More than the sum of its parts: compact preference representation over combinatorial domains

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Chapter 7
Voting

7.1 Introduction

The problem of electing a committee which satisfies as many voters as possible is one for which good solutions are scarce. There have been numerous attempts to devise methods for committee election, some which have their origins in single-winner voting methods [Brams, Kilgour, and Sanver, 2004, 2006, 2007], others of which are intended to produce outcomes which are proportional in some way [Chamberlin and Courant, 1983; Monroe, 1995]. Single-winner voting systems frequently fail to respect voter preferences when extended to a multi-winner setting, due mainly to the fact that they deny voters the ability to express interdependence among candidates. Moreover, the way in which such systems measure the “representativeness” of committees may not be at all similar to the way in which voters measure it.

In order to tackle the interdependence problem, we propose a voting method which uses goalbases as ballots, in the spirit of combinatorial vote as proposed by Lang [2004].

We begin in Section 7.2 by recalling a bit of voting theory. We present some known methods for committee selection in Section 7.3.1, and then in Section 7.3.2 find fault with these due to their lack of expressive power. In Section 7.4 we will show how the election methods presented earlier can be simulated and extended using goalbases, and in Section 7.5 we consider the computational complexity of finding winning committees using this method. In Section 7.6, we give an example of extending approval voting using goalbases as ballots, and in Section 7.7 we touch on several avenues for further investigation.

7.2 Background

Voting theory, as its name implies, deals with the formal properties of systems of voting. Prior to the 18th century, there were few, isolated attempts to study
voting systems, two of which resulted in the discovery of the Borda count: In the 13th century, Ramon Llull devised an iterative version of it to be used for selecting the abbess of a monastery [Llull, 1926], and in 1433 Nicholas of Cusa proposed it for electing the Holy Roman Emperor [Hägele and Pukelsheim, 2008]. It was not until the two decades prior to the French Revolution that members of the French Academy—Jean-Charles de Borda and Nicolas de Condorcet, in particular—began a sustained and systematic study of voting methods. Since then, there has been extensive investigation of voting rules and their properties. We recount here only as much as we need for the present chapter. For much, much more on the subject, see [Taylor, 2005].

In voting theory, the dramatis personae are a set of candidates and a set of voters. The voters cast ballots indicating their preferences over the candidates; the ballots are fed as input to a voting rule, the output of which indicates which candidate or candidates are winners. Numerous voting rules have been devised. Here we describe some which may be familiar, and others which figure in our discussion later in this chapter.

Some voting rules solicit relatively little preference information from voters:

**unanimity** Each voter may cast at most one vote for one candidate. A candidate is a winner iff every voter selects that candidate.

**plurality** Each voter may cast at most one vote for one candidate. A candidate is a winner iff no other candidate receives more votes.

**approval** Each voter may cast at most one vote for each candidate. A candidate is a winner iff no other candidate receives more votes.

The unanimity and plurality rules ask voters only to register their top choice, while approval voting permits voters to make no finer distinction than preferred versus nonpreferred. The plurality rule is the familiar first-past-the-post rule used in virtually all elections in the United States. Historically, a complex iterated version of approval voting was used during 1268–1789 by the Venetians to elect their doge [Lines, 1986], and another iterated version was used to elect new popes by the papal conclaves held between 1294 and 1621 [Colomer and McLean, 1998]. Modernly, numerous professional societies\(^1\) have adopted approval voting for their elections, and the Secretary-General of the United Nations is elected by approval voting [Brams, 2007, Sections 1.2–1.4]. Unanimity is not practical for large groups—Poland’s Sejm (diet) demonstrated this during the latter half of the

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\(^1\)Among them: the Mathematical Association of America (MAA), the American Mathematical Society (AMS), the Institute for Operations Research and Management Sciences (INFORMS), the American Statistical Association (ASA), the Institute of Electrical and Electronics Engineers (IEEE), the Public Choice Society, the Society for Judgment and Decision Making, the Social Choice and Welfare Society, the European Association for Logic, Language and Information, the Game Theory Society, the Econometric Society, and the International Joint Conference on Artificial Intelligence (IJCAI).
17th century [Roháč, 2008]—though a similar rule is frequently used in criminal trials, where the jury or panel of judges are the voters and the “candidates” are guilty and innocent.

Other voting rules require voters to supply full preference orders over the candidates:

**Condorcet** Each voter gives a strict linear order over the candidates. A candidate \( c \) is a winner iff for each other candidate \( c' \), a majority of voters rank \( c \) above \( c' \) (\( c > c' \)).

**Borda** Each voter gives a strict linear order over the candidates. From each ballot, a candidate \( c \) receives one point for each other candidate \( c' \) above whom he is ranked. A candidate is a winner iff no other candidate scores more points.

The Condorcet and Borda rules incorporate much more of the voters’ preference information into their results than do unanimity, plurality, and approval.

The plurality and Borda rules are instances of a class of voting rule known as positional scoring rules. A positional scoring rule is one where voters submit strict linear orders and ballots are scored using a scoring vector \( \langle s_1, s_2, \ldots, s_{n-1}, s_n \rangle \). A candidate ranked \( i \)-th by a voter receives \( s_i \) points; the winner(s) are the highest-scoring candidate(s). The scoring vector for the plurality rule is \( \langle 1, 0, \ldots, 0 \rangle \), while for Borda it is \( \langle n - 1, n - 2, \ldots, 1, 0 \rangle \). We need not require that all voters use the same scoring vector. General scoring rules are ones where each ballot induces a score for each candidate, and winners may be determined by summing candidate scores across all ballots. (That is, positional scoring rules use one scoring vector globally, while general scoring rules permit each ballot to have its own scoring vector.) Approval voting is a general scoring rule, but not a positional one; the Condorcet rule is not a scoring rule at all.

The Borda rule does not preserve any intensity information which a voter might provide. Borda treats a voter who strongly prefers candidate \( a \) to candidate \( b \) the same as one who has a slight preference for \( a \) over \( b \). A rule which we will return to later in this chapter, known as cumulative voting, preserves intensity of preference by asking voters to express cardinal rather than ordinal preferences:

**cumulative** Each voter is given \( k \) points which may be distributed among the candidates. A candidate is a winner iff no other candidate scores more points.

Cumulative voting is a general scoring rule, but it is not a positional scoring rule, since each voter is free to choose his own scoring vector. (If we take the ordering of the candidates as fixed by their order of appearance on the ballot, then a cumulative ballot essentially is a scoring vector.)

We now move on to some properties of voting rules. The Condorcet rule is rather weak, in the sense that it is easy to devise situations in which it will produce
no winners; however, this means that when a candidate is the winner according to the Condorcet rule—known as the Condorcet winner—then that candidate is quite strong. By definition, a Condorcet winner would defeat every other candidate in a head-to-head vote. Always electing a Condorcet winner, if one exists, is an intuitively desirable property for a voting rule to have, and so this is one property for which new voting rules are always examined. (Similarly, a Condorcet loser is a candidate who would lose every head-to-head vote, and we would also like for our voting rules never to elect a Condorcet loser.) Unfortunately, many voting rules fail to elect the Condorcet winner in some circumstances: Young [1975] proved that every positional scoring rule will sometimes fail to elect the Condorcet winner.

Other properties of interest are resoluteness, anonymity, neutrality, monotonicity, unanimity, non-imposition, Pareto, and strategyproofness. A rule is resolute if it always chooses a single winner. A rule is anonymous if all voters are treated the same; a rule is neutral if all candidates are treated the same. A rule is monotone if winners continue to be winners if their ranking on some ballot improves. A rule is unanimous if, when all voters have the same candidate as their first choice, that candidate wins; a rule is non-imposing when every candidate has some configuration of ballots which would cause him to win. A rule is Pareto if there is never a candidate which all voters prefer to the winner. A rule is strategyproof if voters have no incentive to misrepresent their preferences; rules which are not strategyproof are said to be manipulable.

Finally, we mention one more voting rule, one which plays a much larger role in voting theory than most voting theorists would like:

**dictatorship** Each voter may cast at most one vote for one candidate. A candidate is a winner iff the voter predesignated as the dictator votes for that candidate.

It is an unfortunate fact that the preponderance of results in voting theory are negative, sometimes of the form:

> Any voting rule which satisfies desirable properties $X_1, \ldots, X_k$ when there are at least three candidates is a dictatorship.

Famous instances of this include Arrow’s Theorem [Arrow, 1970] and the Gibbard-Satterthwaite Theorem [Gibbard, 1973; Satterthwaite, 1975]: For Arrow, the properties are Pareto and independence of irrelevant alternatives; for Gibbard-Satterthwaite the properties are resoluteness, non-imposition, and strategyproofness.

There are many other voting rules not mentioned here, as well as many other properties of interest. For a thorough overview of voting rules, see [Brams and Fishburn, 2002].

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2While Arrow’s Theorem is usually stated for social welfare functions, the version for voting rules is equivalent. See [Taylor, 2005, Section 3.4].
7.3 Multi-Winner Elections

The voting rules mentioned in the previous section are generally intended to be used for electing single winners, despite that some of them will occasionally produce ties. Less studied are voting rules intended for the election of multiple winners. In this section, we discuss some of the challenges associated with multi-winner elections.

7.3.1 Some Methods for Committee Election

Various methods for committee elections have been proposed. (For a general discussion of the difficulties of committee elections, see [Chevaleyre, Endriss, Lang, and Maudet, 2008b] and [Lang and Xia, 2009].) The naïve (and perhaps for that reason, most widely used) approach is an extension of the single-vote plurality method. With \( k \) seats to fill from a slate of \( n \) candidates, each voter may cast up to \( k \) votes, no more than one vote per candidate, and the top \( k \) candidates win. A similar naïve extension of approval voting to a multi-winner setting is possible: Again with \( k \) seats to fill from a slate of \( n \) candidates, each voter may cast up to \( n \) votes, no more than one per candidate, and again the top \( k \) candidates win. These two methods lie along a spectrum of voting methods where the maximum number of votes cast per voter is varied—the approval version anchoring one end, and a single-vote top-\( k \) method anchoring the other. We now give a formal definition:

**Definition 7.3.1** \((m\text{-vote, top-}k)\). Call a voting method \( m\text{-vote} \) if each voter may cast single votes for up to \( m \) candidates, and top-\( k \) if the \( k \) candidates receiving the most votes are the winners.

Standard plurality voting as used in many elections for public office is a 1-vote top-1 method.

Top-\( k \) methods all share a feature which makes them rather unsuitable for committee elections, namely that they tend to quash minority representation.

**Theorem 7.3.2.** If there are \( v \) voters in an \( m\text{-vote top-}k \) election, then a coordinated block of \( \left\lfloor \frac{v}{2} + 1 \right\rfloor \) voters is sufficient to dictate the top \( m \) candidates.

**Proof.** Strategy: \( \left\lfloor \frac{v}{2} + 1 \right\rfloor \) voters cast votes for the same \( m \) candidates, \( c_1, \ldots, c_m \). Result: Each \( c_i \) must receive at least \( \left\lfloor \frac{v}{2} + 1 \right\rfloor \) votes, and no other candidate can receive more than \( \left\lfloor \frac{v}{2} - 1 \right\rfloor \) votes, thus ensuring that \( c_1, \ldots, c_m \) are the top \( m \) vote-getters. \( \square \)

(Note that while the coordinated block of voters can dictate the top \( m \) candidates, it cannot dictate order among them: If the majority block skimps on votes for one of its candidates and the minority block is also coordinated, one of the majority’s \( m \) candidates could tie with one or more candidates receiving votes
only form the minority block. Hence the votes which determine the order among the top \( m \) all come from voters outside of the majority block.)

If representation of minority views on a committee is important, this fact displays a flaw in \( m \)-vote top-\( k \) committee voting: When \( m \geq k \), a majority block essentially has veto power over candidates. Only candidates supported by the majority block will receive seats. When \( m < k \), the majority block are guaranteed to have all of their \( m \) candidates on the committee, which may allow voters outside that block to succeed in electing candidates as well, but setting the number of votes each voter may cast to be less than the number of seats available has its own drawbacks in terms of permitting voters to express their preferences. The fewer votes a voter is permitted to cast, the less information is gathered from his preference order by the voting method, and moreover, we run the risk of collecting misleading preference information. For example, if we want to fill five seats but permit only three votes per voter (i.e., we are using a 3-vote top-5 method), then it is hard to predict how voters will behave. Will a voter cast a ballot for the three candidates he thinks best, or for the three-candidate subcommittee he thinks best, or for something else? It’s easy to envision a situation in which a voter’s preferred three-candidate subcommittee is not a part of the same voter’s preferred five-candidate full committee.

We must not lose sight of the fact that committees are not just elected, but elected for some purpose. Often, a committee—rather than an individual—is chosen to carry out some task so that the diversity of views held by the committee members may be brought to bear on the given task, or in order to have a decision-making body which is representative of the voters as a whole, and not just some subset of them. An \( m \)-vote top-\( k \) voting method will fail to produce a diverse or representative committee in the face of a coordinated majority block unless that majority block is itself committed to producing a diverse or representative committee. As we cannot hope for this in all but the most collegial circumstances, if we want a diverse or representative committee, then we should not elect our committee using an \( m \)-vote top-\( k \) method.

We have seen that outcomes of \( m \)-vote top-\( k \) elections are dictated by majority blocks regardless of what \( m \) and \( k \) are. The alternatives which we will now consider vary the winning criterion instead of the number of votes each voter may cast. Brams et al. [2004] describe what is known as the “minisum” method. Voters cast ballots as in approval voting, but we do not declare the top \( k \) vote-getters to be the winners—candidates do not win individually. Instead, winning committees are ones which minimize the sum of the Hamming distances to the votes cast, hence the name “minisum”. More precisely:

Consider each ballot as a binary vector \( b_1 \ldots b_n \), where \( b_i = 1 \) if the voter casting the ballot approves of candidate \( i \) and \( b_i = 0 \) otherwise. The Hamming distance \( H \) between two ballots is the least number of bits which must be flipped to transform one ballot into the other. (For example, \( H(01010,01101) = 3 \).) Thus
we can state the minisum rule as
\[
c \text{ is a winner } \iff \forall c' \in C: \sum_{b \in B} H(c, b) \leq \sum_{b \in B} H(c', b),
\]
where \( C \) is the set of \( k \)-seat committees and \( B \) is the multiset of ballots cast, both with their members expressed as binary vectors. (Here, we say \( c \in C \ldots \) because the minimum need not be unique.)

Intuitively, any winning committee is one which is as similar in membership as it can be to as many ballots as it can be. However, this intuition is wrong. Brams et al. [2007, Appendix, Proposition 4] show that any \( k \)-candidate minisum solution will consist of the \( k \) candidates receiving the most approval votes. Hence, we have the following:

**Theorem 7.3.3.** The voting methods \( m \)-vote \( k \)-minisum and \( m \)-vote top-\( k \) are equivalent.

Thus the minisum method, despite that it *sounds* more accommodating, is no more promising for electing representative committees than any of the \( m \)-vote top-\( k \) plurality methods are.

Brams et al. [2007] suggest minimax as an alternative to minisum. Rather than selecting committees which minimize the sum of Hamming distances to the ballots, minimax minimizes the maximum Hamming distance to any ballot. Formally, the minimax rule is
\[
c \text{ is a winner } \iff \forall c' \in C: \max_{b \in B} H(c, b) \leq \max_{b \in B} H(c', b),
\]
where, as before, \( C \) is the set of \( k \)-seat committees and \( B \) is the multiset of ballots cast, both with their members expressed as binary vectors.

The intuitive effect of minimax as a winning criterion is that it antagonizes outliers the least; or, rather, it antagonizes the farthest outlier the least. This gives rise to the following feature of the pure minimax criterion: The exact number of voters casting any particular ballot is irrelevant to the outcome; only whether particular ballots are cast matters. If one incorrigible voter casts the ballot 00101 while all others choose 11101, then it makes no difference if there are two, ten, or a million ballots cast in total—the winning three-member committees are 10101 and 01101. Usually this is not a desirable feature, though it arguably is appropriate in some circumstances (e.g., multilateral treaty negotiation, as described by Brams et al. [2004]). Brams et al. [2007] suggest a useful refinement which avoids this problem without reintroducing the tyranny of the majority, namely the addition of *proximity weights*.

The proximity weight \( w_b \) of a ballot \( b \) is defined as
\[
w_b = \frac{m_b}{\sum_{b' \in B} m_{b'} H(b, b')},
\]
where \( m_b \) is the number of voters casting ballot \( b \). The minimax criterion with proximity weights becomes

\[
\text{c is a winner } \iff \forall c' \in C: \max_{b \in B} w_b H(c, b) \leq \max_{b \in B} w_b H(c', b).
\]

Weighting the Hamming distance to a ballot by its proximity to other ballots diminishes the influence of outliers while at the same time reintroducing the proportionality which pure minimax lacks.

Many other weightings are possible, but these are representative, and sufficient to illustrate our point in the next section.

### 7.3.2 Similar Committees Need Not Be Similarly Preferable

As Brams et al. [2006, pp. 83–84] say, ‘[w]e view the problem of identifying the most representative committee as that of identifying the subset that is “closest” to the collection of subsets specified by the voters.’ In \( m \)-vote top-\( k \) methods, proximity is tied to support of individual candidates; the minisum and minimax criteria equate proximity with average and maximum Hamming distance, respectively. Taking the pure minimax criterion as our example, it is not hard to see that the intent is to minimize the dissatisfaction of the farthest outlier, while the rule is to minimize the dissimilarity between the farthest outlier’s ballot and the winning committee. But why should we suppose that the farthest outlier (or any voter, for that matter) actually cares about his ballot’s similarity to the winning committee? As we shall see now, it is quite reasonable to think that for many voters their committee preferences will not track the Hamming distance at all.

Taking the Hamming distance as a measure of similarity, we have the following two properties: If \( c \) is a voter \( v \)’s preferred committee, then

- any substitution of \( n \) members in \( c \) is strictly better according to \( v \) than every substitution of \( m \) members, for \( n < m \), and
- all committees \( c' \) which are Hamming-equidistant from \( c \) are equally preferred by \( v \),

both of which are dubious when applied to voters electing real committees.

For example, suppose that we are electing a three-seat committee from the five candidates Alice, Bob, Charlie, Dave, and Elaine. Suppose further that one of the voters believes that

- Alice and Bob are the best candidates, so any committee with one of them is better than any committee with neither,
- Alice and Bob will fight if they are on the committee together, so any committee with both is worse than any committee with neither,
Figure 7.1: Order on ballots induced by Hamming distance from 10110.

Figure 7.2: A more realistic order on ballots.
and that this voter is otherwise indifferent among potential committees.

Thus our voter ranks the committees in preference order as

$$ACD, ACE, ADE, BCD, BCE, BDE > CDE > ABC, ABD, ABE$$

as seen in Figure 7.2. This preference ordering is sensitive to small changes in committee composition. Each of the best committees is only one substitution away from some worst committee. (Notice that this is neither due to the size of the committee nor to the number of candidates, but to the way that two candidates interact in our example voter’s preferences. We use a three-seat committee with five candidates only to keep the example manageable.) Put another way, the Hamming distance between some pairs of best and worst committees is 2, which is always the minimum Hamming distance between two committees of the same size.\(^3\) From our voter’s point of view, $ACD \sim BDE > ABC$; but $H(10110, 01011) = 4$ while $H(10110, 11100) = 2$, and so the ordering induced by the Hamming distance from $ACD$ is $ACD > ABC > BDE$, thus putting an optimal committee last and one of our voter’s least favored committees in second place. (Cf. Figures 7.1 and 7.2.) If we use a minisum or minimax procedure and have many voters with preferences like this one, we risk an outcome that is similar in composition to voters’ first choices, yet is widely disliked.

The problem we have identified is that the question of committee membership for one candidate is not necessarily independent of the question of committee membership for some other candidate. In the language of utility functions, some voters have nonmodular preferences. (Benoit and Kornhauser [1991, 1994, 2006] identify a similar problem with the election of representative assemblies—not only might a voter have complex preferences over the composition of the assembly, but preferences over candidates for his district might depend on or involve candidates for other districts where he isn’t even able to cast a vote.) It could be argued that the problem is caused not by the voting methods we examined, but rather because of the way they are applied: The candidates are individuals rather than committees. If committees were raised to the status of first-class citizens—that is, if voters were to vote for whole committees rather than for individuals—perhaps we would not have this problem. However, this approach is unhelpful implemented one way, and scales poorly implemented another. Brams et al. [2007] report that the 2003 Game Theory Society election filled 12 seats from a slate of 24 candidates, giving 2704156 possible committees for voters to consider. This is still manageable if the voter is queried for his most preferred committee only—but then we are left with no information about candidate interdependence, and so we are no better off than before. It is also likely that no two voters will share the same committee

\[^3\] If $c, c'$ are distinct $k$-member committees, then the least difference there could be is that $c'$ is $c$ but with one member replaced. Since committees are represented as bit vectors, a single-member substitution turns one ‘on’ bit off and one ‘off’ bit on. Hence the Hamming distance between two distinct committees is always even and at least 2.
as their first choice, so the result will be a many-way tie. If we ask voters for rankings, we begin to sink into the combinatorial morass: Voters would balk at ranking their top 0.001% of the possible committees let alone all 2.7 million of them, and even if the voters were able to rank all of the possible committees the vote tabulator would be overwhelmed by the data.

A more general argument for the unsuitability of single-candidate voting systems for electing committees can be given, by way of comparing how many voter profiles single-candidate voting methods can accommodate. The most expressive voting method considered above is approval voting, where every subset of candidates is a valid ballot. (All other methods mentioned restrict the set of valid ballots to a proper subset of the powerset of candidates.) In comparison, there are

\[ \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} \binom{n}{i} \]

distinct voter profiles over \( k \)-seat committees chosen from \( n \) candidates, which, for any useful value of \( n \), dwarfs the \( 2^n \) distinct approval ballots.\(^4\) Taking the Game Theory Society election as our example, the largest term in the sum is \( \binom{24}{12} = 2704156 \), which is rather large.\(^5\) Clearly, we need a different approach.

### 7.4 Simulating Voting Methods Using Goalbases

Many common election methods may be simulated by using goalbases as ballots, summing them to get a single goalbase, and then using some method for finding optimal models over the resulting goalbase.

Recall the goalbase summation operator \( \oplus \) (see Definition 2.2.10). Suppose that there are voters \( 1, \ldots, n \). Then we can straightforwardly simulate the following voting methods:

- **plurality:** Each voter casts a single vote, with the candidate(s) receiving the most votes as the winner(s). Define a goalbase \( G_i = \{(c_j, 1)\} \) for each voter \( i \), where \( c_j \) is the candidate for which \( i \) casts his vote. Then find an optimal model for \( G_1 \oplus \ldots \oplus G_n \) considering single-atom models only. Alternatively, let \( G_i = \{(c_j \land \bigwedge_{j \neq k} \neg c_k, 1)\} \) and place no constraint on models.

\(^4\)A set of size \( n \) may be partitioned into \( k \) nonempty subsets \( \binom{n}{k} \) ways (where \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \) denotes a Stirling number of the second kind [Graham, Knuth, and Patashnik, 1994, Section 6.1]), and in each case these \( k \) subsets may themselves be strictly ordered in \( k! \) ways. Thus the number of (not necessarily strict) linear orders on \( n \) items is \( \sum_{k=1}^{n} k! \binom{n}{k} = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \).

\(^5\)For comparison, Haub [2002] estimates that 106 billion people had ever lived as of the year 2002. The profile space is more than adequate to permit every person who has ever lived a unique opinion on the 2003 Game Theory Society committee election.
• **unanimity**: Each voter casts a single vote. Any candidate receiving all \( n \) votes is the unique winner; otherwise, all candidates tie. Define \( G_i \) as for plurality, and find an optimal model having \( n \) utility, considering single-atom models only.

More generally, we can simulate the following parametrized voting rule:

- **\( m \)-vote top-\( k \)**: Each voter casts up to \( m \) votes. The \( k \) candidates receiving the most votes win. Define the goalbase \( G_i = \{(c, 1)\} \) where \( V_i \) is the set of \( m \) or fewer candidates favored by voter \( i \). Find an optimal model for \( \bigoplus_i G_i \), considering \( k \)-sized models only.

Many other election procedures mentioned by Taylor [2005], such as near-unanimity, omninomination, dictatorship, and oligarchy, may also be simulated in this fashion. In fact, we will now show that any **positional scoring rule** may be simulated using goalbases as ballots.

First, we need some notation for referring to (sets of) optimal models.

**Definition 7.4.1 (Optimal Models).** Given a goalbase \( G \), define \( \text{opt}(G) \) and \( \text{opt}_k(G) \) to be the sets such that

\[
\text{opt}(G) = \arg\max_{M \subseteq PS} u_G(M), \\
\text{opt}_k(G) = \arg\max_{M \subseteq PS, |M| = k} u_G(M).
\]

Clearly,

\[
\text{opt}(G) = \arg\max \left\{ u_G(M) \bigg| M \in \bigcup_{k=0}^{|PS|} \text{opt}_k(G) \right\},
\]

because each model in the set of optimal models must also be an optimal model among the models of its own size. Note also that the problem of generating some member of the set \( \text{opt}(G) \) is an instance of the function problem corresponding to the decision problem \( \text{MAX-UTIL} \) (see Definition 5.3.1).

The next theorem shows that every single-winner positional scoring rule may be simulated by casting goalbases as ballots and finding the set of utility-maximizing models:

**Theorem 7.4.2.** Let \( V \) be a single-winner (possibly nonresolute) positional scoring rule with scoring vector \( \langle s_1, \ldots, s_m \rangle \). Let \( b_1, \ldots, b_n \) be a sequence of ballots where \( \text{rank}_i(j) \) is the rank given to candidate \( j \) (denoted by \( c_j \)) by ballot \( b_i \). Let \( G_i = \{(c_j, s_{\text{rank}_i(j)}) \} \) where \( 1 \leq j \leq m \). Then \( V(b_1, \ldots, b_n) = \text{opt}_1(\bigoplus_{i=1}^{n} G_i) \).

**Proof.** Let \( S(x) \) be the score of candidate \( x \) under rule \( V \) with ballots \( b_1, \ldots, b_n \). If \( x \in V(b_1, \ldots, b_n) \), then since \( V \) is a positional scoring rule, this implies that \( S(x) \geq S(y) \) for all \( y \in A \). Similarly, if \( \{x\} \in \text{opt}_1(\bigoplus_{i=1}^{n} G_i) \), then \( u_{\bigoplus_{i=1}^{n} G_i}(\{x\}) \geq u_{\bigoplus_{i=1}^{n} G_i}(\{y\}) \) for all singleton models \( \{y\} \). Finally, observe that for all \( x \in A \), \( S(x) = u_{\bigoplus_{i=1}^{n} (\{x\})} \).
Finding the set of utility-maximizing models is related to functional version of MAX-UTIL seen in Chapter 5. Though in general this is NP-complete, determining winners for positional scoring rules is always in P, so there is clearly no complexity-theoretic point to be made here. (The class of goalbases corresponding to positional scoring rules represents only modular utility functions.) Rather, what is noteworthy is that if a voting rule can be simulated using goalbases as ballots, then that voting rule can be extended by loosening the restrictions we imposed in order to simulate it.

Though we have defined ballots for positional scoring rules to be total preorders, we could also have defined them cardinally. Suppose that each voter were given a supply of points which he may assign to the candidates as he wishes, and as with positional scoring rules, the winners are the set of candidates receiving the maximal number of points. This voting rule is known as cumulative voting. Any positional scoring rule may be seen as a special case of cumulative voting, wherein the voters are not given free reign as to the assignment of points, but rather required to award points only in predefined, indivisible blocks. (For example, the plurality rule gives each voter a single, indivisible one-point block of votes, while the Borda rule gives each voter blocks of size $m, m-1, \ldots, 1$ when there are $m$ candidates.) Positional scoring rules are able to use total preorders as ballots because each voter has the same scoring vector; hence, any positional scoring rule can use cardinal ballots simply by moving the rule’s scoring vector into the ballot in this way.

**Fact 7.4.3.** If no restrictions are placed on the ballot goalbases $G_i$, then $\text{opt}_1(\bigoplus_{i=1}^n G_i)$ corresponds to cumulative voting without point limits.

Cumulative voting without point limits is not a practical voting method, since for the voters it is equivalent to playing the game of which voter can write down the largest number. However, if we restrict the weights in the ballots $G_i$ to some closed interval of the nonnegative reals, then we have the following correspondence:

**Fact 7.4.4.** If each $G_i \in \mathcal{L}(\text{forms}, [0, m], \Sigma)$ and $\Sigma_{(\varphi, w) \in G_i} w \leq m \in \mathbb{R}^+$, then $\text{opt}_1(\bigoplus_{i=1}^n G_i)$ corresponds to $m$-vote cumulative voting.

Note that it is essential that the interval from which weights are chosen is closed rather than open on the right—otherwise, the voters are again playing the write-the-largest-number game, this time approaching $m$ rather than infinity.

Many undesirable properties of voting rules—the failure to always elect a Condorcet winner, the possibility of electing a Condorcet loser, nonmonotonicity—are existence properties: A voting rule has them by virtue of some set of permissible ballots for which the rule yields a pathological result. Cumulative vote contains every positional scoring rule, in the sense that any collection of ballots which is permissible input for some positional scoring rule is also permissible input for cumulative voting, and on those ballots both rules will generate the same outcome.
As a result, a collection of ballots which is pathological for some positional scoring rule will yield the same pathological result under cumulative voting. For example: Borda ballots are legal cumulative ballots and Borda can fail to elect the Condorcet winner, so if all voters happen to submit ballots which would cause this defect under the Borda rule, then the same result will occur with those ballots under the cumulative voting rule. There is a minor subtlety here, in that what we are holding constant when comparing positional scoring rules with cumulative voting is the ballots cast, rather than the voters’ preferences. It might well be the case that the same group of voters would not submit identical ballots under, say, Borda, and cumulative voting, due to the less constrained ballot space which the latter affords.

Though cumulative voting lacks some desirable properties, it also has a much larger ballot space than its subrules. How this affects the likelihood of encountering pathological ballot profiles in practice is unknown. Finally, the fact that cumulative voting can fail to elect the Condorcet winner does not obviously preclude there being some anonymous subrule which does always elect the Condorcet winner. That subrule cannot be a positional scoring rule, as proved by Young [1975], but there are many subrules of cumulative voting which are not positional scoring rules; we have not yet eliminated the possibility that some subrule of cumulative voting is a Condorcet rule. Whether any such rule exists we leave for future investigation.

7.5 The Complexity of Deciding Winning Slates

Determining the winner of an election where goalbases are ballots is related to solving \textsc{max-util} for the sum of those goalbases. \textsc{max-util} for languages with straightforward definitions tends to be either trivial or \textsc{NP}-complete. (For a thorough treatment of the complexity of \textsc{max-util}, see Section 5.5.)

Because we are concerned here with electing committees of a size fixed prior to the election (as opposed to the open-ended committees discussed by Brams et al. [2007]), we cannot apply \textsc{max-util} directly to the sum of voters’ goalbases in order to determine the winners of the election. Doing that might yield a model with the wrong number of winners. We must do something to ensure that only models which fill \(k\) seats are potentially optimal. One approach to adapting our winner determination problem to \textsc{max-util} is to augment the sum of the voters’ goalbases with formulas which increase the utility of \(k\)-sized models (or decrease the utility of non-\(k\)-sized models).

First, some notation is required. Define the following formulas:

\[
\varphi_{\geq k} = \bigvee \left\{ \bigwedge X \mid X \subseteq \mathcal{P}S \text{ and } |X| = k \right\}
\]

\[
\varphi_{\leq k} = \neg \varphi_{\geq k+1}
\]

\[
\varphi_{=k} = \varphi_{\leq k} \land \varphi_{\geq k}
\]
and the quantity
\[ \delta = \sum \{|w| \mid (\varphi, w) \in \bigoplus G_i \} . \]

The formula \( \varphi = k \) is such that a model \( M \models \varphi = k \) iff \( |M| = k \). The quantity \( \delta \) is a (not necessarily tight) upper bound on the utility change between arbitrary models for \( u_{\bigoplus G_i} \). Note that \( \varphi \geq k \) has \( \binom{n}{k} \) disjuncts, and so is potentially a very long formula.\(^6\) However, in the context of committee elections \( n \) and \( k \)—the numbers of candidates and seats—will tend to be small, and, as will be seen below, \( \varphi = k \) appears exactly once in the goalbase which represents the voters’ preferences.

Suppose that \( \bigoplus_i G_i \) is the sum of voter goalbases in a \( k \)-seat committee election. Let
\[ G = \left( \bigoplus_i G_i \right) \oplus \{ (\varphi = k, \delta + 1) \} . \]

Since \( \delta \) is an upper bound on utility change between models for \( u_{\bigoplus G_i} \), we can say the following: If \( M, N \) are models such that \( M \models \varphi = k \) and \( N \models \varphi = k \), then \( u_G(M) > u_G(N) \), as the greatest possible utility loss of moving from \( N \) to \( M \) in \( u_{\bigoplus G_i} \) is \( \delta \), and making \( \varphi = k \) true results in a gain of \( \delta + 1 \). Thus, since any model of size \( k \) is strictly better than every model of any other size, we are guaranteed that all models which yield maximal utility are of size \( k \). Moreover, since \( \varphi = k \) is true on every model of size \( k \), it does not affect their utility relative to one another, so augmenting \( \bigoplus_i G_i \) with \( (\varphi = k, \delta + 1) \) preserves the ordering of (relevant) models.

Therefore, we may easily adapt the input to force size-\( k \) models to the top of the ordering and use an off-the-shelf algorithm for deciding \( \text{MAX-UTIL} \) at most an exponential number of times:

**Theorem 7.5.1.** If \( G \in \mathcal{L} \), \( G' \notin \mathcal{L} \), and \( \mathcal{L} \) is closed under substitution of logical constants for atoms, then \( \text{MAX-UTIL} \) for \( G \oplus G' \) may be solved with no more than \( 2^{\text{Var}(G')} \) calls to a \( \text{MAX-UTIL} \) oracle for \( \mathcal{L} \).\(^6\)

---

\(^6\)While it is not possible to shorten \( \varphi \geq k \) using standard Boolean connectives, we can write it more concisely if we are willing to augment our language with a cardinality operator. For example, Benhamou, Sais, and Siegel [1994] consider a variant of propositional logic in which there are pair formulas \( (\rho, \mathcal{L}) \), where \( \mathcal{L} \) is a multiset of literals and \( \rho \) specifies how many elements of the multiset must be true in order for the \( (\rho, \mathcal{L}) \) to be true. Clearly \( \left( \frac{\rho \mathcal{P} \mathcal{S}}{2} \right) \mathcal{P} \mathcal{S} \) is equivalent to \( \varphi \geq \frac{\rho \mathcal{P} \mathcal{S}}{2} \), but exponentially shorter. Hoos and Boutilier [2000] propose a similar, though less powerful, \( k \)-of operator—less powerful due to the fact that their (bidding) language lacks negation, and so any \( k \)-of operates on atoms only.
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Proof. There are $2^{|\text{Var}(G')|}$ models on just the variables occurring in formulas in $G'$. For each model over the variables in $G'$, we substitute $\top$ and $\bot$ into $G \oplus G'$ as per the model and carry out MAX-UTIL on the modified $G \oplus G'$.

If $\text{Var}(G')$ is small and does not depend on $\mathcal{PS}$, decomposing a goalbase containing alien formulas in this way is potentially feasible. However, the formula $\phi_k$ contains every atom in $\mathcal{PS}$ at least once and hence the upper bound we get is exponential in $|\mathcal{PS}|$, which is unhelpful. (For a discussion of the substitution closure condition, see Section 5.7.1.)

An alternative approach is to modify the decision problem instead of the goalbase. Perhaps MAX-UTIL is not the right decision problem unless we have the same number of candidates as seats—in which case, why vote? Instead, we define a variant of MAX-UTIL where exactly $k$ atoms must be true in any solution:

**Definition 7.5.2** ($k$-MAX-UTIL). The decision problem $k$-MAX-UTIL($\Phi, W, F$) is defined as: Given a goalbase $G \in \mathcal{L}(\Phi, W, F)$ and an integer $K$, is there is a model $M \in 2^{\mathcal{PS}}$ such that $u_G(M) \geq K$ and $|M| = k$?

$k$-MAX-UTIL is the decision-problem version of finding members of opt$_k$, just as MAX-UTIL is the decision-problem version of finding members of opt.

Fortunately, having a fixed number of seats we are trying to fill dramatically reduces the complexity of finding a voter’s preferred ballot:

**Theorem 7.5.3.** $k$-MAX-UTIL($\text{forms}, \mathcal{R}, \Sigma$) $\in$ P, for fixed $k \in \mathbb{N}$.

*Proof.* For any given $k$ and $\mathcal{PS}$, there are $\binom{|\mathcal{PS}|}{k}$ models of size $k$ to check. It is always the case that $\binom{n}{k} \leq \frac{n^k}{k!}$, which grows polynomially in $n = |\mathcal{PS}|$ for any fixed $k$.

This makes *whatever* language we want for representing our voters’ ballots computationally tractable (though not necessarily trivial) so long as the number of seats and candidates is not too large. In particular, it is well within the capabilities of contemporary desktop computers to determine the winners in committee elections of a size similar to that conducted by Game Theory Society in 2003, where there would be only 2.7 million models to check.

7.6 Extending Single-Winner Voting Methods

In this section, we consider ways in which single-winner voting methods may be extended using goalbase ballots, and provide a concrete example where we extend approval voting from the approval of single candidates to the approval of properties of outcomes.

The fact that we can easily simulate many single-step voting procedures by using goalbases and solving MAX-UTIL on them suggests a way of extending
these methods to better register the preferences of voters. Let us extend the expressiveness of plurality (in our terminology, 1-vote top-1) as an example. In a standard plurality election, each voter casts a single vote for a single candidate. This permits voters to express only single-peaked, modular, monochromatic utility functions—that is, the only voters who can accurately express their preferences using such a method are those who prefer all candidates equally, except for one candidate who is preferred over the others. This is an unusual preference ordering for a voter to have. (Think of the 2000 U.S. Presidential election: What sort of voter would most prefer Gore, but at the same time be indifferent between Nader and Bush?)

The goalbase simulation of plurality allows voters to weight a single atom each. What if, instead, voters were subject to fewer restrictions on the goalbases they submit? Suppose that we ease the restriction on our voting language so that instead of just one, voters may specify up to \( n \{0, 1\} \)-weighted atoms. Now we can additionally express preference orderings where more than one candidate is maximally preferred, and indeed, solving \( \text{MAX-UTIL} \) over singleton models will give us approval voting instead of plurality voting.

If we move to multi-winner voting as we have when electing committees, the fit between voter preferences and the expressivity of the voting language grows worse, as argued above. Using goalbases, it is not difficult to simulate top-\( k \) voting methods—in order to find the top \( k \) candidates in the aggregate preference order, we need only solve \( \text{MAX-UTIL} \) on the sum of voter goalbases, ignoring models electing more or fewer than \( k \) candidates. In order to gain more expressivity, we can further relax the restrictions on the formulas which may be weighted. Examples:

- Suppose that we restrict voters to positive clauses with binary weights. This language is sufficient for expressing any weak linear ordering of candidates, as we shall see later this section.

- Suppose that we restrict voters to positive cubes with binary weights. This language permits voters to assign a bonus to committees which contain favored combinations of candidates. If a voter believes that, \textit{ceteris paribus}, committees with both \( A \) and \( B \) are preferable, then he may have a goalbase such that \((a \land b, 1) \in G\).

We mention here a several classes of formulas which voters electing committees might find useful:

- **Literals:** \( a \) and \( \neg b \) are useful for expressing simple preferences, e.g., “I want Alice on the committee”, or “I don’t want Bob on the committee”.

- **Positive cubes:** \( a \land b \) is useful when the combination of some candidates is better than those candidates individually.
• **Negative Horn clauses:** \( -a \lor -b \) is useful when the combination of some candidates is worse than those candidates individually.

This last class, negative Horn clauses, is exactly what is needed to overcome the difficulty described in Section 7.3.2, where two candidates may be individually desirable but collectively undesirable. Or, equivalently, we could use positive cubes with negative weights. The voter in our example who preferred Alice-committees and Bob-committees over neither-committees over both-committees could represent his preferences as \( G = \{ (a, 1), (b, 1), (a \land b, -3), (\top, 1) \} \). It is easily checked that \( u_G \) respects the voter’s preference ordering:

\[
   u_G(X) = \begin{cases} 
   2 & \text{if } a \in X, b \notin X \text{ or vice versa} \\
   1 & \text{if } a, b \notin X \\
   0 & \text{if } a, b \in X 
   \end{cases}
\]

In the general case where voters cast arbitrary goalbases \( G_i \) as ballots, we can determine a winning committee by solving MAX-UTIL for \( \bigoplus_i G_i \) on \( k \)-seat models only.

Now we offer one example of how a single-winner voting method may have its expressivity extended through goalbase voting.

Call Property Approval Voting (PAV) the voting method in which properties of the outcome (rather than individual candidates) are the objects of approval or disapproval. Any goalbase \( G \in \mathcal{L}(\text{forms}, \{1\}, \Sigma) \) constitutes an admissible PAV ballot. However, some formulas will be useless: Any formula which implies a positive cube longer than the intended number of winners, and any formula which implies a negative cube longer than the intended number of losers, will effectively be equivalent to \( \perp \). A significant difference between AV and PAV is the range of preorders of which they permit representation. Every AV ballot induces a dichotomous order, while PAV supports much more. In the case where there are three candidates \( a, b, c \), the PAV ballot \( \{ (a, 1), (a \lor b, 1) \} \) induces the (non-dichotomous) order \( a > b > c \), since the state \( \{a\} \) receives two points, \( \{b\} \) one point, and \( \{c\} \) zero points. (Only singleton states are relevant here, since we are considering the single-winner case.)

In fact, there is a general way of representing any strict linear order \( a_1 > a_2 > \ldots > a_n \) with a PAV ballot:

\[
   (a_1 \lor \ldots \lor a_{n-2} \lor a_{n-1}, 1) \\
   (a_1 \lor \ldots \lor a_{n-2}, 1) \\
   \vdots \\
   (a_1 \lor a_2, 1) \\
   (a_1, 1)
\]
The clause which ends with $a_i$ is the one which causes $a_i$ to be ordered strictly above $a_{i+1}$, so by omitting that clause we can get a ballot where $a_i \sim a_{i+1}$. This is sufficient to induce any weak linear order over the candidates. Thus, in the single-winner case PAV is something like a nonresolute version of the Borda rule.

### 7.7 Future Work

There are a number of paths yet to be explored regarding voting with goalbases. In this section we give an overview of those of which we are aware.

In order to use goalbase ballots for multi-winner cumulative voting, we must place some restriction on the weights which are available to voters. As noted after Fact 7.4.3, cumulative voting without point limits is not a sensible voting method. Having established that restrictions are needed, we are now faced with the problem of selecting some—it is not presently obvious which restrictions are most suitable. The restriction which cumulative voting itself suggests is to limit the sum of weights in any goalbase: $\sum_{(\varphi, w) \in G} w \leq K$.\(^7\) This is a limit on the input space. Another approach is to restrict the output space: For example, we might limit the utility of any admissible state: $u_G(M) \leq K$ for all $M \subseteq \mathcal{P}S$ where $|M| = k$.

There are advantages and disadvantages to both methods. If our voter is a person, then he will find it easier to cast a valid sum-limited ballot than a valid state-limited one. Input limits are not uncommon. For example, in the U.S., the State of Illinois used cumulative voting (over atoms) with a 3-point limit for electing members of its House of Representatives from 1870 to 1980 [Moore, 1909; Yale Law Journal, 1982]. Corporate boards of directors are usually elected using cumulative voting, where the point limit for each voter is the number of shares he owns. We know of no uses of output limits: Presumably this is because it is hard to see when working in the input space whether output limits are being respected; output limits expect too much of the average voter. However, output limits on elections of the size human voters are likely to face will not be difficult for machines to enforce, so might be useful if the voters are using a computer-aided voting system. This is a user-interface issue.

Point limits also raise a fairness issue. For simplicity, we use a single-winner example, though the problem it illustrates is general. The sum-limit $\sum_{(\varphi, w) \in G} w \leq K$ will not always produce utility functions which have equal sums for singleton models. E.g., consider the goalbase ballots $G_1 = \{(a, 10)\}$ and $G_2 = \{(a \lor b, 10)\}$. The latter has a singleton state sum of 20 ($u_{G_2}(\{a\}) = 10$, $u_{G_2}(\{b\}) = 10$), while singleton states for the former sum only to 10 ($u_{G_1}(\{a\}) = 10$). The effect of

---

\(^7\)If we permit negative weights, then we would need to place an upper bound on the sum of the absolute values of the weights instead of on the sum of the weights. In this way we avoid ballots like $\{(a, -2^{1000}), (b, 2^{1000} + 10)\}$ which the voter could claim is a 10-point ballot according to the latter method.
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the sum-limit is to give voters with top-heavy preferences more influence on the outcome than voters with balanced or bottom-heavy preferences. Note that this is not a failure of anonymity, as it has nothing to do with voters' names or order. We could try to account for this by “normalizing” formulas based on the number of states they affect, e.g., \((a \lor b, 10)\) could be translated to \((a, 5), (b, 5)\), but this would seem to disadvantage voters who have top-heavy preferences. If \((\bigvee \mathcal{PS} \setminus \{a\}, 10)\), after normalization, gives one point to everyone but candidate \(a\), that is not likely to be very effective for voters who dislike \(a\) but otherwise do not distinguish among the other candidates. Or we could try other ways of normalizing—Lafage and Lang [2000, Section 3.2.3] suggest postprocessing (dis)utilities to equalize entropy across agents—which will potentially have some other differential effect on voters.

The basic question here seems to be how to set the value of preferences which are not over single states against those which are. What is an appropriate measure of voting power here? Input limits seem to favor top-heavy voters, output limits seem to favor bottom-heavy voters. One way of quantifying the effect that a proposed weight limit could have is by considering the efficacy of voters with different preferences under that weight limit. (The efficacy of a ballot for a voter is a measure of how often that voter will be pivotal if he casts that ballot.) Ideally, all voters would have equally efficacious ballots to cast. Brams and Fishburn [2007, Chapter 5] calculate the efficacy of ballots for approval voting and find that not all ballots are equally effective. If we assume that our voters are truthful, what this means is that approval voting is advantageous for voters with some kinds of preference orders and disadvantageous for others. A similar analysis could be done for cumulative voting with goalbase ballots, with an eye to which weight restrictions treat voters most equitably.

With any voting system, there are questions about whether it encourages or discourages strategic voting. The manipulability of a voting system must always be considered in the context of a notion of sincerity, for we cannot say whether a voter is misrepresenting his preferences if we cannot first say what it would be for a voter to represent his preferences accurately.

Consider, first, voting systems with ordinal ballots. Many standard systems—e.g., plurality, approval, Borda—use ballots which contain purely ordinal information. In the case where there is an allowable ballot which induces the same preorder over outcomes as the voter’s true preorder, then any reasonable notion of sincerity should deem that ballot sincere (and any ballot which does not induce that same preorder, insincere). This means, for example, that for voters with strict linear orders, there will always be unique sincere plurality and Borda ballots; and similarly, for voters whose preferences are dichotomous (and not wholly indifferent), there will be a unique sincere approval ballot. However, this will not be the case for voters with other kinds of preorders. There are no approval ballots which express nondichotomous preferences (e.g., \(x > y > z\)); standardly, Borda does not permit ties, so voters with weak (instead of strict) orders will have no ballots which express their preferences exactly.
What should count as sincere in the space of cardinal ballots is not immediately obvious. A voter’s preferences may be inexpressible as a result of restrictions on the ballot language, and this can result in the existence of multiple sincere ballots which the voter could cast. Endriss [2007] explores the existence of multiple sincere ballots for approval voting and shows that the Gibbard-Satterthwaite Theorem is avoidable in that context; Endriss, Pini, Rossi, and Venable [2009] present several measures of sincerity for languages where ballots are preorders, and examine the consequences for strategyproofness under these. This line of research could be continued for goalbase ballots, first by developing reasonable notions of sincerity, and secondly by determining which language restrictions induce sincerity in rational voters. Meir, Procaccia, Rosenschein, and Zohar [2008] avoid the problem of sincerity in multi-winner voting altogether by defining manipulation as an optimization problem asking whether, given the ballots of some other voters, there is a ballot which the manipulating voter may cast which yields him at least $t$ utility. The question of whether a better ballot exists is more general than, and serves as a proxy for, the question of whether a better insincere ballot exists—though this still leaves open the possibility that some ballot which is optimal is nonetheless also sincere, and so does not exactly capture classical manipulability.

Finally, we might consider questions about the difficulty of finding a sincere ballot given a voter’s preferences. It would not be surprising to learn that for some languages, it is always in the voter’s best interests to cast a sincere ballot, but nonetheless quite difficult for him to determine which ballots are sincere for him. Strategyproofness is not worth much in this case. A method for constructing sincere ballots will be essential for any language intended for human voters.

### 7.8 Conclusion

In this chapter we introduced some methods for electing committees and demonstrated that they lack certain properties which are desirable when conducting multi-winner elections. In particular, single-winner voting methods lack the expressivity to extend well to the multi-winner case. The observation that it is possible to simulate many single-winner voting methods using goalbases and MAX-UTIL suggests one way of extending the expressivity of existing voting methods for use in a multi-winner setting. Because multi-winner elections tend to have the number of winners fixed beforehand, the complexity of MAX-UTIL is limited, even when the goalbase language is not. Along these lines, we suggest a multi-winner extension of approval voting, which we call Property Approval Voting. Finally, we discuss the possibilities for future work: the need to find useful limits on weights in goalbase ballots; the fairness of these limits, since they may differentially affect voters with dissimilar preferences; and issues related to sincerity and strategic voting.