Appendix A Risk factors for automobile insurance

Table A.1: Description of the key variables (top) and risk factors (bottom) used in both the frequency and severity components for MTPL insurance.

Variable	Values	Description
`Count`	Integer	The number of claims filed by the policyholder.
`Exposure`	Continuous	The exposure to risk in years.
`Size`	Continuous	The size of the claim in euros.
`Cust_Age`	Continuous	Age of the policyholder in years.
`Cust_Residence`	10 categories	Residential area of the policyholder.
`Veh_Age`	Continuous	Age of the insured vehicle in years.
`Veh_BodyDoors`	10 categories	Bodywork of the insured vehicle, accounting for its number of doors.
`Veh_CatValue`	Continuous	Catalogue value of the insured vehicle in multiples of five hundred euro.
`Veh_FuelType`	7 categories	Type of fuel used by the insured vehicle.
`Veh_Mileage`	3 categories	Mileage of the insured vehicle.
`Veh_PowerWeight`	Continuous	Horsepower of the insured vehicle, accounting for its weight.
`Veh_Region`	10 categories	Geographical region in which the insured vehicle is used.
`Veh_Weight`	Continuous	Weight of the insured vehicle in multiples of fifty kilogram.

Appendix B Numerical optimization of M-step

Given the posterior expectations from the E-step and the previous parameter estimates, we numerically maximize over the remaining parameters in the M-step of the Baum-Welch algorithm. Under the Bayesian experience rating convention of $\left(b_U^{(j)}, b_V^{(j)}\right) = \left(a_U^{(j)}, a_V^{(j)} - 1\right)$, the r-th M-step in the full representation entails for every j

$$\arg\max_{\boldsymbol{\vartheta}_N^{(j)}} \sum_{i=1}^{M} \sum_{t=1}^{T_i} {}^{(r)}\gamma_{i,t}^{(j)} \ln\left[\mathbb{P}^{(j)}\left(N_{i,t} \middle| \boldsymbol{\vartheta}_N^{(j)}\right)\right] \quad \text{and} \quad \arg\max_{\boldsymbol{\vartheta}_X^{(j)}} \sum_{i=1}^{M} \sum_{t=1}^{T_i} {}^{(r)}\gamma_{i,t}^{(j)} \sum_{n=0}^{N_{i,t}} \ln\left[\mathbb{P}^{(j)}\left(X_{i,t,n} \middle| \boldsymbol{\vartheta}_X^{(j)}\right)\right].$$

In the sparse HMM where $\left(\boldsymbol{\delta}_A^{(j)}, \varphi^{(j)}, \boldsymbol{\delta}_B^{(j)}\right) = (\boldsymbol{\delta}_A, \varphi, \boldsymbol{\delta}_B)$ for every j, these $2K$ optimizations reduce to the 2 problems

$$\arg\max_{\boldsymbol{\vartheta}_N} \sum_{i=1}^{M} \sum_{t=1}^{T_i} \sum_{j=1}^{K} {}^{(r)}\gamma_{i,t}^{(j)} \ln\left[\mathbb{P}^{(j)}\left(N_{i,t} \middle| \boldsymbol{\vartheta}_N\right)\right] \quad \text{and} \quad \arg\max_{\boldsymbol{\vartheta}_X} \sum_{i=1}^{M} \sum_{t=1}^{T_i} \sum_{j=1}^{K} {}^{(r)}\gamma_{i,t}^{(j)} \sum_{n=0}^{N_{i,t}} \ln\left[\mathbb{P}^{(j)}\left(X_{i,t,n} \middle| \boldsymbol{\vartheta}_X\right)\right].$$

Regardless of the characterization, these optimization problems do not allow a full closed-form solution and need to be solved numerically using, for instance, the Newton-Raphson or Fisher scoring method.

Both of these optimization methods rely on the gradient vectors and Hessian matrices of the log-likelihoods. Based on the expected marginal complete log-likelihoods shown earlier, we find that the gradients are given by $\boldsymbol{g}_N^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right)$ and $\boldsymbol{g}_X^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right)$ or $\boldsymbol{g}_N(\boldsymbol{\vartheta}_N)$ and $\boldsymbol{g}_X(\boldsymbol{\vartheta}_X)$ with respective elements

$$\boldsymbol{g}_{N,1}^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right) = \sum_{i=1}^{M} \sum_{t=1}^{T_i} \gamma_{i,t}^{(j)} \frac{a_U^{(j)}\left(N_{i,t} - e_{i,t}\lambda_{i,t}^{(j)}\right)}{a_U^{(j)} + e_{i,t}\lambda_{i,t}^{(j)}} \boldsymbol{A}_{i,t},$$

$$g_{N,2}^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right) = \sum_{i=1}^{M} \sum_{t=1}^{T_i} \gamma_{i,t}^{(j)} \left[\psi\left(N_{i,t} + a_U^{(j)}\right) - \psi\left(a_U^{(j)}\right) + \ln\left(a_U^{(j)}\right) - \ln\left(a_U^{(j)} + e_{i,t}\lambda_{i,t}^{(j)}\right) + \frac{e_{i,t}\lambda_{i,t}^{(j)} - N_{i,t}}{a_U^{(j)} + e_{i,t}\lambda_{i,t}^{(j)}}\right]$$

or

$$
\begin{aligned}
\boldsymbol{g}_{N,1}\left(\boldsymbol{\vartheta}_N\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\frac{N_{i,t}b_U^{(j)} - e_{i,t}\lambda_{i,t}a_U^{(j)}}{b_U^{(j)} + e_{i,t}\lambda_{i,t}}\boldsymbol{A}_{i,t}, \\
g_{N,2}^{(j)}\left(\boldsymbol{\vartheta}_N\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left[\psi\left(N_{i,t} + a_U^{(j)}\right) - \psi\left(a_U^{(j)}\right) + \ln\left(b_U^{(j)}\right) - \ln\left(b_U^{(j)} + e_{i,t}\lambda_{i,t}\right)\right], \\
g_{N,3}^{(j)}\left(\boldsymbol{\vartheta}_N\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}a_U^{(j)} - N_{i,t}b_U^{(j)}}{b_U^{(j)}\left(b_U^{(j)} + e_{i,t}\lambda_{i,t}\right)}
\end{aligned}
$$

for the number of claims and

$$
\begin{aligned}
g_{X,1}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left\{\frac{\mu_{i,t}^{(j)}}{\varphi^{(j)}}\left[N_{i,t}\left(\psi\left[\mu_{i,t}^{(j)} + a_V^{(j)}\right] - \psi\left[\mu_{i,t}^{(j)}\right] + \ln\left[\varphi^{(j)}\right] + 1\right)\right.\right. \\
&\quad \left.\left. + \sum_{n=0}^{N_{i,t}}\left(\ln\left[X_{i,t,n}\right] - \ln\left[a_V^{(j)} - 1 + \varphi^{(j)}X_{i,t,n}\right]\right)\right] - \sum_{n=0}^{N_{i,t}}\frac{\left[\mu_{i,t}^{(j)} + a_V^{(j)}\right]X_{i,t,n}}{a_V^{(j)} - 1 + \varphi^{(j)}X_{i,t,n}}\right\}, \\
\boldsymbol{g}_{X,2}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\mu_{i,t}^{(j)}\left\{N_{i,t}\left[\psi\left(\mu_{i,t}^{(j)} + a_V^{(j)}\right) - \psi\left(\mu_{i,t}^{(j)}\right) + \ln\left(\varphi^{(j)}\right)\right]\right. \\
&\quad \left. + \sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right) - \ln\left(a_V^{(j)} - 1 + \varphi^{(j)}X_{i,t,n}\right)\right]\right\}\boldsymbol{B}_{i,t}, \\
g_{X,3}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left\{N_{i,t}\left[\psi\left(\mu_{i,t}^{(j)} + a_V^{(j)}\right) - \psi\left(a_V^{(j)}\right) + \ln\left(a_V^{(j)} - 1\right) + \frac{a_V^{(j)}}{a_V^{(j)} - 1}\right]\right. \\
&\quad \left. - \sum_{n=0}^{N_{i,t}}\left[\ln\left(a_V^{(j)} - 1 + \varphi^{(j)}X_{i,t,n}\right) + \frac{\mu_{i,t}^{(j)} + a_V^{(j)}}{a_V^{(j)} - 1 + \varphi^{(j)}X_{i,t,n}}\right]\right\}
\end{aligned}
$$

or

$$
\begin{aligned}
g_{X,1}\left(\boldsymbol{\vartheta}_X\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\left\{\frac{\mu_{i,t}}{\varphi}\left[N_{i,t}\left(\psi\left[\mu_{i,t} + a_V^{(j)}\right] - \psi\left[\mu_{i,t}\right] + \ln\left[\varphi\right] + 1\right)\right.\right. \\
&\quad \left.\left. + \sum_{n=0}^{N_{i,t}}\left(\ln\left[X_{i,t,n}\right] - \ln\left[b_V^{(j)} + \varphi X_{i,t,n}\right]\right)\right] - \sum_{n=0}^{N_{i,t}}\frac{\left[\mu_{i,t} + a_V^{(j)}\right]X_{i,t,n}}{b_V^{(j)} + \varphi X_{i,t,n}}\right\}, \\
\boldsymbol{g}_{X,2}\left(\boldsymbol{\vartheta}_X\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\mu_{i,t}\left\{N_{i,t}\left[\psi\left(\mu_{i,t} + a_V^{(j)}\right) - \psi\left(\mu_{i,t}\right) + \ln\left(\varphi\right)\right]\right. \\
&\quad \left. + \sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right) - \ln\left(b_V^{(j)} + \varphi X_{i,t,n}\right)\right]\right\}\boldsymbol{B}_{i,t}, \\
g_{X,3}^{(j)}\left(\boldsymbol{\vartheta}_X\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left\{N_{i,t}\left[\psi\left(\mu_{i,t} + a_V^{(j)}\right) - \psi\left(a_V^{(j)}\right) + \ln\left(b_V^{(j)}\right)\right] - \sum_{n=0}^{N_{i,t}}\ln\left[b_V^{(j)} + \varphi X_{i,t,n}\right]\right\}, \\
g_{X,4}^{(j)}\left(\boldsymbol{\vartheta}_X\right) &= \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\sum_{n=0}^{N_{i,t}}\frac{\varphi X_{i,t,n}a_V^{(j)} - \mu_{i,t}b_V^{(j)}}{b_V^{(j)}\left(b_V^{(j)} + \varphi X_{i,t,n}\right)}
\end{aligned}
$$

for the claim sizes, for all $j \in \{1, \ldots, K\}$ and where $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\Gamma(x)\right)$ denotes the digamma function. The respective Hessians $\boldsymbol{H}_N^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right)$ and $\boldsymbol{H}_X^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right)$ or $\boldsymbol{H}_N\left(\boldsymbol{\vartheta}_N\right)$ and $\boldsymbol{H}_X\left(\boldsymbol{\vartheta}_X\right)$ are given by

$$\boldsymbol{H}_{N,1,1}^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right) = -\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}^{(j)}a_U^{(j)}\left(a_U^{(j)}+N_{i,t}\right)}{\left(a_U^{(j)}+e_{i,t}\lambda_{i,t}^{(j)}\right)^2}\boldsymbol{A}_{i,t}\boldsymbol{A}_{i,t}',$$

$$\boldsymbol{H}_{N,1,2}^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}^{(j)}\left(N_{i,t}-e_{i,t}\lambda_{i,t}^{(j)}\right)}{\left(a_U^{(j)}+e_{i,t}\lambda_{i,t}^{(j)}\right)^2}\boldsymbol{A}_{i,t}',$$

$$H_{N,2,2}^{(j)}\left(\boldsymbol{\vartheta}_N^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left[\psi_1\left(N_{i,t}+a_U^{(j)}\right)-\psi_1\left(a_U^{(j)}\right)+\frac{1}{a_U^{(j)}}-\frac{a_U^{(j)}+2e_{i,t}\lambda_{i,t}^{(j)}-N_{i,t}}{\left(a_U^{(j)}+e_{i,t}\lambda_{i,t}^{(j)}\right)^2}\right]$$

or

$$\boldsymbol{H}_{N,1,1}\left(\boldsymbol{\vartheta}_N\right) = -\sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}b_U^{(j)}\left(a_U^{(j)}+N_{i,t}\right)}{\left(b_U^{(j)}+e_{i,t}\lambda_{i,t}\right)^2}\boldsymbol{A}_{i,t}\boldsymbol{A}_{i,t}',$$

$$\boldsymbol{H}_{N,1,2}^{(j)}\left(\boldsymbol{\vartheta}_N\right) = -\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}}{b_U^{(j)}+e_{i,t}\lambda_{i,t}}\boldsymbol{A}_{i,t}',$$

$$\boldsymbol{H}_{N,1,3}^{(j)}\left(\boldsymbol{\vartheta}_N\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}\left(a_U^{(j)}+N_{i,t}\right)}{\left(b_U^{(j)}+e_{i,t}\lambda_{i,t}\right)^2}\boldsymbol{A}_{i,t}',$$

$$H_{N,2,2}^{(h,j)}\left(\boldsymbol{\vartheta}_N\right) = \mathbb{1}\left[h=j\right]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left[\psi_1\left(N_{i,t}+a_U^{(j)}\right)-\psi_1\left(a_U^{(j)}\right)\right],$$

$$H_{N,2,3}^{(h,j)}\left(\boldsymbol{\vartheta}_N\right) = \mathbb{1}\left[h=j\right]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}}{b_U^{(j)}\left(b_U^{(j)}+e_{i,t}\lambda_{i,t}\right)},$$

$$H_{N,3,3}^{(h,j)}\left(\boldsymbol{\vartheta}_N\right) = -\mathbb{1}\left[h=j\right]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{e_{i,t}\lambda_{i,t}a_U^{(j)}\left(e_{i,t}\lambda_{i,t}+2b_U^{(j)}\right)-N_{i,t}{b_U^{(j)}}^2}{{b_U^{(j)}}^2\left(b_U^{(j)}+e_{i,t}\lambda_{i,t}\right)^2}$$

for the number of claims and

$$\begin{aligned}H_{X,1,1}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = &\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left\{\frac{\mu_{i,t}^{(j)}}{\varphi^{(j)}}\left[\frac{N_{i,t}}{\varphi^{(j)}}\left(\mu_{i,t}^{(j)}\left[\psi_1\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\psi_1\left(\mu_{i,t}^{(j)}\right)\right]+1\right)\right.\right.\\&\left.\left.-\sum_{n=0}^{N_{i,t}}\frac{2X_{i,t,n}}{a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}}\right]+\sum_{n=0}^{N_{i,t}}\frac{\left[\mu_{i,t}^{(j)}+a_V^{(j)}\right]X_{i,t,n}^2}{\left[a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}\right]^2}\right\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{H}_{X,1,2}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = &\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\frac{\mu_{i,t}^{(j)}}{\varphi^{(j)}}\left\{N_{i,t}\left[\psi\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\psi\left(\mu_{i,t}^{(j)}\right)+\mu_{i,t}^{(j)}\left(\psi_1\left[\mu_{i,t}^{(j)}+a_V^{(j)}\right]-\psi_1\left[\mu_{i,t}^{(j)}\right]\right)\right.\right.\\&\left.\left.+\ \ln\left(\varphi^{(j)}\right)+1\right]+\sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right)-\ln\left(a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}\right)-\frac{\varphi^{(j)}X_{i,t,n}}{a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}}\right]\right\}\boldsymbol{B}_{i,t}',\end{aligned}$$

$$H_{X,1,3}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\frac{\gamma_{i,t}^{(j)}}{\varphi^{(j)}}\left[N_{i,t}\mu_{i,t}^{(j)}\psi_1\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\sum_{n=0}^{N_{i,t}}\frac{\varphi^{(j)}X_{i,t,n}\left(\varphi^{(j)}X_{i,t,n}-1\right)+\mu_{i,t}^{(j)}\left(a_V^{(j)}-1\right)}{\left(a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}\right)^2}\right],$$

$$\boldsymbol{H}_{X,2,2}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\mu_{i,t}^{(j)}\left\{ N_{i,t}\left[\psi\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\psi\left(\mu_{i,t}^{(j)}\right)+\mu_{i,t}^{(j)}\left(\psi_1\left[\mu_{i,t}^{(j)}+a_V^{(j)}\right]-\psi_1\left[\mu_{i,t}^{(j)}\right]\right)\right.\right.$$

$$\left.\left. +\ \ln\left(\varphi^{(j)}\right)\right]+\sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right)-\ln\left(a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}\right)\right]\right\}\boldsymbol{B}_{i,t}\boldsymbol{B}_{i,t}',$$

$$\boldsymbol{H}_{X,2,3}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\mu_{i,t}^{(j)}\left[N_{i,t}\psi_1\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\sum_{n=0}^{N_{i,t}}\frac{1}{a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}}\right]\boldsymbol{B}_{i,t}',$$

$$H_{X,3,3}^{(j)}\left(\boldsymbol{\vartheta}_X^{(j)}\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left\{N_{i,t}\left[\psi_1\left(\mu_{i,t}^{(j)}+a_V^{(j)}\right)-\psi_1\left(a_V^{(j)}\right)+\frac{a_V^{(j)}-2}{\left(a_V^{(j)}-1\right)^2}\right]\right.$$

$$\left. -\sum_{n=0}^{N_{i,t}}\frac{a_V^{(j)}-2+2\varphi^{(j)}X_{i,t,n}-\mu_{i,t}^{(j)}}{\left[a_V^{(j)}-1+\varphi^{(j)}X_{i,t,n}\right]^2}\right\}$$

or

$$H_{X,1,1}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\left\{\frac{\mu_{i,t}}{\varphi}\left[\frac{N_{i,t}}{\varphi}\left(\mu_{i,t}\left[\psi_1\left(\mu_{i,t}+a_V^{(j)}\right)-\psi_1\left(\mu_{i,t}\right)\right]+1\right)-\sum_{n=0}^{N_{i,t}}\frac{X_{i,t,n}}{b_V^{(j)}+\varphi X_{i,t,n}}\right]\right.$$

$$\left. -\sum_{n=0}^{N_{i,t}}\frac{\mu_{i,t}b_V^{(j)}-\varphi X_{i,t,n}a_V^{(j)}}{\varphi\left[b_V^{(j)}+\varphi X_{i,t,n}\right]^2}X_{i,t,n}\right\},$$

$$\boldsymbol{H}_{X,1,2}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\frac{\mu_{i,t}}{\varphi}\left\{N_{i,t}\left[\psi\left(\mu_{i,t}+a_V^{(j)}\right)-\psi\left(\mu_{i,t}\right)+\mu_{i,t}\left(\psi_1\left[\mu_{i,t}+a_V^{(j)}\right]-\psi_1\left[\mu_{i,t}\right]\right)\right.\right.$$

$$\left.\left. +\ \ln\left(\varphi\right)+1\right]+\sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right)-\ln\left(b_V^{(j)}+\varphi X_{i,t,n}\right)-\frac{\varphi X_{i,t,n}}{b_V^{(j)}+\varphi X_{i,t,n}}\right]\right\}\boldsymbol{B}_{i,t}',$$

$$H_{X,1,3}^{(j)}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\left[N_{i,t}\frac{\mu_{i,t}}{\varphi}\psi_1\left(\mu_{i,t}+a_V^{(j)}\right)-\sum_{n=0}^{N_{i,t}}\frac{X_{i,t,n}}{b_V^{(j)}+\varphi X_{i,t,n}}\right],$$

$$H_{X,1,4}^{(j)}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\sum_{n=0}^{N_{i,t}}\frac{\varphi X_{i,t,n}a_V^{(j)}-\mu_{i,t}b_V^{(j)}}{\varphi\left(b_V^{(j)}+\varphi X_{i,t,n}\right)^2},$$

$$\boldsymbol{H}_{X,2,2}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\sum_{j=1}^{K}\gamma_{i,t}^{(j)}\mu_{i,t}\left\{N_{i,t}\left[\psi\left(\mu_{i,t}+a_V^{(j)}\right)-\psi\left(\mu_{i,t}\right)+\mu_{i,t}\left(\psi_1\left[\mu_{i,t}+a_V^{(j)}\right]-\psi_1\left[\mu_{i,t}\right]\right)\right.\right.$$

$$\left.\left. +\ \ln\left(\varphi\right)\right]+\sum_{n=0}^{N_{i,t}}\left[\ln\left(X_{i,t,n}\right)-\ln\left(b_V^{(j)}+\varphi X_{i,t,n}\right)\right]\right\}\boldsymbol{B}_{i,t}\boldsymbol{B}_{i,t}',$$

$$\boldsymbol{H}_{X,2,3}^{(j)}\left(\boldsymbol{\vartheta}_X\right) = \sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}N_{i,t}\mu_{i,t}\psi_1\left(\mu_{i,t}+a_V^{(j)}\right)\boldsymbol{B}_{i,t}',$$

$$\boldsymbol{H}_{X,2,4}^{(j)}\left(\boldsymbol{\vartheta}_X\right) = -\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\sum_{n=0}^{N_{i,t}}\frac{\mu_{i,t}}{b_V^{(j)}+\varphi X_{i,t,n}}\boldsymbol{B}_{i,t}',$$

$$H_{X,3,3}^{(h,j)}\left(\boldsymbol{\vartheta}_X\right) = \mathbb{1}\left[h=j\right]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}N_{i,t}\left[\psi_1\left(\mu_{i,t}+a_V^{(j)}\right)-\psi_1\left(a_V^{(j)}\right)\right],$$

$$H_{X,3,4}^{(h,j)}\left(\boldsymbol{\vartheta}_X\right) = \mathbb{1}\left[h=j\right]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\sum_{n=0}^{N_{i,t}}\frac{\varphi X_{i,t,n}}{b_V^{(j)}\left(b_V^{(j)}+\varphi X_{i,t,n}\right)},$$

$$H_{X,4,4}^{(h,j)}(\boldsymbol{\vartheta}_X) = -\mathbb{1}[h=j]\sum_{i=1}^{M}\sum_{t=1}^{T_i}\gamma_{i,t}^{(j)}\sum_{n=0}^{N_{i,t}}\frac{\varphi X_{i,t,n}a_V^{(j)}\left(\beta X_{i,t,n}+2b_V^{(j)}\right)-\mu_{i,t}{b_V^{(j)}}^2}{{b_V^{(j)}}^2\left(b_V^{(j)}+\varphi X_{i,t,n}\right)^2}$$

for the claim sizes, for all $h,j\in\{1,\dots,K\}$ and where $\psi_1(x)=\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)$ denotes the trigamma function. Since the Hessians involve complicated expressions of $N_{i,t}$ or $X_{i,t,n}$, it becomes infeasible to calculate the Fisher information, or negative expected Hessian, matrices necessary for the Fisher scoring method. We therefore adopt the Newton-Raphson method, where ${}^{(s+1,r)}\boldsymbol{\vartheta} = {}^{(s,r)}\boldsymbol{\vartheta} - \boldsymbol{H}\left({}^{(s,r)}\boldsymbol{\vartheta}\right)^{-1}\boldsymbol{g}\left({}^{(s,r)}\boldsymbol{\vartheta}\right)$ in iteration $s+1$, to overcome this issue. Finally, the asymptotic variance of the parameter estimates also depends on the Fisher information matrix, but a sample analogue can be determined from the diagonal entries of the inverse of the observed information, or negative Hessian, matrix.

Appendix C Supplementary material

Table C.1: Log-likelihood, information criteria and portfolio loss ratio of the MTPL insurance HMMs.

					Loss ratio (%)	
Mixture	**Parameters**	**Log-likelihood**	**AIC**	**BIC**	**Prior**	**Posterior**
Independent (K = 1)	82	-77,384.62	154,933.23	155,767.52	130.38	129.10
K = 2						
- Full	170	-77,014.92	154,369.85	156,099.48	88.99	117.52
- Sparse	92	-77,208.80	154,601.60	155,537.60	97.20	110.18
K = 3						
- Full	258	-76,987.57	154,491.14	157,116.10	117.24	118.96
- Sparse	102	-77,119.25	154,442.49	**155,480.27**	**99.12**	122.32
K = 4						
- Full	348	**-76,755.70**	**154,207.40**	157,748.00	96.07	117.89
- Sparse	114	-77,060.86	154,349.71	155,509.58	65.75	**92.44**

Table C.2: Estimated prior parameters for each MTPL insurance HMM.

	Full representation		Sparse representation			
Mixture	$a_U^{(j)}$	$a_V^{(j)}$	$a_U^{(j)}$	$b_U^{(j)}$	$a_V^{(j)}$	$b_V^{(j)}$
Independent (K = 1)	0.2924	1.7597	0.2924	6.1742	1.7597	1,814.0806
K = 2						
- j = 1	0.3893	1,541.9100	0.2737	5.5105	1.5522	1,213.37
- j = 2	0.9713	1.4317	1.0743	40.3253	1.0144	82.4183
K = 3						
- j = 1	16.0000	231.9674	0.2717	115.2963	4.7075	423.5172
- j = 2	1.2907	174.3674	0.0763	4.9981	1.0070	41.9193
- j = 3	0.8251	1.4286	1.0873	10.6704	1.5563	1,166.6399
K = 4						
- j = 1	0.1128	1,124.7680	0.0032	46.4526	35.0477	295.0512
- j = 2	77.9492	151.8134	4.9798	136.7600	3.5451	2,038.4914
- j = 3	33.9498	1,263.3680	7.9950	123.2833	1.0703	77.2851
- j = 4	1.3160	1.4524	0.6714	5.3886	1.2336	779.7081

Table C.3: Estimated prior transition probabilities in percentages for each MTPL insurance HMM.

	Full representation				**Sparse representation**			
Mixture	**To $j = 1$**	**To $j = 2$**	**To $j = 3$**	**To $j = 4$**	**To $j = 1$**	**To $j = 2$**	**To $j = 3$**	**To $j = 4$**
$K = 2$								
- *From* $h = 0$	64.7900	35.2100			89.4018	10.5982		
- *From* $h = 1$	91.9563	8.0437			99.1273	0.8727		
- *From* $h = 2$	25.1172	74.8828			13.6503	86.3497		
$K = 3$								
- *From* $h = 0$	28.6298	33.2533	38.1170		51.5818	4.5401	43.8781	
- *From* $h = 1$	98.7127	0.0291	1.2582		89.8854	0.2112	9.9034	
- *From* $h = 2$	1.8680	67.0011	31.1309		0.1656	3.3070	96.5274	
- *From* $h = 3$	3.1558	38.4530	58.3911		17.0302	10.7982	72.1717	
$K = 4$								
- *From* $h = 0$	43.6139	16.7018	9.8774	29.8069	34.3907	35.7452	4.6674	25.1968
- *From* $h = 1$	92.1153	7.8847	<0.0000	<0.0000	96.6924	3.3076	<0.0000	<0.0000
- *From* $h = 2$	15.6389	26.3936	32.5300	25.4375	0.0035	83.9551	1.0175	15.0239
- *From* $h = 3$	18.6846	13.0733	<0.0000	68.2421	<0.0000	0.0002	42.9188	57.0810
- *From* $h = 4$	<0.0000	31.7293	24.0639	44.2067	10.6968	22.8832	6.9203	59.4997

Table C.4: Mean [median] of MTPL insurance observations and predictions for each sparse mixture component, where the claim frequencies and risk premia are weighted by their corresponding exposures to risk.

	Claim frequency			**Claim severity**			**Risk premium**		
Mixture	**Observed**	**Prior**	**Posterior**	**Observed**	**Prior**	**Posterior**	**Observed**	**Prior**	**Posterior**
Independent	0.0524	0.0559	0.0562	3,445.43	2,475.57	2,525.16	180.48	138.43	139.80
($K = 1$)	[0.0000]	[0.0514]	[0.0443]	[1,100.00]	[2,462.14]	[2,444.35]	[0.00]	[125.21]	[107.47]
$K = 2$	0.0524	0.0479	0.0563	3,445.43	4,509.14	3,147.61	180.48	185.68	163.81
	[0.0000]	[0.0438]	[0.0441]	[1,100.00]	[3,887.16]	[3,091.25]	[0.00]	[163.58]	[125.30]
- $j = 1$		0.0588	0.0586		2,721.04	2,756.99		160.22	157.20
		[0.0361]	[0.0325]		[2,695.96]	[2,669.02]		[142.40]	[118.29]
- $j = 2$		0.0315	0.0324		7,087.61	6,706.65		223.81	211.73
		[0.0289]	[0.0287]		[7,022.28]	[6,935.12]		[198.92]	[186.84]
$K = 3$	0.0524	0.0633	0.0551	3,445.43	2,462.12	1,656.95	180.48	182.09	147.55
	[0.0000]	[0.0642]	[0.0459]	[1,100.00]	[2,484.10]	[1,489.85]	[0.00]	[175.88]	[116.01]
- $j = 1$		0.0030	0.0038		147.17	427.42		0.45	2.92
		[0.0028]	[0.0028]		[146.07]	[147.87]		[0.40]	[0.41]
- $j = 2$		0.0196	0.0275		7,686.65	7,247.72		151.03	146.15
		[0.0181]	[0.0150]		[7,629.23]	[7,533.46]		[135.19]	[109.31]
- $j = 3$		0.1310	0.1220		2,701.73	2,739.94		354.25	327.49
		[0.1206]	[0.1093]		[2,681.55]	[2,656.25]		[317.09]	[276.18]
$K = 4$	0.0524	0.0691	0.0546	3,445.43	2,121.38	1,836.25	180.48	274.49	195.24
	[0.0000]	[0.0618]	[0.0460]	[1,100.00]	[2,003.32]	[1,692.04]	[0.00]	[236.51]	[156.88]
- $j = 1$		<0.0000	0.0020		13.96	206.48		<0.00	3.20
		[<0.0000]	[<0.0000]		[13.79]	[14.01]		[<0.00]	[<0.00]
- $j = 2$		0.0466	0.0467		1,290.41	1,458.20		60.24	64.76
		[0.0430]	[0.0430]		[1,275.03]	[1,271.20]		[53.50]	[52.88]
- $j = 3$		0.0830	0.0826		1,772.12	1,900.42		147.34	151.22
		[0.0765]	[0.0761]		[1,751.00]	[1,738.05]		[130.86]	[127.70]
- $j = 4$		0.1594	0.1400		5,377.23	5,160.02		858.96	701.86
		[0.1470]	[0.1231]		[5,313.15]	[5,241.05]		[762.93]	[600.31]

Table C.5: Average Bonus-Malus correction of prior risk premium in terms of total previous number of claims and claim amount in independent model ($K = 1$) and each sparse HMM in MTPL insurance. These corrections coincide with those shown in Figure 9 but are represented in tabular form in this table.

	Average Bonus-Malus correction of prior risk premium (×€100)									
	Total number of claims					**Total claim amount (×€100)**				
Independent (K = 1)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.16	4.96	8.25	24.15	26.07	-0.16	4.87	10.65	19.28	21.68
- *2 years*	-0.33	3.20	6.85	15.18	5.09	-0.33	3.33	7.06	12.82	18.77
- *3 years*	-0.43	3.18	3.52	9.19	12.78	-0.43	2.60	5.34	7.74	11.63
- *4 years*	-0.51	2.19	3.26	7.64	16.33	-0.51	2.19	5.08	7.34	11.60
- *5 years*	-0.55	1.42	3.35	9.97	20.08	-0.55	1.77	4.11	5.98	7.40
- *6 years*	-0.59	1.76	12.42	4.74	2.29	-0.59	1.39	3.62	6.75	
- *7 years*	-0.59	1.09	8.09	3.44	2.08	-0.59	1.15	2.88	6.12	6.78
- *8 years*	-0.60	0.80	3.46	3.86	2.118	-0.60	1.16	2.72	3.93	6.53
- *9 years*	-0.63	0.60	5.90	0.867		-0.63	1.04	3.89	5.90	
- *10 years*	-0.44	-0.21				-0.44				
	Total number of claims					**Total claim amount (×€100)**				
Sparse HMM (K = 2)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.41	4.87	8.28	26.29	27.63	-0.41	4.66	12.17	23.43	25.03
- *2 years*	-0.55	2.88	5.9	15.73	4.85	-0.56	3.02	7.87	15.32	21.55
- *3 years*	-0.63	3.02	3.03	9.19	12.62	-0.64	2.26	5.72	8.05	13.26
- *4 years*	-0.69	1.94	2.75	7.71	16.63	-0.70	1.79	5.39	7.89	12.13
- *5 years*	-0.71	1.07	3.01	10.03	20.40	-0.71	1.44	4.24	5.53	8.85
- *6 years*	-0.74	1.57	13.24	4.42	2.12	-0.74	1.06	3.38	7.98	
- *7 years*	-0.72	0.80	8.28	3.04	1.91	-0.72	0.81	2.78	6.21	7.09
- *8 years*	-0.72	0.57	3.35	3.37	1.95	-0.72	0.92	2.60	4.31	6.77
- *9 years*	-0.74	0.33	6.15	-0.30		-0.75	0.81	4.00	6.04	
- *10 years*	-0.46	-0.60				-0.46				
	Total number of claims					**Total claim amount (×€100)**				
Sparse HMM (K = 3)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.79	3.34	5.05	15.62	15.87	0.80	2.86	8.79	16.99	18.86
- *2 years*	-0.71	1.91	4.24	9.85	2.43	-0.72	1.95	5.3	12.03	16.73
- *3 years*	-0.67	2.19	1.89	6.11	9.54	-0.68	1.33	3.83	6.2	9.09
- *4 years*	-0.67	1.19	1.76	5.23	12.86	-0.67	1.22	3.61	5.3	9.44
- *5 years*	-0.62	0.58	1.92	6.89	14.34	-0.63	0.92	2.75	3.5	8.03
- *6 years*	-0.64	0.85	9.67	2.17	0.60	-0.65	0.87	1.63	6.42	
- *7 years*	-0.60	0.59	5.26	1.52	0.30	-0.60	0.51	1.41	7.64	6.18
- *8 years*	-0.63	0.15	2.09	1.23	0.16	-0.63	0.53	0.40	3.92	5.11
- *9 years*	-0.69	0.08	3.77	-1.32		-0.69	0.52	1.76	3.93	
- *10 years*	-0.25	-0.81				-0.25				

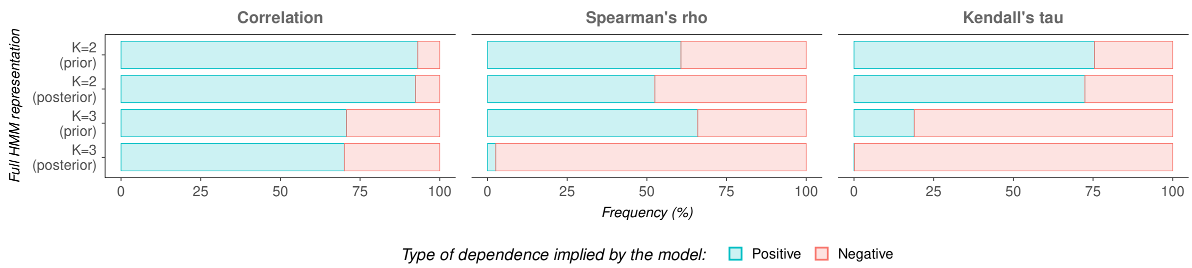
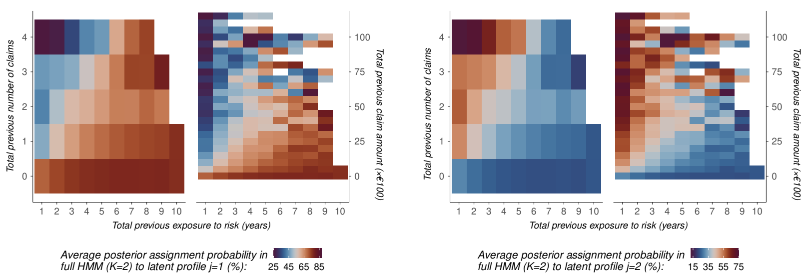


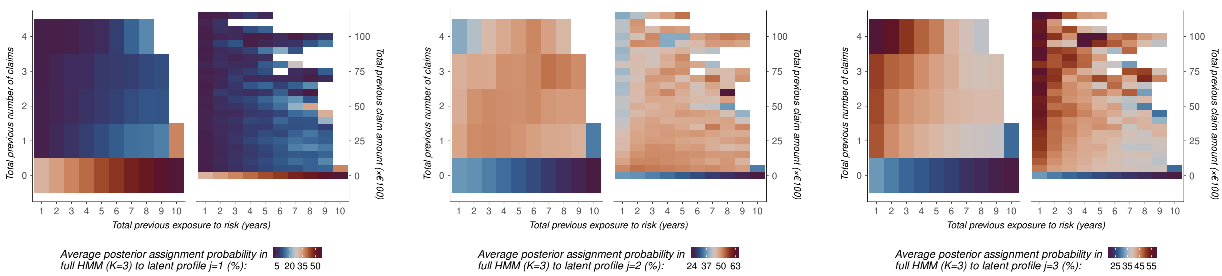
Figure C.1: Frequency of the type of dependence, i.e., positive or negative correlation, Spearman's rho, and Kendall's tau, as implied by the estimated full HMM representation in MTPL insurance.

Table C.6: Mean [median] of MTPL insurance observations and predictions for each full mixture component, where the claim frequencies and risk premia are weighted by their corresponding exposures to risk.

	Claim frequency			Claim severity			Risk premium		
Mixture	**Observed**	**Prior**	**Posterior**	**Observed**	**Prior**	**Posterior**	**Observed**	**Prior**	**Posterior**
Independent	0.0524	0.0559	0.0562	3,445.43	2,475.57	2,525.16	180.48	138.43	139.80
$(K = 1)$	[0.0000]	[0.0514]	[0.0443]	[1,100.00]	[2,462.14]	[2,444.35]	[0.00]	[125.21]	[107.47]
$K = 2$	0.0524	0.0651	0.0554	3,445.43	2,080.79	1,776.92	180.48	202.81	153.58
	[0.0000]	[0.0555]	[0.0451]	[1,100.00]	[1,898.45]	[1,665.33]	[0.00]	[160.36]	[116.01]
- $j = 1$		0.0249	0.0263		795.53	795.75		22.34	23.40
		[0.0195]	[0.0192]		[762.52]	[762.68]		[15.42]	[15.32]
- $j = 2$		0.1250	0.1156		3,977.88	3,865.56		474.76	422.55
		[0.1076]	[0.0963]		[3,702.59]	[3,650.74]		[403.01]	[340.47]
$K = 3$	0.0524	0.0517	0.0547	3,445.43	3,101.17	2,984.73	180.48	148.27	150.38
	[0.0000]	[0.065]	[0.0450]	[1,100.00]	[1,841.63]	[1,859.48]	[0.00]	[124.94]	[115.47]
- $j = 1$		<0.0000	<0.0000		4,491.50	4,328.27		<0.00	<0.00
		[<0.0000]	[<0.0000]		[1,017.98]	[1,037.33]		[<0.00]	[<0.00]
- $j = 2$		0.0496	0.0487		789.79	791.84		44.44	43.44
		[0.0455]	[0.0452]		[757.01]	[757.95]		[30.77]	[30.61]
- $j = 3$		0.1147	0.1061		3,944.18	3,862.51		438.99	389.72
		[0.0970]	[0.0864]		[3,768.19]	[3,716.65]		[370.57]	[310.80]
$K = 4$	0.0524	0.0638	0.0547	3,445.43	4,896.41	3,298.50	180.48	187.86	153.10
	[0.0000]	[0.0565]	[0.0467]	[1,100.00]	[3,273.96]	[2,366.81]	[0.00]	[147.18]	[115.13]
- $j = 1$		0.0019	0.0026		223.48	551.60		2.32	6.55
		[0.0006]	[0.0007]		[124.33]	[139.02]		[0.08]	[0.10]
- $j = 2$		0.1070	0.1068		862.27	864.76		100.62	100.56
		[0.0863]	[0.0863]		[827.04]	[827.88]		[73.48]	[73.49]
- $j = 3$		0.0047	0.0047		20,125.76	18,532.41		107.10	101.53
		[0.0011]	[0.0011]		[11,989.07]	[10,528.14]		[14.06]	[12.36]
- $j = 4$		0.1408	0.1304		3,912.16	3,808.69		522.29	468.40
		[0.1172]	[0.1071]		[3,642.41]	[3,595.60]		[436.11]	[374.31]



(a) Based on $K = 2$ latent risk profiles



(b) Based on $K = 3$ latent risk profiles

Figure C.2: Distribution of average posterior assignment probabilities over total previous number of claims and claim amount for profile $j = 1$ (left), $j = 2$ (middle), and $j = 3$ (right) with $K = 2$ (panel (a)) and $K = 3$ (panel (b)) latent risk profiles for the full HMM representation in MTPL insurance.

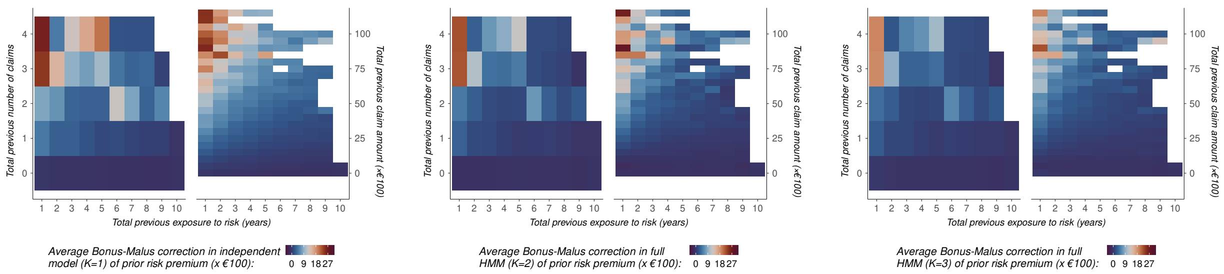


Figure C.3: Average Bonus-Malus correction of prior risk premium in terms of total previous number of claims and claim amount in independent model ($K = 1$, left) and with $K = 2$ (middle) and $K = 3$ (right) latent risk profiles for the full HMM representation in MTPL insurance.

Table C.7: Average Bonus-Malus correction of prior risk premium in terms of total previous number of claims and claim amount in independent model ($K = 1$) and each full HMM in MTPL insurance. These corrections coincide with those shown in Figure C.3 but are represented in tabular form in this table.

	Average Bonus-Malus correction of prior risk premium (×€100)									
	Total number of claims					**Total claim amount (×€100)**				
Independent (K = 1)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.16	4.96	8.25	24.15	26.07	-0.16	4.87	10.65	19.28	21.68
- *2 years*	-0.33	3.20	6.85	15.18	5.09	-0.33	3.33	7.06	12.82	18.77
- *3 years*	-0.43	3.18	3.52	9.19	12.78	-0.43	2.60	5.34	7.74	11.63
- *4 years*	-0.51	2.19	3.26	7.64	16.33	-0.51	2.19	5.08	7.34	11.60
- *5 years*	-0.55	1.42	3.35	9.97	20.08	-0.55	1.77	4.11	5.98	7.40
- *6 years*	-0.59	1.76	12.42	4.74	2.29	-0.59	1.39	3.62	6.75	
- *7 years*	-0.59	1.09	8.09	3.44	2.08	-0.59	1.15	2.88	6.12	6.78
- *8 years*	-0.60	0.80	3.46	3.86	2.118	-0.60	1.16	2.72	3.93	6.53
- *9 years*	-0.63	0.60	5.90	0.867		-0.63	1.04	3.89	5.90	
- *10 years*	-0.44	-0.21				-0.44				
	Total number of claims					**Total claim amount (×€100)**				
Full HMM (K = 2)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.63	3.31	5.02	22.23	22.49	-0.64	2.70	8.07	14.03	17.66
- *2 years*	-0.61	1.56	3.09	10.94	3.18	-0.62	1.53	3.75	11.85	15.57
- *3 years*	-0.56	1.42	1.65	4.42	7.14	-0.57	0.99	3.09	5.28	6.46
- *4 years*	-0.55	0.73	1.38	4.27	8.85	-0.55	0.48	2.78	3.36	9.58
- *5 years*	-0.51	0.32	1.54	5.15	12.27	-0.51	0.39	2.04	1.80	5.50
- *6 years*	-0.53	0.4	7.15	1.75	1.76	-0.54	0.23	1.55	3.89	
- *7 years*	-0.53	0.45	3.19	1.27	1.70	-0.53	0.19	0.99	3.96	4.98
- *8 years*	-0.55	0.08	1.78	0.87	1.18	-0.55	0.26	0.53	3.23	3.76
- *9 years*	-0.68	-0.22	2.69	-0.33		-0.68	0.2	1.3	2.69	
- *10 years*	-0.35	-0.20				-0.35				
	Total number of claims					**Total claim amount (×€100)**				
Full HMM (K = 3)	**0**	**1**	**2**	**3**	**4**	**0**	**25**	**50**	**75**	**100**
Total exposure to risk										
- *1 year*	-0.06	2.89	4.19	18.62	18.95	-0.06	2.48	6.73	11.80	14.04
- *2 years*	-0.25	1.36	3.01	9.11	2.51	-0.25	1.36	3.27	9.04	12.10
- *3 years*	-0.35	1.20	1.34	3.45	5.66	-0.35	0.94	2.61	4.48	5.03
- *4 years*	-0.41	0.71	1.10	3.44	6.81	-0.42	0.55	2.41	3.02	6.97
- *5 years*	-0.44	0.35	1.31	4.20	9.97	-0.44	0.45	1.74	1.91	4.09
- *6 years*	-0.48	0.50	6.07	1.75	1.72	-0.48	0.25	1.32	2.94	
- *7 years*	-0.50	0.40	3.29	1.29	1.57	-0.50	0.19	0.95	3.01	3.39
- *8 years*	-0.53	0.24	1.63	1.21	1.19	-0.53	0.36	0.70	2.32	3.02
- *9 years*	-0.63	0.05	2.64	-0.20		-0.63	0.37	1.41	2.62	
- *10 years*	-0.29	-0.25				-0.29				

Table C.8: Distribution of observed claim sizes and GB2 predictions for independent model ($K = 1$) after truncating (top) and removing (bottom) outliers, where $[\text{Q1} - 1.5 \cdot \text{IQR}, \text{Q3} + 1.5 \cdot \text{IQR}] = [-1405, 4110]$.

Outlier truncation	**Total**	$\leq$ **€100,000**	$\leq$ **€50,000**	$\leq$ **€25,000**	$\leq$ **€10,000**	$\in$ **€[Q1 − 1.5 · IQR, Q3 + 1.5 · IQR]**
Observed						
- *Number of claims*	5,662	5,662	5,662	5,662	5,662	5,662
- *Mean claim size*	3,445.43	2,936.23	2,632.56	2,310.33	1,903.03	1,530.26
- *Median claim size*	1,100.00	1,100.00	1,100.00	1,100.00	1,100.00	1,100.00
Predicted size (GB2)						
- *Min. prediction*	1,415.76	1,392.29	1,357.43	1,290.87	1,136.62	949.71
- *Mean prediction*	2,475.57	2,429.55	2,360.76	2,228.33	1,932.21	1,531.53
- *Median prediction*	2,462.14	2,416.38	2,347.98	2,215.91	1,923.20	1,526.38
- *Max. prediction*	3,801.38	3,718.99	3,595.21	3,355.67	2,830.62	2,083.39
Outlier removal						
Observed						
- *Number of claims*	5,662	5,639	5,611	5,556	5,431	5,069
- *Mean claim size*	3,445.43	2,540.33	2,202.02	1,877.45	1,558.63	1,228.51
- *Median claim size*	1,100.00	1,095.68	1,090.55	1,084.11	1,057.02	997.33
Predicted size (GB2)						
- *Min. prediction*	1,415.76	1,305.33	1,220.29	895.53	776.21	634.88
- *Mean prediction*	2,475.57	2,244.76	2,068.49	1,840.44	1,559.00	1,229.35
- *Median prediction*	2,462.14	2,232.92	2,057.80	1,830.98	1,553.66	1,224.61
- *Max. prediction*	3,801.38	3,455.81	3,135.69	2,789.96	2,345.96	1,810.99

Table C.9: Estimated effects for the risk factors in Table A.1 in the frequency and severity component of the independent model ($K = 1$) and each sparse HMM in MTPL insurance. Estimates that are statistically significant at a significance level of 5% or less are marked by an asterisk.

	Frequency component sparse HMM				**Severity component sparse HMM**			
Risk factor	**Indep. (K = 1)**	**K = 2**	**K = 3**	**K = 4**	**Indep. (K = 1)**	**K = 2**	**K = 3**	**K = 4**
φ					2.85	95.37	297.26	33.41
`Cust_Age`	-0.01*	-0.01*	-0.01*	-0.01*	<0.00*	<0.00*	<0.00*	<0.00*
`Veh_Weight`	0.02*	0.02*	0.02*	0.02*	<0.00*	0.01*	0.01*	0.01*
`Veh_Mileage`								
- *Category 1*	0.01*	0.01*	0.01*	0.01*	-0.03*	-0.03*	-0.01*	-0.03*
- *Category 2*	0.07*	0.07*	0.07*	0.07*	-0.02*	0.01*	0.04*	0.02*
`Veh_FuelType`								
- *Category 1*	0.24*	0.24*	0.23*	0.23*	0.11*	0.14*	0.12*	0.11*
- *Category 2*	0.30*	0.30*	0.28*	0.29*	-0.04*	-0.05*	-0.07*	-0.05*
- *Category 3*	-0.09	-0.11	-0.11	-0.03	-0.34	-0.77	-0.75	-0.68
- *Category 4*	0.18	0.17	0.23	0.23	0.30	0.22	0.23	0.23
- *Category 5*	32.96*	-17.86*	-8.33*	-8.33*				
- *Category 6*	-33.26*	-18.03*	-6.64*	-6.60*				
`Veh_BodyDoors`								
- *Category 1*	0.01*	0.01*	0.01*	0.01*	-0.02*	-0.01*	-0.02*	-0.03*
- *Category 2*	0.08*	0.08*	0.09*	0.10*	-0.03*	-0.01*	-0.03*	-0.05*
- *Category 3*	0.18*	0.18*	0.16*	0.15*	0.05*	0.01*	0.04*	<0.00*
- *Category 4*	0.03*	0.03*	0.04*	0.04*	-0.01*	0.01*	0.01*	0.01*
- *Category 5*	-0.24*	-0.25*	-0.24*	-0.24*	-0.09*	-0.08*	-0.04*	-0.09*
- *Category 6*	0.30*	-0.31*	-0.31*	-0.28*	-0.01*	<0.00*	<0.00*	-0.05*
- *Category 7*	-0.10*	-0.09*	-0.10*	-0.09*	0.11*	0.13*	0.12*	0.04*
- *Category 8*	-0.08*	-0.08*	-0.08*	-0.07*	-0.14*	-0.02*	-0.05*	-0.04*
- *Category 9*	-0.24*	-0.25*	-0.24*	-0.23*	0.15*	0.05*	0.04*	0.01*
`Veh_CatValue`	<0.00*	<0.00*	<0.00*	<0.00*	<0.00*	<0.00*	<0.00*	<0.00*
`Veh_PowerWeight`	0.01*	-0.01*	-0.01*	-0.01*	<0.00*	0.01*	0.01*	0.01*
`Veh_Age`	-0.01*	-0.01*	-0.01*	-0.01*	0.01*	0.01*	0.01*	0.01*
`Veh_Region`								
- *Category 1*	0.08*	0.08*	0.08*	0.07*	0.01*	0.06*	0.09*	0.11*
- *Category 2*	0.10*	0.10*	0.10*	0.09*	<0.00*	0.05*	0.05*	0.09*
- *Category 3*	0.04*	0.04*	0.03*	0.03*	0.01*	0.07*	0.07*	0.10*
- *Category 4*	0.10*	0.10*	0.09*	0.08*	0.03*	0.08*	0.08*	0.12*
- *Category 5*	0.02*	0.02*	0.01*	0.01*	-0.01*	0.08*	0.09*	0.12*
- *Category 6*	-0.01*	-0.01*	-0.01*	-0.01*	<0.00*	0.11*	0.12*	0.15*
- *Category 7*	0.11*	0.12*	0.11*	0.10*	-0.02*	0.04*	0.05*	0.09*
- *Category 8*	-0.08*	-0.08*	-0.08*	-0.08*	-0.08*	-0.01*	0.01*	0.02*
- *Category 9*	0.03*	0.03*	0.02*	0.02*	-0.04*	0.04*	0.06*	0.10*
`Cust_Residence`								
- *Category 1*	0.33*	0.33*	0.34*	0.34*	<0.00*	0.06*	0.05*	0.06*
- *Category 2*	0.14*	0.14*	0.14*	0.14*	-0.01*	-0.05*	-0.05*	-0.04*
- *Category 3*	0.37*	0.37*	0.37*	0.37*	0.02*	-0.02*	-0.02*	<0.00*
- *Category 4*	-0.08*	-0.09*	-0.08*	-0.08*	-0.02*	-0.01*	<0.00*	0.01*
- *Category 5*	-0.15*	-0.15*	-0.15*	-0.15*	0.02*	0.03*	0.03*	0.06*
- *Category 6*	0.63*	0.63*	0.63*	0.63*	-0.07*	-0.06*	-0.06*	-0.05*
- *Category 7*	0.92*	0.92*	0.91*	0.92*	0.37	0.26	0.27	0.21
- *Category 8*	-0.73	-0.74	-0.77	-0.77	0.19*	0.17*	0.16*	0.19*
- *Category 9*	-33.34*	-18.07*	-6.96*	-6.97*				

Table C.10: Estimated effects for the risk factors in Table A.1 in the frequency component of the independent model ($K = 1$) and each full HMM in MTPL insurance. Estimates that are statistically significant at a significance level of 5% or less are marked by an asterisk.

	Indep.	**Full (K = 2)**		**Full (K = 3)**			**Full (K = 4)**			
Risk factor	**(K = 1)**	**j = 1**	**j = 2**	**j = 1**	**j = 2**	**j = 3**	**j = 1**	**j = 2**	**j = 3**	**j = 4**
(Intercept)	-3.05*	-5.55*	-1.71*	7.45	-1.72*	-4.97*	-7.83	-4.01*	-1.44*	-9.43
Cust_Age	-0.01*	0.03*	-0.03*	-1.83	-0.03*	0.03*	0.01*	0.03*	-0.03*	0.05*
Veh_Weight	0.02*	0.02*	0.03*	0.56	0.03*	0.02*	0.09*	0.02*	0.03*	0.01*
Veh_Mileage										
- *Category 1*	0.01*	0.16*	-0.08*	-7.99	-0.05*	0.11*	-0.35	0.16*	-0.06*	-1.32
- *Category 2*	0.07*	0.32*	-0.06*	-1.93	-0.06*	0.30*	0.73	0.29*	-0.12*	0.78
Veh_FuelType										
- *Category 1*	0.24*	0.05*	0.35*	-3.47	0.36*	0.02*	-0.07	0.02*	0.38*	1.11
- *Category 2*	0.30*	0.34*	0.29*	-0.10	0.33*	0.31*	-0.14	0.22*	0.27*	3.01
- *Category 3*	-0.09	0.14	-0.43	<0.00	-0.33	0.14	4.51	-10.33	-9.93	-6.09
- *Category 4*	0.18	0.87	-1.09	<0.00	0.02	0.43	2.93	0.50	-9.26	3.98
- *Category 5*	-32.96*	-32.96	-32.96	<0.00	-7.64	-8.40	-3.37	-6.38	-8.38	-2.36
- *Category 6*	-33.26*	-33.26	-33.26	<0.00	-5.69	-5.78	-2.02	-5.12	-6.46	-0.90
Veh_BodyDoors										
- *Category 1*	0.01*	0.24*	-0.09*	-2.58	-0.09*	0.23*	-1.39	0.25*	-0.06*	-0.62
- *Category 2*	0.08*	0.66*	-0.23*	-2.36	-0.20*	0.64*	-0.74	0.63*	-0.19*	-0.06
- *Category 3*	0.18*	0.49*	-0.03*	-14.49	-0.03*	0.49*	-0.82	0.47*	-0.02*	0.21
- *Category 4*	0.03*	-0.02*	0.09*	-7.46	0.09*	-0.03*	-1.10	-0.03*	0.12*	0.30
- *Category 5*	-0.24*	0.05*	-0.33*	-0.14	-0.34*	0.07*	0.79	-0.21*	-0.36*	1.34
- *Category 6*	-0.30*	0.12*	-0.52*	-0.01	-0.46*	0.05*	-3.30	0.17*	-0.59*	0.98
- *Category 7*	-0.10*	0.45*	-0.46*	-0.49	-0.47*	0.48*	-0.26	0.44*	-0.54*	0.84
- *Category 8*	-0.08*	-0.29*	0.09*	-0.20	0.11*	-0.31*	-1.36	-0.27*	0.14*	0.55
- *Category 9*	-0.24*	-0.29	-0.17*	-2.92	-0.28*	-0.04*	-0.10	-0.08*	-0.33*	1.50
Veh_CatValue	<0.00*	<0.00*	<0.00*	-3.72	<0.00*	<0.00*	0.01*	<0.00*	<0.00*	-0.01*
Veh_PowerWeight	-0.01*	-0.01*	-0.01*	0.17	-0.01*	-0.01*	-0.11*	-0.01*	-0.01*	-0.02*
Veh_Age	-0.01*	-0.04*	0.01*	-1.85	<0.00*	-0.03*	-0.03*	-0.03*	0.01*	-0.04*
Veh_Region										
- *Category 1*	0.08*	0.17*	0.04*	-1.26	0.09*	0.07*	1.94	0.02*	0.04*	0.63
- *Category 2*	0.10*	0.02*	0.14*	-6.26	0.16*	-0.03*	1.77	-0.10*	0.17*	0.18
- *Category 3*	0.04*	0.15*	-0.03*	-0.76	0.02*	0.04*	-0.48	<0.00*	0.02*	0.62
- *Category 4*	0.10*	0.47*	-0.15*	20.42	-0.18*	0.44*	1.34	0.36*	-0.16*	-1.68
- *Category 5*	0.02*	0.09*	-0.03*	-0.80	0.02*	-0.01*	1.02	-0.11*	0.06*	-1.10
- *Category 6*	-0.01*	-0.07*	0.03*	-0.64	0.03*	-0.07*	0.09	-0.15*	0.07*	-0.69
- *Category 7*	0.11*	-0.15*	0.22*	-1.09	0.26*	-0.22*	-0.30	-0.22*	0.25*	1.18
- *Category 8*	-0.08*	-0.32*	0.02*	-0.68	0.04*	-0.34*	1.45	-0.41*	0.05*	0.44
- *Category 9*	0.03*	0.19*	-0.05*	-1.01	-0.01*	0.08*	0.32	0.09*	-0.03*	0.34
Cust_Residence										
- *Category 1*	0.33*	0.26*	0.39*	-0.81	0.41*	0.25*	-0.32	0.28*	0.39*	2.25
- *Category 2*	0.14*	-0.02*	0.24*	15.27	0.22*	0.02*	-1.57	0.06*	0.23*	0.72
- *Category 3*	0.37*	0.22*	0.45*	-1.09	0.47*	0.20*	-0.50	0.21*	0.47*	2.03
- *Category 4*	-0.08*	-0.08*	-0.13*	-0.73	-0.12*	-0.08*	-0.57	-0.05*	-0.14*	0.58
- *Category 5*	-0.15*	-0.34*	-0.06*	-0.26	-0.06*	-0.33*	0.03	-0.42*	-0.02*	-0.05
- *Category 6*	0.63*	0.15*	0.82*	-0.72	0.85*	0.09*	-0.74	0.23*	0.85*	-0.06
- *Category 7*	0.92*	0.19	1.14*	-0.12	1.20*	<0.00	0.88	-0.17	1.20*	3.29
- *Category 8*	-0.73	-0.13	-2.01	<0.00	-2.02	-0.15	2.66	-1.85	-1.89	-7.21
- *Category 9*	-33.34*	-33.34	-33.34	<0.00	-6.13	-6.49	-2.62	-5.11	-6.68	-0.67

Table C.11: Estimated effects for the risk factors in Table A.1 in the severity component of the independent model ($K = 1$) and each full HMM in MTPL insurance. Estimates that are statistically significant at a significance level of 5% or less are marked by an asterisk.

	Indep.	**Full (K = 2)**		**Full (K = 3)**			**Full (K = 4)**			
Risk factor	**(K = 1)**	**j = 1**	**j = 2**	**j = 1**	**j = 2**	**j = 3**	**j = 1**	**j = 2**	**j = 3**	**j = 4**
$\varphi^{(j)}$	<0.00*	<0.00*	38.98	71.73	21.22	<0.00*	65.36	<0.00*	18.53	62.9
`Cust_Age`	7.78*	6.34*	7.85*	0.07	7.81*	6.33*	2.57*	6.53*	7.76*	6.51*
`Veh_Weight`	<0.00*	0.01*	<0.00*	0.08	<0.00*	0.01*	0.01*	0.01*	<0.00*	-0.01*
`Veh_Mileage`										
- *Category 1*	<0.00*	<0.00*	0.01*	0.14	0.01*	<0.00*	<0.00*	<0.00*	0.01*	0.05*
- *Category 2*	-0.03*	0.01*	0.02*	<0.00	<0.00*	-0.01*	0.02*	-0.02*	0.03*	-0.25*
`Veh_FuelType`										
- *Category 1*	-0.02*	-0.03*	0.12*	<0.00	0.11*	-0.04*	-0.02*	0.03*	0.09*	0.05*
- *Category 2*	0.11*	0.01*	0.10*	-0.02	0.09*	0.01*	-0.06*	-0.01*	0.07*	0.12*
- *Category 3*	-0.04*	0.01*	0.01*	<0.00	<0.00*	-0.02*	-1.99*	-0.02*	-0.09*	-0.73*
- *Category 4*	-0.34	-0.34	3.11	<0.00	-0.34	-0.41	0.78*	-0.13	3.10	0.10*
`Veh_BodyDoors`										
- *Category 1*	0.30	0.50	0.06	<0.00	0.06	0.52	2.86*	0.54	-0.01	<0.00
- *Category 2*	0.02*	-0.02*	-0.04*	-0.07	-0.05*	-0.03*	0.31*	-0.08*	-0.06*	1.19*
- *Category 3*	-0.03*	0.06*	0.02*	0.02	<0.00*	0.07*	-0.94*	0.02*	-0.01*	0.24*
- *Category 4*	0.05*	0.09*	0.19*	<0.00	0.19*	0.10*	0.78*	-0.09*	0.20*	-1.46*
- *Category 5*	-0.01*	0.10*	0.06*	-0.07	0.07*	0.10*	0.21*	0.04*	0.07*	0.08*
- *Category 6*	-0.09*	0.13*	0.05*	<0.00	0.03*	0.13*	-0.38*	0.07*	0.01*	-0.53*
- *Category 7*	-0.01*	-0.06*	-0.06*	0.01	-0.08*	-0.06*	-0.39*	-0.06*	-0.12*	0.46*
- *Category 8*	0.11*	0.30*	0.11*	<0.00	0.09*	0.29*	-1.52*	0.28*	0.05*	-0.55*
- *Category 9*	-0.14*	-0.03*	0.39*	<0.00	0.38*	<0.00*	-0.60*	-0.01*	0.36*	0.62*
`Veh_CatValue`	0.15*	0.49*	-0.10*	<0.00	-0.09*	0.46*	0.46*	0.35*	-0.10*	-0.79*
`Veh_PowerWeight`	<0.00*	<0.00*	<0.00*	<0.00	<0.00*	<0.00*	0.01*	<0.00*	<0.00*	<0.00*
`Veh_Age`	<0.00*	-0.01*	0.01*	-0.03	0.01*	-0.01*	-0.05*	-0.01*	0.01*	0.04*
`Veh_Region`										
- *Category 1*	0.01*	<0.00*	0.01*	0.02	0.01*	<0.00*	0.10*	<0.00*	0.01*	0.06*
- *Category 2*	0.01*	-0.04*	0.05*	0.05	0.06*	-0.04*	1.19*	-0.04*	0.07*	1.16*
- *Category 3*	<0.00*	-0.06*	<0.00*	0.21	0.03*	-0.04*	0.82*	-0.04*	0.05*	1.31*
- *Category 4*	0.01*	-0.06*	-0.02*	<0.00	<0.00*	-0.06*	1.02*	0.08*	-0.01*	1.65*
- *Category 5*	0.03*	0.03*	0.03*	-0.10	0.03*	0.02*	1.90*	0.03*	0.06*	0.85*
- *Category 6*	-0.01*	<0.00*	0.12*	-0.21	0.16*	0.03*	1.46*	0.01*	0.19*	0.17*
- *Category 7*	<0.00*	-0.21*	0.06*	0.04	0.07*	-0.21*	0.20*	-0.17*	0.07*	0.52*
- *Category 8*	-0.02*	-0.10*	-0.02*	0.06	<0.00*	-0.07*	0.26*	-0.07*	0.02*	0.50*
- *Category 9*	-0.08*	-0.14*	0.02*	0.05	0.02*	-0.15*	0.79*	-0.12*	0.05*	0.91*
`Cust_Residence`										
- *Category 1*	-0.04*	-0.20*	-0.02*	0.04	<0.00*	-0.19*	0.52*	-0.08*	<0.00*	-0.83*
- *Category 2*	<0.00*	-0.09*	0.05*	0.05	0.04*	-0.09*	0.20*	-0.12*	0.02*	0.10*
- *Category 3*	-0.01*	0.04*	-0.14*	0.01	-0.13*	0.04*	0.04*	0.02*	-0.15*	0.99*
- *Category 4*	-0.02*	<0.00*	-0.12*	0.07	-0.12*	0.01*	-0.45*	-0.03*	-0.14*	0.65*
- *Category 5*	-0.02*	-0.09*	-0.06*	0.01	-0.07*	-0.10*	-0.37*	-0.11*	-0.10*	0.40*
- *Category 6*	0.02*	-0.04*	-0.10*	0.06	-0.11*	-0.06*	0.95*	-0.08*	-0.10*	-3.00*
- *Category 7*	-0.07*	-0.12*	-0.13*	-0.05	-0.14*	-0.11*	0.05*	-0.10*	-0.17*	-0.26*
- *Category 8*	0.37	0.58	2.11	<0.00	2.11	0.58	1.73*	1.10	2.09	<0.00