UvA-DARE (Digital Academic Repository)

Optimal demographic risk sharing with funded pensions in a general equilibrium model

Beetsma, R.M.W.J.; Romp, W.E.; Vos, S.J.

Publication date
2009

Document Version
Submitted manuscript

Link to publication

Citation for published version (APA):
Beetsma, R. M. W. J., Romp, W. E., & Vos, S. J. (2009). *Optimal demographic risk sharing with funded pensions in a general equilibrium model*. Faculteit Economie en Bedrijfskunde.

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:24 Jun 2021

Optimal demographic risk sharing with funded pensions in a general equilibrium model

Roel M. W. J. Beetsma
University of Amsterdam, Tinbergen Institute, Netspar, CEPR and CESifo[*]

Ward E. Romp
University of Amsterdam and Netspar[†]

Siert J. Vos
University of Amsterdam and Netspar[‡]

This version: April 20, 2009

Abstract

In this paper we assess the general equilibrium effects of a two-tier pension system in intergenerational risk sharing in the presence of productivity, financial market and demographic risks. We find relatively large welfare gains from the presence of a two-tier pension system with a defined benefit second pillar when compared to laissez-faire. The first, PAYG pillar takes care of appropriate redistribution between the young and the old generation. Benefits from the fully funded second pillar allow for risk sharing between the two generations.

The exact form of the defined benefit second pillar is not very important, as different specifications of the second pillar lead to almost identical welfare gains compared to laissez-faire.

[*]Mailing address: Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands; Phone: +31.20.5255280; fax: +31.20.5254254; e-mail: R.M.W.J.Beetsma@uva.nl.

[†]Mailing address: Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands; Phone: +31.20.5257178; fax: +31.20.5254254; e-mail: W.E.Romp@uva.nl.

[‡]Mailing address: Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands; Phone: +31.20.5257178; fax: +31.20.5254254; e-mail: S.J.Vos@uva.nl.

1 Introduction

Pension arrangements will generally affect the risks faced by workers and retirees. One type of risk is financial market risk. Those risks are relatively large for funded pension systems in which workers accumulate assets via their pension fund for future retirement. Another important source of risk is demographic risk.

In this paper we explore optimal pension system design in the presence of several sources of risk, in particular productivity risk, financial market risk and demographic risk. To keep the set-up as simple as possible, we consider a two-period model with two generations whose lifes overlap only in the second period. The young generation is born after those shocks have materialized. Hence, under laissez-faire, that is in the absence of a pension system, they are unable to participate in financial markets and thus unable to share risks with the old generation. The pension systems we consider consist of a PAYG first tier and a fully funded second tier. A far as the second tier is concerned, benefits can be defined in several ways. They can be defined in real terms, in terms of wages and/or in demographic terms.

We explore optimal pension arrangements under the assumption that the arrangement's parameters are kept constant. Hence, in particular, they are not contingent on the shocks. Generally, one would like to see these parameters constant in order to avoid that confidence in the pension system is undermined. Second, each time the system's parameter values are to be reset, a conflict may arise between the generations as they will be affected in different ways. Finally, it may be hard to fine-tune the parameters in response to shocks. With all sources of risk present, it is socially optimal for the pension system to pay out a combination of wage and demography-linked benefits.

2 The command economy

Our model is deliberately kept as simple as possible, in order to describe the intuitions as clearly as possible.

2.1 Individuals and preferences

The model represents a closed economy. It incorporates two periods (0 and 1) and two generations. In period 0, a generation of mass 1 is born. This generation consumes only in period 1 when it is old, and hence features the following utility function:

$$\psi \mathrm{E}_0 \left[u \left(c_o \right) \right], \tag{1}$$

where c_o represents consumption when the agent is old and ψ is the probability that an individual old person survives period 0. Further, $\mathrm{E}_0 [.]$ is the expectations operator conditional on information before any of the shocks (see below) have occurred. We assume that $u_c > 0$ and $u_{cc} < 0$, where subscripts denote partial derivatives.

In period 1, a new generation of mass γ is born. The size of γ is uncertain in period 0. The representative individual of the young generation features the utility function:

$$u\left(c_y\right), \tag{2}$$

which is defined over their consumption c_y in period 1.

The are two sources of demographic risk in the model: the life expectancy of the old generation, as captured by ψ, and fertility as captured by the size of the young generation, γ.

2.2 Production

In period 0 each old generation member receives an exogenous non-storable endowment η_0. Production is endogenous only in period 1, when the two generations co-exist. Labour supply is exogenous in our model and given by 1 unit of labour per young person. Hence, production is given by

$$Y = AF\left(K,\gamma\right), \tag{3}$$

where A denotes stochastic total factor productivity, K represents the aggregate capital stock and γ is the stochastic aggregate labor input. The production function exhibits constant returns to scale. In our closed economy, the capital stock K is the result of investment in the previous period 0.

2.3 Resource constraints

The resource constraints in periods 0 and 1 are given by, respectively,

$$\eta_0 = K, \tag{4}$$

$$AF\left(K,\gamma\right) + \left(1-\delta\right)K = \gamma c_y + \psi c_o, \tag{5}$$

where $0 \leq \delta \leq 1$ is the stochastic depreciation rate of the capital stock. The left-hand sides of (4) and (5) are the available resources in the two periods. Specifically, the right-hand side of (5) equals total production plus what is left over of the capital stock after taking depreciation into account. The right-hand sides of (4) and (5) describe the use of these resources. The entire endowment in period 0 is spent on investment in physical capital (equation (4)), while the right-hand side of (5) is total consumption in the economy.

2.4 The social planner's solution

The vector of the stochastic shocks hitting the command economy is $\xi \equiv \{A, \delta, \gamma, \psi\}$. It is unknown in period 0, but becomes known before period 1 variables are determined. As a benchmark, we consider a (utilitarian) social planner who aims to maximize the sum of the discounted expected utilities of all individuals. In period 0, this planner commits to an optimal state-contingent plan. Hence, the consumption levels are functions of the shocks, so that we write $c_o = c_o(\xi)$ and $c_y = c_y(\xi)$. We can write the planner's problem as:

$$\pounds = \int \begin{bmatrix} \psi \left[u\left(c_o\left(\xi\right)\right)\right] + \gamma \left[u\left(c_y\left(\xi\right)\right)\right] \\ +\lambda\left(\xi\right)\left[AF\left(K,\gamma\right) + \left(1-\delta\right)K - \gamma c_y\left(\xi\right) - \psi c_o\left(\xi\right)\right] \end{bmatrix} f\left(\xi\right) d\xi \tag{6}$$

$$\tag{7}$$

Here, $f(\xi)$ stands for the probability density function of the vector of stochastic shocks ξ. The Lagrange multiplier on the resource constraint in period is denoted by λ. Maximization of the planner's program with respect to $c_y(\xi)$ and $c_o(\xi)$ for all ξ yields the following first order conditions:

$$\begin{aligned} \frac{\partial \pounds}{\partial c_y} &= 0 \Rightarrow u_c\left(c_y\left(\xi\right)\right) = \lambda\left(\xi\right), \forall \xi, \\ \frac{\partial \pounds}{\partial c_o} &= 0 \Rightarrow u_c\left(c_o\left(\xi\right)\right) = \lambda\left(\xi\right), \forall \xi. \end{aligned}$$

By eliminating the Lagrange multiplier from these first-order conditons, we obtain

$$u_c\left(c_y\right) = u_c\left(c_o\right), \forall \xi. \tag{8}$$

which reduces under equal utility functions to:

$$c_y = c_o, \forall\ \xi,$$

If a decentralized equilibrium is to replicate the planner's solution, condition (8) needs to be met in addition to the resource constraints (4) and (5). Condition (8) equalizes the marginal utilities of the two generations.

3 The decentralized economy

3.1 The pension systems

The first pillar in the pension system is a pay-as-you-go (PAYG) part composed of a lump-sum part and a wage-indexed part. Each young person pays a (possibly negative) amount $\frac{\psi}{\gamma}\theta^p$ and a fraction $\frac{\psi}{\gamma}\theta^w$ of his wage income. Hence, the systematic transfer per surviving old member via the first pillar is $\theta^p + \theta^w w$.

The second pillar of the pension system consists of a pension fund that collects contributions θ^f per old-generation member in period 0, invests these contributions and pays out benefits to the ψ surviving old generation members in period 1. If we denote the rate of return that a surviving member of the old generation receives on their contributions by r^f, then the per-old payout is $(1 + r^f)\theta^f$.

The pension fund can invest the old generation's contribution in price-indexed bonds (b^f) or in physical capital (k^f):

$$\theta^f = b^f + k^f. \tag{9}$$

The real bonds provide a non-stochastic return of r and physical capital provides the net-of-depreciation rate of return $r^{kn} \equiv AF_K - \delta$, where F_K is the marginal product of capital. The average return on the assets held by the fund is denoted by r^a. Therefore, the total value of the fund before the payout is

$$(1 + r^a)\theta^f = (1 + r)\, b^f + \left(1 + r^{kn}\right) k^f. \tag{10}$$

Depending on the pension scheme and the fund's investment scheme, there can be a difference between the value of the fund and the payout equal to $\left[(1 + r^a) - \psi\left(1 + r^f\right)\right]\theta^f$, where $\psi\left(1 + r^f\right)\theta^f$ is the total pension payout to the old generation. The young are the residual claimants of the fund and receive this difference. If we denote the generational account per old person by G we have

$$G = \theta^p + \theta^w w + \left[\left(1 + r^f\right) - \frac{1}{\psi}\left(1 + r^a\right)\right]\theta^f. \tag{11}$$

3.1.1 Defined contribution (DC)

In a DC system, the pension fund invests the contributions of the old generation in period 0 and pays out whatever these investments turn out to be worth in period 1:

$$\left(1 + r^f\right)\theta^f = \frac{1}{\psi}(1 + r^a)\theta^f$$

which we can rewrite as:

$$r^f = \frac{1}{\psi}(1 + r^a) - 1 \tag{12}$$

Therefore, the payout and the value of the pension fund always coincide and there is no residual claim left for the young generation. This implies that the generational account under a DC system simplifies to:

$$G = \theta^p + \theta^w w. \tag{13}$$

3.1.2 Defined real benefit (DRB)

Under a DRB system, the pension fund benefit is as follows linked to the return on bonds:

$$\left(1 + r^f\right)\theta^f = \frac{1}{\bar{\psi}}(1 + r)\theta^f \tag{14}$$

where $\bar{\psi}$ is the expected value of the realization of the mortality rate ψ in period 0. Hence, the return to the old generation on its contribution to the pension fund is known in advance, while consequences of an unexpectedly high or low life expectancy of the old generation (as captured by the realisation of ψ) are born by the young generation. In period 1 the pension fund will have accumulated assets equal to $(1+r)\,b^f + \left(1+r^{kn}\right)k^f$. Hence, the difference between the assets and the payout of the pension fund equals:

$$\begin{aligned} & (1+r)\,b^f + \left(1+r^{kn}\right)k^f - \frac{\psi}{\bar{\bar{\psi}}}(1+r)\,\theta^f \\ = \; & \left(1-\frac{\psi}{\bar{\bar{\psi}}}\right)(1+r)\,b^f + \left[\left(1+r^{kn}\right) - \frac{\psi}{\bar{\bar{\psi}}}(1+r)\right]k^f, \end{aligned}$$

which implies a generational account per old person of:

$$G = \theta^p + \theta^w w + \left(\frac{1}{\bar{\bar{\psi}}} - \frac{1}{\psi}\right)(1+r)\,b^f + \left[\frac{1}{\bar{\bar{\psi}}}(1+r) - \frac{1}{\psi}\left(1+r^{kn}\right)\right]k^f \tag{15}$$

3.1.3 Defined wage-indexed benefit (DWB)

Under a DWB system, the pension fund benefit is indexed to the wage rate, with a return to the pension system equal to the return to wage indexed bonds. The pension fund benefit per old in period 1 is:

$$\left(1+r^f\right)\theta^f = \frac{1}{\bar{\bar{\psi}}}\theta^{dwb}\gamma w, \tag{16}$$

where γw is the total wage sum and parameter θ^{dwb} links the aggregate pension payout to the total wage bill. The return to the old generation on their pension contribution is stochastic, because the size of the young generation is unknown when the contribution is made, while, moreover, the wage rate is only determined by market forces after the shocks have materialised. Hence, if the realised total wage bill exceeds the expected total wage bill, the payout of the pension fund is higher than expected.

The difference between the assets and the total payout of the pension fund is:

$$\left[(1+r^a) - \psi\left(1+r^f\right)\right]\theta^f = (1+r)\,b^f + \left(1+r^{kn}\right)k^f - \frac{\psi}{\bar{\bar{\psi}}}\theta^{dwb}\gamma w.$$

Hence, the generational account under the DWB system is:

$$G = \theta^p + \theta^w w + \frac{1}{\bar{\bar{\psi}}}\theta^{dwb}\gamma w - \frac{1}{\psi}\left[(1+r)\,b^f + \left(1+r^{kn}\right)k^f\right]. \tag{17}$$

3.1.4 Defined wage- and demography-indexed benefit (DWDB)

Under a DWDB system, the second pillar pension benefits are indexed to both wages and the demographic shocks. This way, the pension system allows the two generations to share both wage risks and demographic risks. The pension fund benefit per old persion in period 1 is:

$$\left(1+r^f\right)\theta^f = \frac{1}{\bar{\bar{\psi}}}\theta^{dwb}\gamma w + \frac{1}{\bar{\bar{\psi}}}\theta^{\gamma}\frac{\gamma}{1+\gamma} + \theta^{\psi}\frac{1}{1+\psi} \tag{18}$$

The difference between the value of the assets and the payout of the pension fund is:

$$\left[(1+r^a) - \psi\left(1+r^f\right)\right]\theta^f = (1+r)\,b^f + \left(1+r^{kn}\right)k^f - \left[\frac{\psi}{\bar{\psi}}\theta^{dwb}\gamma w + \frac{\psi}{\bar{\psi}}\theta^{\gamma}\frac{\gamma}{1+\gamma} + \theta^{\psi}\frac{\psi}{1+\psi}\right].$$

Given that the young generation bears the mismatch risk of the pension fund, we expect that an efficient sharing of demographic risks between the two generations requires θ^{γ} and θ^{ψ} to be positive. A larger young generation (an increase in γ) reduces the burden on each individual young person of the pension payout to the old generation and allows for an increase in the pensions of the old generation. Similarly, an increase in the size of the old generation (ψ rises) means that the young generation faces a larger pension burden and, hence, the pension $\left(1/\bar{\psi}\right)\left(1+r^f\right)\theta^f$ of an old person should fall. Of course, the total pension payout $\left(\psi/\bar{\psi}\right)\left(1+r^f\right)\theta^f$ will increase as ψ rises.

Under the DWDB system, the generational account becomes:

$$G = \theta^p + \theta^w w + \frac{1}{\bar{\psi}}\gamma\theta^{dwb}w + \frac{1}{\bar{\psi}}\theta^{\gamma}\frac{\gamma}{1+\gamma} + \theta^{\psi}\frac{1}{1+\psi} - \frac{1}{\psi}\left[(1+r)\,b^f + \left(1+r^{kn}\right)k^f\right] \quad (19)$$

3.2 Individual budget constraints and generational accounts

Each individual old person receives his deterministic period 0 endowment η_0, pays the mandatory pension fund contribution θ^f, and makes a private investments k and b in physical capital and bonds, respectively:

$$\eta_0 = b + k + \theta^f, \quad (20)$$

In period 1, the surviving members of the old generation receive the returns on their private investments plus their pension system benefits, while the members of the young generation receive their wage plus the residual (potentially negative) value of the pension fund. In addition, the (non-pension) assets of those old generation members that do not survive until period 1 are distributed equally across all individuals alive in period 1. Hence, each one of them receives an accidental bequest of $\left(\frac{1-\psi}{\gamma+\psi}\right)\left[\left(1+r^{kn}\right)k + (1+r)\,b\right]$, implying that:

$$\begin{aligned} c_o &= \frac{1+\gamma}{\gamma+\psi}\left[\left(1+r^{kn}\right)k + (1+r)\,b\right] + \frac{1}{\psi}\left(1+r^a\right)\theta^f + G, && (21)\\ c_y &= w + \frac{1-\psi}{\gamma+\psi}\left[\left(1+r^{kn}\right)k + (1+r)\,b\right] - \frac{\psi}{\gamma}G. && (22)\end{aligned}$$

For the specific case of the DWDB system, consumption of the old and the young can be written as:

$$c_o = \left\{\begin{array}{c} \frac{1+\gamma}{\gamma+\psi}\left[\left(1+r^{kn}\right)k + (1+r)\,b\right] + \theta^p + \theta^w w \\ +\frac{1}{\bar{\psi}}\left[\theta^{dwb}\gamma w + \theta^{\gamma}\frac{\gamma}{1+\gamma}\right] + \theta^{\psi}\frac{1}{1+\psi} \end{array}\right\} \quad (23a)$$

$$c_y = \left\{\begin{array}{c} \left(1 - \frac{\psi}{\gamma}\theta^w - \frac{\psi}{\bar{\psi}}\theta^{dwb}\right)w - \frac{\psi}{\gamma}\theta^p - \frac{\psi}{\bar{\psi}}\theta^{\gamma}\frac{1}{1+\gamma} - \frac{1}{\gamma}\theta^{\psi}\frac{\psi}{1+\psi} \\ +\frac{1}{\gamma}\left[\left(1+r^{kn}\right)k^f + (1+r)\,b^f\right] + \frac{1-\psi}{\gamma+\psi}\left[\left(1+r^{kn}\right)k + (1+r)\,b\right] \end{array}\right\} \quad (23b)$$

3.3 Individual and firm optimization

We solve the model by backwards induction. A continuum of perfectly competitive representative firms, with mass normalized to unity, produce according to (3) and maximize profits $AF(K,L) - wL - r^k K$ over L and K, taking the wage rate and rental rate on capital as given. The first-order conditions are:

$$AF_L = w, \tag{24}$$

$$AF_K = r^k. \tag{25}$$

In period 0, the old generation decides on the allocation of its savings over the various assets. Subject to (20), they maximize (1) over b and k, where c_o is given by (21). The first-order condition is:

$$(1+r)\,\mathrm{E}_0\left[u_c(c_o)\right] = \mathrm{E}_0\left[\left(1+r^k-\delta\right)u_c(c_o)\right]. \tag{26}$$

The full funding condition states that, at the margin the old generation individual should be indifferent between putting an extra dollar into the pension fund or investing the extra dollar in risk-free bonds or equity (or any other asset potentially traded on the market). This way the second pillar of the pension system avoid systematic redistribution between the two generations. Based on the indifference about investing in risk-free bonds, the full funding condition becomes:

$$(1+r)\,\mathrm{E}_0\left[\frac{1+\gamma}{\gamma+\psi}\psi u_c(c_o)\right] = \mathrm{E}_0\left[\left(1+r^f\right)\psi u_c(c_o)\right], \tag{27}$$

where the left-hand side is the expected marginal utility of investing one more dollar in risk-free bonds, while the right-hand side is the expected marginal utility increasing the persion contribution by one more dollar.

3.4 Market equilibrium conditions

The goods market equilibrium conditions in periods 0 and 1 are (4) and (5), respectively. Given that the mass of the old generation in period 0 is 1, equilibrium in the capital market requires that

$$K = k + k^f, \tag{28}$$

while, with zero net outstanding debt, debt market equilibrium in period 0 requires that:

$$b + b^f = 0. \tag{29}$$

4 Optimal pension policy

4.1 Optimal solution

For the market system to replicate the planner's allocation, (8) needs to be fulfilled. This is achieved by setting the pension parameters as:

$$\theta^p = \frac{1+\gamma}{\gamma+\psi}(1+r)\,b^f,\ k^f = \frac{\gamma}{1+\gamma}K,\ \theta^w + \frac{\gamma}{\bar{\psi}}\theta^{dwb} = \frac{\gamma}{\gamma+\psi},\ \theta^w = \theta^\gamma = \theta^\psi = 0,$$

Hence, there exists an infinite number of paramete combinations that achive the planner's allocation. However, in reality a pension arrangement with shock-contingent parameters does not seem to be a very attractive option. Frequent changes in the pension parameters may undermine the confidence in the pension system, while, moreover, each time that the system's parameter combinations are to be reset, a intergenerational conflict may be re-opened on how the parameters should exactly be set. Finally, it may be hard to find the most appropriate parameter values if the size of the shocks is difficult to precisely establish.

4.2 Linearizing around the median

Under the assumption that the parameters characterizing the pension arrangement are constant (not contingent on the shocks) we can no longer analytically solve for the optimal arrangement. We can, however, construct a first-order approximation of what the non-shock-contingent pension system parameters should look like by linearizing around the median system (the point where all shocks happen to be zero). This might give some indication of the direction in which the pension system parameters should go, but we will miss out on the important non-linearities in the model because the approximation is of first-order. Therefore, we also do some numerical analysis, which allows us to optimize the pension system parameters given some calibration of our economy.

The condition for optimal risk sharing in the linearized version of the model becomes:

$$\hat{c}_y = \frac{\sigma_o}{\sigma_y}\,\hat{c}_o \tag{30}$$

where a hat denotes the logarithmic deviation from the point around which the approximation is taken and $\sigma_i \equiv -\bar{c}_i u''(\bar{c}_i)/u'(\bar{c}_i)$ is the coefficient of relative risk aversion of generation i.

4.2.1 The solution to the general case

We start with giving the solution for the case when both demographic risks are present. In this solution, we normalize $\bar{\gamma} = \bar{\psi} = 1$ and we assume equal risk aversion for both generations, i.e. $\frac{\sigma_o \bar{c}_y}{\sigma_y \bar{c}_o} = \nu = 1$. We will develop the intuition for this result by first looking at the case when there is no demographic uncertainties, then at the two cases where there is only one source of demographic uncertainty and finally by looking in detail at the full result.

To replicate condition (30), we can show (see appendix) that we have to set

Proposition 1 *For an economy featuring productivity, depreciation, fertility and mortality risks and a DWDB second pillar pension fund, there are multiple settings in which we can obtain optimal risk sharing. One such setting is* $k^f = k$, $\theta^w = 1/2 - \theta^{dwb}$, $\theta^p = -\frac{1}{2}\bar{w}$, $\theta^\gamma = \theta^\psi = 0$ *and* $\theta^{dwb} = \frac{(1+r)b}{2\bar{w}} - \frac{\left(1+\bar{r}^{kn}\right)k}{2\bar{w}}$

4.2.2 No demographic risk

In the special case of only productivity and depreciation risk, the proposition simplifies to:

Proposition 2 *For an economy featuring productivity and depreciation risk and a DWDB second pillar pension fund, we obtain optimal risk sharing by setting* $k^f = k$ *and* $\theta^w + \theta^{dwb} = 1/2$

This result tells us that the pension fund should mimick the position of the old generation in capital ($k^f = k$) and that total exposure of the old to wages should be 1/2 (see Beetsma & Bovenberg, 2009). The result of this set-up is that both generations have the same exposure to the two shocks in their income.

4.2.3 Only fertility risk

In the special case where we have technological risk, depreciation risk and fertility risk, to replicate condition (30) we have to set the pension system parameters as follows.

Proposition 3 *For an economy featuring productivity, depreciation and fertility risks and a DWDB second pillar pension fund, there are multiple ways to obtain optimal risk sharing. One special case is given by setting:*
$k^f = k$, $\theta^w + \theta^{dwb} = 1/2$, $\theta^p = (1+r)\, b^f + \left(1+\bar{r}^{kn}\right) k^f$, $\theta^\gamma = \theta^\psi = 0$

The results for k^f and the wage-linked parameters θ^w and θ^{dwb} are equal to the simpler case: the pension fund mimicks the individual investments in capital and the total wage linked transfer is 1/2, so that both generations have an equal exposure to the productivity and depreciation shocks.

The solution for θ^p implies that the median income the young receive from the pension fund is transferred back to the old generation again. The reason is that on the one hand the (per old) lumpsum payments θ^p decrease for each member of the young generation if their total number grows, but on the other hand the transfer from the pension fund to each young member becomes smaller if their total number grows. Since these amounts decrease at exactly the same speed, one can insulate against fluctuations in the number of young by setting these amounts equal to each other.

4.2.4 Only mortality risk

In the case where we have technological risk, depreciation risk and mortality risk, to replicate (30) we have to set the pension system parameters as follows.

Proposition 4 *For an economy featuring productivity, depreciation and mortality risks and a DWDB second pillar pension fund, thera are multiple ways to obtain optimal risk sharing. One special case of the general solution is:* $k^f = k$, $\theta^w = 0$, $\theta^{dwb} = 1/2$, $\theta^p = -\frac{1}{2}\bar{w}$ *and* $\theta^\gamma = \theta^\psi = 0$.

Intuition here is again similar as in the previous section for the investment in capital by the pension fund and the wage-linked pension parameters.

Concerning θ^p, it is set such that the old generation transfers to the young the median value of their second pillar pension benefit. This is done to offset the simultaneous effects of a mortality shock on consumption of the young through on the one hand the lumpsum amount they have to transfer/receive (which is per old person) an on the other hand the wage-linked benefits per old.

4.2.5 Both demographic risks

In case both demographic risks are present, we still have more instruments than shocks, so there is still an infinite number of combinations with which we can replicate the social planner up to a first order approximation. Intuition here is again similar as in the previous section for the investment in capital by the pension fund, the lumpsum transfer θ^p and the wage-linked parameter θ^w.

The form that θ^{dwb} takes is $\theta^{dwb} = \frac{(1+r)b}{2\bar{w}} - \frac{\left(1+\bar{r}^{kn}\right)k}{2\bar{w}}$.

These results indicate that to replicate the social planner solution up to a first order approximation we need to set some of our instruments in a complicated way already, if we don't want to set the pension system parameters in a shock dependent way. That's why we are going to check numerically how close we can get to the social optimum with simple rules.

4.2.6 Different risk aversion parameters ($\nu \neq 1$)

In this case, the expressions for the general case become:
$k^f = \nu k$, $\theta^w = \frac{1}{1+\nu} - \theta^{dwb}$, $\theta^p = -\frac{1}{1+\nu}\bar{w}$, $\theta^\gamma = (\nu - 1)\left[\left(1+\bar{r}^{kn}\right)k + (1+r)\,b\right]$, $\theta^{dwb} = \frac{\nu^2+3}{8\bar{w}}(1+r)\,b - \frac{1+4\nu-\nu^2}{8\bar{w}}\left(1+\bar{r}^{kn}\right)k$ and $\theta^\psi = 0$.

If the old generation is more risk averse than the young generation ($\nu > 1$), then the young should hold a larger claim on risky capital income ($k^f > k$) and also on risky wage

income ($\theta^w + \theta^{dwb} < 1/2$). The effect of this is that for a given shock, the income of the young generation will fluctuate more than the income of the old generation.
The fertility linked parameter θ^γ becomes positive, so the young transfer a fraction from the actual fertility realization to the old. To compensate for this, the lumpsum transfer from the old to the young decreases in size.

5 Numerical analysis

Under the assumption that the parameters characterizing the pension arrangement are constant (not contingent on the shocks) we can no longer analytically solve for the optimal arrangement. Therefore, we solve numerically for the optimal solution. The optimal solution will also no longer replicate the planner's solution. However, the optimal solution is the one that gets social welfare as close as possible to that under the planner. Hence, the pension designer numerically solves the following problem

$$\max_{k^f,\theta^f,\theta^w,\theta^\gamma,\theta^p} \mathrm{E}_0 \left[\psi u\left(c_o\right) + \gamma u\left(c_y\right)\right]$$

subject to the resource constraints (4) and (5) and the market equilibrium conditions (28) and (29). In its choice of the optimal pension parameters, the pension designer internalizes how individuals react to changes in those parameters.

We solve this model in a recursive way. Given a set of pension system parameters k^f, θ^f, θ^w, θ^γ and θ^p, the capital market condition (28) determines k, the arbitrage equation (26) determines r and the full-funding constraint of the pension fund (27)determines θ^{dwb}. As explained before, the pension system parameters and market solution together pin down $b^f = \theta^f - k^f$ and $b = -b^f$. Then, depending on the realizations for the shocks A, γ, ψ and δ, we can calculate r^k, w, c_o and c_y.

5.1 Calibration

For the numerical analysis of the model, we have to specify functional forms for the utility and production functions and we have to make assumptions on the distributions of the different shocks in the model.
The utility function features constant relative risk aversion (CRRA)

$$u\left(c_i\right) = \frac{c_i^{1-\varphi}}{1-\varphi}, \quad i = y, o,$$

where φ captures the degree of risk aversion. Our baseline value is $\varphi = 2.5$, which is probably close to the median of the values offered by the literature. However, we shall vary φ to see how the results are affected by changes in risk attitude. Production is given

Table 1: Parameters

parameter	mean	distance from mean
α	0.3	
η	1	
φ	2.5	
A	3	0.3
δ	0.5	0.1
γ	1	0.25

by the Cobb-Douglas function

$$AF(K,\gamma) = AK^{\alpha}\gamma^{1-\alpha},$$

where α is the share of national income for the equity providers. We assume that $\alpha = 0.3$, a value commonly used in the literature.
For now, we assume that ψ is deterministic and is equal to 1, i.e. every member of the old generation survives until period 1. For each of the other three shocks, we assume a 2-point distribution with probability $\frac{1}{2}$ for each of the two points. The mean of the demographic shock is normalized to 1, such that in expectation both generations are equally large. The mean of the depreciation shock is 0.5, corresponding with an annual depreciation rate of around 2%. The mean of the productivity shock is set to 3. This mean was chosen so that the numerical simulations yield a reasonable net-of-depreciation return on capital and a reasonable return on bonds. In our experiments we will consider mean-preserving increases in the variances of the shocks.

The remaining parameter to be calibrated is η. It's choice is only relevant in relation to the average value of A. These two parameters determine the scale of the economy and we can fix one of them to unity. So we set $\eta = 1$. We summarise our calibration in Table 1, together with the assumed distributions of the shocks.

6 Results

We explore how optimal institutional arrangements change when we change the uncertainty about the demography, while keeping the uncertainty about depreciation and productivity constant.

6.1 Varying uncertainty in fertility

In the most extreme case, there is no uncertainty about the demography and, hence, $\gamma = 1$ with probability one. In that case, the optimal pension arrangement is able to produce to

Table 2: Different pension systems without demographic uncertainty

$\gamma = 1$	DRB	DWB	DWDB
θ^p	-0.6868	0	0
$\theta^w + \theta^{dwb}$	0.5	0.5	0.5
θ^f	0.5	0.5	0.5
k^f	0.5	0.5	0.5
b^f	0	0	0
b	0	0	0
r	0.3736	0.3736	0.3736
θ^γ	0	0	0
k	0.5	0.5	0.5
K	1	1	1
W	-0.584892	-0.584892	-0.584892

social optimum (this is the case analyzed in Beetsma and Bovenberg, 2008). The optimal arrangement is reported in Table 2. There exists an infinite number of combinations θ^w and θ^{dwb} that all deliver the same optimal allocation as long as $\theta^w + \theta^{dwb} = \frac{1}{2}$. All these combinations are equivalent, because they all leave the old generation with the same exposure to wage risk. Furthermore, for the DRB system the choices for θ^f, b^f and b are arbitrary in the sense that their magnitude doesn't matter in the social optimum. We have two equations - the pension fund equation (9) and the debt market equilibrium (29) to determine these three variables. Thus, we can freely pick the value for one of these variables, which will determine the value of the other two. We choose b to be equal to 0, which implies $b^f = 0$ and $\theta^f = 0.5$.

For the DWB and DWDB systems the same holds, but there the optimal value of θ^p depends on the chosen value for b^f since in the optimum without demographic uncertainty $\theta^p = (1 + r)\, b^f$. So our choice of $b^f = 0$ implies $\theta^p = 0$.

After the introduction of uncertainty into the demography, θ^w and θ^{dwb} are no longer perfectly substitutable. In our base scenario we assume the two point distribution $\gamma = [0.75;1.25]$. Table 3 reports the numerically computed optimal pension arrangements outcomes when we vary the uncertainty in γ relative to the uncertainty in the base scenario (which is indicated in italics).

We see that it is optimal to have a negative wage-independent PAYG transfer from the young to the old generation. It is further optimal to expose the old to wage risk, both via the first pillar and via the funded pillar of the pension system. We see that an increase in uncertainty about γ reduces the optimal values of θ^w and θ^{dwb} and also reduces the amount invested in capital by the pension fund, k^f. To see the intuition, notice that in the social optimum, the wage indexation parameters together would equal $\theta^w + \theta^{dwb} = \frac{\gamma}{1+\gamma}$, while $k^f = \frac{\gamma}{1+\gamma} K$. As the uncertainty about γ increases, the expected value of the term

Table 3: Varying fertility risk without mortality risk under DWDB

$\Delta\gamma$	0.01	0.05	0.15	*0.25*	0.35	0.5
θ^p	-0.3952	-0.3941	-0.3844	*-0.3646*	-0.3339	-0.2647
θ^w	0.3581	0.3575	0.3521	*0.3411*	0.3243	0.2866
θ^f	0.0370	0.0371	0.0381	*0.0401*	0.0430	0.0490
k^f	0.5000	0.4992	0.4924	*0.4783*	0.4559	0.4027
b^f	-0.4630	-0.4620	-0.4543	*-0.4382*	-0.4129	-0.3537
θ^{dwb}	0.1418	0.1417	0.1409	*0.1382*	0.1342	0.1238
θ^γ	-0.4814	-0.4814	-0.4813	*-0.4812*	-0.4811	-0.4809
b	0.4630	0.4620	0.4543	*0.4382*	0.4129	0.3537
$1+r$	1.3736	1.3730	1.3678	*1.3570*	1.3392	1.2942
k	0.5000	0.5008	0.5076	*0.5217*	0.5441	0.5973
K	1	1	1	*1*	1	1
$\mathrm{E}(c_o)$	1.7500	1.7494	1.7444	*1.7342*	1.7179	1.6795
$\mathrm{E}(c_y)$	1.7500	1.7494	1.7445	*1.7343*	1.7181	1.6797
EV^{lf}	0.0539	0.0544	0.0583	*0.0662*	0.0782	0.1045
EV^{fb}	3.36e-08	6.64e-07	4.85e-06	*1.63e-05*	3.49e-05	8.09e-05
$\rho\left(c_o,\gamma\right)$	0.0111	0.0562	0.1704	*0.2892*	0.4132	0.6009
$\rho\left(c_o,w\right)$	0.9479	0.9279	0.7772	*0.5367*	0.2644	-0.1381
$\rho\left(c_o,r^{kn}\right)$	0.8692	0.8585	0.8007	*0.7639*	0.7705	0.8336
$\rho\left(w,r^{kn}\right)$	0.6665	0.6096	0.2569	*-0.1249*	-0.4045	-0.6590
$\rho\left(c_o,\theta^\gamma\frac{\gamma}{1+\gamma}\right)$	-0.0112	-0.0562	-0.1704	*-0.2892*	-0.4132	-0.6009
$\rho\left(c_o,\theta^{dwb}\gamma w\right)$	0.9471	0.9121	0.7655	*0.697*	0.7025	0.7775
$\rho\left(c_o,\theta^w w\right)$	0.9478	0.9279	0.7772	*0.5367*	0.2644	-0.1381

Table 4: Correlations varying γ but keeping the pension system parameters fixed

$\Delta\gamma$	0.01	0.05	0.15	*0.25*	0.35	0.5
$\rho\left(c_o,\gamma\right)$	0.0112	0.056	0.1636	0.2593	0.3371	0.4099
$\rho\left(c_o,w\right)$	0.9478	0.9281	0.7829	0.566	0.3485	0.0944
$\rho\left(c_o,r^{kn}\right)$	0.8692	0.8581	0.7951	0.7404	0.7105	0.6822
$\rho\left(w,r^{kn}\right)$	0.6665	0.6096	0.2569	-0.1249	-0.4045	-0.659
$\rho\left(c_y,\gamma\right)$	0.0109	0.0551	0.1805	0.348	0.5632	0.8557
$\rho\left(c_y,w\right)$	0.9479	0.9278	0.7692	0.4797	0.0882	-0.5113
$\rho\left(c_y,r^{kn}\right)$	0.8692	0.8584	0.8061	0.7995	0.8634	0.973
$\rho\left(c_o,c_y\right)$	1	0.9999	0.9986	0.9918	0.9589	0.7993
$\rho\left(c_o,ddb\right)$	-0.0112	-0.056	-0.1636	-0.2593	-0.3371	-0.4099
$\rho\left(c_o,dwb\right)$	0.9471	0.9122	0.7626	0.6775	0.6466	0.6272
$\rho\left(c_o,wl\right)$	0.9478	0.9281	0.7829	0.566	0.3485	0.0944

$\frac{\gamma}{1+\gamma}$ decreases.

Hence, to stay as close to the social optimum as possible, the government sets the wage links and the pension fund investment in capital lower as uncertainty about demographic developments becomes larger.

The negative value of θ^{γ} can be interpreted as a hedge of the old generation against demographic risk. This negative value allows the pension fund to issue bonds (i.e. invest a negative amount in bonds), which the old generation can buy as a safe investment. The price the old generation pays for this is that in the second period, a transfer will have to be paid to the young generation. If γ is high this transfer is high, but then the return on their capital investments is high and the total wage sum γw is also high, which implies that the wage linked transfer they receive from the pension fund is high. In such a situation, the old can then afford to pay a high demography linked transfer to the young.
As far as the total wage bill is concerned, increasing γ lowers the wage per capita for the young generation but increases the total number of young. The effect on the total wage bill is positive, however:

$$\gamma w = \gamma A (1-\alpha) K^{\alpha} \gamma^{-\alpha}, \; = A (1-\alpha) K^{\alpha} \gamma^{1-\alpha}$$

$$\frac{\partial \gamma w}{\partial \gamma} = A (1-\alpha)^2 K^{\alpha} \gamma^{-\alpha} > 0$$

The return on bonds falls as demographic uncertainty increases. To see this, consider the arbitrage equation (26) between equity and bond investment:

$$\mathrm{E}_0 \left[\left(1 + r^k - \delta\right) u_c (c_o) \right] = (1+r) \, \mathrm{E}_0 \left[u_c (c_o) \right]$$

Notice that there is a positive correlation between r^k and c_o. However, because of decreasing marginal utility, high values of r^k are weighed with smaller values of marginal utility. By producing a higher variance in equity returns, more uncertainty in demography reduces the left-hand of the expression. This left-hand side will in addition fall as a result of the reduction in the average equity return. Hence, the return on bonds required by investors goes down as the risk of investing in equity goes up and its average return goes down.

Compared to a laissez-faire situation, the welfare gains from having an optimal pension system are substantial. To equalize social welfare under DWDB and laissez-faire, the latter system requires a uniform increase in consumption of 5.39% for both generations in all states of the world when demographic uncertainty is at its minimum. This amount of compensation rises to 10.45 % for the highest degree of demographic uncertainty.

Compared to the social planner solution, the welfare loss associated with the optimal pension arrangement (with constant parameters) is extremely small. To bring welfare up to the level under the planner the required consumption increase rises from 3.36*10^{-6} % for the lowest level of demographic uncertainty to 8.1*10^{-4} % for the highest level of demographic uncertainty.

Table 5: Varying mortality risk under DWDB

$\Delta\psi$	0.01	0.05	0.1	*0.2*	0.3	0.5
θ^p	-0.6709	-0.4051	-0.2818	*-0.1893*	-0.1286	-0.3775
θ^w	0.5468	0.3776	0.3017	*0.2516*	0.2220	0.3411
θ^f	0.0502	-0.1294	-0.2021	*-0.2274*	-0.2281	0.0331
k^f	0.4830	0.4675	0.4612	*0.4591*	0.4592	0.4783
b^f	-0.4328	-0.5968	-0.6633	*-0.6865*	-0.6874	-0.4452
θ^{dwb}	0.0976	0.2548	0.3157	*0.3283*	0.3193	0.1382
θ^γ	-0.2662	-1.4184	-1.8735	*-1.9961*	-1.9604	-0.4812
b	0.4328	0.5968	0.6633	*0.6865*	0.6874	0.4452
$1+r$	1.3735	1.3733	1.3729	*1.3717*	1.3697	1.3647
k	0.5170	0.5325	0.5388	*0.5409*	0.5408	0.5217
K	1	1	1	*1*	1	1
E(c_o)	2.3259	2.3268	2.3339	*2.3653*	2.4201	2.4351
E(c_y)	2.3264	2.3298	2.3389	*2.3747*	2.4361	2.6528

6.2 Varying uncertainty in mortality

In this section we investigate the effects of increasing the uncertainty about mortality while keeping the uncertainties about the other sources of risk fixed at their mean.

6.3 Other forms of the pension fund

In this section, we present the results of an analysis for the other possible set-ups of the second pillar pension fund. We analyze four different cases: the case where we have a DC pension fund, the case where we have a DRB pension fund and the two cases where we have only one part of the DWDB system. That is, we will have results for a system with a pension fund which has benefits indexed to wages (a DWB system as discussed in section 3) and results for a pension fund with benefits indexed to demography (a defined demography- indexed benefit, or DDB, fund).

For each of these four different set-ups, we do a similar analysis as in the previous section. For brevity, we only report the consumption and welfare results in table 6. As in the last section, we report the expected consumption of the young and the old under the optimal set-up in that system, and the corresponding welfare comparisons with the laissez- faire and the first best situations.

From table 6 we see that under a DC system expected consumption of the young goes down as uncertainty about demographic developments goes up, but expected consumption of the old goes up first and only declines as uncertainty becomes very large. The reason for this is ...

Under all three DB pension systems expected consumption of both the young and the old goes down, although under the DWB system risks are shared slightly better than under the other two systems as can be seen from the fact that the difference between expected consumption of the young and the old remains smallest as uncertainty increases.
Regarding welfare, all four systems provide a large improvement on the laissez-faire situation. However, there is a marked difference between the DC system and the three DB systems. Consumption in the laissez-faire economy would have to be increased by 5.39% for both generations to replicate the welfare that is attained under all three DB systems in the case of quite small demographic risk ($\Delta\gamma = 0.01$), while this amount is smaller, 5.28%, under the DC system. For large uncertainty about demographics ($\Delta\gamma = 0.5$), the compensation goes up to 10.12% for the DC system and 10.44%/10.45% for the DB systems.

Finally, when compared to the first best solution, all three DB systems come up with an allocation of consumption to both generations that is quite close to the first best. When demographic uncertainty is small, the required compensation to obtain the same welfare as under the first best is smaller than $1*10^{-5}$ %. As uncertainty goes up, this required compensation becomes larger and starts to diverge slightly more across the different DB systems. For the case where uncertainty about demographics is largest, the required increase in consumption under a DRB system is $1.91*10^{-2}$ %. Under a DDB system this compensation is somewhat smaller, $1.38*10^{-2}$ % and for a DWB system it is somewhat smaller again, $9.94*10^{-3}$ %. These required compensations are larger than under the DWDB system (where the required increase was only $8.1*10^{-4}$ % under this amount of uncertainty), but the differences are fairly small when compared to the welfare difference with the laissez-faire economy.

Under a DC system, the difference with the first best solution is markedly higher. When demographic uncertainty is low, the required compensation in consumption is 0.11% and this goes up to 0.31% for high uncertainty. This indicates that a system with a DC second pillar is significantly less able to have both generations share in the risks, which leads to lower welfare.

6.4 Varying the uncertainty in the productivity and depreciation shocks

We now explore the consequences of changes in the uncertainty about productivity and depreciation. In this section we fix the uncertainty about γ at the following distribution: $\gamma = [0.75; 1.25]$

6.4.1 Varying the uncertainty in the productivity shocks

The effects of changing the uncertainty about productivity are very small, because the technology shock enters linearly into the production function and thus (given that the

Table 6: Welfare comparison under different pension funds

System		0.01	0.05	0.15	*0.25*	0.35	0.5
DC	E(c_o)	1.755	1.7556	1.7583	*1.7574*	1.7482	1.7155
	E(c_y)	1.7449	1.7439	1.7354	*1.7206*	1.7007	1.6587
	EV^{lf}	0.0528	0.0532	0.0566	*0.0639*	0.0754	0.1012
	EV^{fb}	0.0011	0.0011	0.0016	*0.0022*	0.0027	0.0031
DRB	E(c_o)	1.7500	1.7491	1.7432	*1.7326*	1.7161	1.6772
	E(c_y)	1.7500	1.7494	1.7443	*1.7340*	1.7177	1.6792
	EV^{lf}	0.0539	0.0544	0.0582	*0.0661*	0.0781	0.1044
	EV^{fb}	2.41e-08	2.51e-05	9.78e-05	*1.32e-04*	1.55e-04	1.91e-04
DWB	E(c_o)	1.7500	1.7494	1.7444	*1.7340*	1.7176	1.6787
	E(c_y)	1.7500	1.7494	1.7445	*1.7343*	1.7181	1.6798
	EV^{lf}	0.0539	0.0544	0.0583	*0.0662*	0.0782	0.1045
	EV^{fb}	3.25e-08	8.13e-07	6.22e-06	*2.02e-05*	4.31e-05	9.94e-05
DDB	E(c_o)	1.7500	1.7494	1.7442	*1.7337*	1.7170	1.6779
	E(c_y)	1.7500	1.7494	1.7445	*1.7343*	1.7181	1.6796
	EV^{lf}	0.0539	0.0544	0.0583	*0.0662*	0.0782	0.1044
	EV^{fb}	6.13e-08	1.53e-06	1.24e-05	*3.60e-05*	7.01e-05	1.38e-04

Table 7: Varying uncertainty about A

ΔA	0.01	0.05	0.1	*0.3*	0.5	0.75
θ^p	-0.3672	-0.3672	-0.3669	*-0.3646*	-0.3602	-0.3519
θ^w	0.3411	0.3411	0.3411	*0.3411*	0.3411	0.3412
θ^f	0.0440	0.0439	0.0435	*0.0401*	0.0332	0.0196
k^f	0.4781	0.4781	0.4782	*0.4783*	0.4786	0.4792
b^f	-0.4341	-0.4343	-0.4346	*-0.4382*	-0.4454	-0.4597
θ^{dwb}	0.1381	0.1381	0.1381	*0.1382*	0.1383	0.1385
θ^γ	-0.4813	-0.4814	-0.4814	*-0.4812*	-0.4810	-0.4805
b	0.4341	0.4343	0.4346	*0.4382*	0.4454	0.4597
$1+r$	1.3756	1.3751	1.3735	*1.3570*	1.3250	1.2662
k	0.5219	0.5219	0.5218	*0.5217*	0.5214	0.5208
K	1	1	1	*1*	1	1
E(c_o)	1.7342	1.7342	1.7342	*1.7342*	1.7342	1.7342
E(c_y)	1.7343	1.7343	1.7343	*1.7343*	1.7343	1.7343
EV^{lf}	0.06780	0.06776	0.06762	*0.06622*	0.06344	0.05816
EV^{fb}	1.55e-05	1.56e-05	1.56e-05	*1.63e-05*	1.76e-05	2.17e-05
$\rho\left(c_o, A\right)$	0.0698	0.3300	0.5729	*0.9017*	0.9603	0.9810
$\rho\left(c_o, w\right)$	-0.6675	-0.5504	-0.2771	*0.5367*	0.7903	0.8951
$\rho\left(c_o, r^{kn}\right)$	0.9627	0.9334	0.8704	*0.7639*	0.8059	0.8670
$\rho\left(w, r^{kn}\right)$	-0.8436	-0.8064	-0.7026	*-0.1249*	0.2804	0.5580

production factors are also fixed) expected production does not change if the variance of the shock changes. Indirectly, there is a negative effect of larger output variance on the utility of consumption.

As far as the first pillar is concerned, the lump sum net transfer to the young slightly decreases, while the wage indexation parameter θ^w remains virtually unchanged. This may not be surprising, because an increase in the variance of productivity raises the variances of the wage and the equity return in the same proportions ceteris paribus and, hence, the consumption of the two generations is affected in the same proportion. Hence, no reallocation of risk between the two generations is needed.

Further, because the uncertainty about the equity returns goes up, the demand of the old generation for the safe asset rises, resulting in a fall in the bond interest rate .The larger amount of bonds held by the old generation has to be issued by the pension fund, thus decreasing the overall size of the pension fund θ^f given that its equity investment k^f hardly changes. As regards to the other parameters characterizing the pension fund, the degree of wage indexation of the benefit θ^{dwb} is essentially unchanged as explained above, while the demography-linked component of the benefit θ^γ also hardly changes.

As in the case where we varied the uncertainty surrounding the demographic shock, the welfare gains from the pension systems when compared to the laissez- faire economy are large. Interestingly, the larger the uncertainty about the technology shock becomes, the *less* compensation both generations require in the laissez-faire situation. It diminishes from 6.78% when uncertainty around the technology shock is very small to 5.82% as this uncertainty becomes very large. Apparently, the risk sharing with respect to the technology shock that the pension fund generates is very small and the 'protection' that the pension fund offers against bad outcomes diminshes in relative size as the technology shock starts dominating the outcomes in the different states of the world more.

When compared to the first best, we see the usual pattern of increasing required compensation as uncertainty increases. For the smallest amount of uncertainty around the technology shock, both generations require $1.55*10^{-3}\%$ extra consumption to be equally well off as under the first best, for large uncertainty around A this amount increases slightly to $2.17*10^{-3}\%$.

6.4.2 Varying the uncertainty in the depreciation shocks

Increasing the variance of the δ-shock, the shock to the depreciation rate, we obtain the following results, holding the distributions of A and γ fixed at respectively [2.7;3.3] and [0.75;1.25]

Just as in the case of productivity risk, increasing depreciation risk does not change total production. Again, all effects that we see come from the expected utility of consumption. From the table,we can see that the larger the uncertainty about δ becomes,

Table 8: Varying the degree of uncertainty about δ

$\Delta\delta$	0.01	0.05	*0.1*	0.2	0.3
θ^p	-0.3663	-0.3659	*-0.3646*	-0.3594	-0.3501
θ^w	0.3401	0.3404	*0.3411*	0.3440	0.3484
θ^f	0.0347	0.0360	*0.0401*	0.0564	0.0838
k^f	0.4782	0.4783	*0.4783*	0.4785	0.4789
b^f	-0.4435	-0.4422	*-0.4382*	-0.4221	-0.3951
θ^{dwb}	0.1392	0.1389	*0.1382*	0.1353	0.1308
θ^γ	-0.4998	-0.4953	*-0.4812*	-0.4264	-0.3395
b	0.4435	0.4422	*0.4382*	0.4221	0.3951
$1+r$	1.3649	1.3629	*1.3570*	1.3333	1.2943
k	0.5218	0.5217	*0.5217*	0.5215	0.5211
K	1	1	*1*	1	1
$\mathrm{E}(c_o)$	1.7341	1.7342	*1.7342*	1.7343	1.7345
$\mathrm{E}(c_y)$	1.7341	1.7342	*1.7343*	1.7346	1.7352
EV^{lf}	0.0626	0.0634	*0.0662*	0.0777	0.0979
EV^{fb}	1.76e-07	3.15e-06	*1.63e-05*	6.77e-05	1.49e-04
$\rho\left(c_o,\delta\right)$	-0.0335	-0.1655	*-0.3178*	-0.5553	-0.7052
$\rho\left(c_o,w\right)$	0.5689	0.5606	*0.5367*	0.4629	0.3837
$\rho\left(c_o,r^{kn}\right)$	0.7328	0.7410	*0.7639*	0.8266	0.8807
$\rho\left(w,r^{kn}\right)$	-0.1422	-0.1374	*-0.1249*	-0.0960	-0.0740

the smaller the lump sum transfer becomes. The reason is that the old bear more of the increase in depreciation risk (in effect, they hold slightly more equity than the young) and need to be compensated through a smaller lump-sum transfer to the young. The effect is small, though. Wage risk is not influenced at all, so the total link of pension benefits to wages does not change, although the division between first and second pillar does. Surprising is that the amount invested in bonds goes down: the amount of bonds issued by the pension fund also falls, which increases the total size of the pension fund. The link to demographics θ^γ shrinks in absolute value with the decline of the pension fund.

The welfare analysis doesn't reveal any surprises: as uncertainty about the depreciation shock becomes larger, the welfare gain from having the pension system becomes larger. In the laissez-faire economy both generations require 6.26% more consumption in the case of very low uncertainty about δ, which increases to 9.79% for high uncertainty. Thus, as uncertainty increases, risk-sharing provided by the pension fund becomes more valuable. Similarly, compared to the first best both generations require 1.76*10^{-5}% extra consumption under a DWDB pension system when uncertainty about depreciation is low. This compensation increases to 0.0149% for high uncertainty about the δ shock, indicating that the higher the uncertainty about δ is, the further away we get from the first best solution.

Table 9: Varying the degree of risk aversion

φ	0.1	1	*2.5*	5	10
θ^p	-0.3776	-0.3727	*-0.3646*	-0.3517	-0.3279
θ^w	0.3405	0.3407	*0.3411*	0.3418	0.3428
θ^f	0.0399	0.0400	*0.0401*	0.0406	0.0433
k^f	0.4821	0.4806	*0.4783*	0.4747	0.4684
b^f	-0.4421	-0.4406	*-0.4382*	-0.4341	-0.4251
θ^{dwb}	0.1405	0.1396	*0.1382*	0.1360	0.1320
θ^γ	-0.4940	-0.4892	*-0.4812*	-0.4685	-0.4456
b	0.4421	0.4406	*0.4382*	0.4341	0.4251
$1+r$	1.3925	1.3791	*1.3570*	1.3219	1.2614
k	0.5179	0.5194	*0.5217*	0.5253	0.5316
K	1	1	*1*	1	1
$\mathrm{E}(c_o)$	1.7341	1.7342	*1.7342*	1.7343	1.7346
$\mathrm{E}(c_y)$	1.7342	1.7342	*1.7343*	1.7343	1.7342
EV^{lf}	0.0025	0.0255	*0.0662*	0.1331	0.2335
EV^{fb}	6.32e-07	6.90e-06	*1.63e-05*	3.27e-05	1.00e-03

6.5 Varying the coefficient of risk aversion φ

Now we check what happens to the optimal solution if risk aversion changes. To do so, we fix the shock distributions at their initial values and vary the parameter of risk aversion φ. The results are presented in table 9.

We see that as one would expect, the cost of uncertainty goes up as risk aversion increases. In the case where people are almost risk neutral ($\varphi = 0.1$) people in the laissez-faire economy require only 0.25% extra consumption to be equally well off as in our economy with a DWDB fund. As risk aversion increases the required compensation goes up, leading to a required compensation of 23.35% of consumption in the case that people are very risk averse ($\varphi = 10$).

The same holds for the comparison between our economy and the first best: if risk aversion is very low the difference people require only $6.32{*}10^{-5}$% extra consumption to be equally well off as under the first best, while this amount goes up to 0.1% if risk aversion is very high.

7 Conclusion

In this paper we have explored intergenerational risk sharing within two-tier pension systems in the presence of productivity, financial market and demographic risks. Relative to laissez-faire, we find relatively large welfare gains from the presence of a two-tier pension system with a defined benefit second pillar. The first, PAYG pillar takes care of appropriate redistribution between the young and the old generation takes place while also allowing for some sharing of wage risks. Benefits from the fully funded second pillar are defined. This pillar allows for risk shring between the two generations. This is accomplished by the mismatch between the assets and the liabilities of the pension fund, where the young are the residual claimants of the pension fund.

Our simulation results suggest that while having a defined benefit pension two-tier fund yields large welfare gains, the exact form that this pension fund takes is less relevant, as the welfare differences among the various defined benefit systems are small, irrespective of the degree of uncertainty and the degree of risk aversion. Hence, while complicating the benefit system yields higher welfare (the optimal DWDB system dominates the optimal DWB system, which in turn dominates the optimal the optimal DRB system), the welfare differences in terms of certain consumption equivalents are small. Given that more complicated systems tend to be less transparent and possibly more subject to, for example, political influence, an overall trade-off may tip the balance to simple forms of two-tier funded systems.

Appendix: Derivation of (23a) and (23b)

Using (21) and (19) we can write

$$
\begin{aligned}
c_o &= \frac{1+\gamma}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right]+\frac{1}{\psi}\left(1+r^{a}\right)\theta^{f}+G \\
&= \frac{1+\gamma}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right]+\frac{1}{\psi}\left[(1+r)\,b^{f}+\left(1+r^{kn}\right)k^{f}\right]+ \\
&\quad \theta^{p}+\theta^{w}w+\frac{1}{\bar{\psi}}\gamma\theta^{dwb}w+\frac{1}{\bar{\psi}}\theta^{\gamma}\frac{\gamma}{1+\gamma}+\theta^{\psi}\frac{1}{1+\psi} \\
&\quad -\frac{1}{\psi}\left[(1+r)\,b^{f}+\left(1+r^{kn}\right)k^{f}\right] \\
&= \frac{1+\gamma}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right]+\theta^{p}+\theta^{w}w \\
&\quad +\frac{1}{\bar{\psi}}\left[\gamma\theta^{dwb}w+\theta^{\gamma}\frac{\gamma}{1+\gamma}\right]+\theta^{\psi}\frac{1}{1+\psi}
\end{aligned}
$$

Using (22) and (19) we can write

$$
\begin{aligned}
c_y &= w+\frac{1-\psi}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right]-\frac{\psi}{\gamma}G \\
&= w+\frac{1-\psi}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right] \\
&\quad -\frac{\psi}{\gamma}\left[\begin{array}{c} \theta^{p}+\theta^{w}w+\frac{1}{\bar{\psi}}\gamma\theta^{dwb}w+\frac{1}{\bar{\psi}}\theta^{\gamma}\frac{\gamma}{1+\gamma}+\theta^{\psi}\frac{1}{1+\psi} \\ -\frac{1}{\psi}\left[(1+r)\,b^{f}+\left(1+r^{kn}\right)k^{f}\right] \end{array}\right] \\
&= \left(1-\frac{\psi}{\gamma}\theta^{w}-\frac{\psi}{\bar{\psi}}\theta^{dwb}\right)w-\frac{\psi}{\gamma}\theta^{p}-\frac{\psi}{\bar{\psi}}\theta^{\gamma}\frac{1}{1+\gamma}-\frac{1}{\gamma}\theta^{\psi}\frac{\psi}{1+\psi} \\
&\quad +\frac{1-\psi}{\gamma+\psi}\left[\left(1+r^{kn}\right)k+(1+r)\,b\right]+\frac{1}{\gamma}\left[(1+r)\,b^{f}+\left(1+r^{kn}\right)k^{f}\right].
\end{aligned}
$$