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### An inequality between perpendicular least-squares and ordinary least-squares

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94.2.4. *An Inequality Between Perpendicular Least-Squares and Ordinary Least-Squares*, proposed by H. Peter Boswijk and Heinz Neudecker. Let  $Z = [Z_1 : Z_2]$  be an  $n \times p$  matrix with  $n > p$ , and with  $Z_i$  matrices of order  $n \times p_i$ ,  $i = 1, 2$ . Define  $M_2 := I_n - Z_2 Z_2^+$ , where  $Z_2^+$  is the Moore–Penrose inverse of  $Z_2$ ; if  $\text{rank}(Z_2) = p_2$ , then  $M_2 = I_n - Z_2 (Z_2' Z_2)^{-1} Z_2'$ . Moreover, let  $\lambda_1 \leq \dots \leq \lambda_p$  denote the eigenvalues of  $Z'Z$ .

1. Prove that

$$\sum_{i=1}^{p_1} \lambda_i \leq \text{tr}(Z_1' M_2 Z_1), \tag{1}$$

where  $\text{rank}(Z_2) = l \leq p_2$ .

2. Let  $S$  denote a  $p \times p$  random matrix with a standard central Wishart distribution with  $n$  degrees of freedom, i.e.,  $S \sim W(I_p, n)$ , let  $\mu_1 \leq \dots \leq \mu_p$  denote its eigenvalues, and define

$$s_1 := \sum_{i=1}^{p_1} \mu_i. \tag{2}$$

Finally, let  $s_2$  denote a random variable with a  $\chi^2$  distribution with  $p_1(n - p_2)$  degrees of freedom. Use (1) to prove that  $s_1$  is stochastically dominated by  $s_2$ , i.e., for any  $c > 0$ ,

$$P\{s_1 > c\} \leq P\{s_2 > c\}. \tag{3}$$

Remarks.

(i) The right-hand side of (1) is equal to the trace of the residual sum of squares from a multivariate ordinary least-squares (OLS) regression of  $Z_1$  on  $Z_2$ . The left-hand side is the corresponding quantity for a multivariate version of perpendicular least-squares (PLS). Hence, (1) entails that PLS yields a smaller residual sum of squares than OLS.

(ii) An application of (3) can be found in reduced rank regression, cf. Anderson [1]. Consider the multivariate normal linear regression  $Y = XB + U$ , with  $Y, U: n \times g$ ,  $X: n \times k$ , and  $B: k \times g$ , and with  $n > g \geq k$ . The likelihood ratio (LR) statistic for the hypothesis  $\text{rank}(B) \leq r$ ,  $r < k$ , has an asymptotic  $\chi^2$  distribution with  $(g - r)(k - r)$  degrees of freedom under the null hypothesis, provided that  $\text{rank}(B) = r$ . On the other hand, if  $\text{rank}(B) = q < r$ , then the LR statistic has the same asymptotic distribution as the sum of the  $(k - r)$  smallest eigenvalues of a  $W(I_{k-q}, g - q)$  matrix. Result (3) shows that the latter distribution is more concentrated towards the origin, so that if  $\chi_{\alpha}^2$  is the  $100\alpha\%$  critical value of the  $\chi^2((g - r)(k - r))$  distribution, then under the null hypothesis and as  $n \rightarrow \infty$ ,  $P\{LR > \chi_{\alpha}^2\} \rightarrow \alpha_q \leq \alpha$ . Thus the size of the test is controllable (cf. Cragg and Donald [2, Theorem 2] for a related result).

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1. Anderson, T.W. Estimating linear restrictions on regression coefficients for multivariate normal distributions. *Annals of Mathematical Statistics* 22 (1951): 327–351.
2. Cragg, J.G. & S.G. Donald. Testing identifiability and specification in instrumental variable models. *Econometric Theory* 9 (1993): 223–240.