Topological strings and quantum curves
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Appendix A

Level-rank duality

The affine algebras $\hat{su}(N)_k$ and $\hat{su}(k)_N$ are related by the so-called level-rank duality \cite{66,87,319,83,88}, which maps to each other orbits of their irreducible integrable representations under outer automorphism groups. Let us explain this in more detail. The Dynkin diagram of $\hat{su}(N)_k$ consists of $N$ nodes permuted in a cyclic order by the outer automorphism group $\mathbb{Z}_N$. This also induces an action on affine irreducible integrable representations. There are

$$\frac{(N + k - 1)!}{(N - 1)! k!}$$

(A.1)

such representations of $\hat{su}(N)_k$, which can be identified in a standard way with Young diagrams $\rho$ with at most $N - 1$ rows and at most $k$ columns. We denote the set of such diagrams by $\mathcal{Y}_{N-1,k}$. In particular, the generator of the outer automorphism group $\sigma_N$, the so-called basic outer automorphism, has a simple realization in terms of a Young diagram $\rho = (\rho_1, \ldots, \rho_{N-1})$ corresponding to a given integrable representation. The action of $\sigma_N$ amounts to adding a row of length $k$ as a first row of $\rho$, and then reducing the diagram, i.e. removing $\rho_{N-1}$ columns which acquired a length $N$ (so that indeed $\sigma_N(\rho) \in \mathcal{Y}_{N-1,k}$),

$$\sigma_N(\rho_1, \ldots, \rho_{N-1}) = (k - \rho_{N-1}, \rho_1 - \rho_{N-1}, \ldots, \rho_{N-2} - \rho_{N-1}).$$

(A.2)

It follows that $\sigma_N^N(\rho) = \rho$, as expected for $\mathbb{Z}_N$ symmetry. All $N$ irreducible integrable representations related by an action of $\sigma_N$ constitute an orbit denoted as $[\rho] \subset \mathcal{Y}_{N-1,k}$. As an example, the $\mathbb{Z}_4$ orbit generated from $\hat{su}(4)_3$ irreducible integrable representation corresponding to a diagram $\rho = (2, 1) \in \mathcal{Y}_{3,3}$ is given by

\[
\begin{array}{ccc}
\begin{array}{c}
\end{array} & \rightarrow & \begin{array}{c}
\end{array} & \rightarrow & \begin{array}{c}
\end{array} & \rightarrow & \begin{array}{c}
\end{array}
\end{array}
\]
The number of such \( \mathbb{Z}_N \) orbits is given by (A.1) divided by \( N \). For both \( \widehat{su}(N)_k \) and \( \widetilde{su}(k)_N \) this number is the same, therefore a bijection between orbits of their integrable irreducible representations exists. The level-rank duality is a statement that for \( \widehat{su}(N)_k \) orbit represented by a diagram \( \rho \in \mathcal{Y}_{N-1,k} \) there is a canonical bijection realized as

\[
\mathcal{Y}_{N-1,k} \supset [\rho] = \{ \sigma_N^j(\rho) \mid j = 0, \ldots, N - 1 \} \mapsto \{ \sigma_k^a(\rho^t) \mid a = 0, \ldots, k - 1 \} = [\rho^t] \subset \mathcal{Y}_{k-1,N}, \quad (A.3)
\]

where \( ^t \) denotes a transposition and a diagram \( \rho^t \) should be reduced (i.e. all columns of length \( k \) should be removed if \( \rho_1 \) was equal to \( k \)).

The level-rank duality can also be formulated in terms of the embedding

\[ \widehat{u}(1)_{Nk} \times \widehat{su}(N)_k \times \widehat{su}(k)_N \subset \widehat{u}(Nk)_1. \]

The \( \widehat{u}(Nk)_1 \) affine Lie algebra can be realized in terms of \( Nk \) free fermions, so that their total Fock space \( \mathcal{F}^{\otimes Nk} \) decomposes under this embedding as

\[ \mathcal{F}^{\otimes Nk} = \bigoplus_{\rho} U_{||\rho||} \otimes V_{\rho} \otimes \bar{V}_{\bar{\rho}}, \quad (A.4) \]

where \( U_{||\rho||} \), \( V_{\rho} \) and \( \bar{V}_{\bar{\rho}} \) denote irreducible integrable representations of \( \widehat{u}(1)_{Nk} \), \( \widehat{su}(N)_k \), and \( \widehat{su}(k)_N \) respectively. In the above decomposition only those pairs \( \rho, \bar{\rho} \) arise, which represent orbits mapped to each other by the duality (A.3). For a given \( \widehat{su}(N)_k \) orbit \( [\rho] \) represented by \( \rho \), these pairs are therefore of the form \( (\sigma_N^j(\rho), \sigma_k^a(\rho^t)) \), where \( \sigma_N \) and \( \sigma_k \) are appropriate outer automorphism groups. The \( U(1) \) charge corresponding to such a pair is \( ||\rho|| = (|\rho| + jk + aN) \mod Nk \), where \( |\rho| \) is the number of boxes in the Young diagram \( \rho \). With such identifications, the decomposition (A.4) can be written in terms of characters as [88]

\[ \chi_{\widehat{u}(Nk)}(u, v, \tau) = \sum_{[\rho] \subset \mathcal{Y}_{N-1,k}} \sum_{j=0}^{N-1} \sum_{a=0}^{k-1} \chi_{\widehat{u}(1)_{Nk}}(N|u| + k|v|, \tau) \chi_{\sigma_N^j(\rho)}(\bar{u}, \tau) \chi_{\sigma_k^a(\rho^t)}(\bar{v}, \tau). \]

Here \( u = (u_j)_{j=1 \ldots N} \) are elements of the Cartan subalgebra of \( u(N) \), \( |u| = \sum_j u_j \) and \( \bar{u} \) denotes the traceless part. Similarly \( v = (v_a)_{a=1 \ldots k} \) are elements of Cartan subalgebra of \( u(k) \). \( \chi_{\widehat{su}(N)_k}(\bar{u}, \tau) \) are characters of \( \widehat{su}(N)_k \) at level \( k \) for an integrable irreducible representation specified by a Young diagram \( \rho \), and \( \chi_{\widehat{u}(1)_{N}} \) characters are defined as

\[ \chi_{\widehat{u}(1)_{N}}(z, \tau) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2(n+j/N)^2} e^{2\pi iz(n+j/N)}, \]

where \( \eta(q) \) is the Dedekind \( \eta \) function.
for $q = e^{2\pi i \tau}$.

As an example of a decomposition (A.4) let us consider the case of $\hat{u}(1)_{12} \times \hat{su}(4)_3 \times \hat{su}(3)_4 \subset \hat{u}(12)_1$, with $N = 4$ and $k = 3$. From (A.1) we deduce there are 5 orbits of outer automorphism groups $\mathbb{Z}_4$ and $\mathbb{Z}_3$. Let us consider $\hat{su}(4)_3$ integrable representation related to a diagram $\rho = \square$, and the corresponding $\hat{su}(3)_4$ diagram $\rho' = \square$. The two orbits under $\sigma_4$ and $\sigma_3$ are shown respectively in the first row and column of a table below. All 12 pairs of representations appear in the decomposition (A.4) with $\hat{u}(1)_{12}$ charges given in the table. Note that acting with $\sigma_4$ takes us to another pair of weights given by a step to the right in the table, and increases $\hat{u}(1)_{12}$ charge by 3 (modulo 12). The action of $\sigma_3$ takes us a step to the bottom in the table and increases $\hat{u}(1)_{12}$ charge by 4 (modulo 12). Of course the same table is generated when we build it starting from any other element of these two orbits.

Pairs of $\hat{su}(4)_3 \times \hat{su}(3)_4$ integrable weights with the same fixed $\hat{u}(1)_{12}$ charge, arising in the decomposition of $\hat{u}(12)_1$, are easily found if all 5 such tables of orbits are drawn. For example for charge 0 we then get

$$
\bullet \otimes \bullet + \quad + \quad \quad + \quad + \quad + \quad +
$$

$$
\begin{array}{cccc}
\square & \rightarrow & \square & \rightarrow \\
1 & 4 & 7 & 10 \\
5 & 8 & 11 & 2 \\
9 & 0 & 3 & 6 \\
\end{array}
$$