On the functioning of markets with frictions

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CHAPTER 2

Persistence and volatility of Beveridge cycles

2.1 Introduction

The dynamic relation between vacancies and unemployment is characterized by counterclockwise cycles in the *unemployment, vacancy rate*-plane. After removing any long-term trends with an HP-filter, the cycles for the United States are presented in Panel 1a of Figure 1. It shows that the cycles mostly consist of movements parallel to an almost perfectly inverse relationship between vacancies and unemployment - the Beveridge curve - where downturns trace out a lower path than recoveries. The figure also shows that vacancies are about as volatile and persistent as unemployment.

These observations are confirmed in the summary statistics of the unemployment and vacancy rate. Table 1 reports standard deviations, autocorrelations, and cross-correlations of these and other variables: market tightness v/u, the job finding rate f, the destruction rate δ, and labor productivity y. The standard deviations of the unemployment and vacancy rate v are similar, their autocorrelations are about 0.95, and their correlation is almost -0.9. Note also that the standard deviation of productivity is almost ten times smaller than that of the unemployment and vacancy rate. Finally, the average (unfiltered) unemployment and quarterly job finding and destruction rates are 0.0587, 1.738, and 0.100 respectively.

In the canonical search and matching model of Pissarides (1985), unemployment is a state variable while vacancies respond to shocks immediately. As a result, the

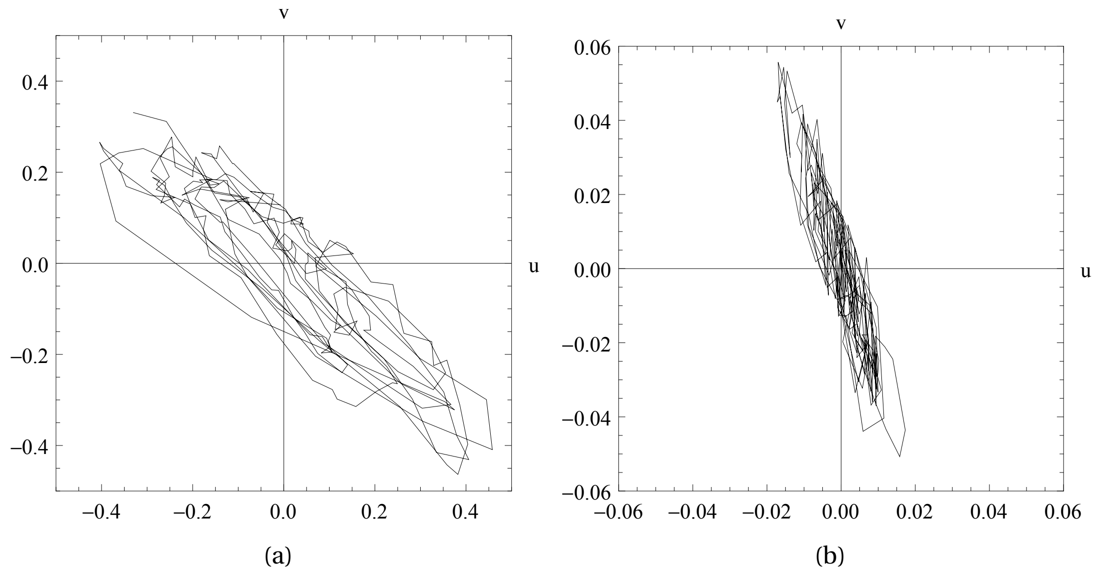

Figure 1: Cyclical component of the Beveridge curve for (a) the United States, 1951-2014, and (b) a simulation of Shimer (2005).

Notes: The seasonally adjusted unemployment rate u is constructed by the Bureau of Labor Statistics (BLS) from the Current Population Survey (CPS). The seasonally adjusted vacancy rate v is obtained by dividing a measure of vacancies by the labor force from the CPS. Following Daly et al. (2012), the former is the Composite Help-Wanted Index constructed by Barnichon (2010) for the 1951-2000 period, rescaled to equal the JOLTS in December 2000, which is used from that month onwards. I thank Bart Hobijn for providing me with these data. The data are quarterly averages of a monthly series. The simulation is a representative realization with productivity shocks only. Both data and simulation are expressed in logs as deviations from an HP trend with smoothing parameter 10^5.

model captures the counterclockwise direction of the cycles along the Beveridge curve. However, the model fails to describe these cycles in at least two dimensions. First, as is well-known, the model lacks sufficient amplification to generate the observed volatility in unemployment and vacancies in response to realistic exogenous shocks in productivity and/or the destruction rate (Shimer, 2005). Second, the model lacks cyclical responses, as can be seen comparing the panels of Figure 1. Panel 1b plots a realization of a detrended Beveridge curve as simulated by Shimer (2005), in his calibration with productivity shocks only. The simulation not only lacks unemployment volatility (note the scaling differences across the panels), but it also mainly features near

		u	v	v/u	f	δ	y
Standard deviation		0.195	0.178	0.362	0.175	0.073	0.02
Quarterly autocorr.		0.946	0.946	0.948	0.926	0.735	0.894
Correlation matrix	u	1	−0.888	−0.974	−0.962	0.64	−0.35
	v		1	0.969	0.887	−0.647	0.327
	v/u			1	0.953	−0.662	0.349
	h				1	−0.522	0.333
	s					1	−0.504
	y						1

Table 1: Summary statistics, quarterly US data, 1951-2014.

Notes: Labor market tightness v/u is the ratio of the seasonally adjusted quarterly unemployment rate u and vacancy rate v, both described below Figure 1. Appendix 2.7.A describes the construction of the job finding rate f and destruction rate δ from the monthly seasonally adjusted employment, unemployment, and short-term unemployment rate as provided by the BLS from the CPS. Average labor productivity y is seasonally adjusted real average output per person in the non-farm business sector constructed by the BLS from the National Income and Product Accounts and the Current Employment Statistics. All variables are reported in logs as deviations from an HP trend with smoothing parameter 10^5.

vertical dynamics, in contrast to the data in Panel 1a. As a result, vacancies in the model are neither as persistent as in the data, nor as persistent as unemployment.[4]

The current paper explains the cyclical behavior of unemployment and vacancies with an endogenous cycle driven by demand externalities. Demand externalities result from spillovers across a monopolistic goods market and a labor market with search frictions, in which high aggregate unemployment feeds back to low demand for output and thus a low revenue per worker-firm match. Combined with the delay in matching and the congestion externality that are standard in search models of the labor market, such feedback can give rise to equilibrium paths in vacancies and unemployment that never converge to a steady state, but converge to a deterministic periodic closed path. I refer to these rational expectations limit cycles in vacancies and unemployment as Beveridge cycles. Expectations of high revenue per match are self-fulfilling because

[4]The lack of propagation in Shimer (2005) can also be understood from a univariate regression of labor market tightness (or the job finding rate) on productivity, which results in an R^2 of 1.00.

such expectations make firms open vacancies, resulting in higher employment and - via demand externalities - higher revenue per match in the future. However, congestion in a tight labor market may make hiring so costly that firms may reduce vacancies before reaching a steady state. In that case, the equilibrium path turns and job destruction takes over from job creation, unemployment increases and revenue per match falls. Hiring picks up again when the labor market becomes sufficiently slack.

Rather than modeling the goods market explicitly, I assume that revenue per match is a function of aggregate employment. I add this reduced-form relationship to a Pissarides (2000) search and matching model with variable search intensity and analyze the global dynamics of the model. Theoretically, the occurrence of a Bogdanov-Takens bifurcation shows that Beveridge cycles exist and are stable for a range of parameter values. This implies that a small perturbation to a parameter, or introducing a small amount of risk aversion, would not eliminate cycles. Quantitatively, I investigate whether these cycles can match the empirically observed standard deviation and autocorrelation of unemployment and vacancies. Calibrating the cycle to the average duration of the business cycle, the simulated autocorrelations of both unemployment and vacancies closely match their observed autocorrelations. Persistence is an endogenous feature of the Beveridge cycle, and results from the neighborhood of a steady state with saddle-path stability. In particular, it is not a result of a persistent stochastic process for productivity, and it does not compromise volatility. Indeed, the model is subject to the same (lack of) amplification mechanisms as the standard model, but generates its volatility in revenue per match endogenously, reducing the need for exogenous shocks. Variable search intensity and a high positive value of leisure are important for the empirical performance of the model. The flow value of unemployment does not only contribute to amplification, but also brings the size of the required demand externalities in line with the literature. The existence of limit cycles does not rely on variable search intensity or a positive value of leisure.

My model is formally equivalent to the model of Mortensen (1999), who shows the existence of limit cycles in employment and the surplus of a match. He gives the intuition of the counterclockwise cycles along the Beveridge curve, but does not perform a calibration. I present the model in the standard search and matching variables - unemployment and labor market tightness - and add a positive value of leisure, which has a non-trivial impact on the existence and characteristics of equilibria in this nonlinear model. In Mortensen (1999), following Mortensen (1989), the feedback from aggregate employment on revenue per match results from increasing returns in production. Alternatively, this feedback can result from increasing returns in the matching function for the goods market as in Diamond (1982) and Howitt and McAfee (1987). Diamond and Fudenberg (1989) and Boldrin et al. (1993) show that such increasing returns can result in endogenous cycles, but their models do not contain vacancies. In a reduced form, however, the three interpretations of the relationship between aggregate employment and revenue per match are equivalent, and capture the three examples of Cooper and John (1988) that can result in strategic complementarities: production technology, matching technology, and agents' demands.

With respect to the latter, Heller (1986) and Roberts (1987) show that demand externalities can generate multiple equilibria. Exploiting equilibrium multiplicity, Howitt and McAfee (1992) propose belief shocks to switch from one equilibrium path to another. Drazen (1988) shows that demand externalities can generate endogenous cycles in firm match value and unemployment. However, he assumes an equal number of vacancies and unemployed and thus rules out cycles in the two. Recently, novel interactions between frictional labor and product markets have been proposed to generate endogenous cycles or multiplicity. Beaudry et al. (2015) obtain demand externalities from unemployment risk and precautionary savings, and embed the resulting limit cycle in a model with exogenous disturbances. They estimate this model and show that the endogenous cycle can explain U.S. business cycle fluctuations in output and employment well, provided that the productivity shocks make the cycle sufficiently irregular. They do not investigate the dynamics of vacancies. Kaplan and Menzio (2016) argue that the unemployed do not only spend less, but are also more likely to pay low prices for the same goods. They use the combination of these effects - referred to as shopping externalities - to explain the outward shift of the Beveridge curve since the Great Recession. In particular, this shift results from the transitional dynamics after a belief shock switches coordination to an equilibrium with higher unemployment and smaller markups.

The literature on the outward shift of the Beveridge curve is growing. In Ravn and Sterk (2012), lower aggregate demand results from precautionary savings in the wake of higher unemployment, and further reduces job finding prospects. Heterogeneity in search efficiency introduces negative duration dependence and an outward shift in the Beveridge curve. In Eeckhout and Lindenlaub (2015), sorting makes on-the-job search improve the composition of job searchers, boosting labor demand and justifying on-the-job search. The shift in the Beveridge curve results from a switch from an equilibrium without to an equilibrium with on-the-job search. Consistent with my model and suggested by Panel 1a of Figure 1, Diamond and Sahin (2015) show that outward shifts in the Beveridge curve after a trough are common in U.S. historical data, and not likely to be persistent.

My paper is also related to the large literature that proposes mechanisms that increase the amplification of the standard search and matching model. Mortensen and Nagypal (2007) provide an overview. Recently, Gomme and Lkhagvasuren (2015) have shown that procyclical search intensity increases amplification by making the net flow value of leisure countercyclical, and as the result of a strategic complementarity in search and recruiting activity. My model features procyclical search intensity, but, given the elasticity of the matching function, constrains its effect to the estimated elasticity of the job finding rate with respect to labor market tightness. A few other papers address the persistence of the standard model. Fujita and Ramey (2007) show that cyclical responses can be generated by the introduction of sunk costs of vacancy creation. They show, however, that spreading out the impact of a shock in such a way results in a counterfactually high cross correlation between labor market variables and productivity across time. Coles and Kelishomi (2011) achieve persistence of vacancies by replacing the free entry condition by search for business opportunities and time to build an identified opportunity into a vacancy. Dromel et al. (2010) and Petrosky-Nadeau (2014) address propagation by credit frictions, and therefore follow Fujita and Ramey in focusing on the costs of vacancy creation. In contrast, persistence in my model results from a self-reinforcing effect on the benefits of vacancy creation, while maintaining free entry of vacancies.

The calibration of Chéron and Decreuse (2016) is most closely related to mine. They show that phantoms - traders that are still present in the market but that are no longer available for trade - can result in a labor market matching function with increasing returns to scale in the short run and constant returns in the long run, which results in excess volatility and self-fulfilling fluctuations. They also calibrate a deterministic limit cycle, and can explain the persistence of unemployment and labor market tightness if wages are rigid, if phantoms mostly consist of vacancies, and if they ‘haunt’ the market for a long time. Note that the demand externalities in my model are similar to decreasing, not increasing, returns to scale in the labor market. Ellison et al. (2014) show that such locally decreasing returns to scale in labor market matching can increase amplification and persistence. However, with respect to their quantitative results, they focus on saddle-path dynamics. Sterk (2016) shows that skill losses upon unemployment can result in multiple steady states, and slower dynamics in their neighborhoods result in more persistence. Technically, the same mechanism delivers persistence for the Beveridge cycle, but the model of Sterk (2016) features a unique equilibrium. Only Chéron and Decreuse (2016) calibrate a deterministic model to quantify its performance in explaining labor market dynamics, while Beaudry et al. (2015) show that fundamental shocks on top of a deterministic cycle can reproduce the spectrum of the data.

Finally, my paper is related to applications of the Bogdanov-Takens bifurcation in economics. Using this bifurcation, Benhabib et al. (2001) show that an active monetary policy rule in the presence of a zero lower bound results in indeterminacy and can direct an economy into a liquidity trap. In models with search frictions, phase diagrams in Howitt and McAfee (1988), Coles and Wright (1998), and Kaplan and Menzio (2016) also suggest the occurrence of a Bogdanov-Takens bifurcation, although these authors do not explore this possibility.

2.2 Model

This section presents a Pissarides (2000) equilibrium search and matching model with variable search intensity and feedback from employment to revenue per match. Time is continuous and lasts forever, and there is no aggregate uncertainty.

2.2.1 Preferences, markets, and choices

The economy consists of a measure one of infinitely lived workers and an endogenous measure of firms owned by workers. All firms have access to the same technology. They maximize expected profits and discount future profits at rate r. Workers are endowed with an indivisible unit of homogeneous labor every period. They are risk-neutral too and maximize lifetime utility, discounted at the same rate r. At time t, an endogenous measure n_t of workers is employed, and the remainder $u_t = 1 - n_t$ is unemployed. Unemployed workers receive a flow utility $z > 0$ that is independent of labor market conditions. It captures the combination of the unemployment benefit, the stigma of unemployment, the value of home production and the pure value of leisure that come with unemployment.

Matches of a single worker and firm produce consumption goods that are sold for a one-period IOU in a goods market. A firm's receipts are split in a wage w_t and profits, are immediately transferred to its employee and owners respectively, and must be spent in the same period on consumption goods produced by other worker-firm matches. Rather than modeling the goods market explicitly, I propose a reduced-form relationship between the flow revenue y_t of a worker-firm match and *aggregate* employment. Assuming a constant elasticity, $y_t = \phi(n_t) = \phi(1-u_t) = \phi_0(1-u_t)^{\alpha}$. I normalize ϕ_0 to one and assume that $\alpha > 0$, so that revenue is increasing in aggregate employment. As explained in the introduction, several interpretations can be given to the effect of aggregate employment on revenue per match, but throughout this paper I refer to this effect as a demand externality. In this interpretation the goods market is characterized by imperfect competition. Revenue per match increases in aggregate employment because when employment is high, customers spend more, possibly also in any given trade.

The labor market is characterized by search frictions, so that the formation of worker-firm matches takes both time and resources. The total measure of matches m_t formed in a certain period is given by the common Cobb-Douglas matching function $m(v_t, s_t u_t) = m_0 v_t^{\eta}(s_t u_t)^{1-\eta}$, with $0 < \eta < 1$. Inputs of this function are aggregate recruiting activity represented by the vacancy rate v_t (scaled by the labor force), and aggregate search effort given by the unemployment rate u_t, times the search intensity s_t of the unemployed. Normalize m_0 to one and define labor market tightness θ_t as the ratio of the inputs of the matching function: $\theta_t \equiv v_t/(s_t u_t)$.[5] An individual unemployed worker finds a job at the Poisson rate $v_t^{\eta}(s_t u_t)^{1-\eta}/u_t = s_t\theta_t^{\eta}$. Similarly, individual vacancies are filled at a rate $v_t^{\eta}(s_t u_t)^{1-\eta}/v_t = \theta_t^{\eta}/\theta_t$. Matches are destroyed at an exogenous rate $\delta > 0$. The surplus of a match is divided by Nash bargaining.

There are two choices to be made in this economy. First, at any moment in time t, unemployed workers choose the intensity s_t at which they search for jobs. Search intensity comes at increasing and strictly convex costs $c(s_t) = c_0 s_t^{\gamma}$, with $\gamma > 1$. Simply scaling search intensity, I normalize c_0 to one.[6] The discouraged worker effect - unemployed workers stop looking for a job if the prospect of finding one is very bad - is modeled by s_t, because the labor force is fixed while search intensity will be seen to

[5] In this paper I only study the cases where $s_t, u_t > 0$, so that θ is always defined.

[6] The model allows for normalization of m_0 and c_0, because the level of the ratio of vacancies to unemployment v_t/u_t and the value of search intensity s_t are intrinsically meaningless.

increase with labor market tightness. Combined with the gross value of leisure z, the net flow utility of the unemployed is $z - s_t^{\gamma}$.[7] Second, at any moment in time, potential firms freely decide whether to enter the labor market by opening a single vacancy at a flow cost $k > 0$. Simultaneously any existing firm decides freely whether to withdraw its (unfilled) vacancy from the market.

2.2.2 Asset values of workers and firms

Free entry of vacancies implies that in equilibrium the value of opening an additional vacancy cannot be positive. As described above, an individual vacancy costs k per period and is filled at a rate θ_t^{η}/θ_t. Defining J_t as the value of a worker to a firm, it must hold that

$$k \geq \frac{\theta_t^{\eta}}{\theta_t} J_t, \tag{2.1}$$

and $\theta_t \geq 0$ with complementary slackness. The value of opening an additional vacancy is therefore only negative if the stock of vacancies is zero. In the remainder of this paper I focus on equilibria with economic activity, i.e. with a positive level of labor market tightness.

An unemployed worker enjoys a flow utility $z - s_t^{\gamma}$ and finds a job at the Poisson rate $s_t\theta_t^{\eta}$. The value U_t of unemployment to an individual worker is therefore described by

$$rU_t = z - s_t^{\gamma} + s_t\theta_t^{\eta}\left(W_t - U_t\right) + \dot{U}_t, \tag{2.2}$$

where W_t is the value of a job to a worker.

The flows to a firm are revenues $(1 - u_t)^{\alpha}$ minus wage w_t, which is the periodical income to the worker. The asset price equations of a job J_t and W_t, to a firm and a worker respectively, are then

$$rJ_t = (1 - u_t)^{\alpha} - w_t - \delta J_t + \dot{J}_t, \tag{2.3}$$

$$rW_t = w_t - \delta\left(W_t - U_t\right) + \dot{W}_t. \tag{2.4}$$

The wage is determined by Nash bargaining over the surplus of a match $p_t \equiv J_t + W_t - U_t$, with worker's bargaining power equal to $\beta \in (0,1)$ and separation $(U_t, 0)$ as threat point. The firm's rent is therefore equal to its share of the surplus:

$$J_t = (1 - \beta)\left(J_t + W_t - U_t\right), \ \text{ and } \ \dot{J}_t = (1 - \beta)\left(\dot{J}_t + \dot{W}_t - \dot{V}_t\right), \tag{2.5}$$

[7] Note that z does not affect $c(s_t)$, which seems a reasonable assumption for risk-neutral agents. This assumption highlights the effect of the value of leisure on vacancy creation rather than labor supply.

where the latter follows because wages are continuously renegotiated.[8]

As in (2.2), an unemployed worker's net expected income from search activity g_t is $s_t\theta_t^{\eta}(W_t - U_t) - s_t^{\gamma}$. He optimally chooses his search intensity s_t, taking into account that the surplus of a match will be divided by Nash bargaining. Using (2.1) with equality, net expected income from search activity can be expressed in terms of labor market tightness:

$$g(\theta_t) = \max_{s_t}\left[\frac{\beta}{1-\beta}s_t k\theta_t - s_t^{\gamma}\right]. \tag{2.6}$$

Finally, add (2.3) to (2.4), subtract (2.2), and substitute $g(\theta_t)$, to obtain the law of motion for match surplus

$$\dot{p}_t = (r+\delta)p_t - (1-u_t)^{\alpha} + z - g(\theta_t). \tag{2.7}$$

2.2.3 Equilibrium behavior and dynamics

Unemployed workers choose the intensity s_t at which they search for jobs. Balancing the benefits of search with the costs, from (2.6) optimal search intensity $s^*(\theta_t)$ is given by

$$s^*(\theta_t) = \left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\theta_t\right)^{\frac{1}{\gamma-1}}. \tag{2.8}$$

As referred to above, optimal intensity increases in tightness, and is positive for $\theta > 0$. Together with the stocks of vacancies and unemployed workers, it determines the measure of matches formed at any instant. Combined with the destruction of existing jobs, unemployment evolves according to a differential equation in unemployment and tightness:

$$\dot{u}_t = \delta(1-u_t) - s^*(\theta_t)\theta_t^{\eta}u_t. \tag{2.9}$$

By opening or closing vacancies, firms translate changes in expectations about the value of a worker to the firm in changes in labor market tightness with perfect foresight. Indeed, for any positive level of tightness, (2.1) implies

$$J_t = \frac{k\theta_t}{\theta_t^{\eta}}. \tag{2.10}$$

[8] Pissarides (2009) shows that the crucial assumption for job creation is that wages of *new* matches are given by this rule. How rents in ongoing jobs are split is inconsequential for job creation, and thus for the dynamics in this model. Coles and Wright (1998) have shown that outside a steady state, Nash bargaining no longer necessarily corresponds to the outcome of strategic bargaining with appropriately defined threat points as the time between offers goes to zero, as it would in a stationary environment (Binmore et al., 1986). Consequently, the division rule in this paper should not be interpreted as the outcome of strategic bargaining, but as an axiomatic solution.

Differentiating with respect to time, the value of a worker to a firm and labor market tightness move in tandem:

$$\dot{J}_t = \frac{k\dot{\theta}_t(1-\eta)}{\theta_t^\eta}. \tag{2.11}$$

Substituting (2.5) and (2.10) into (2.7), the value of a worker to a firm evolves according to

$$\dot{J}_t = (r+\delta)\frac{k\theta_t}{\theta_t^\eta} - (1-\beta)\left[(1-u_t)^\alpha - z + g(\theta_t)\right]. \tag{2.12}$$

Finally, combine (2.11) and (2.12) into a second differential equation in unemployment and tightness:

$$\dot{\theta}_t = (r+\delta)\frac{\theta_t}{1-\eta} + (1-\beta)\frac{\theta_t^\eta}{k(1-\eta)}\left[g(\theta_t) + z - (1-u_t)^\alpha\right]. \tag{2.13}$$

Equilibrium can now be defined as in:

Definition 2.1. A perfect foresight equilibrium with economic activity is a pair of functions $\{u_t, \theta_t\}$ such that:

1. For all $t \geq 0$, unemployment $u_t \in [0,1]$ evolves according to (2.9);
2. For all $t \geq 0$, labor market tightness $\theta_t > 0$ evolves according to (2.13);
3. $\lim_{t\to\infty}\theta_t$ is finite and u_0 is given.

In the presence of search frictions, an equilibrium is not necessarily efficient. Positive externalities of search and recruiting activity occur for trading partners for whom matching is *more* likely because of the availability of more (effective) trading partners. Negative externalities of search and recruiting activity occur for searchers of the same type, for whom matching is *less* likely because of increased congestion for trading partners. These externalities only cancel once the net private returns from search and recruiting activity equal the net social returns. Proposition 2.2 states this happens if the familiar Hosios (1990) condition is satisfied. Note that the Hosios condition only concerns the search externalities, not the demand externalities. The proof in Appendix 2.7.B extends the efficiency results of the Pissarides (2000) model with variable search intensity to out-of-steady state dynamics.

Proposition 2.2. *Suppose revenue per match is independent of unemployment and constant, i.e. $y_t = y$. Then search intensity s_t and labor market tightness θ_t are efficient if and only if the bargaining power of firms $1-\beta$ is equal to the elasticity of the matching function η.*

Thus, $1-\beta=\eta$ is the efficient sharing rule for a social planner that takes the demand externalities as given, just as firms and workers do, but does internalize search externalities. In fact, the demand externalities result in multiple equilibria for the same fundamentals, and Mortensen (1999) shows that these equilibria can be Pareto-ranked. The next section presents steady-state equilibria, whereas the section after that presents the non-stationary equilibrium of the Beveridge cycle that encloses one of these steady states.

2.3 Steady state equilibria

In this section, I present the steady-state equilibria of the model economy, and study their stability. I show that if there exists a steady state with economic activity for $z>0$, then there are generically multiple of them. Knowledge of the stability of these steady states helps to understand the Beveridge cycle that ultimately explains the data.

2.3.1 Nullclines

A steady state is a pair of functions (u_t, θ_t) in which both unemployment and labor market tightness are constant. Unemployment is constant on the $\dot{u}_t = 0$-locus or unemployment nullcline, where (2.9) is equal to zero:

$$u_t = \frac{\delta}{\delta + s^*(\theta_t)\theta_t^\eta}, \tag{2.14}$$

with $s^*(\theta_t)$ as given in (2.8). One can see that in steady state all workers are unemployed if tightness is zero, and that steady-state unemployment decreases in tightness via the job finding rate.

Tightness is constant on the $\dot{\theta}_t = 0$-locus or tightness nullcline. Equalizing (2.13) to zero, the tightness nullcline for equilibria with economic activity can also be expressed as unemployment in terms of tightness:

$$u_t = 1 - \left[(r+\delta)\frac{k\theta_t}{(1-\beta)\theta_t^\eta} + g(\theta_t) + z\right]^{\frac{1}{\alpha}}. \tag{2.15}$$

Again unemployment decreases in tightness: at a lower level of unemployment, revenue will be higher, and therefore in equilibrium firms open more vacancies. From (2.15) with $\theta_t = 0$, we can see that the nullcline crosses the $\theta = 0$-axis at

$$u_{\theta=0} = 1 - z^{1/\alpha}. \tag{2.16}$$

At this level of unemployment, revenue per worker-firm match equals the gross value of leisure, and firms do not want to open any vacancies. For smaller values of leisure, wages are lower. As a result, there is more surplus of a match, and firms open more vacancies. Figure 2 shows representative unemployment and tightness nullclines, and indicates the location of $u_{\theta=0}$ for some $z > 0$.[9]

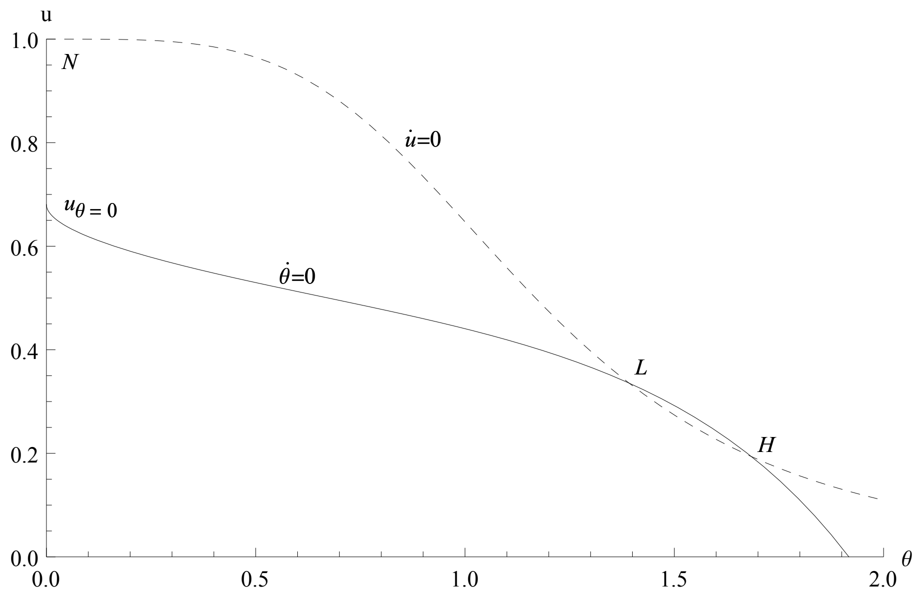

Figure 2: Nullclines of unemployment (dashed) and labor market tightness, resulting in the three steady states N, L, and H.

Notes: Parameters are $k = 0.55$, $\delta = 0.1$, $r = 0.012$, $\eta = 0.5$, $\gamma = 1.29$, $\alpha = 0.3$, $\beta = 0.5$, and $z = 0.71$.

2.3.2 Existence of steady states

A steady state with economic activity exists where the two downward-sloping nullclines intersect. Figure 2 shows two steady states with activity: a steady state L with a relatively low but positive employment and labor market tightness, and a steady state H with high employment and tightness. Due to the complementary slackness condition, there is also always a steady state N without economic activity at $\theta = 0$ and $u = 1$. However, it is

[9] Since unemployment can be expressed more easily as a function of tightness than the reverse, I plot unemployment on the vertical axis and tightness on the horizontal. Once I later plot vacancies to unemployment, unemployment will be on the horizontal axis as is common in the literature. With these conventions, counterclockwise cycles in unemployment and vacancies thus correspond to clockwise cycles in unemployment and tightness.

not generally possible to give explicit solutions for steady states with economic activity, and they may not always exist. In particular, for $z > 0$ there are only steady states with economic activity if recruiting costs k are small enough. If not, the unemployment nullcline will lie entirely above the tightness nullcline, and the steady state without activity is the only steady state.[10]

As soon as recruiting costs become small enough, the two nullclines touch and a *saddle-node bifurcation* occurs. In a bifurcation the qualitative properties of a dynamical system change as the result of a change in one or more parameters, k in this case.[11] Bifurcations are of interest because regions in the parameter space delimited by bifurcations are therefore structurally stable, i.e. the qualitative dynamics are invariant to small perturbations of the parameters. The qualitative change as the result of a saddle-node bifurcation is the emergence of two additional steady states: a saddlepoint and an *antisaddle* (node or focus).[12] Depending on the shape of the nullclines, saddle-node bifurcations can happen multiple times. As a result, if any steady state with economic activity exists for $z > 0$, there is generically an even number of them. Focusing on this empirically relevant case, Proposition 2.3 states a sufficient condition for the existence of exactly two steady states with economic activity. The proof is in Appendix 2.7.B.

Proposition 2.3. *Suppose k is small enough to guarantee the existence of a steady state with economic activity. Then if $\alpha \leq 1$ and $z > \left(1 - \frac{\delta}{\delta+(r+\delta)(1-\eta)\eta(\gamma-1)}\right)^{\alpha}$, exactly two of them exist.*

The sufficient condition in Proposition 2.3 ensures that the unemployment nullcline is convex and the tightness nullcline is concave on the relevant sections, but my calibrations as presented in Subsection 2.5.1 shows that this condition is not necessary. In the next subsection I characterize the stability of the set of steady-state equilibria with activity.

[10]If $z = 0$, at least one steady state with activity exists if $\eta + (\gamma - 1)^{-1} < 1$, independent of other parameters.

[11]Alternatively, for a given k and α, the gross value of leisure z must be small enough. See e.g. Kuznetsov (2004) for more on bifurcation theory.

[12]An antisaddle is a focus if its eigenvalues are complex, thus if $\mathrm{tr}^2 < 4\det$, and a node otherwise.

2.3.3 Stability of steady states

The local stability of the steady states with economic activity can be studied by linearizing the dynamical system given by (2.9) and (2.13). In a steady state with economic activity, the Jacobian matrix is

$$\left.\begin{pmatrix} \frac{\partial\dot{\theta}_t}{\partial\theta_t} & \frac{\partial\dot{\theta}_t}{\partial u_t} \\ \frac{\partial\dot{u}_t}{\partial\theta_t} & \frac{\partial\dot{u}_t}{\partial u_t} \end{pmatrix}\right|_{\dot{\theta},\dot{u}=0} = \begin{pmatrix} r+\delta+s^*(\theta_t)\theta_t^{\eta}\frac{\beta}{(1-\eta)} & (1-\beta)\frac{\theta_t^{\eta}}{k(1-\eta)}\alpha(1-u_t)^{\alpha-1} \\ -s'(\theta_t)\theta_t^{\eta}u_t - s^*(\theta_t)\eta\theta_t^{\eta-1}u_t & -\delta-s^*(\theta_t)\theta_t^{\eta} \end{pmatrix}, \tag{2.17}$$

with

$$s'(\theta_t) = \frac{1}{\theta_t(\gamma-1)}\left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\theta_t\right)^{\frac{1}{\gamma-1}}.$$

We see that $\partial\dot{\theta}_t/\partial\theta_t > 0$, $\partial\dot{\theta}_t/\partial u_t > 0$, $\partial\dot{u}_t/\partial\theta_t < 0$, and $\partial\dot{u}_t/\partial u_t < 0$ in steady state. Depending on whether the product of the diagonal elements $(\partial\dot{\theta}_t/\partial\theta_t)(\partial\dot{u}_t/\partial u_t)$ or the product of the cross-diagonal elements $(\partial\dot{\theta}_t/\partial u_t)(\partial\dot{u}_t/\partial\theta_t)$ is more negative, the determinant is negative or positive respectively. If and only if the determinant of the Jacobian matrix at a steady state is negative, it has eigenvalues of different signs and thus saddle path dynamics (see e.g. Kuznetsov (2004, p. 49)). If and only if the determinant is positive, the eigenvalues are of equal sign and the steady state is an antisaddle. Since only an antisaddle can feature surrounding oscillatory dynamics, Proposition 2.4 states a necessary condition for endogenous cycles.

Proposition 2.4. *A steady state with economic activity is an antisaddle if and only if the unemployment nullcline crosses the tightness nullcline from above.*

Proof. The slopes of the nullclines in any of the steady states with economic activity are given by

$$\frac{du_t}{d\theta_t}|_{\dot{u}_t=0} = -\frac{\partial\dot{u}_t}{\partial\theta_t}/\frac{\partial\dot{u}_t}{\partial u_t} \quad \text{and} \quad \frac{du_t}{d\theta_t}|_{\dot{\theta}_t=0} = -\frac{\partial\dot{\theta}_t}{\partial\theta_t}/\frac{\partial\dot{\theta}_t}{\partial u_t}.$$

Now we see that

$$\frac{\partial\dot{\theta}_t}{\partial\theta_t}\frac{\partial\dot{u}_t}{\partial u_t} > \frac{\partial\dot{\theta}_t}{\partial u_t}\frac{\partial\dot{u}_t}{\partial\theta_t}$$

if and only if the former is steeper than the latter. Since both terms are negative by the sign restrictions, this corresponds to the unemployment nullcline crossing the tightness nullcline from above. □

As can be seen in Figure 2, steady state L is the intersection of the unemployment nullcline crossing the tightness nullcline from above. As a result, only this steady state can feature endogenous oscillatory dynamics enclosing the steady state. Steady state H is a saddlepoint, so that there is an equilibrium saddlepath leading to it.

The trace of the Jacobian matrix in the antisaddle indicates whether the dynamics locally converge to it (the trace is negative and the antisaddle is a sink), diverge from it (positive trace; source), or neither (zero trace; center). The trace is given by the sum of the diagonal elements of the Jacobian matrix (2.17), and is thus

$$\frac{\partial \dot{\theta}_t}{\partial \theta_t} + \frac{\partial \dot{u}_t}{\partial u_t} = r + s^* \left(\theta_t\right) \theta_t^{\eta} \left(\frac{\beta}{1-\eta} - 1\right). \tag{2.18}$$

We see that the trace is exactly $r > 0$ if $\beta = 1 - \eta$, thus if the Hosios condition is satisfied. The trace is also always positive for $\beta > 1 - \eta$. As a result, the antisaddle is unstable in these cases. However, as we will again see in the next section, the trace can take either sign for $\beta < 1 - \eta$.

2.4 Beveridge cycle equilibria

In this section I show that the steady states and equilibrium paths leading to them are not the only equilibria. In particular, I show that there exists a stable *limit cycle* for a range of values for workers' bargaining power. A limit cycle is a periodic orbit enclosing an antisaddle such that at least one other path converges to it as time approaches positive infinity (the cycle is stable) or negative infinity (the cycle is unstable). Since the limit cycle in this paper results in enduring endogenous fluctuations in vacancies and unemployment, I refer to it as a Beveridge cycle.

The existence of a stable Beveridge cycle follows from the occurrence of a *Bogdanov-Takens bifurcation*. This bifurcation generically occurs in a system of two or more parameters, in which a Hopf bifurcation, a saddle-loop bifurcation, and a saddle-node bifurcation occur in a single point in the parameter space. This Bogdanov-Takens point is important because it is an organizing center for the dynamics: it characterizes the qualitative dynamics of the system in the neighborhood of this point. The next subsection presents examples of the relevant kinds of qualitative dynamics, as demarcated by a Hopf and a saddle-loop bifurcation respectively. These examples also show the existence of multiple equilibria for the same initial unemployment rate. Next, I show the occurrence of the bifurcations more formally. Finally, I describe the economics of the Beveridge cycle.

2.4.1 Examples of oscillatory dynamics

Figure 3 plots phase diagrams for four different values of the workers' bargaining power β. These phase diagrams zoom in on the steady states with economic activity, and the nullclines are dashed. Starting off from a small β in Panel 3a, the antisaddle is a stable focus, so that it attracts oscillating paths from initial conditions for unemployment outside itself. Notice that the equilibrium path towards the antisaddle is locally indeterminate, and that, in between the two outer paths of the panel there must also be an equilibrium saddlepath leading to the saddlepoint.

Increasing β, at some critical value the eigenvalues of the Jacobian matrix at the antisaddle become purely imaginary, and a Hopf bifurcation occurs. In a Hopf bifurcation, a periodic orbit emerges out of an antisaddle, and inherits its stability. Because the antisaddle of Panel 3a is stable, in this Hopf bifurcation it becomes unstable and gives rise to a stable limit cycle. Panel 3b presents this limit cycle. Equilibrium is still locally indeterminate and in between the two depicted paths there is again a saddlepath.

The limit cycle grows for a larger and larger workers' bargaining power until it coincides with the stable and unstable manifolds of the saddlepoint. When this happens, a saddle-loop bifurcation occurs, which is depicted in Panel 3c. At this bifurcation the periodic orbit connects the saddlepoint with itself, and is therefore called a *homoclinic orbit*. At the saddle-loop bifurcation the basin of attraction outside the periodic orbit has disappeared, except for the remaining saddlepath below the saddlepoint.

For larger values of β as in Panel 3d, periodic orbits do not exist any longer. The antisaddle remains unstable, but now the paths originating from its neighborhood will no longer be bounded, except for the saddlepath. Equilibrium is, however, still locally indeterminate.

2.4.2 Hopf, saddle-loop and Bogdanov-Takens bifurcations

The phase diagrams suggest the occurrence of a Hopf and a saddle-loop bifurcation. In this subsection I confirm the occurrence of these bifurcations. Moreover, I show that parameter combinations for which these bifurcations occur come together in a Bogdanov-Takens point. As a result, these bifurcations fully describe the behavior of the dynamical system in the neighborhood of this point in the parameter space.

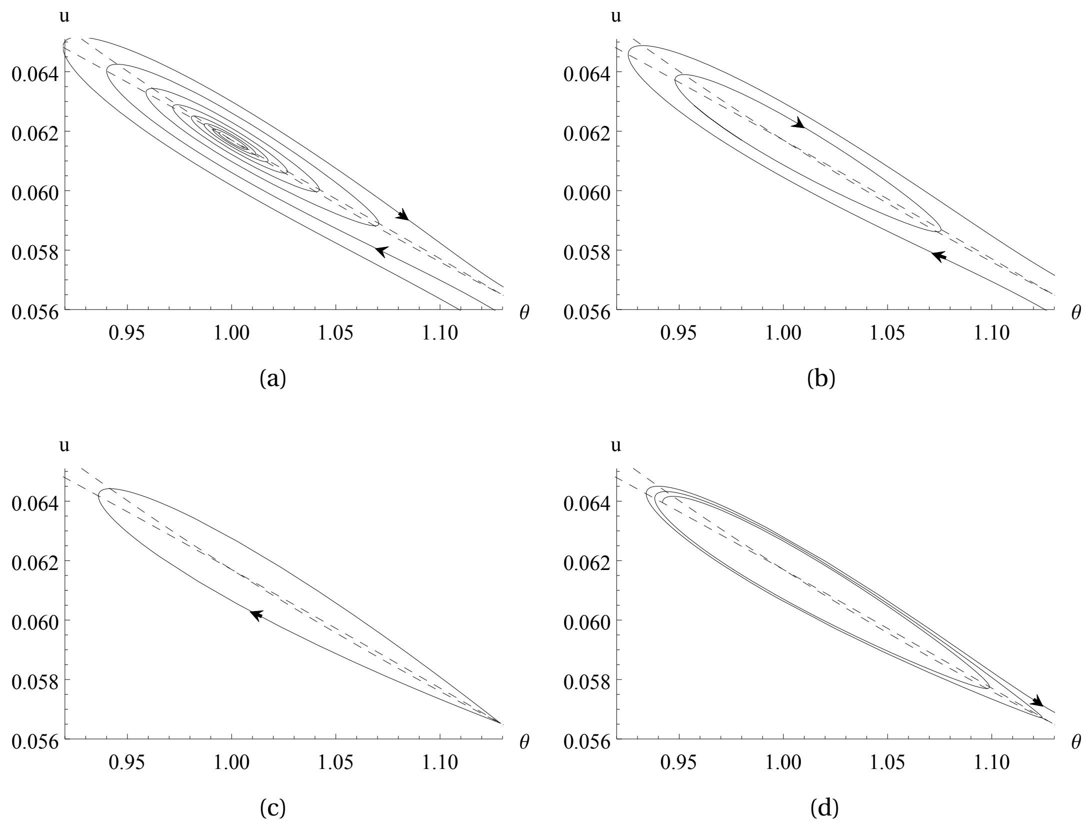

Figure 3: Representative phase diagrams for different values for β.

Notes: (a) Stable steady state for $\beta = 0.85$; (b) Stable limit cycle for $\beta = 0.89294$; (c) Homoclinic orbit of the saddle-loop bifurcation for $\beta \approx \beta_{SL} \approx 0.8930$; (d) No closed orbits at efficient bargaining for $\beta = 0.9$. Nullclines are dashed, intersecting twice. k and c_0 used to normalize steady state L tightness to 1. Other parameter values from Table 2, first column. $\beta_{Hopf} \approx 0.89290$.

A *Hopf bifurcation* occurs when the eigenvalues of the Jacobian matrix at the antisaddle cross the imaginary axis, so that the trace becomes zero. Remember from discussion of Equation 2.18 that the trace is always positive for $\beta \geq 1-\eta$. Proposition 2.5 shows that it can become zero for a sufficiently small workers' bargaining power.

Proposition 2.5. *Under the regularity condition that the job finding rate in the antisaddle exceeds the discount rate $r > 0$, there exists a $\beta_{Hopf} \in (0, 1-\eta)$ for which the antisaddle undergoes a Hopf bifurcation. As a result, there is a limit cycle in a one-sided neighborhood of β_{Hopf}.*

Proof. In any antisaddle the real parts of the eigenvalues are of the same sign, so that if the trace has a simple zero in the antisaddle, the eigenvalues cross the imaginary axis. From (2.18) we see that the trace has a simple zero at $\beta_{Hopf} = (1-\eta)\left(1 - r/\left(s^*(\theta_t)\theta_t^{\eta}\right)\right) \in (0, 1-\eta)$ if $s^*(\theta_t)\theta_t^{\eta} > r > 0$. Therefore, for $s^*(\theta_t)\theta_t^{\eta} > r$ in the antisaddle there exists a $0 < \beta_{Hopf} < 1-\eta$ for which a Hopf bifurcation occurs. As a result, a limit cycle exists in a one-sided neighborhood of β_{Hopf}. □

Proposition 2.5 implies that the antisaddle is a sink for $\beta < \beta_{Hopf}$ and a source for $\beta > \beta_{Hopf}$. For only one of these inequalities there exists a limit cycle enclosing the antisaddle for β sufficiently close to β_{Hopf}, but the proposition does not tell in which case. If the limit cycle exists for the left-sided neighborhood of β_{Hopf} it must be unstable, because the antisaddle is stable, and vice versa for the right-sided neighborhood.

While the limit cycle coincides with the antisaddle and the period of the cycle approaches zero as $\beta \to \beta_{Hopf}$, in a so-called *saddle-loop bifurcation* the cycle assumes its maximal size. A saddle-loop bifurcation occurs when the stable and the unstable manifolds of a saddlepoint connect to form a homoclinic orbit, a path that connects a steady state with itself. Because of the neighborhood of the saddlepoint, the period of the cycle approaches infinity as the cycle approaches the homoclinic orbit. In Hamiltonian systems (where all orbits are level curves) homoclinic orbits are a generic phenomenon, but in systems where the trace is generically nonzero the existence of a homoclinic orbit is not robust to small perturbations of a single parameter. In such systems the existence of a homoclinic orbit can be proven by perturbing a Hamiltonian system, and then the Andronov-Leontovich theorem (see e.g. Kuznetsov (2004, p. 200)) states that a limit cycle bifurcates on one side of the homoclinic orbit. Proposition 2.6 extends the result of Mortensen (1999) on the existence of a homoclinic orbit to the case of a positive value of leisure.[13] The proof is in Appendix 2.7.B.

Proposition 2.6. *Suppose that parameters are such that two steady states with economic activity exist for* $z > 0$, $r = 0$, *and* $\beta = 1-\eta$, *and define* θ_H *and* u_H *as market tightness and unemployment in the saddlepoint* H *respectively. Suppose also that* $(1-u_H+\alpha z^{1/\alpha})^{\alpha+1} < (1-\alpha)\left[(1-u_H)\left(z + k\theta_t^{1-\eta}/(1-\beta)\right) - u_H g(\theta_H)\right]$. *Then there exist a* $\beta_{SL} < 1-\eta$ *such that*

[13] Mortensen (1999) shows that (2.18) holds globally, not only in steady states, and as a result his system can be described by a Hamiltonian function when the Hosios condition holds and $r = 0$. Moreover, Bendixson's criterion (Guckenheimer and Holmes, 1983, p. 44) then rules out limit cycles whenever both $r > 0$ and $\beta \geq 1-\eta$.

for a sufficiently small $r > 0$ a homoclinic orbit exists. Moreover, there exists a family of stable limit cycles for a one-sided neighborhood of β_{SL}.

Proposition 2.6 states that there exists an orbit that connects the saddlepoint with itself for $\beta = \beta_{SL}$, and $(1-u_H+\alpha z^{1/\alpha})^{\alpha+1} < (1-\alpha)\left[(1-u_H)\left(z+k\theta_t^{1-\eta}/(1-\beta)\right)-u_H g(\theta_H)\right]$ is necessary and sufficient to guarantee that $\theta_t > 0$ on this homoclinic orbit. While the homoclinic orbit itself is not a robust phenomenon, it is of interest because it implies that a family of limit cycles can be found in the neighborhood of β_{SL}. However, although Proposition 2.6 tells us that these limit cycles are stable, it cannot tell us whether these cycles occur for $\beta > \beta_{SL}$ or $\beta < \beta_{SL}$. Moreover, we do not know yet whether $\beta_{Hopf} < \beta_{SL}$, or the other way around.

Given the limited number of ways that limit cycles can appear or disappear, if $\beta_{Hopf} < \beta_{SL}$ stable limit cycles would exist for the right-sided neighborhood of β_{Hopf}, delimited by β_{SL}. If $\beta_{Hopf} > \beta_{SL}$, stable limit cycles would exist for the right-sided neighborhood of β_{SL}, upon collision (in a saddle-node bifurcation of cycles) annihilating an unstable limit cycle that would exist for the left-sided neighborhood of β_{Hopf}. To rule out this latter case, and to show that a stable limit cycle exists for a range of values for the workers' bargaining power $\beta \in (\beta_{Hopf}, \beta_{SL})$, I show the occurrence of a Bogdanov-Takens bifurcation. A necessary condition for this bifurcation is Bogdanov-Takens singularity, which requires that both eigenvalues are zero in a steady state. Proposition 2.7 states that such a point in parameter space exists. The proof is in Appendix 2.7.B.

Proposition 2.7. *Define Bogdanov-Takens singularity as a steady state with two zero eigenvalues but a nonzero Jacobian matrix. There exists a point in parameter space that satisfies this singularity, and it is unique for $\alpha < 1$.*

Under certain genericity conditions this singularity is sufficient for the occurrence of a Bogdanov-Takens bifurcation. These conditions require that one can 'travel' through the Bogdanov-Takens point by varying the parameters. To suggest that this is the case, I present the bifurcation diagram of Figure 4.[14] The figure shows combinations of parameters α and β for which bifurcations occur. The regions bounded by these bifurcations can be represented by qualitatively similar phase diagrams, where (a), (b),

[14] A formal proof requires the construction of a normal form and is too involved to present here, see e.g. Kuznetsov (2004, p. 322).

and (d) correspond to the respective panels of Figure 3, reprinted for convenience. The gray solid curve represents combinations of α and β for which a saddle-node bifurcation occurs, so that above the curve (in region (0)) no steady states with economic activity exist while below it there are two of them. The bold curve depicts the occurrence of a saddle-loop bifurcation as in Panel 3c. Right of this curve is the familiar region (d) in which all equilibria are either steady states or saddlepaths, and in which the dots indicate efficient bargaining. The dashed curve represents combinations of α and β for which a Hopf bifurcation occurs, so that left of this curve (in region (a)) there is a continuum of equilibria spiraling towards antisaddle L. Region (b), bounded by the Hopf and the saddle-loop bifurcations, features the Beveridge cycle. This region increases in r. Finally, there is a Bogdanov-Takens point BT where the Hopf bifurcation curve and the saddle-loop bifurcation curve connect tangentially to the saddle-node bifurcation curve, as they should for a Bogdanov-Takens bifurcation to occur.

Consequently, for an elasticity of the demand externalities α in the left-sided neighborhood of the Bogdanov-Takens point, there exists a family of limit cycles enclosed by a Hopf and saddle-loop bifurcation. The Bogdanov-Takens bifurcation rules out another bifurcation that could change the stability of these cycles. Because Proposition 2.6 shows that the limit cycles born at the saddle-loop bifurcation are stable, the entire family is stable. This implies that $\beta_{Hopf} < \beta_{SL}$, because a stable limit cycle requires a positive trace in antisaddle L, which only occurs for $\beta > \beta_{Hopf}$. As a result, a stable Beveridge cycle exists for $\beta \in (\beta_{Hopf}, \beta_{SL})$. Because the Beveridge cycle exists for a range of values for the parameters, it is structurally stable. The cycle will thus continue to exist under small perturbations of the parameters, or upon introduction of some risk aversion.

2.4.3 The economics of the Beveridge cycle

The analysis above implies that for $\beta < \beta_{SL}$, starting from a sufficiently low initial condition for unemployment, expectations select which steady state will be reached in the long run. Moreover, also for $\beta > \beta_{SL}$ multiple equilibria exist in the neighborhood of the antisaddle due to local indeterminacy. Obviously, multiple self-fulfilling expectations are the result of the feedback from aggregate employment to the revenue per match generated by the demand externality, but they also require delays in matching. An expectation that unemployment will be low makes firms open vacancies, because low

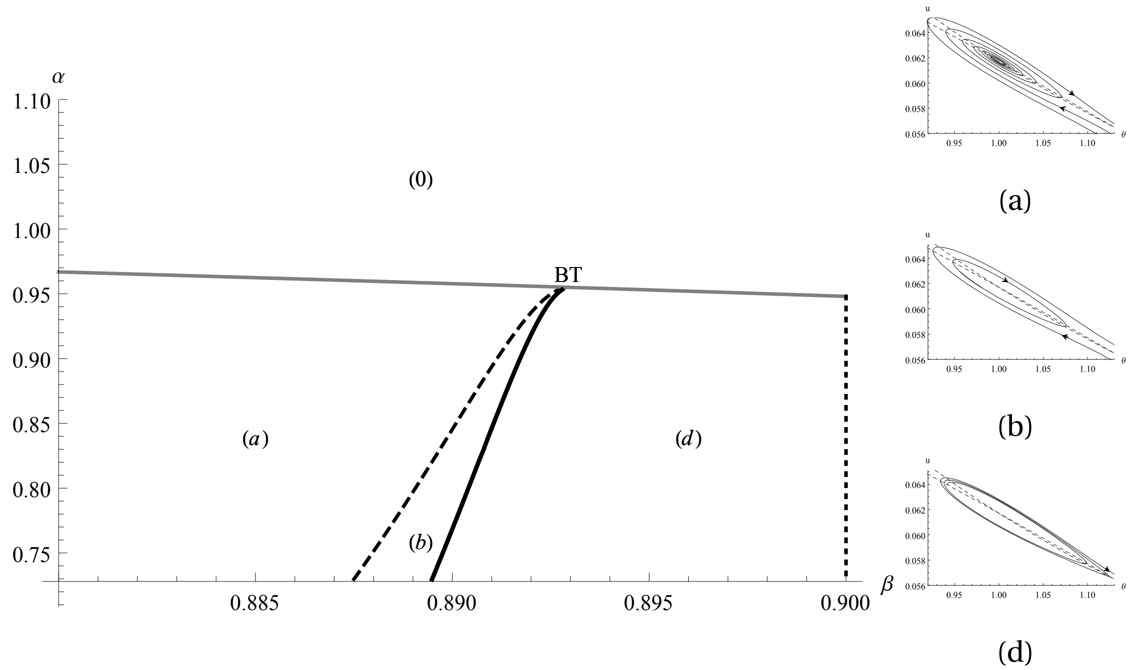

Figure 4: Bifurcation diagram of the Bogdanov-Takens bifurcation, with α and β as bifurcation parameters.

Notes: Other parameters from Table 3, first column. The gray curve corresponds to the saddle-node bifurcation, the bold curve to the saddle-loop bifurcation, and the dashed curve to the Hopf bifurcation. Regions (a), (b), and (d) refer to the panels of Figure 3, reprinted for convenience, and (d) extends beyond efficient bargaining at $\beta = 0.9$ (dotted curve). BT refers to the Bogdanov-Takens-point, and there are no steady states with economic activity in region (0). The homoclinic orbit hits the u-axis for $\alpha \approx 0.73$.

unemployment implies a high revenue per match. This expectation is self-fulfilling because more vacancies now result in lower future unemployment.

However, rational expectations Beveridge cycles do not only require positive feedback and a propagation mechanism, but also a congestion externality. Howitt and McAfee (1988) argue that in models with only a positive externality, because of positive discounting antisaddles are unstable and no cycles exist. However, they show that a negative externality can overturn this effect. This negative externality is a standard ingredient in search models of the labor market, where matching is less likely for a potential trader when many other potential traders search on the same side of the market. The congestion externality makes firms with expectations of low unemployment open vacancies immediately, not only because revenue per match will be higher in the future and matching takes time, but also because hiring will be more costly in the future. Proposition 2.2 shows that when $\beta = 1-\eta$, thick-market externalities on one side of the market exactly cancel congestion externalities on the other side of the market. It follows from Propositions 2.5 and 2.6 that Beveridge cycles exist for a range of value for workers' bargaining power that satisfy $\beta < 1-\eta$. This condition is required because free entry of firms drives the evolution of market tightness, and if $\beta < 1-\eta$, congestion occurs primarily on the firms' side of the market. Besides, variable search intensity is not essential for the existence of the cycle, but is only helpful for the calibration.

It follows from Proposition 2.4 that oscillations enclose a steady state where the unemployment nullcline crosses the tightness nullcline from above. At this antisaddle, the strategic complementarity from the demand externality dominates the strategic substitute from the congestion externality: when firms increase their vacancies, this results in such a large decrease in unemployment in the future, that revenue per match will increase so much that firms would want to open even more vacancies. As a result, firms overshoot the steady state level of tightness of the antisaddle for the alluringly high revenue per match at high aggregate employment levels. However, with so many vacancies, unemployment decreases fast and it starts to take a long time to fill an individual vacancy. Consequently, while unemployment still decreases but firms foresee an end to the boom, they do not want to spend valuable resources on vacancies that are hard to fill. Expecting higher unemployment in the future and thus smaller benefits of a filled vacancy, firms reduce labor market tightness. As a result, at some point fewer matches are made than jobs are destroyed, and unemployment increases. Higher unemployment feeds back to lower revenue per match, so that firms also overshoot the steady state level of tightness in the trough. With so few vacancies, however, a single vacancy is filled very fast. So while unemployment still increases but firms foresee an end to the trough, they are willing to spend some resources on vacancies that will be filled very soon and pay off at the higher employment levels of the future. Job matching takes over from job destruction again and unemployment decreases, completing the cycle.

The relative size of the discount rate r and the distortion of the Hosios condition, $\beta < 1-\eta$, determines whether oscillations diverge, converge, or form a closed orbit. The asset pricing equation of the system, (2.13), shows that for a high r and given flow values, equilibrium requires large capital gains or losses, i.e. fast movements in market tightness. For that reason, for a higher r and a given η, stability of the antisaddle requires a smaller β, or more congestion. Alternatively, for a given r the oscillations converge for a small β, form closed orbits for an intermediate β, and diverge for a large β, as in Figure 3. Remember that for a given r, the cycle is largest for β_{SL}. Otherwise, fix $\beta \in (\beta_{Hopf}, \beta_{SL})$ and compare Beveridge cycles for different discount rates. The cycle is small for a small discount rate because when firms are patient, they heavily respond to expected future changes in the revenue per match. A limit cycle is then only consistent with equilibrium if revenue per match does not vary too much over the cycle, thus if the cycle is small.

As the result of self-fulfilling expectations, the model features multiple equilibria, which results in the problem of equilibrium selection. Because of the evidence on the cyclical dynamics as presented in Panel 1a of Figure 1, I take the dynamics of the Beveridge cycle to be the relevant dynamics to explain the actual data. The stability of the limit cycle further supports its plausibility as a data-generating process, although its basin of attraction may be small. Figure 4 shows that the set of β's that give rise to a Beveridge cycle has a positive but small measure, so that the proposed data-generating process is not very robust to changes in β. On the other hand, for virtually all $\beta < \beta_{Hopf}$ the dynamics oscillate but eventually settle down in the antisaddle, as in Panel 3a of Figure 3. Especially if β is smaller than but close to β_{Hopf}, it may take a very long time to reach the steady state, so that many business cycles could be explained with one exogenous shock in fundamentals or beliefs. Kaplan and Menzio (2016) provide an example of the latter. I focus on the limit cycle and therefore I do not exploit the additional degrees of freedom that exogenous shocks provide. Instead, I use the fixed period of the limit cycle to calibrate the model. The next section presents the quantitative contribution of this paper.

2.5 Quantitative results

In this section I calibrate the model to the average duration of the business cycle and assess its quantitative performance in describing unemployment and vacancies over the business cycle. I compare the model-generated data to the actual data and to data generated by the canonical search and matching model of Pissarides (1985) with productivity shocks. Because the latter features constant search intensity, I also calibrate a model of the Beveridge cycle without variable search intensity. Next, I discuss robustness to alternative calibrations, including alternative targets for the duration of the business cycle. Finally, I show that the model-generated data move in the expected direction upon changes in unemployment benefits.

2.5.1 Calibration

I calibrate the parameters of the model using data on the duration of the business cycle. Depending on its measurement, the typical cycle in Panel 1a of Figure 1 lasts between 18 and 28 quarters. Because productivity in the model moves with employment, the duration of the NBER business cycle provides an alternative calibration target that avoids some arbitrary choices. Figure 5 shows the duration of the NBER business cycles falling entirely in the sample period from 1951 to 2014. Irrespective of whether the cycle is measured from peak to peak or from trough to trough, the average cycle lasts roughly 24 quarters.

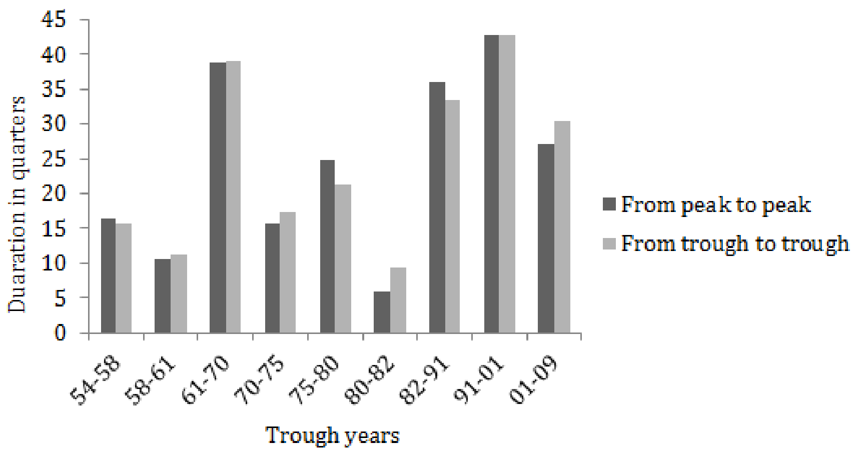

Figure 5: Duration in quarters of all completed NBER business cycles between 1951 and 2014.

Source: http://www.nber.org/cycles.html

The parameters that describe workers' preferences are discount rate r, gross value of leisure z, and the elasticity of the search cost function γ. Firms' technology is described by the elasticity of revenue per match with respect to aggregate employment α, the vacancy creation cost k, and the job destruction rate δ. Matching and bargaining are described by the elasticity of the matching function with respect to vacancies η, and workers' bargaining power β.

The bifurcation analysis of the preceding section makes clear that β is important for the existence of the Beveridge cycle. I choose the β that is closest to efficient bargaining but still gives rise to a limit cycle with a computationally significant basin of attraction outside its path. This limit cycle is close to the homoclinic orbit, the largest closed orbit possible taking the other parameters as given.[15] By moving β further away from efficient bargaining in the direction of the Hopf bifurcation, the limit cycle can be made arbitrarily small, but then the Beveridge cycle can explain little volatility. My calibration strategy therefore approaches an upper bound to deterministic volatility.

I choose a period in the model to be one quarter. Given β, I set α such that the duration of the Beveridge cycle corresponds to the average duration of the NBER business cycle over the sample period. The value of r corresponds to an annual interest rate of 4 percent. I choose the value for δ to equal the observed average quarterly job destruction rate, and the value for k so that the average unemployment rate is the same in the model and in the data.

In the model, the elasticity ε of the job finding rate with respect to the vacancy-to-unemployment ratio is $\varepsilon = \eta + (1-\eta)/\gamma$. Therefore I choose η and γ such that their combination results in an estimated elasticity of 0.46. To distinguish the role of variable search intensity from the role of the matching function, I exploit that (by definition) non-participants do not search for jobs but still find jobs at cyclical rates. Assuming that non-participants and unemployed workers find jobs via the same matching function, and that the higher job finding rate of the unemployed is the result of their search effort, allows me to disentangle η and γ. I use a matching function with ranking (similar to Blanchard and Diamond (1994)) to ensure that non-participants do not create congestion for unemployed workers, as in the model. Appendix 2.7.A describes this procedure and the data used in more detail.

I choose z such that I match an average flow value of leisure of 0.71 as in Hall and Milgrom (2008). In my model, however, the flow benefit of leisure consists of a gross value of leisure z and variable search costs s_t^{γ}. On top of that, while average productivity is commonly normalized to 1, productivity in my model never reaches 1. For that reason, the relevant calibration target of my model is not z, but the net value of leisure relative to output:

$$\zeta = \frac{z - s^* (\theta_t)^{\gamma}}{(1-u_t)^{\alpha}}.$$

[15]As pointed out in section 2.4, the period of the homoclinic orbit approaches infinity, but because it has no basin of attraction outside itself, sampling from this cycle is computationally unstable.

I calibrate the parameters of the Pissarides (1985) model using the same targets. I use the same β as for the Beveridge cycle, because in the standard model it is unrelated to the size of the cycle. However, I choose the parameters of the stochastic process for productivity such that the standard deviation and autocorrelation of revenue per match are the same for the Beveridge cycle and the Pissarides (1985) model. Following the RBC literature's convention for stochastic processes, I match the statistics in logs as deviations from a linear trend.[16] I repeat this procedure to match the targets of the Beveridge cycle without variable search intensity.

The calibration strategy described above results in the parameters of Table 2. All calibrations of the Beveridge cycle result in two steady states with economic activity. My calibration procedure to disentangle γ and η results in an estimated γ of 2.5, which is a bit higher than the quadratic cost function that the literature (e.g. Gomme and Lkhagvasuren (2015)) generally assumes.[17] The target for ε then implies $\eta = 0.1$, so that the Hosios condition corresponds to $\beta = 0.9$. In both the Pissarides (1985) model and the model of the Beveridge cycle without variable search intensity, $\eta = \varepsilon$. When on top of that average revenue per match is equal to one, as in the Pissarides (1985) model, $z = \zeta$.

To match the average unemployment rate over the cycle, my benchmark calibration results in a demand externality of 6.9. This value is considerably higher than Kaplan and Menzio (2016)'s estimate of shopping and demand externalities that implies $\alpha = 1.15$, and lies completely out of the range of estimates of increasing returns to scale, which provide an upper bound of $\alpha = 0.17$ (Harrison, 2001). As can be seen in the last column of Table 2, this problem only deteriorates for a fixed search intensity. The following two alternative calibration strategies show that either a high value of leisure, or a high elasticity of the job finding rate with respect to tightness, result in demand externalities that are of the same order of magnitude as Kaplan and Menzio (2016).

The estimation of the elasticity of the job finding rate ε is very sensitive to the choices made and the data selected. As a result, in the literature it takes almost any value between zero and one, see Petrongolo and Pissarides (2001). Because the quantitative results are also sensitive to ε, I consider an alternative calibration, exploiting that from December 2000 onwards the JOLTS provides an alternative source of the job finding

[16] An HP-filter with a lower smoothing parameter biases autocorrelation coefficients to such an extent that an Ornstein-Uhlenbeck process cannot match the filtered autocorrelation of the Beveridge cycle.

[17] Note, however, that $\gamma = 2$ is not compatible with a positive elasticity of the matching function for an estimated elasticity $\varepsilon < 0.5$.

	Beveridge cycle	Pissarides I	$\gamma \to \infty$	Pissarides II
r	0.012	0.012	0.012	0.012
z	0.527	0.71	0.345	0.71
γ	2.5	-	-	-
α	6.887	-	11.05	-
k	$2.079 * 10^{-8}$	$1.155 * 10^{-2}$	$4.561 * 10^{-2}$	$7.974 * 10^{-2}$
δ	0.1	0.1	0.1	0.1
η	0.1	0.46	0.46	0.46
β	0.893	0.893	0.536	0.536

Table 2: Calibrated parameters for the benchmark calibration, the Pissarides (1985) model (Pissarides I), the Beveridge cycle without variable search intensity ($\gamma \to \infty$), and the Pissarides (1985) model that uses the same targets as the the Beveridge cycle without variable search intensity (Pissarides II)

Notes: For Pissarides I, the Ornstein-Uhlenbeck volatility parameter is $\sigma = 0.02139$, and its persistence parameter is $\gamma = 0.032$. For Pissarides II, $\sigma = 0.0546$ and $\gamma = 0.0459$.

probability. Regressing the log of this job finding probability on the log of labor market tightness for the relevant sample period results in $\varepsilon = 0.84$. To disentangle γ and η, I set $\gamma = 1.29$ as estimated by Burdett et al. (1984).

Similarly, the literature disagrees on the flow value of leisure. For that reason, I also consider Hagedorn and Manovskii (2008)'s target of 0.955 for the average ζ over the cycle. Finally, for the benchmark calibration targets of $\varepsilon = 0.46$ and $\zeta = 0.71$, I also calibrate the duration of the Beveridge cycle to a lower and upper bound of 18 and 28 quarters, respectively, as suggested by the evidence in Panel 1a of Figure 1.

The parameters resulting from these alternative calibrations are presented in Table 3. Remember that the estimate of Kaplan and Menzio (2016) corresponds to $\alpha = 1.15$, so that for $\zeta = 0.955$ the calibrated demand externalities are smaller than theirs. Of course, in this case the calibrated value for z is relatively high. For $\varepsilon = 0.84$, α is somewhat above the estimate of Kaplan and Menzio (2016), but in the same order of magnitude. Besides, the calibration strategy with $\gamma = 1.29$ results in $\eta = 0.3$, which is more common in the literature without variable search intensity. Finally, a Beveridge cycle of shorter duration requires demand externalities that are slightly higher than the benchmark parameter value, while a longer duration is obtained for a smaller α.

	$\zeta = 0.955$	$\varepsilon = 0.84$	18 quarters	28 quarters
r	0.012	0.012	0.012	0.012
z	0.909	0.839	0.490	0.538
γ	2.5	1.29	2.5	2.5
α	1.064	1.342	7.500	6.706
k	$1.936 * 10^{-11}$	$1.226 * 10^{-3}$	$2.943 * 10^{-8}$	$1.862 * 10^{-8}$
δ	0.1	0.1	0.1	0.1
η	0.1	0.3	0.1	0.1
β	0.893	0.695	0.893	0.893

Table 3: Calibrated parameters for four alternative targets.

Notes: $\varepsilon = 0.45$ and $\zeta = 0.71$ unless specified otherwise.

A high value of leisure and a high elasticity of the job finding rate reduce the calibrated value for α, because both decrease the unemployment rate, or increase the job finding rate, in the antisaddle. Consequently, smaller demand externalities are required to target an average unemployment rate of 0.587. A higher elasticity and more variable search intensity change the unemployment nullcline so that the same market tightness results in lower unemployment. A higher value of leisure shifts the tightness nullcline to the left, because for a high ζ, the surplus of a match is lower and firms open fewer vacancies. The next subsection presents the model-generated data.

2.5.2 Performance

In this subsection I present the data generated by the Beveridge cycle, and compare it to the actual data, the data generated by the calibrated Pissarides (1985) model, and the data that result from a Beveridge cycle without variable search intensity. To assess the quantitative performance of the Beveridge cycle, I sample 256 (quarterly) observations from the calibrated Beveridge cycle, and compute time series for unemployment, vacancies, the ratio of vacancies to unemployment, the job finding rate $f_t = s^*(\theta_t)\theta_t^{\eta}$, and revenue per match $y_t = (1-u_t)^{\alpha}$. More specifically, for each variable I draw time series for each of the 24 different starting points around the cycle, and report the average statistics. As for the original data, I take logs and deviations from an HP trend with smoothing parameter 10^5. Table 4 presents the standard deviation, autocorrelation and cross-correlations of the variables of the model.

		u	v	v/u	f	y
Standard deviation		0.043	0.060	0.101	0.047	0.019
Quarterly autocorr.		0.942	0.934	0.939	0.939	0.940
Correlation matrix	u	1	−0.938	−0.979	−0.979	−1.000
	v		1	0.989	0.989	0.938
	v/u			1	1.000	0.979
	h				1	0.979
	y					1

Table 4: Summary statistics of the Beveridge cycle calibrated to $\zeta = 0.71$ and $\varepsilon = 0.45$.

Notes: f stands for the job finding rate, y for revenue per match. Statistics are averages from 24 samples of 256 quarters across the cycle. All variables are in logs as deviations from an HP trend with smoothing parameter 10^5.

Comparing the model-generated data to the actual data in Table 1, some features stand out. First, the endogenous fluctuations in revenue per match almost account for the full observed volatility of productivity. However, in logs as deviations from a linear trend the standard deviation of revenue per match is also only 0.019, compared to 0.041 in the data, so that the demand externality amounts to less than half of the fluctuations in productivity. The HP-filtered time series for revenue per match is somewhat more persistent than in the data, but in logs as deviations from a linear trend the autocorrelation coefficient is also only 0.940, compared to 0.976 in the data. These differences can be understood from the fact that the Beveridge cycle features only fluctuations in the business cycle frequency domain. Besides, by construction, revenue per match is much more negatively correlated with unemployment than observed productivity.

Second, the Beveridge cycle explains about 22 per cent of the observed volatility in unemployment, and more than one-third of the observed volatility in vacancies. Due to the strong negative correlation between vacancies and unemployment, both in the model and in the data, the standard deviation of the ratio of vacancies to unemployment is about the same as the sum of the standard deviations of vacancies and unemployment, both in the model and in the data. As a result, the model-generated standard deviation of the ratio of vacancies to unemployment and the job finding rate also fall short compared to the data.

Third, the autocorrelation of unemployment, vacancies, the ratio of vacancies to unemployment, and the job finding rate are about the same in the model and in the data. It is important to note that this persistence is an endogenous result of calibrating the Beveridge cycle to the duration of the average business cycle, and is not generated by feeding in a persistent stochastic process. The existence of a saddlepoint in the neighborhood of the Beveridge cycle slows down the dynamics and therefore allows the cycle to be calibrated to the duration of the business cycle. To assess to what extent the persistence of the labor market variables is driven by the Beveridge cycle rather than simply the persistence of the fluctuations in revenue per match, it is useful to compare the model-generated data of the Beveridge cycle and the Pissarides (1985) model.

Table 5 reports the summary statistics of the Pissarides (1985) model, using the code from Shimer (2005) but the parameters from Table 2, second column. In particular, the standard deviation and autocorrelation of revenue per match in logs as deviations from a linear trend are the same as for the Beveridge cycle. Note that the autocorrelation of unemployment approaches that of the Beveridge cycle and the observed data, but that vacancies, and to a smaller extent the ratio of vacancies to unemployment and the job finding rate, are not as persistent. Similarly, the standard deviation of unemployment lags behind that of the Beveridge cycle, in absolute terms, but also relative to filtered productivity. The ratios of the standard deviations across the labor market variables are the same for the Beveridge cycle and the Pissarides (1985) model, resulting in a similar slope of the Beveridge curve. Finally, in both models the cross-correlations are generally higher than in the data.

The differences between the Beveridge cycle and the Pissarides (1985) model can also be seen graphically in Figure 6. Panel 6a contains 25 simulated quarterly observations that complete a Beveridge cycle, HP-filtered and connected by straight lines. The shape of the Beveridge cycle is similar to the observed cycles in Panel 1a of Figure 1, and it rotates counterclockwise. Panel 6b of Figure 6 plots a representative realization of 256 quarters of the Pissarides (1985) model. The model-generated Beveridge curve is somewhat less volatile than the Beveridge cycle, but, most importantly, lacks cyclical dynamics. Although there are some swings parallel to the inverse relationship between vacancies and unemployment, most of the dynamics is almost vertical.

Figure 6 also shows that the Beveridge curve generated by both the Beveridge cycle and the Pissarides (1985) model is too steep compared to the data. It is important to note, however, that both the Beveridge cycle and the Pissarides (1985) model feature a

		u	v	v/u	f	y
Standard deviation		0.018	0.027	0.044	0.02	0.013
Quarterly autocorr.		0.931	0.781	0.873	0.873	0.873
	u	1	−0.900	−0.963	−0.963	−0.962
	v		1	0.984	0.984	0.983
Correlation matrix	v/u			1	1.000	0.999
	f				1	0.999
	y					1

Table 5: Summary statistics of the Pissarides (1985) model calibrated to the same targets as the Beveridge cycle.

Notes: Code from Shimer (2005), where the Ornstein-Uhlenbeck volatility parameter is $\sigma = 0.02139$ and its persistence parameter is $\gamma = 0.032$, to target the standard deviation and autocorrelation of productivity in logs as deviations from a linear trend of the 24 quarter Beveridge cycle. All variables are in logs as deviations from an HP trend with smoothing parameter 10^5. The table reports averages across 10000 simulations.

constant job destruction rate and no transitions into employment from employment or non-participation. It is well known that job destruction shocks can flatten out the Beveridge curve, but only at the cost of a counterfactually low correlation between vacancies and unemployment. Other mechanisms can also increase the volatility of unemployment relative to the volatility of vacancies. It follows from Elsby et al. (2015) that the participation margin contributes to unemployment volatility and thus reduces the slope of the Beveridge curve. Menzio and Shi (2011) show that on-the-job search affects firms' incentives to post vacancies and can sharply reduce the volatility of vacancies over the cycle. However, Eeckhout and Lindenlaub (2015) show that in the presence of sorting, on-the-job search can also contribute to the volatility of vacancies and even result in self-fulfilling fluctuations.

Both the absence of fluctuations in job destruction, and the absence of transitions into employment from employment or non-participation in the model of the Beveridge cycle imply that the model-generated data should not exactly match the observed volatility of both unemployment and vacancies. For instance, Pissarides (2009) argues that in reality one-third to one-half of the volatility in unemployment is driven by fluctuations in the inflow into unemployment rather than the outflow. As a result, a

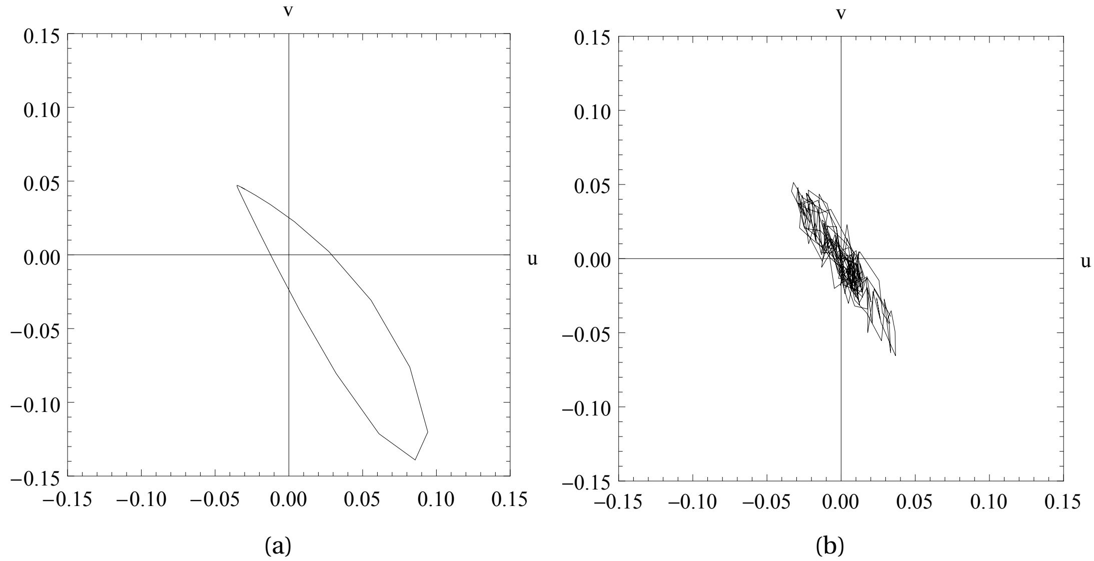

Figure 6: Simulated dynamics of unemployment and vacancies for (a) the Beveridge cycle, and (b) the Pissarides model.

Notes: Panel a plots 25 quarterly datapoints from the calibrated Beveridge cycle. Panel b plots one simulation of 256 quarters based on code from Shimer (2005), calibrated to the same targets as the Beveridge cycle. Simulated data are in logs as deviations from an HP trend with smoothing parameter 10^5, connected by straight lines.

model with constant job destruction should explain at most two-thirds of the volatility in unemployment.

Summing up, vacancies are more persistent over the Beveridge cycle than in the Pissarides (1985) model, without being less volatile. In fact, both unemployment and vacancies are somewhat more volatile over the Beveridge cycle than in the Pissarides (1985) model, also relative to HP-filtered productivity. To investigate the source of additional volatility, I also compare the data generated by the Pissarides (1985) model with data generated by the Beveridge cycle without variable search intensity. Table 6 presents the standard deviations and autocorrelations from the Beveridge cycle without variable search intensity, and the Pissarides (1985) model calibrated to the same targets. The former features counterfactually large fluctuations in revenue per match as the result of the large demand externalities. However, the table shows that *relative* to the HP-filtered volatility of revenue per match, the volatility of the labor market variables is about the same for the Pissarides (1985) model and the Beveridge cycle without variable search intensity. As a result, the Beveridge cycle does not feature any new amplification mechanisms. Variable search intensity contributes to amplification because it makes the net flow value of leisure countercyclical and because search and recruiting activity are strategic complements.

		u	v	v/u	f	y
Bev. cycle, $\gamma \to \infty$	Standard deviation	0.061	0.087	0.145	0.067	0.044
	Quarterly autocorr.	0.931	0.918	0.926	0.926	0.927
Pissarides II	Standard deviation	0.045	0.066	0.109	0.05	0.031
	Quarterly autocorr.	0.926	0.763	0.864	0.864	0.863

Table 6: Summary statistics of the Beveridge cycle without variable search intensity ($\gamma \to \infty$), and the Pissarides (1985) model recalibrated to feature the same β and the same standard deviation and autocorrelation of revenue per worker in logs as deviations from a linear trend (Pissarides II).

Regarding persistence, in both the Pissarides (1985) model and the Beveridge cycle (with or without variable search intensity), the autocorrelation of labor market tightness and the job finding rate is about the same as the autocorrelation of the (HP-filtered) revenue per match. As a result, the higher persistence of these variables may be attributed to the higher persistence of revenue per match after filtration. However, in the Pissarides (1985) model the autocorrelation of vacancies is substantially smaller than that of labor market tightness and the job finding rate, unlike the Beveridge cycle. The feature of the Beveridge cycle that vacancies are almost as persistent as unemployment (and the other variables) is exactly what results in the counterclockwise cycles in unemployment and vacancies. The next subsection shows that this feature is robust to alternative targets for the duration of the cycle.

2.5.3 Robustness

This subsection presents the model-generated data resulting from the four alternative calibration strategies. Table 7 presents the standard deviation and autocorrelations of the data generated by the Beveridge cycle calibrated to 18 and 28 quarters. Not surprisingly, the 18-quarter Beveridge cycle results in a smaller autocorrelation coefficient than the 24-quarter cycle, and larger demand elasticities result in a smaller standard deviation. Equivalently, the 28-quarter Beveridge cycle features smaller volatility but more persistence. Note, however, that the relative persistence of vacancies compared to the other variables survives under alternative targets for the duration of the cycle, although it is higher for longer cycles.

		u	v	v/u	f	y
18 quarters	Standard deviation	0.095	0.143	0.231	0.106	0.047
	Quarterly autocorr.	0.888	0.855	0.875	0.875	0.879
28 quarters	Standard deviation	0.027	0.037	0.063	0.029	0.011
	Quarterly autocorr.	0.960	0.957	0.959	0.959	0.960

Table 7: Summary statistics of the Beveridge cycle calibrated to 18 and 28 quarters, respectively.

The calibration results in Table 3 have shown that either a high value of leisure, or a high elasticity of the job finding rate with respect to tightness, result in demand externalities that are of the same order of magnitude as Kaplan and Menzio (2016). The summary statistics of Table 8 show the familiar result that both also contribute to amplification. Although smaller demand externalities result in only a fraction of the volatility in revenue per match, the volatility of unemployment falls to a much smaller extent. The standard deviation of unemployment is about the same in the benchmark calibration and the calibration with $\varepsilon = 0.84$, and still more than three-quarters of the benchmark result in the calibration with $\zeta = 0.955$. Moreover, the persistence of vacancies does not suffer from additional amplification. Note, however, that a high elasticity of the job finding rate with respect to tightness results in a low volatility of vacancies compared to unemployment.

As discussed above, a high value of leisure reduces the required externalities because it lowers the steady state L unemployment rate. Although such comparative statics might seem counterintuitive, I show that the dynamics of the Beveridge cycle move in the opposite 'intuitive' direction. Increasing the value of leisure to $z = 0.5277$ while keeping all other parameters fixed, steady state unemployment decreases from 0.062 to 0.060. This is the comparative statics effect described above. However, I claim that the dynamical system of the Beveridge cycle is the data-generating process. Sampling from

		u	v	v/u	f	y
$\zeta = 0.955; \varepsilon = 0.45$	Standard deviation	0.033	0.046	0.079	0.036	0.002
	Quarterly autocorr.	0.946	0.940	0.944	0.944	0.945
$\zeta = 0.71; \varepsilon = 0.84$	Standard deviation	0.042	0.015	0.054	0.045	0.004
	Quarterly autocorr.	0.952	0.934	0.951	0.951	0.951

Table 8: Summary statistics of the Beveridge cycle calibrated to $\zeta = 0.955$ and $\varepsilon = 0.45$, and $\zeta = 0.71$ and $\varepsilon = 0.84$, respectively.

the slightly displaced Beveridge cycle, the average unemployment rate over the cycle increases from 0.059 to 0.060. Consequently, my model predicts a positive effect of the unemployment benefit on observed unemployment, as most economists would expect.

This argument does not rely on adjustment dynamics, but compares datapoints on two different Beveridge cycles. Figure 7 shows a time series of unemployment over 256 quarters, connected by straight lines, resulting from the Beveridge cycle with the calibrated value of leisure. As can be seen in this figure, the calibrated cycle spends most of its time on segments of the cycle with low unemployment rates. A Beveridge cycle for a higher value of leisure spends its time more evenly over the cycle. This nonlinear effect dominates the displacement of steady state L and its enclosing Beveridge cycle. As a result, a Beveridge cycle with a high value of leisure produces a lower average unemployment rate than a cycle with a high value of leisure, even though steady state L moves in the opposite direction.

Finally, the time series in Figure 7 is very regular, much more so than actual data. However, exogenous shocks in fundamentals or beliefs can cause variations in amplitude and period of the cycle, without altering its driving mechanism. Beaudry et al. (2015) show that adding exogenous shocks on top of a deterministic cycle can reproduce the spectrum of business cycle fluctuations in output and employment. Whether the persistence of vacancies survives such additional shocks is a question for future research.

2.6 Conclusion

Mortensen (1999) presents a parsimonious model to show that multiple Pareto-ranked cycles and steady states can coexist, and that different expectations can be self-fulfilling

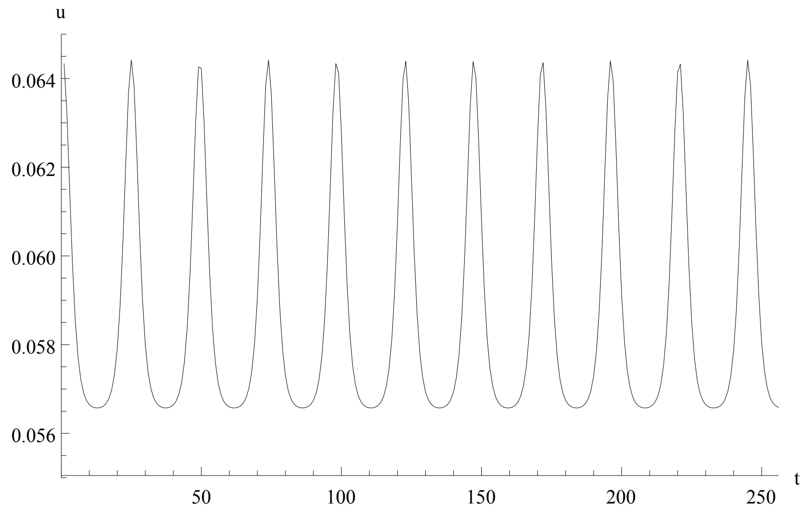

Figure 7: Simulated time series of unemployment.

Notes: 256 quarterly datapoints from the calibrated Beveridge cycle, connected by straight lines.

and result in each of these equilibria. By presenting a Bogdanov-Takens bifurcation, I show that a stable limit cycle - the Beveridge cycle - exists for a range of values for the workers' bargaining power enclosed by a Hopf and a saddle-loop bifurcation. I calibrate this Beveridge cycle to the average duration of the business cycle. The calibrated cycle looks qualitatively similar to the observed counterclockwise cycles in the *unemployment, vacancy rate*-plane. In addition, it can account for the persistence of unemployment and vacancies for plausible parameter values, but suffers from the same lack of amplification as the Pissarides (1985) model. However, volatility can be generated endogenously by the interplay of demand and congestion externalities.

A limitation of this study is that the range of parameter values that results in a limit cycle is small. Although my calibration focuses on this purely deterministic Beveridge cycle, for a much bigger set of parameter values a single shock can result in counterclockwise fluctuations that are able to explain many business cycles but eventually settle down into a steady state. Separation rate and productivity shocks provide a natural complement to the endogenous mechanism of this paper. On top of that, the indeterminacy of equilibrium allows for belief shocks, which directly result in another level of labor market tightness by the opening or closing of vacancies by firms. However, while additional exogenous shocks can result in more irregular time series than those generated by my calibration, they also provide additional degrees of freedom.

2.7 Appendix

2.7.A Data and calibration

I follow Shimer (2005) in the construction of the monthly job finding probability F_t and job destruction probability Δ_t. In particular,

$$F_t = 1 - \frac{u_{t+1} - u^s_{t+1}}{u_t},$$
$$\Delta_t = \frac{u^s_{t+1}}{e_t\left(1 - \frac{1}{2}F_t\right)},$$

where u^s denotes the short-term unemployment rate and e denotes the employment rate. Following Elsby et al. (2009), I inflate the short-term unemployment rate by 1.16 from January 1994 onwards to correct for changes in the way the CPS measures unemployment duration. These probabilities are subsequently transformed in job finding and job destruction rates according to

$$f_t = -\log(1 - F_t),$$
$$\delta_t = -\log(1 - \Delta_t),$$

respectively. I add three monthly rates to obtain the quarterly rate. Regressing the HP-filtered job finding rate on the HP-filtered vacancy-to-unemployment ratio results in an estimate of 0.46.

In my model, the elasticity of the job finding rate with respect to the vacancy-to-unemployment ratio is $\eta + (1-\eta)/\gamma$. I exploit the cyclicality in the job finding rate of non-participants (that by definition do not search) to isolate the elasticity of the matching function. I assume that non-participants and unemployed workers find jobs according to a matching function with the same parameters, but that the unemployed have a superior ranking. In particular, denoting the number of non-participants, unemployed workers and vacancies by N, U and V, respectively, the total number of matches in any period is given by $\mu_0 V^\eta (N + sU)^{1-\eta}$. Consistent with the model, the number of unemployed workers finding a job is given by $\mu_0 V^\eta (sU)^{1-\eta}$, which is not affected by the number of non-participants. Consequently, the number of non-participants finding a job is given by $\mu_0 V^\eta (N + sU)^{1-\eta} - \mu_0 V^\eta (sU)^{1-\eta} = \mu_0 V^\eta \left((N + sU)^{1-\eta} - (sU)^{1-\eta}\right)$. Note that unless $\eta = 0$, unemployed workers do create congestion for non-participants. This variant of Blanchard and Diamond (1994), similar in spirit to Blanchard and Diamond

(1989, p. 32), can be justified by the search effort of the unemployed that allows them to form all potential matches before non-participants arrive.

I take data on the job finding rate of non-participants from Elsby et al. (2015), using their classification error adjusted ("deNUNified") and time-aggregation adjusted hazard rates. These hazard rates are based on monthly gross worker flows, which the BLS provides from February 1990 onwards. The data from June 1967 and December 1975 were tabulated by Joe Ritter and made available by Hoyt Bleakley. This leaves a gap of fifteen years. This data was constructed by Robert Shimer. For additional details, please see Shimer (2012). Extending the series of Elsby et al. (2015) with recent data from the BLS, I have monthly job finding rates from 1967 to 2014.

A first-stage regression of the job finding rate of the *unemployed* from these same sources on the vacancy-to-unemployment ratio results in an elasticity of 0.45, virtually the same elasticity as for job-finding rate based on short-term unemployment available for 1951-2014. In my nonlinear second-stage regression of the job finding rate of non-participants I therefore impose that $\eta + (1-\eta)/\gamma = 0.45$. Moreover, I impose that the scale parameter of the matching function is the same, so that any differences in the *level* of the job finding rate of the unemployed from that of non-participants results from the search intensity and the superior ranking of the unemployed. Using the matching function with ranking above and the expression for optimal search intensity in (2.8), the logarithm of the job finding rate of non-participants is given by

$$\mu - \frac{1-\eta}{\gamma}c + \eta\log(V) - \log(N) + \log\left(\left(U\left(\frac{\mathrm{e}^c V}{U}\right)^{1/\gamma} + N\right)^{1-\eta} - \left(U\left(\frac{\mathrm{e}^c V}{U}\right)^{1/\gamma}\right)^{1-\eta}\right),$$

where μ is the estimated constant in the first-stage regression of the job finding rate of the *unemployed* on the vacancy-to-unemployment ratio, and where c is a constant of the second-stage regression that takes out the difference in the level of the job finding rates of the unemployed and non-participants that results from the search intensity of the former. Minimizing the sum of squared residuals under the restriction that $\eta + (1-\eta)/\gamma = 0.45$, results in $\eta = 0.0975$ and $\gamma = 2.537$. Rounding off at one decimal results in a combination of η and γ that is consistent with the estimated elasticity for the job-finding rate based on short-term unemployment available for 1951-2014: $0.1 + (1-0.1)/2.5 = 0.46$.

2.7.B Proofs

Proof of Proposition 2.2

Social welfare is given by

$$\int_0^\infty \mathrm{e}^{-rt}\left[(1-u_t)\,y+u_t(z-s_t^\gamma)-k\theta_t s_t u_t\right]dt. \tag{2.19}$$

The social planner maximizes this function by choosing both the socially efficient level of labor market tightness and search intensity, subject to the law of motion of unemployment given in (2.9). First-order conditions for the optimal θ_t and s_t, where μ_t denotes the co-state variable for the constraint on the dynamics of u_t, are

$$-\mathrm{e}^{-rt}\left[y-z+s_t^\gamma+k\theta_t s_t\right]+\mu_t\left[\delta+s_t\theta_t^\eta\right]-\dot{\mu}_t=0, \tag{2.20}$$

$$-\mathrm{e}^{-rt}ks_t u_t+\mu_t s_t u_t\eta\theta_t^{\eta-1}=0, \tag{2.21}$$

$$-\mathrm{e}^{-rt}\left[\gamma s_t^{\gamma-1}u_t+k\theta_t u_t\right]+\mu_t u_t\theta_t^\eta=0. \tag{2.22}$$

Both (2.21) and (2.22) can be rewritten to yield μ_t, so that

$$\mu_t=\frac{\mathrm{e}^{-rt}\theta_t k}{\eta\theta_t^\eta}=\frac{\mathrm{e}^{-rt}k\theta_t}{\eta\theta_t^\eta} \tag{2.23}$$

$$=\frac{\mathrm{e}^{-rt}\left[\gamma s_t^{\gamma-1}+k\theta_t\right]}{\theta_t^\eta}. \tag{2.24}$$

The efficient search intensity s_t is therefore given by

$$\frac{1-\eta}{\eta}k\theta_t=\gamma s_t^{\gamma-1}.$$

Comparing this expression with the privately chosen intensity in (2.8), search intensity is efficient if and only if $\beta=1-\eta$, the Hosios condition.

The expression for μ_t in (2.23) can be used to derive

$$\dot{\mu}_t=\frac{\mathrm{e}^{-rt}k\dot{\theta}_t-r\mathrm{e}^{-rt}k\theta_t}{\eta\theta_t^\eta}-\frac{\mathrm{e}^{-rt}k\theta_t\eta^2\frac{\theta_t^\eta}{\theta_t}\dot{\theta}_t}{\eta^2(\theta_t^\eta)^2}=\frac{\mathrm{e}^{-rt}k[(1-\eta)\dot{\theta}_t-r\theta_t]}{\eta\theta_t^\eta}. \tag{2.25}$$

Substituting (2.23) and (2.25) into (2.20) and rearranging, yields

$$\begin{aligned}
\frac{(1-\eta)k\dot{\theta}_t-rk\theta_t}{\eta\theta_t^\eta}&=\frac{k\theta_t\left[\delta+s_t\theta_t^\eta\right]}{\eta\theta_t^\eta}-\left[y-z+s_t^\gamma+k\theta_t s_t\right],\\
\Leftrightarrow\frac{(1-\eta)k\dot{\theta}_t}{\theta_t^\eta}&=\frac{k\theta_t\left[\delta+r\right]}{\theta_t^\eta}-\eta\left[y-z+s_t^\gamma-\frac{1-\eta}{\eta}k\theta_t s_t\right],\\
\Leftrightarrow\dot{\theta}_t&=\frac{\theta_t}{1-\eta}\left[\delta+r\right]-\frac{\eta\theta_t^\eta}{(1-\eta)k}\left[y-z+s_t^\gamma-\frac{1-\eta}{\eta}k\theta_t s_t\right].
\end{aligned}$$

Comparing this expression with the privately chosen tightness in (2.13), taking into account the definition of $g(\theta_t)$ in (2.6), labor market tightness is efficient if and only if $\beta = 1-\eta$.

Proof of Proposition 2.3

Proof. The second derivative with respect to θ_t of the tightness nullcline in (2.15) is

$$\frac{d^2 u_t}{d\theta_t^2} = -\frac{1}{\alpha}\left[\frac{(r+\delta)k\theta_t}{(1-\beta)\theta_t^{\eta}} + g(\theta_t) + z\right]^{\frac{1-\alpha}{\alpha}}\left[\frac{k\beta s^*(\theta_t)}{(\gamma-1)(1-\beta)\theta_t} - \frac{(r+\delta)k(1-\eta)\eta}{(1-\beta)\theta_t^{\eta}\theta_t}\right]$$
$$-\frac{1-\alpha}{\alpha^2}\left[\frac{(r+\delta)k\theta_t}{(1-\beta)\theta_t^{\eta}} + g(\theta_t) + z\right]^{\frac{1}{\alpha}-2}\left[\frac{(r+\delta)k(1-\eta)}{(1-\beta)\theta_t^{\eta}} + \frac{k\beta s^*(\theta_t)}{(1-\beta)}\right]^2.$$

One can see that for $\alpha \leq 1$, the *tightness* nullcline is concave at least on the segment of the nullcline for which

$$s^*(\theta_t)\theta_t^{\eta} > (r+\delta)(1-\eta)\eta(\gamma-1).$$

Define ξ as the job finding rate equal to $(r+\delta)(1-\eta)\eta(\gamma-1)$, and $\chi \equiv \eta + (\gamma-1)^{-1}$. Now note that the *unemployment* nullcline is convex or has the shape of a negative logistic function. In particular, differentiate (2.14) twice with respect to θ, to obtain

$$\frac{d^2 u_t}{d\theta_t^2} = \frac{\delta\chi\left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\theta_t^{\chi-2}\left[\delta(1-\chi) + (1+\chi)\left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\theta_t^{\chi}\right]}{\left[\delta + \left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\theta_t^{\chi}\right]^3}.$$

For $\chi \leq 1$ the second derivative is positive for all $\theta_t > 0$, so that the unemployment nullcline is convex. For $\chi > 1$, the second derivative can be positive or negative, depending on θ_t. More specifically, for $\chi > 1$ there exists a unique inflection point at the positive labor market tightness given by

$$\theta^* = \left[\frac{\delta(\chi-1)}{(1+\chi)\left(\frac{\beta}{1-\beta}\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}}\right]^{\frac{1}{\chi}}.$$

Consequently, for $\chi > 1$ the unemployment nullcline is concave for $0 < \theta_t < \theta^*$, and convex for all $\theta_t > \theta^*$, so that as a whole it has the shape of a negative logistic function.

Given that the unemployment nullcline is convex or negative logistic, if any steady state with economic activity exists, generically exactly two steady states with economic activity exist if the tightness nullcline lies below the unemployment nullcline for any potential non-concave segment of the former. In that case, the concave segment of the tightness nullcline intersects at most twice with the unemployment nullcline. A sufficient condition for any non-concave segment of the tightness nullcline to lie below the unemployment nullcline is the maximum unemployment rate giving rise to any vacancy creation ($u_{\theta=0}$ as given by (2.16)) to be lower than the unemployment rate consistent with the job finding rate ξ. Consequently, assuming the existence of a steady state in the positive quadrant, for $\alpha \leq 1$ generically exactly two steady states exist if

$$z > \left(1 - \frac{\delta}{\delta + \xi}\right)^{\alpha}.$$

□

Proof of Proposition 2.6

This proof and the next can be more concisely written after a change in coordinates from labor market tightness θ_t to match surplus p_t. To point out the similarities with Mortensen (1999), I also change u_t to n_t. Lemma 2.8 then first shows equivalence between Mortensen's system in p_t and n_t and the one presented here. It is proven by the recognition that there is a smooth one-to-one correspondence between employment and unemployment, and surplus and tightness respectively. Following the definition of Kuznetsov (2004, p. 42), two smooth systems $\dot{x} = \mu(x)$, $x \in \mathbb{R}^n$ and $\dot{y} = \nu(y)$, $y \in \mathbb{R}^n$ are not only topologically equivalent, but also smoothly equivalent if (1) an invertible map $f : \mathbb{R}^n \to \mathbb{R}^n$ exists such that $y = f(x)$, if (2) this map is smooth together with its inverse, and if (3) f can be used to change coordinates such that holds that $\mu(x) = M^{-1}(x)\nu(f(x))$, where $M(x) = df(x)/dx$ is the Jacobian matrix of $f(x)$ at x. As a result, f is not only a homeomorphism, but also a diffeomorphism.

Lemma 2.8. *The dynamical system in unemployment u_t and labor market tightness θ_t and Mortensen (1999)'s dynamical system in employment n_t and surplus p_t for $\beta(\theta_t) = \beta$ and a positive value of leisure z are smoothly equivalent for all equilibria with economic activity.*

Proof. My dynamical system in u_t and θ_t is for all equilibria with economic activity given by the two smooth differential equations in (2.13) and (2.9), for convenience reprinted below

$$\begin{aligned}\dot{\theta}_t &= (r+\delta)\frac{\theta_t}{1-\eta} + (1-\beta)\frac{\theta_t^\eta}{k(1-\eta)}\left[g(\theta_t) + z - (1-u_t)^\alpha\right],\\ \dot{u}_t &= \delta(1-u_t) - s^*(\theta_t)\, u_t\theta_t^\eta,\end{aligned}$$

with $u_t \in [0,1]$ and $\theta_t > 0$. The dynamical system of Mortensen (1999) extended with z follows from (2.7) and the definition of the labor force. For all interior equilibria, it is given by the following two smooth differential equations

$$\dot{p}_t = (r+\delta)p_t + g\left(p_t\right) + z - n_t^\alpha, \tag{2.26}$$

$$\dot{n}_t = h\left(p_t\right)(1-n_t) - \delta n_t, \tag{2.27}$$

with $p_t > 0$ and $n_t \in [0,1]$, and where $h\left(p_t\right) = s^*(\theta_t)\theta_t^\eta$ and $g\left(p_t\right) = g(\theta_t)$, for the invertible map defined by

$$p_t = \frac{k\theta_t}{(1-\beta)\theta_t^\eta}, \tag{2.28}$$

$$n_t = 1 - u_t. \tag{2.29}$$

Nash bargaining implies $J_t = (1-\beta)p_t$, so that the first equation follows from the free-entry condition in (2.1), while the second is true by definition. Both equations are smooth together with their inverses, so that they satisfy the second requirement as well. The Jacobian matrix of this diffeomorphism is given by

$$M(x) = \begin{pmatrix} \frac{k(1-\eta)}{(1-\beta)\theta_t^\eta} & 0 \\ 0 & -1 \end{pmatrix}.$$

If we apply the map in (2.28) and (2.29), then indeed

$$\begin{pmatrix} (r+\delta)\frac{\theta_t}{1-\eta} + (1-\beta)\frac{\theta_t^\eta}{k(1-\eta)}\left[g(\theta_t) + z - (1-u_t)^\alpha\right] \\ \delta(1-u_t) - s^*(\theta_t)\, u_t\theta_t^\eta \end{pmatrix} = \begin{pmatrix} \frac{(1-\beta)\theta_t^\eta}{k(1-\eta)} & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} (r+\delta)\frac{k\theta_t}{(1-\beta)\theta_t^\eta} + g(\theta_t) + z - (1-u_t)^\alpha, \\ s^*(\theta_t)\, u_t\theta_t^\eta - \delta(1-u_t) \end{pmatrix},$$

so that the two systems also satisfy the last of the three requirements. As a result, they are smoothly equivalent as long as $\theta > 0$. □

Lemma 2.8 shows that the two systems is the same system written in different coordinates, retaining the same eigenvalues of the corresponding equilibria and the same periods of the corresponding limit cycles (Kuznetsov, 2004, p. 42). I can thus prove Proposition 2.6 in n_t and p_t.

Proof. The system in (2.26) and (2.27) is characterized by Hamiltonian dynamics if the discount rate r is zero and the sharing rule is efficient. The Hamiltonian function is

$$H\left(p_t, n_t\right) = \int_0^{n_t} \phi(x)dx + (1-n_t)[g\left(p_t\right) + z] - \delta p_t n_t, \tag{2.30}$$

as can be checked by noting that $\partial H/\partial p_t = \dot{n}_t$ and $\partial H/\partial n_t = -\dot{p}_t$ for $\beta = 1-\eta$ and $r = 0$. Indeed, remember that $h\left(p_t\right) = s^*\left(\theta_t\right)\theta_t^{\eta}$ and $g\left(p_t\right) = g\left(\theta_t\right)$ for the map in (2.28), so that $g\left(p_t\right) = \int_0^p h(q)dq$.

Although a homoclinic orbit generically exists in this Hamiltonian system, for $z > 0$ part of this homoclinic orbit may fall outside the positive quadrant. Define $n_{\theta=0} \equiv 1 - u_{\theta=0}$ as the employment level at the intersection of the tightness null-cline with the unemployment axis (thus with $u_{\theta=0}$ as defined in (2.16)). Note that $(1-u_H+\alpha z^{1/\alpha})^{\alpha+1} < (1-\alpha)\left[(1-u_H)\left(z + k\theta_t^{1-\eta}/(1-\beta)\right) - u_H g\left(\theta_H\right)\right]$ is equivalent to $H\left(p_H, n_H\right) < H\left(0, n_{\theta=0}\right)$. If and only if the latter holds, the homoclinic orbit is entirely situated in the positive quadrant. Because in a Hamiltonian system all equilibrium paths are level curves, combinations of n and p on the homoclinic orbit have the same value of the Hamiltonian as the saddlepoint on it. The laws of motion in (2.26) and (2.27) show that the antisaddle L is a local minimum in the system. Moving along the continuum of surrounding closed orbits, the largest possible closed orbit in the positive quadrant lies on $n_{\theta=0}$. Consequently, with $H\left(p_H, n_H\right) < H\left(0, n_{\theta=0}\right)$, a homoclinic orbit connecting H to itself lies entirely in the positive quadrant.[18]

For a small perturbation towards positive discounting and a smaller than efficient β two steady states with economic activity continue to exist by continuity. Melnikov perturbation (see e.g. Guckenheimer and Holmes (1983, p. 184)) shows that the same

[18]Substituting (2.16) into (2.7) it can be checked for $z, \alpha > 0$ that $H\left(0, n_{\theta=0}\right) < H(0,0) = z$, so that $H\left(p_H, n_H\right) < H\left(0, n_{\theta=0}\right)$ implies $H\left(p_H, n_H\right) < H(0,0)$. The latter ensures that the saddlepoint on the homoclinic orbit is steady state H rather than the no-trade steady state.

holds for the homoclinic orbit. The differential vector system allowing for a small distortion such that $r > 0$ and $\beta < 1-\eta$ is defined by

$$\dot{x}_t = F(x_t) + \varepsilon G(x_t) \text{ with } x_t = \begin{pmatrix} p_t \\ n_t \end{pmatrix},$$

$$F(x_t) = \begin{bmatrix} F_1(x_t) \\ F_2(x_t) \end{bmatrix} = \begin{bmatrix} -\frac{\partial H(p_t,n_t)}{\partial n_t} \\ \frac{\partial H(p_t,n_t)}{\partial p_t} \end{bmatrix} = \begin{bmatrix} \delta p_t + \int_0^{p_t} h(q)dq - n_t^{\alpha} + z \\ h(p_t)[1-n_t] - \delta n_t \end{bmatrix},$$

$$G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix} = \begin{bmatrix} rp_t + g(p_t) - \int_0^{p_t} h(q)dq \\ 0 \end{bmatrix},$$

where ε is a small positive number. $F(x_t)$ is the Hamiltonian vector field, and $\varepsilon G(x_t)$ is a perturbation attributable to positive discounting and a smaller than efficient bargaining power.

Because the perturbation is time independent, the Melnikov function $M(p_t, n_t)$ is simply

$$M(p_t, n_t) = \int_{\Gamma} \left[r + h(p_t)\left(\frac{\beta}{1-\eta} - 1\right) \right] dp_t dn_t,$$

where $\Gamma = \{x \in \mathbb{R}^2 | H(x) \le H(p_H, n_H)\}$ is the area enclosed by the homoclinic orbit in the Hamiltonian system. Note that the Melnikov function is independent of ε. Now $\beta_{SL} < 1-\eta$ can be chosen to target any sufficiently small $r = \hat{r} > 0$ with

$$\hat{r} = \frac{\int_{\Gamma} \left[h(p_t)\left(1 - \frac{\beta_{SL}}{1-\eta}\right) \right] dp_t dn_t}{\int_{\Gamma} dp_t dn_t}.$$

For $r = \hat{r}$ the Melnikov function has a simple zero at β_{SL}, so that for a sufficiently small distortion a homoclinic orbit in p_t and n_t continues to exist and remains in the positive quadrant. By Lemma 2.8 the same must hold for the system in θ_t and u_t.

According to the Andronov-Leontovich theorem, a family of limit cycles bifurcates on one side of this homoclinic orbit, and these are stable if the trace of the Jacobian matrix at saddlepoint H is negative. Because the homoclinic orbit is proven for a perturbed Hamiltonian system, the trace is only based on the distortion, and is simply equal to ε times the integrand of the Melnikov function at H:

$$\text{tr}(H) = \varepsilon \left[r + h(p_H)\left(\frac{\beta}{1-\eta} - 1\right) \right].$$

Given that $\beta < 1-\eta$, the integrand of the Melnikov function is monotonically decreasing in p_t. Consequently, when the Melnikov function is zero, the tr(H) is negative for values of β in the neighborhood of β_{SL}, so that the limit cycles are stable. □

Proof of Proposition 2.7

Proof. Because the proof is more concisely written in surplus than in tightness, I present it for Mortensen (1999)'s system extended with $z > 0$. Remember that by Lemma 2.8 the two systems are smoothly equivalent so that the eigenvalues are the same. The nullclines of the dynamical system in (2.26) and (2.27) are

$$(r+\delta)p_t + g(p_t) + z = (n_t)^\alpha \tag{2.31}$$

$$n_t = \frac{h(p_t)}{h(p_t)+\delta}, \tag{2.32}$$

and its nonzero Jacobian matrix is

$$J = \begin{pmatrix} r+\delta+g'(p_t) & -\alpha\frac{(n_t)^\alpha}{n_t} \\ h'(p_t)(1-n_t) & -h(p_t)-\delta \end{pmatrix}.$$

Both eigenvalues are zero if and only if both the determinant and the trace are zero, so that

$$\text{tr} = r + g'(p_t) - h(p_t) = 0, \tag{2.33}$$

$$\det = \alpha\frac{(n_t)^\alpha}{n_t} h'(p_t)(1-n_t) - (r+\delta+g'(p_t))(h(p_t)+\delta) = 0. \tag{2.34}$$

Remember that $g(p_t) = \beta/(1-\eta)\int_0^{p_t} h(q)dq$, and moreover that $h(p_t) = s^*(\theta_t)\theta_t^\eta$ so that using the map in (2.28) $h'(p_t)p_t/h(p_t) = (1-\eta+\eta\gamma)/((1-\eta)(\gamma-1)) \equiv \kappa$. Substituting (2.33) and the elasticities into (2.34) yields

$$\alpha\frac{(n_t)^\alpha}{n_t}\kappa\frac{h(p_t)}{p}(1-n) = (h(p_t)+\delta)^2.$$

Substituting the nullclines of (2.31) and (2.32),

$$\alpha\kappa\delta\frac{(r+\delta)p_t + g(p_t) + z}{p_t} = (h(p_t)+\delta)^2.$$

Consequently, both eigenvalues are non-degenerately zero in steady state if the function

$$B(p_t) = (h(p_t)+\delta)^2 - \alpha\kappa\delta\left[r+\delta+\frac{h(p_t)-r}{1+\kappa}+\frac{z}{p_t}\right], \tag{2.35}$$

has a simple zero. Because $\lim_{p_t\to\infty} B(p_t) = \infty$, $\lim_{p_t\to 0} B(p_t) = -\infty$, and $B(p_t)$ is continuous, this condition is satisfied by the Intermediate Value Theorem. Moreover, for $\alpha < 1$ the condition is satisfied only once, because $B'(p_t) > 0$ if $2 > \alpha\kappa/(1+\kappa)$.[19] □

[19]The same holds if $z = 0$, but then $\lim_{p_t\to 0} B(p_t) = \delta^2 - \alpha\delta\kappa(\delta + \kappa r/(\kappa+1))$, so that (2.35) can only be zero for a sufficiently large α and κ. A sufficient condition met by Mortensen's numerical example and my calibration is $\alpha\kappa > 1$.