Seiberg-Witten geometries

More than once \( \mathcal{N} = 2 \) supersymmetric gauge theories have proved to provide an important theoretical framework to test new ideas in physics. The most important advances in this context are the solution of Seiberg and Witten in terms of a family of hyperelliptic curves, as well as the explicit solution of Nekrasov and Okounkov in terms of two-dimensional partitions. In what follows we will provide a novel perspective on these results, by wrapping an I-brane around a Seiberg-Witten curve. The \( B \)-field on the I-brane quantizes the curve, and a fermionic state is obtained from the corresponding \( D \)-module. As we will see, this state sums over all possible fermion fluxes through the Seiberg-Witten geometry, and may be interpreted as a sum over geometries. First we briefly review the Seiberg-Witten and Nekrasov-Okounkov approaches.

The solution of the \( U(N) \) Seiberg-Witten theory is encoded in its partition function \( Z(a_i, \lambda, \Lambda) \), which is a function of the scale \( \Lambda \), the coupling \( \lambda \) and boundary conditions for the Higgs field denoted by \( a_i \) for \( i = 1, \ldots, N \) (with \( \sum_i a_i = 0 \) for the \( SU(N) \) theory). The partition function is related to the free energy \( F \) as

\[
Z(a_i, \lambda, \Lambda) = e^F = e^{\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(a_i, \Lambda)}.
\]

In the above expansion \( F_0 \) is the prepotential which contains in particular an instanton expansion in powers of \( \Lambda^{2N} \), while higher \( F_g \)'s encode gravitational corrections. The \( U(N) \) Seiberg-Witten solution identifies the \( a_i \)'s and the derivatives of the prepotential \( \frac{1}{2\pi i} \frac{\partial F}{\partial a_i} \) as the \( A_i \) and \( B_i \) periods of the meromorphic differential

\[
\eta_{SW} = \frac{1}{2\pi i} \frac{dt}{v}
\]

on the hyperelliptic curve (4.10)

\[
\Sigma_{SW} : \Lambda^N(t + t^{-1}) = P_N(v) = \prod_{i=1}^{N} (v - \alpha_i).
\]

Despite great conceptual advantages, extracting the instanton expansion of the prepotential from this description is a non-trivial task. However, an explicit formula for the partition function, encoding not only the full prepotential but also entire expansion in higher \( F_g \) terms, was postulated by Nekrasov in [207]. Subsequently this formula was derived rigorously jointly by him and Okounkov.
in [147] and independently by Nakajima and Yoshioka in [208, 209]. For $U(N)$ theory this partition function is given by a sum over $N$ partitions $\vec{R} = (R_{(1)}, \ldots, R_{(N)})$

$$Z(a_i, \lambda, \Lambda) = Z_{\text{pert}}(a_i, \lambda) \sum_{\vec{R}} \Lambda^{2N|\vec{R}|} \mu^2_{\vec{R}}(a_i, \lambda), \quad (6.24)$$

where

$$\mu^2_{\vec{R}}(a_i, \lambda) = \prod_{(i,m) \neq (j,n)} \frac{a_i - a_j + \lambda(R_{(i),m} - R_{(j),n} + n - m)}{a_i - a_j + \lambda(n - m)}, \quad (6.25)$$

and

$$Z_{\text{pert}}(a_i, \lambda) = \exp \left( \sum_{i,j} \gamma(\lambda(a_i - a_j, \Lambda)) \right). \quad (6.26)$$

The function $\gamma(a_i, \Lambda)$ is related to the free energy of the topological string theory on the conifold, and its various representations and properties are discussed extensively in [147] in Appendix A. The vevs $a_i$ are quantized in terms of $\lambda$, so that for $p_i \in \mathbb{Z}$,

$$a_i = \lambda(p_i + \rho_i), \quad \rho_i = \frac{2i - N + 1}{2N}.$$

The approach of [210] is based on the localization technique in presence of the so-called $\Omega$-background. In general this background provides a two-parameter generalization of the prepotential: the coupling $\lambda$ is replaced by two geometric parameters $\epsilon_1$ and $\epsilon_2$. The prepotential, as given above, is recovered for $\lambda = \epsilon_1 = -\epsilon_2$. By the duality web Fig. 1.6 supersymmetric gauge theories are related to intersecting brane configurations. The Nekrasov-Okounkov solution must therefore have an interpretation in terms of a quantum Seiberg-Witten curve, where $\lambda$ plays the role of the non-commutativity parameter.

### 6.3.1 Dual partition functions and fermionic correlators

For a relation to the I-brane partition function (4.53), it is necessary to consider the dual of the partition function (6.24). This is introduced in [147] as the Legendre dual

$$Z^D(\xi, p, \lambda, \Lambda) = \sum_{\sum_i p_i = p} Z(\lambda(p_i + \rho_i), \lambda, \Lambda) e^{\frac{i}{\hbar} \sum_i p_i \xi_i}. \quad (6.27)$$

An important observation of Nekrasov and Okounkov is that this dual partition function can be elegantly written as a free fermion correlator. This is a con-
sequence of the correspondence between fermionic states and two-dimensional partitions described in Section 5.3.1. For $U(1)$ there is no difference between the partition function and its dual and both can be written as

$$Z_{U(1)}^D(p, \lambda, \Lambda) = \langle p | e^{-\frac{1}{\lambda} \alpha_1 \Lambda^2 L_0} e^{\frac{1}{\lambda} \alpha_{-1}} | p \rangle,$$

(6.28)

where $|p\rangle$ is the fermionic vacuum whose Fermi level is raised by $p = a/\lambda$ units and $L_0$ measures the energy of the state. A version of the boson-fermi correspondence implies the following decomposition

$$e^{\frac{1}{\lambda} \alpha_{-1}} | p \rangle = \sum_R \frac{\mu_R}{\lambda |R|} | p; R \rangle$$

(6.29)

in terms of partitions $R$, where $\mu_R$ is the Plancherel measure

$$\mu_R = \prod_{1 \leq m < n < \infty} \frac{R_m - R_n + n - m}{n - m} = \prod_{\square \in R} \frac{1}{h(\square)}$$

which can be written equivalently as a product over hook lengths $h(\square)$.

For general $N$ the dual partition function (6.27) looks very similar

$$Z_{U(N)}^D(\xi_i; p, \lambda, \Lambda) = \langle p | e^{-\frac{1}{\lambda} \alpha_1 \Lambda^2 L_0} e^{\frac{1}{\lambda} \alpha_{-1}} | p \rangle,$$

(6.30)

however, now this expression is obtained by blending $N$ free fermions $\psi^{(i)}$ into a single fermion $\psi$, as explained in Section 5.3.1. In particular

$$H_{\xi_i} = \frac{1}{\lambda} \sum_r \xi_{(r+1/2)} \mod N \psi_r \psi^\dagger_{-r},$$

while the bosonic mode $\alpha_{-1}$ arises from the bosonization of the single blended fermion $\psi$. In formula (6.29) the Plancherel measure of a blended partition $R$ can be decomposed into $N$ constituent partitions as

$$\mu_R = \sqrt{Z_{\text{pert}}(a_i, \lambda) \mu_{\hat{R}}(a_i, \lambda)},$$

(6.31)

with $\mu_{\hat{R}}$ and $Z_{\text{pert}}$ given in (6.25) and (6.26). When read in terms of the $N$ twisted fermions $\psi^{(i)}$, the correlator (6.30) involves a sum over the individual fermion charges $p_i$.

Our aim in this section is to derive the above fermionic expressions for the dual partition function from the $D$-module perspective. In the next subsections we will see how canonically quantizing the Seiberg-Witten curve in terms of a $D$-module elegantly reproduces to the fermionic correlators (6.28) and (6.30).
6.3.2 Fermionic correlators as $\mathcal{D}$-modules

In this section we compute the I-brane partition function for $U(N)$ Seiberg-Witten geometries. We start with the simpler $U(1)$ and $U(2)$ examples and then generalize this to $U(N)$. As a first principal step we notice that the $U(N)$ Seiberg-Witten geometry

$$\Sigma_{SW} : \Lambda^{N}(t + t^{-1}) = P_N(v) = \prod_{i=1}^{N}(v - \alpha_i), \quad (6.32)$$

can be rewritten as

$$(P_N(v) - \Lambda^{N}t)(P_N(v) - \Lambda^{N}t^{-1}) = \Lambda^{2N}.$$ 

This shows that the Seiberg-Witten surface may be seen as a transverse intersection of a left and a right half-geometry defined by

$$\Sigma_L : \Lambda^{N}t = P_N(v) \quad \text{resp.} \quad \Sigma_R : \Lambda^{N}t^{-1} = P_N(v),$$

which are connected by a tube of size $\Lambda^{2N}$. The left geometry parametrizes the asymptotic region where both $t \to \infty$ and $v \to \infty$, whereas the right geometry describes the region where $v \to \infty$ while $t \to 0$. This is illustrated in Fig. 6.1.

![Figure 6.1](image-url)  

**Figure 6.1**: The right-half Seiberg-Witten geometry is distorted around the asymptotic point $(t \to 0, v \to \infty)$. A fermion field on the quantized curve can be described as an element of a $\mathcal{D}$-module, and sweeps out a state $|\mathcal{W}\rangle$ at the $S^1$-boundary where $t \to \infty$. 
Next we wish to associate a subspace in the Grassmannian to both half Seiberg-Witten geometries. This will be swept out by a fermion field on the curve that couples to the holomorphic part of the $B$-field

$$B = \frac{1}{\lambda} ds \wedge dv$$

Since this $B$-field quantizes the coordinate $v$ into the differential operator $\lambda \partial_s$, any subspace in this section is a $D$-module for the differential algebra

$$D_{C^*} = \langle t, \lambda \partial_s \rangle.$$

The free fermions on the Seiberg-Witten curves couple to the gauge field $A = \frac{1}{\lambda} \eta_{SW}$. This determines their flux through the $A_i$ cycles of the Seiberg-Witten geometry as

$$p_i = \frac{1}{\lambda} \int_{A_i} \eta_{SW}.$$

The flux leaking through infinity is $p = \sum_{i=1}^{N} p_i$, which is zero for $SU(N)$. A fermion field with fermion flux $p$ at infinity, will sweep out a fermionic state in the $p$th Fock space. The parameters $\xi_i = \int_{B_i} \eta_{SW}$ are dual to the fermion fluxes. Notice that in the perturbative regime $p_i$ can be written as a $\lambda$-expansion

$$\lambda p_i = \alpha_i + O(\lambda).$$

Since both half Seiberg-Witten geometries are distorted near $v = \infty$ (see Fig. 6.1), while a fermionic subspace can be read off in the neighbourhood where $v$ is finite, both half-geometries parametrize a subspace of $C((v))$:

$$\mathcal{W}_L, \mathcal{W}_R \subset C((v)).$$

The trivial geometry corresponds to a disk with origin at $v = \infty$, whereas its boundary encloses the point $v = 0$. The vacuum state is therefore given by

$$|0\rangle = v^0 \wedge v^{-1} \wedge v^{-2} \wedge \ldots. \quad (6.33)$$

Exponentials in $v^{-1}$ act trivially (as pure gauge transformations in $\Gamma_+$) on this state, whereas exponentials in $v$ transform the vacuum into a non-trivial fermionic state.

Finally, the partition function is recovered by contracting the left and the right fermionic state. Note that $s = -\log t$ is a local spatial coordinate on both half Seiberg-Witten geometries, which tends to $-\infty$ on the left and to $+\infty$ on the right. This makes a huge difference with the $c = 1$ geometry discussed in Sec-
tion 6.2 where the local coordinate is the exponentiated coordinate, which on the left is the inverse of that on the right. While in that example a non-trivial $S$-matrix is required to identify the left and right half-geometries, here we can just glue the fermionic states using the classic Hamiltonian $L_0$. Let us now find these quantum states.

**U(1) theory**

The $U(1)$ Seiberg-Witten curve is embedded in $\mathbb{C}^* \times \mathbb{C}$ as

$$\Lambda(t + t^{-1}) = v - \alpha, \quad (t = e^s \in \mathbb{C}^*, \ v \in \mathbb{C})$$

where $\alpha \in \mathbb{C}$ is a normalizable mode. This geometry may be factorized into a left and a right geometry

$$\Sigma_L : v = \Lambda t + \alpha \quad \text{and} \quad \Sigma_R : v = \Lambda t^{-1} + \alpha,$$

that intersect transversely with degeneration parameter $\Lambda^2$.

The symplectic form $B = \frac{1}{\lambda} ds \wedge dv$ quantizes both half geometries into $D_\lambda$-modules on a punctured disc $\mathbb{C}^*_t$, parametrized by $t$. We claim that these are characterized by the $U(1)$ $\lambda$-connections

$$\nabla_L = -\lambda t \partial_t + \Lambda t + \lambda p \quad \text{and} \quad \nabla_R = \lambda t \partial_t + \Lambda t^{-1} + \lambda p.$$

These are just the canonical quantizations of the classical Seiberg-Witten geometries, where additionally $u$ is quantized into $\lambda p$, with $p \in \mathbb{Z}$. They yield the linear differential equations

\begin{equation}
\begin{aligned}
P_L \psi_L^\lambda(t; p) &= (-\lambda t \partial_t + \Lambda t + \lambda p) \psi_L^\lambda(t; p) = 0, \\
P_R \psi_R^\lambda(t; p) &= (\lambda t \partial_t + \Lambda t^{-1} + \lambda p) \psi_R^\lambda(t^{-1}; p) = 0.
\end{aligned}
\end{equation}

The $D_\lambda$-modules are of the canonical form

$$\mathcal{M}_{L/R} = \frac{D_\lambda}{D_\lambda \cdot P_{L/R}},$$

and are generated by the solutions

$$\psi_L^\lambda(t; p) = t^p e^{\frac{\Lambda}{\lambda} t} \quad \text{and} \quad \psi_R^\lambda(t; p) = t^{-p} e^{\frac{\Lambda}{\lambda} t^{-1}}.$$

From the discussion in Section 5.3.1 it follows that the factor $t^{-p}$ acts on the right Dirac vacuum by raising the Fermi level into $|p\rangle$, while the exponent of $t^{-1}$ translates to the exponentiated $\alpha_{-1}$ operator. With an analogous statement for
6.3. Seiberg-Witten geometries

Figure 6.2: Contracting two Seiberg-Witten half-geometries yields the Nekrasov-Okounkov partition corresponding to a fermion flux $p$ through the surface.

the left state, the modules $\mathcal{M}_{L/R}$ translate into the Bogoliubov states

$$\langle W_L | = \langle p| e^{\frac{A}{\Lambda} \alpha_1} \quad \text{and} \quad |W_R\rangle = e^{\frac{A}{\Lambda} \alpha_{-1}}|p\rangle.$$  \hfill (6.35)

The $U(1)$ Nekrasov-Okounkov partition function with fermion flux $p$ (see Figure 6.2) is found by contracting the above fermion states

$$Z_{NO}^\lambda(p; \Lambda) = \langle p| e^{\frac{A}{\Lambda} \alpha_1} e^{\frac{A}{\Lambda} \alpha_{-1}}|p\rangle.$$  \hfill (6.35)

The factors $\Lambda$ can be pulled out of the exponentials by using the commutator $[L_0, \alpha_{\pm 1}] = \alpha_{\pm 1}$. Up to an extra factor $\Lambda^{-p^2/2}$ we find that

$$Z_{NO}^\lambda(p; \Lambda) \sim \langle p| e^{\frac{A}{\Lambda} \alpha_1} \Lambda^{2L_0} e^{\frac{A}{\Lambda} \alpha_{-1}}|p\rangle.$$  \hfill (6.35)

This has a nice geometrical explanation, since the left and right half geometries are connected by a tube of size $\Lambda^2$ as in the factorized form of the complete $U(1)$ geometry. The factor $\Lambda^{2L_0}$ is the Hamiltonian that describes the propagation of the fermion field along the tube. There is no need to generalize this standard-CFT factor, since both patches are described by the same space-coordinate $s$.

We also note that, as consistent with [128], the solution $\psi_R^\lambda(t; u)$ to $P_R \psi = 0$ equals the one-point-function

$$\langle p-1|\psi(t)|W_R\rangle = \sum_n t^{-p-n}\langle p; R_n|W_R\rangle = t^{-p}e^{\frac{A}{\Lambda} t^{-1}} = \psi_R^\lambda(t; u),$$

where $R_n$ represents a Young tableau consisting of just one row of $n$ boxes.

$U(2)$ theory

We apply now the above strategy for the $U(2)$ geometry. We split the corresponding curve into a left and a right half geometry, and for brevity focus just on the right part defined by

$$\Sigma_R : \quad \Lambda^2 t^{-1} = (v - \alpha_2)(v - \alpha_1).$$  \hfill (6.36)
The $B$-field canonically quantizes this equation into the second order differential equation

$$P_R \psi(t) = \left\{ \lambda^2 (t \partial_t - p_2)(t \partial_t - p_1) - \Lambda^2 t^{-1} \right\} \psi(s) = 0. \quad (6.37)$$

A change of variables $z = 2t^{-1/2}$ followed by the ansatz $\psi(z) = z^{-(p_1 + p_2)} \phi(z)$ and the rescaling $z \rightarrow (\lambda / \Lambda) z$ transforms this differential equation into the familiar Bessel equation

$$\left( z^2 \partial_z^2 + z \partial_z - \nu^2 - z^2 \right) \phi(z) = 0,$$

whose linearly independent solutions are given by modified Bessel functions $I_{\nu}(z)$ and $K_{\nu}(z)$ of the first kind. The total solution in the original $t$-coordinate is therefore a linear combination of

$$\psi_{R}^\lambda(t; p_1, p_2) = \begin{cases} t^{\frac{\nu}{2}} I_{\nu} \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right), \\ t^{\frac{\nu}{2}} K_{\nu} \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right), \end{cases} \quad (6.38)$$

where $p = p_1 + p_2$. These modified Bessel functions have different asymptotics at infinity and relate to each other by going around the punctured disc $\mathbb{C}_t^\times$.

The second order differential operator $P_R$ defines the $\mathcal{D}_\lambda$-module

$$\mathcal{M}_R = \frac{\mathcal{D}_\lambda}{\mathcal{D}_\lambda \cdot P_R},$$

which we claim represents fermions on the quantum $SU(2)$ Seiberg-Witten geometry. To check this statement, we have to find the fermionic state corresponding to $\mathcal{M}_R$. So we asymptotically expand of the modified Bessel functions around $t = 0$ in $\lambda$:

$$I_{\nu} \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \sim t^{1/4} \exp \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \left\{ 1 - \frac{(\mu - 1)}{8} \frac{\lambda \sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu - 9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\}$$

$$K_{\nu} \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \sim t^{1/4} \exp \left( -\frac{2\Lambda}{\lambda \sqrt{t}} \right) \left\{ 1 + \frac{(\mu - 1)}{8} \frac{\lambda \sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu - 9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\},$$

with $\mu = 4\nu^2$.

Recall that equation (6.33) implies that any exponential function in the local coordinate $v^{-1} = \sqrt{t}$ near the puncture acts trivially on the vacuum state. Equivalently, this is true for any asymptotic series in $\sqrt{t}$ that assumes the value 1 at $\sqrt{t} = 0$. In other words, we can forget about the complete expansion in $\sqrt{t}$!
Only the WKB pieces

\[ t^{1/4} \exp \left( \pm \frac{2\Lambda}{\lambda \sqrt{t}} \right) \]

are relevant in writing down the fermionic state. This is exactly opposite to
the matrix model examples, where the WKB-piece can be neglected and the
perturbative series in \( \lambda \) defines the fermionic state.

The derivatives of the above solutions have one term proportional to \( \psi(s) \) (which
we may forget about), and a term proportional to the derivative of the Bessel
functions. The latter may be expanded as

\[
\partial_s I_\nu(t) \sim t^{-1/4} \exp \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \left\{ 1 - \frac{(\mu + 3)}{8} \frac{\lambda \sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu + 15)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\}
\]

\[
\partial_s K_\nu(t) \sim t^{-1/4} \exp \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \left\{ 1 + \frac{(\mu + 3)}{8} \frac{\lambda \sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu + 15)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\}
\]

around \( \sqrt{t} = 0 \). Again with the same reasoning only the WKB piece is necessary
to write down the quantum state. Taking into account the extra factor \( t^{\frac{p}{2}} \) in
(6.38) the subspace \( \mathcal{W}_R^+ \) is thus generated by the \( \mathcal{O}(t) \)-module

\[
t^{\frac{p}{2}} \begin{pmatrix} t^{\frac{1}{4}} \exp \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \\ t^{-\frac{1}{4}} \exp \left( \frac{2\Lambda}{\lambda \sqrt{t}} \right) \end{pmatrix} \mathcal{O}(t),
\]

and blends (via the lexicographical ordering) into the fermionic state

\[
|\mathcal{W}_R^+\rangle = v^{-p} e^{\frac{\lambda}{\bar{\lambda}}} v^{0 \wedge v^{-1} \wedge v^{-2} \wedge v^{-3} \wedge \ldots}
\]

on the cover. Here we used a cover coordinate \( v^{-1} \) obeying \( v^{-2} = t \), and rescaled
the topological string coupling as \( \bar{\lambda} = \lambda / 2 \). \( \mathcal{W}_R^+ \) is thus simply generated by a
single function

\[
\psi^\lambda(v) = v^{-p} e^{\frac{\lambda}{\bar{\lambda}}} v
\]

Hence the fermions blend into the Bogoliubov state

\[
|\mathcal{W}_R^+\rangle = e^{\frac{\Delta}{\bar{\lambda}}} |p\rangle,
\]

when \( p \) is an integer.

Note that the only modulus that appears in this expression is \( p \). This represents
the diagonal \( U(1) \), denoting the total fermion flux through the geometry. The
moduli \( p_1 \) and \( p_2 \) measures the fermion flux through an internal cycle and are
not visible in the result, because the final state sums over all internal momenta.
In general any $SU(2)$ Seiberg-Witten geometry with the same quantized $p$ yields the same fermionic state.

The fermionic (or dual) partition function is found by contracting the left and the right states, similarly as in the $U(1)$ example above. The left state is just the complex conjugate of the right one, so we find

$$Z^D_{NO}(p; \lambda, \Lambda) = \langle p| e^{\frac{\Delta}{\lambda} \alpha_1} e^{\frac{\Delta}{\lambda} \alpha^{-1}} | p \rangle \sim \langle p| e^{\frac{\Delta}{\lambda} \alpha_1} \Lambda^{2L_0} e^{\frac{\Delta}{\lambda} \alpha^{-1}} | p \rangle.$$ 

The result is very similar to the $U(1)$ example, up to the shift $\lambda \mapsto \lambda/2$. But notice that this fermionic state is written in terms of a single blended fermion. Decomposing this fermion into two twisted fermions makes it natural to insert an extra operator in the middle of the correlator, that measures the momenta of the two fermions through the $A$-cycles of the SW geometry. Weighting these momenta with a potential $\xi_i$, for $i = 1, 2$, yields

$$Z^D_{NO}(\xi_i, p; \lambda, \Lambda) \sim \langle p| e^{\frac{\Delta}{\lambda} \alpha_1} e^{H_{\xi_1} \Lambda^{2L_0} e^{\frac{\Delta}{\lambda} \alpha^{-1}} | p \rangle},$$

where $H_{\xi_i} = \frac{1}{\lambda} \sum_r \xi_{(r+1/2)} \mod 2 \psi_r \psi^\dagger_r = \frac{1}{\lambda} (p_1 \xi_1 + p_2 \xi_2)$. This is the answer conjectured by Nekrasov and Okounkov in [147].

$U(N)$ theory

It is not difficult to extend this discussion to the $U(N)$ theory (6.32), whose corresponding right half geometry we write as

$$\Sigma_N : \Lambda^N t^{-1} = \prod_{i=1}^N (v - \alpha_i).$$

(6.40)

Canonically quantizing this geometry and changing the coordinates $z = (\frac{\Lambda}{\lambda})^N t^{-1}$, brings us to the degree $N$ differential equation

$$P_N \psi(z) = \left( \prod_{i=1}^N (z \partial_z - p_i) - z \right) \psi(z) = 0.$$ 

(6.41)

It turns out that a solution to the above equation is given by a particular Meijer G-function, denoted $G^{m,n}_{p,q}(z)$. The Meijer G-function is a complicated special function which was introduced in order to unify a number of standard special function [211, 212, 213], and is defined in terms of a complex integral

$$G^{m,n}_{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| z \right) = \frac{1}{2\pi i} \int_L \prod_{j=m+1}^n \Gamma(1 - b_j - t) \prod_{j=1}^m \Gamma(1 - a_j + t) z^t \ dt,$$

where $L$ is a contour which goes from $-i\infty$ to $+i\infty$ and separates the poles of
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\[ \Gamma(b_j - t), \text{ for } j = 1, \ldots, m, \text{ from those of } \Gamma(1 - a_i + t), \text{ for } i = 1, \ldots, n. \]

It can be shown that the Meijer G-function solves the differential equation

\[
\left( \prod_{i=1}^{q} (z \partial_z - b_i) + (-1)^{p-m-n+1} z \prod_{j=1}^{p} (z \partial_z - a_j + 1) \right) G(z) = 0.
\]

(6.42)

So, indeed the Seiberg-Witten differential equation (6.41) is a special case of Meijer differential equation (6.42) with \( p = n = 0 \) and \( q = N \). Therefore the differential equation (6.41) is solved by

\[ \psi(z) = G_{0,0}^{0,0}(z). \]

Similarly as before we claim that the \( D \)-module corresponding to \( U(N) \) Seiberg-Witten curve is generated by \( P_N \). A subspace \( \mathcal{W} \) corresponding to this \( D \)-module is this generated by a solution \( \psi(t) \) and its derivatives in \( t \partial_t \).

For \( p < q \) the Meyer differential equation (6.42) has a regular singularity at \( z = 0 \) and an irregular one for \( z = \infty \). To extract the I-brane fermionic state, we are interested in the behaviour around the irregular singularity, where \( t \to 0 \). It turns out that one of the independent solutions of the Seiberg-Witten differential equation (6.41) has the asymptotic expansion \([211, 212, 213]\)

\[
\psi(v) \sim e^{\frac{\Lambda}{N} v} v^{-\frac{N}{2}} v^p \sum_{j=0}^{\infty} k_j v^{-j},
\]

around this singularity, which is conveniently written in the cover coordinate \((-v)^N = t^{-1} = (\frac{\lambda}{N})^N z\). The other solutions are found by multiplying the coordinate \( v \) by \( N \)-th roots of unity, and thus behave distinctly at infinity. As before, \( p = \sum_{i=1}^{N} p_i \).

To find the fermionic state corresponding to the \( U(N) \) Seiberg-Witten curve, we act with \( \psi(v) \) on the Dirac vacuum. The positive power of \( v \) in the exponent of \( \psi(v) \) corresponds in the operator language to \( \alpha - 1 \), whereas \( v^p \) lifts the Fermi level. The remaining series just contains negative powers of \( v \) which translate to a trivial action on the vacuum in the operator formalism. Therefore, the above asymptotic solution and its derivatives (in \( t \partial_t \)) blend into the state

\[ |\mathcal{W}_R\rangle = e^{\frac{\Delta}{N} \alpha - 1} |p\rangle, \]

(6.43)

with rescaled topological string coupling \( \tilde{\lambda} = \lambda/N \). Like for the \( U(2) \) Seiberg-Witten geometry the dependence on the individual moduli \( p_i \) has dropped out.

Similarly as in \( U(1) \) and \( U(2) \), in the present case we also find the \( U(N) \) Nekrasov-
Okounkov dual partition function

\[ Z^D_{NO}(\xi_i; \lambda, \Lambda) = \langle p \vert e^{\frac{i}{\Lambda^2} \xi_i} e^{H_{\xi_i} \Lambda^2 L_0} e^{\frac{i}{\Lambda^2} \alpha_1 - 1} \vert p \rangle. \]  

(6.44)

This fermionic correlator is indeed the one postulated in [147]. For \( N = 1 \) or \( N = 2 \) the Meijer G-function specializes respectively to the exponent and Bessel functions, which reproduces the results derived in previous subsections.

Although the normalizable moduli \( p_i \) disappear in the final I-brane partition function, they reappear when the state is unblended in terms of \( N \) single fermions

\[ e^{\frac{i}{\Lambda^2} \alpha_1} \vert p \rangle = \sum_R \frac{H_R}{\chi^{|R|}} \vert p, R \rangle = \sum_{p_i=p} \sum_{R(i)} \sqrt{Z_{\text{pert}}(p)} \frac{H_R(p, \tilde{\lambda})}{\chi^{|R|}} \prod_{l=1}^N \vert p_i, R(i) \rangle, \]

(6.45)

as may be seen from (6.29) and (6.31). The charges \( p_i \) have an interpretation as the fermion fluxes through the \( N \) tubes of the Seiberg-Witten geometry we started with.

Actually, we find the same fermionic state when starting with any other Seiberg-Witten geometry whose fermion flux at infinity is \( p \). Hence one microstate in the total sum (6.45) can be interpreted as a fermion flux through an infinite set of geometries. This gives the state (6.45) as well as the partition function (6.27) the interpretation of a sum over geometries.

### 6.3.3 Topological string theory and quantum groups

Nekrasov and Okounkov also derive a partition function for the 5-dimensional \( U(N) \) Seiberg-Witten theory compactified on the circle of circumference \( \beta \) [210, 147, 209]. It is given by a \( K \)-theoretic generalization of the 4-dimensional formula in equation (6.24).

This 5-dimensional theory is closely related to the topological string theory by geometric engineering on a toric Calabi-Yau background [214, 215]. Namely, the partition function of the topological string theory on an \( A_N \)-singularity fibered over \( \mathbb{P}^1 \) (whose toric diagram consists of \( N - 1 \) meshes as in Fig. 6.3) is equal to the partition function of the 5-dimensional gauge theory given above, when the Kähler sizes of the internal legs are (see Section 4.2.2)

\[ Q_{F_i} = e^{\beta(a_{i+1} - a_i)}, \quad Q_B = \left( \frac{\beta \Lambda}{2} \right)^{2N}, \]  

(6.46)

where \( F_i \) labels the vertical legs and \( B \) the horizontal ones. In the so-called gauge theory limit, when \( \beta \to 0 \), the topological string partition function reduces to the 4-dimensional Seiberg-Witten partition function. The corresponding B-
6.3. Seiberg-Witten geometries

Figure 6.3: On the left we see the five-dimensional $U(2)$ Seiberg-Witten surface with fermion fluxes through its $A$-cycles, and on the right a corresponding toric diagram. The fermion flux deforms the Kähler lengths of the toric diagram as in equation (6.46).

The model mirror geometry is of the form

$$X_{SW} : \quad xy - H(t,v) = 0,$$

where $H(t,v) = 0$ represents a Riemann surface of genus $N - 1$. In the gauge theory limit this surface becomes the Seiberg-Witten curve $\Sigma_{SW}$, parametrized as in the equation (6.23).

In topological string theory it is natural as well to write down a dual partition function [128]. In a local B-model this allows the possibility of arbitrary fermion fluxes through the handles of the Riemann surface. In this setting it has been argued before that turning on a fermion flux is equivalent to deforming the geometry. More precisely, fermion flux parametrized by $P = p_i B_i$ changes the integral of the holomorphic 3-form over any linking 3-cycle $A^3_i$, and thereby shifts the complex structure moduli $S_i = \int_{A^1_i} \Omega$ as

$$S_i \mapsto S_i + \lambda p_i$$

In the A-model fermion flux translates into wrapping D4 branes around 4-cycles, and thereby deforms the Kähler moduli. The I-brane partition function thus equals the dual topological string partition function.

Because the Seiberg-Witten surface is embedded in $\mathbb{C} \times \mathbb{C}^*$, $A^3$ and $B^3$-cycles in the toric threefold will have topologies $S^1 \times S^2$ and $S^3$, respectively (see Fig. 4.8). In particular, a basis of $A^3_i$-cycles can be chosen to reduce to the surface as the combination of 1-cycles $A^1_i - A^1_{i+1}$. Now notice that the 3-cycle $A^3_i$ with topology $S^1 \times S^2$ is mirror to the vertical 2-cycle $F_i$ that connects the $i$-th and the $i+1$-th horizontal leg. So turning on a fermion flux $p_i$ through the $i$-th leg of the Seiberg-Witten geometry changes the complex structure parameter $S_i$ by an amount proportional to $a_i - a_{i+1}$. This explains the Kähler size $Q_{F_i}$.
in (6.46) in terms of fermionic fluxes through the Seiberg-Witten curve, and in reverse why (6.45) may be interpreted as a sum over Seiberg-Witten geometries, or equivalently toric diagrams. So we conclude that the fermionic interpretation in 4d of Nekrasov and Okounkov is dual in 6d to the fermionic interpretation of the topological string, and has a deeper interpretation in terms of $\mathcal{D}$-modules.

**Topological vertex**

An important step to understand Seiberg-Witten curves (as well as other local Calabi-Yau geometries) is the topological vertex, introduced in Section 4.2.1. Recall that its mirror is a genus zero curve with three punctures given by the equation

$$x + y - 1 = 0$$

(6.47)

in $\mathbb{C}^* \times \mathbb{C}^*$. In this case the symplectic form is given by $du \wedge dv$ where $u, v$ are logarithmic coordinates: $x = e^u$ and $y = e^v$. The corresponding $\mathcal{D}$-module is now given by the operator

$$P = e^u + e^{-\lambda \partial_u} - 1.$$  

(6.48)

$P$ is actually a difference operator, instead of a differential operator, so we have to generalize the notion of a $\mathcal{D}$-module somewhat. This is a well-known procedure in the field of quantum groups. These quantum groups appear because in the $\mathbb{C}^*$ case the operators $\hat{x}$ and $\hat{y}$ now satisfy the Weyl algebra or $q$-commutation relation

$$\hat{x} \hat{y} = q \hat{y} \hat{x}, \quad q = e^\lambda.$$

The fundamental solution to $P \Psi = 0$ is the quantum dilogarithm

$$\Psi(u) = \prod_{n=1}^{\infty} (1 - e^u q^n).$$

The corresponding module $\mathcal{M}$ for the Weyl algebra can again be written in terms of the coordinate $u$ or in terms of the dual variable $v$. There is another unitary map $U$ that implements this transformation on the free fermion fields. Because of the hidden cyclic symmetry of the vertex, this can be made transparent by writing it as

$$e^{u_1} + e^{u_2} + e^{u_3} = 0.$$

Up to an overall rescaling of the three variables $u_i$, the map $U$ satisfies $U^3 = 1$. This line of reasoning leads one directly to the formalism of [128], but we will...
not pursue this here in more detail. We reach the important conclusion that
the notion of a quantum curve, as expressed in the concept of a (generalized)
$D$-module, is the right framework to derive the complicated transformations of
\[128\]. We will later use this correspondence in two concrete examples of com-
pact curves, but first make a few remarks about five-dimensional $U(1)$ Seiberg-
Witten theory.

**Five-dimensional $U(1)$ theory**

Quantizing any five-dimensional Seiberg-Witten geometry yields a difference
(instead of differential) equation. Working out $D$-modules for these geo-
metries we leave for future work. Let us treat one example in detail though. The
five-dimensional right-half $U(1)$ Seiberg-Witten half-geometry

$$\Sigma^{5d}_{R} : \beta \Lambda e^{-\beta \lambda t} - e^{-\beta v} - 1 = 0 \quad (6.49)$$

is isomorphic to the topological vertex \[6.47\] and may be drawn as a pair of
pants. In the field theory limit $\beta \to 0$ it reduces to the familiar equation $\Lambda t^{-1} = v$
for the right-half Seiberg-Witten geometry (with $u = 0$).

In the B-model the most general state assigned to a local pair of pants geometry
is given by a Bogoliubov state \[128\]

$$|\mathcal{W}\rangle = \exp \left[ \sum_{i,j} \sum_{m,n=0}^{\infty} a_{ij}^{mn} \psi_{i}^{m-1/2} \psi_{j}^{n-1/2} \right] |0\rangle, \quad (6.50)$$

where the index $i = 1, 2, 3$ describes the fermion field on the three asymptotic
regions of the pair of pants, and the coefficients are determined by a comparison
with the A-model topological vertex. This exponent can be expanded as a sum
over states (see Fig. 6.4)

$$|p_{1}, R_{1}\rangle \otimes |p_{2}, R_{2}\rangle \otimes |p_{3}, R_{3}\rangle,$$

where the fermion flux is conserved: $p_{1} + p_{2} + p_{3} = 0$. To describe the 5d
Seiberg-Witten $U(1)$ geometry we won’t need this state in full generality.

The B-field quantizes this geometry into the difference equation

$$P(t)\Psi(t) = (\beta \Lambda e^{-\beta \lambda t} + e^{\beta \lambda t} - 1) \Psi(t) = 0. \quad (6.51)$$

Like for the topological vertex its fundamental solution is the quantum diloga-
arithm

$$\Psi(t) = \exp \sum_{n>0} \frac{(\beta \Lambda)^{n} t^{-n}}{n(1 - e^{\beta \lambda n})}.$$
Chapter 6. Quantum Curves in Matrix Models and Gauge Theory

Figure 6.4: The B-model vertex (on the left) may be expanded as a sum over fermionic states $|p_1, R_1\rangle \otimes |p_2, R_2\rangle \otimes |p_3, R_3\rangle$, with $p_1 + p_2 + p_3 = 0$, corresponding to a conserved fermion flux through the pair of pants. The five-dimensional right-half Seiberg-Witten geometry (on the right) with charge $p$ only has one partition $R \neq 0$.

As an intermezzo, notice that quantizing the equation

$$\beta v = -\log \left(1 - \beta \Lambda e^{-\beta \lambda t} \right),$$

which is just a rewriting of equation (6.49) for $\Sigma_{5d}^{5d}$, we find a differential equation which may be interpreted as the WKB approximation of difference equation (6.51). A fundamental solution of the differential equation is given by the genus 0 disc amplitude

$$\Psi_0(u) = \exp \sum_{n>0} \frac{(\beta \Lambda)^n t^{-n}}{\lambda n^2 e^{\beta \lambda n}}.$$

Acting with the five-dimensional dilogarithm on the Dirac vacuum state yields the fermionic state

$$|\mathcal{W}\rangle_{U(1)}^{5d} = \exp \sum_{n>0} \frac{(\beta \Lambda)^n \alpha_{-n}}{n(1 - e^{\beta \lambda n})} |0\rangle.$$

This describes a subset of $|\mathcal{W}\rangle$ where only the quantum number $R_1$ is non-trivial. Summing over all external states of the form

$$|-p, R\rangle \otimes |p, \bullet\rangle \otimes |0, \bullet\rangle,$$

incorporates a fermion flux $p$ through the pair of pants. In the field theory limit $\beta \to 0$ the resulting state reduces to the familiar four-dimensional state

$$\exp \left(\alpha_{-1}/\lambda\right) |p\rangle \otimes |p, \bullet\rangle \otimes |0, \bullet\rangle.$$

The partition function is found as the contraction of the left and right 5d half-
geometries. (Or equivalently in the topological B-model by inserting a propagator \[128\].) This yields the fermionic correlator

\[ \langle 0| \tilde{\Gamma}_+ \tilde{\Gamma}_- |0 \rangle = \langle 0| \Gamma_+ (\beta \Lambda)^{2L_0} \Gamma_- |0 \rangle, \]

with

\[ \tilde{\Gamma}_\pm = \exp \sum_{\pm n > 0} \frac{(\beta \Lambda)^{|n|} \alpha_n}{|n|(1 - e^{\beta \lambda n})} \quad \text{and} \quad \Gamma_\pm = \exp \sum_{\pm n > 0} \frac{\alpha_n}{|n|(1 - e^{\beta \lambda n})}. \]

Indeed, the result equals the five-dimensional \(U(1)\) partition function

\[ Z_{5d}^{U(1)}(\lambda, \Lambda, \beta) = \exp \sum_{n=1}^{\infty} \frac{(\beta \Lambda)^{2n}}{4n \sinh^2(\beta \lambda n/2)}, \]

that was found by Nekrasov and Okounkov in \[147\].

### 6.4 Discussion

In this chapter we argued that the fundamental objects underlying various matters in theoretical physics are chiral fermions living on quantum curves. In our formulation the quantum curve is defined, similarly to an affine classical curve, in terms of an equation \(P(z, w) = 0\). Its crucial feature, however, is the non-commutative character of the coordinates \(z, w\). It thereby generalizes the classical curve that comes up in the standard formulation of a given topic. Examples of such classical curves are spectral curves in matrix models, \(c = 1\) string theory, Seiberg-Witten theory, and more generally in topological string theory. Semi-classically their (genus one) free energy is computed as a fermionic determinant on the classical curve. In our approach chiral fermions on the quantum curve generate the all-genus expansion of the free energy with respect to the non-commutativity parameter \(\lambda\).

As we explained in Chapter 5, fermions on a non-commutative curve can be realized physically within string theory as massless states of open strings on an I-brane in the presence of the \(B\)-field. In this chapter we have exploited this system in a few important examples. At the same time we stressed the fundamental importance of \(D\)-modules, which are the appropriate mathematical structures describing non-commutative holomorphic curves. First of all we showed, while reinterpreting the results in \[191\], that I-branes and \(D\)-modules provide an insightful formulation of matrix models. This quite general statement is also appealing when certain matrix model limits are considered, such as a double scaling limits. In this case one recovers an I-brane formulation of minimal string theory, topological gravity and \(c = 1\) string theory.
The I-brane configuration can be related to topological string theory and to Seiberg-Witten theory via a sequence of string dualities Fig. 1.6. In the last part of this chapter we focused on supersymmetric gauge theories. Using the $\mathcal{D}$-module formalism we derived the fermionic expression for the $U(N)$ partition function of the pure $\mathcal{N} = 2$ gauge theory, reproducing the dual all-genus partition function introduced in \cite{147}. We considered mainly 4-dimensional Seiberg-Witten theories with unitary gauge groups, though, and explained only the simplest $U(1)$ example of 5-dimensional theory. It would be insightful to extend these results to other gauge groups and include matter content. It is clear that this should be possible, as these aspects of the 5-dimensional Seiberg-Witten theory are captured by topological strings on toric manifolds. The latter system can be solved in a fermionic B-model formulation of the topological vertex \cite{128} which is equivalent to the I-brane fermions. Nonetheless, finding the quantum I-brane curve representing such configurations appears to be a nontrivial task.

In the process of unraveling the $\mathcal{D}$-module structure in both sets of examples, we noticed some crucial differences. While the WKB piece of the $\mathcal{D}$-module generator can be ignored in finding the matrix model partition function, we discovered that it plays an eminent role for the Seiberg-Witten geometries. Another distinction is the difference in (non-)normalizable modes. While the potential $W$ parametrizes non-normalizable modes that appear in the $\mathcal{D}$-module as parameters, in contrast, the normalizable modes in the Seiberg-Witten geometries are eaten by the $\mathcal{D}$-module, and only visible as a sum over internal fermion fluxes in the geometry. On the other hand, varying the $\mathcal{D}$-module with respect to the non-normalizable modes yields differential equations which relate to isomonodromy and the Stokes phenomenon.

While in this chapter our focus has been to associate a $\lambda$-perturbative quantum state to a spectral curve, we noticed that $\mathcal{D}$-modules in fact contain non-perturbative information. These bits get lost when we turn the $\mathcal{D}$-module in a fermionic state by making an asymptotic expansion of the $\mathcal{D}$-module generators in $\lambda$. This is in line with the discussion on non-perturbative aspects of minimal string theory in \cite{171}, where it is argued that non-perturbative effects drastically modify the non-trivial target space curve into a plain complex plane.

It also agrees with more recent studies of non-perturbative effects in matrix models \cite{148, 216, 217, 218}. These articles revealed that a series of instantons in the matrix model can be summarized in a non-perturbative partition function that sums over all possible filling fractions $p_i = \frac{1}{\lambda} \oint \eta \, a_i$ as

$$Z_{\text{non-pert}}(\mu, \nu) = \sum_{p \in \mathbb{Z}^g} Z_{\text{pert}}(\lambda(p + \mu)) e^{2\pi i p \nu}.$$ 

In this formula $(\mu, \nu)$ is a choice of characteristics on the matrix model spectral curve, that encodes the choice of integration contour in the matrix model. The
integer \( g \) is the genus of the spectral curve. Interestingly, this partition function turns out to have very nice properties. \( Z_{\text{non-pert}}(\mu, \nu) \) is not only holomorphic, but also transforms in a modular fashion under the symplectic group \( Sp(g, \mathbb{Z}) \). Moreover, it satisfies the Hirota equations and can thus be interpreted in terms of a twisted fermion field on \( \Sigma \) with twists \((\mu, \nu)\).

The partition function \( Z_{\text{non-pert}}(\mu, \nu) \) is obviously closely related to the I-brane partition function, that is defined in equation [4.53] and studied from several angles in this chapter. A choice of saddle in the \( \lambda \)-expansion of our \( \mathcal{D} \)-module partition function corresponds to a choice of characteristics in \( Z_{\text{non-pert}}(\mu, \nu) \).

How do these latter matrix model results and our \( \mathcal{D} \)-module insights fit in with other developments that have taken place the last years in the area of topological string theory? Let us start with the observation that the (perturbative) topological string partition function is known to suffer from background dependence [129]. As a result the free energy does not transform as a proper modular form. The modularity can be restored, however, but then the resulting function is not holomorphic anymore [219]. Instead it obeys the holomorphic anomaly equations [117]. It is natural to suggest that a partition function which is both holomorphic and modular, is a candidate for a non-perturbative completion of topological string theory. This is argued in [148].

The claim is strengthened by the following discoveries. In a sequence of papers [220, 221, 131] the Dijkgraaf-Vafa correspondence between matrix models and topological string theory has been extended to arbitrary local Calabi-Yau geometries modeled on a Riemann surface \( \Sigma \). As a result topological string amplitudes can be computed in terms of a simple recursion relation that originates from the theory of matrix models [222]. The Eynard-Orantin formalism is closely related to the Kodaira-Spencer formulation of the B-model, and may be viewed as the bosonized version of our fermionic formulation [223]. It would be valuable to understand this non-commutative version of the familiar boson/fermion correspondence and its interpretation in terms of \( \mathcal{D} \)-modules in more detail.

Moreover, our formalism seems to be closely related to a non-commutative extension of the Eynard-Orantin formalism, that is studied in [224]. The resulting non-commutative invariants depend on two deformation parameters: The first deformation parameter is the usual topological string coupling constant \( \lambda \), whereas the second one is an independent non-commutative deformation. The connection to our \( \mathcal{D} \)-module formalism should arise when we only turn on this second deformation. Turning on either of the deformation parameters is possibly equivalent.

Let us also note that while we mainly studied the web of dualities Fig. 1.6 in the large radius regime, where the topological string partition function has an expansion in terms of the usual Gromov-Witten and Donaldson-Thomas invariants, the \( \mathcal{D} \)-module formalism suggests a relation to invariants in other regimes. Since
we need to make a choice of boundary conditions, when we turn a $\mathcal{D}$-module into a quantum state, the final state is troubled by the Stokes effect: Solutions that decay faster can be added at no cost and the state changes when one crosses certain lines in the moduli space. This suggests that the $\mathcal{D}$-modules we studied may be helpful in understanding the phenomenon of wall-crossing in $\mathcal{N} = 2$ theories \cite{225, 226}. We discuss wall-crossing in $\mathcal{N} = 4$ theories extensively in Chapter 7 where we also make contact with wall-crossing in $\mathcal{N} = 2$ theories.

More mathematically, $\mathcal{D}$-modules play an important role in the geometric Langlands program \cite{227, 156, 228, 229}. In the physical description of this program $\mathcal{D}$-modules enter in the description of eigenbranes of the magnetic ’t Hooft operator in a reduction of 4-dimensional $\mathcal{N} = 4$ gauge theory down to a 2-dimensional sigma model. However, in this sigma model (which is not yet coupled to gravity) the $\mathcal{D}$-modules describe coisotropic A-branes. This is in contrast to their physical appearance in our intersecting brane configuration.

There seems to be a deeper connection of our formalism to quantum integrable systems as they are studied in for example \cite{162, 163, 230}. Quantum curves feature in these quantum systems as so-called opers, that parametrize the base of the integrable system, in the same way that spectral curves parametrize the base of the Hitchin integrable system. It would be enlightening to find out whether the fermions on the quantum curve can be described in a similar way in terms of the quantum integrable system as holds in the semi-classical limit. Does this lead to a better description of the quantum fermion CFT on the quantum curve? Is our set-up related to WZW models based on opers in the geometric Langlands program \cite{227}? Most importantly, we would like to be able to write down a 2-dimensional action for the quantum fermion theory. In Section 7.1 we succeed in writing down the action for a propagator in the I-brane geometry, but not yet for a 3-vertex.