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Ciardelli, I.A.; Roelofsen, F.

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Generalized Inquisitive Logic
—Completeness via Intuitionistic Kripke Models—

Ivano Ciardelli
ILLC, University of Amsterdam
i.ciardelli@student.uva.nl

Floris Roelofsen
ILLC, University of Amsterdam
f.roelofsen@uva.nl

Abstract. This paper investigates a generalized version of inquisitive semantics (Groenendijk, 2008b; Mascarenhas, 2008). A complete axiomatization of the associated logic is established. The connection with intuitionistic logic is clarified and heavily exploited.

1 Introduction

Traditionally, logic is concerned with argumentation. As a consequence, formal semantics usually focusses on the *descriptive* use of language, and the meaning of a sentence is identified with its *informative* content. Stalnaker (1978) gave this informative notion a dynamic and conversational twist by taking the meaning of a sentence to be its potential to change the common ground, where the common ground is viewed as the conversational participants' shared information. Technically, the common ground is taken to be a set of possible worlds, and a sentence provides information by eliminating some of these possible worlds.

Of course, this picture is limited in several ways. First, it only applies to sentences that are used exclusively to provide information. Even in a typical informative dialogue, utterances may serve different purposes as well. Second, the given picture does not take into account that updating the common ground is a *cooperative* process. One speech participant cannot simply change the common ground all by herself. All she can do is *propose* a certain change. Other speech participants may react to such a proposal in several ways. These reactions play a crucial role in the dynamics of conversation.

In order to overcome these limitations, *inquisitive semantics* (Groenendijk, 2008b; Mascarenhas, 2008) starts with an altogether different picture. It views propositions as proposals to change the common ground. These proposals do not always specify just one way of changing the common ground. They may suggest alternative ways of doing so, among which the responder is then invited to choose. Formally, a proposition consists of one or more *possibilities*. Each possibility is a set of possible worlds and embodies a possible way to change the common ground. If a proposition consists of two or more possibilities, it is *inquisitive*: it invites the other participants to respond in a way that will lead to a cooperative choice between the proposed alternatives. Inquisitive propositions raise an issue. They give direction to a dialogue. Thus, inquisitive semantics directly reflects that the primary use of language is communication: the exchange of information in a cooperative dynamic process of raising and resolving issues.

Groenendijk (2008b) and Mascarenhas (2008) defined an inquisitive semantics for the language of propositional logic, focussing on the philosophical and linguistic motivation for the framework, and delineating some of its basic logical properties. The associated logic was axiomatized by Mascarenhas (2009), while a sound and complete sequent calculus was established independently by Sano (2008). Linguistic applications of the framework are discussed by Balogh (2009); Ciardelli *et al.* (2009); Groenendijk and Roelofsen (2009).

In this paper, we consider a generalized version of the semantics proposed by Groenendijk (2008b) and Mascarenhas (2008). This generalized semantics was first discussed by Groenendijk (2008a). Initially, it was thought to give essentially the same results as the original semantics. Upon closer examination, however, Mascarenhas, Groenendijk, and Ciardelli observed that the two systems are different, and Ciardelli (2008) argued that these differences speak in favor of the generalized semantics.

The main aim of the present paper is to establish a sound and complete axiomatization of the logic that the generalized semantics gives rise to. In order to do so, we explore the connection between inquisitive semantics and Kripkean semantics for intuitionistic logic (Kripke, 1965). An axiomatization of inquisitive logic is obtained from an axiomatization of intuitionistic logic by adding two well-known axiom schemes: the

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Kreisel-Putnam axiom scheme (Kreisel and Putnam, 1957) and the law of double negation, $\neg\neg p \to p$, restricted to atomic proposition letters.

The paper is organized as follows. Section 2 presents the generalized version of inquisitive semantics and establishes some basic properties of the system, section 3 explores the connection with intuitionistic semantics, and section 4 presents a sound and complete axiomatization of inquisitive logic. Finally, section 5 suggests a new intuitive interpretation of inquisitive semantics, which sheds further light on some of the results obtained in earlier sections. Proofs of our main results are included in an appendix.

2 Generalized Inquisitive Semantics

Definition 1 (Language)**.** Let $\mathcal{P}$ be a finite set of proposition letters that we will consider fixed throughout the paper. We denote by $\mathcal{L}_\mathcal{P}$ the set of formulas built up from letters in $\mathcal{P}$ and $\bot$ using the binary connectives $\wedge, \vee$ and $\to$. We will refer to $\mathcal{L}_\mathcal{P}$ as the propositional language based on $\mathcal{P}$.

We will also make use of the following abbreviations: $\neg\varphi$ for $\varphi \to \bot$, $!\varphi$ for $\neg\neg\varphi$, and $?\varphi$ for $\varphi \vee \neg\varphi$.

Definition 2 (Indices)**.** An *index* is a function from $\mathcal{P}$ to $\{0, 1\}$. We denote by ω the set of all indices.

Definition 3 (States)**.** A *state* is a set of indices. We denote by $\mathcal{S}$ the set of all states.

Definition 4 (Support)**.**

1. $s \models p \quad \text{iff} \quad \forall w \in s : w(p) = 1$
2. $s \models \bot \quad \text{iff} \quad s = \emptyset$
3. $s \models \varphi \wedge \psi \quad \text{iff} \quad s \models \varphi$ and $s \models \psi$
4. $s \models \varphi \vee \psi \quad \text{iff} \quad s \models \varphi$ or $s \models \psi$
5. $s \models \varphi \to \psi \quad \text{iff} \quad \forall t \subseteq s :$ if $t \models \varphi$ then $t \models \psi$

Notice that formulas are evaluated with respect to arbitrary sets of indices here. In the original version of inquisitive semantics, formulas were evaluated with respect to sets of cardinality at most two. In this sense, the semantics considered here is a generalized version of the original semantics.

It follows from the above definition that the empty state supports any formula φ. Moreover, this clearly does not hold for any other state. Thus, we will refer to $\emptyset$ as *the inconsistent state.*

Proposition 1 (Persistence)**.**
If $s \models \varphi$ then for every $t \subseteq s$: $t \models \varphi$

Proposition 2 (Singleton states behave classically)**.** For any index w and formula φ, $\{w\} \models \varphi$ if and only if φ is classically true in w.

Note in particular that we have $\{w\} \models \varphi$ or $\{w\} \models \neg\varphi$ for any formula φ.

Definition 5 (Possibilities, propositions, truth-sets)**.** Let φ be a formula.

1. A *possibility* for φ is a maximal state supporting φ, i.e., a state that supports φ and is not properly included in any other state supporting φ.
2. The *proposition* expressed by φ, denoted by $[\varphi]$, is the set of possibilities for φ.
3. The *truth set* of φ, denoted by $|\varphi|$, is the set of indices where φ is classically true.

The proposition expressed by φ is conceived of as the *meaning* of φ in an inquisitive setting, while the truth-set of φ represents the *classical meaning* of φ. Notice that $|\varphi|$ is a state, while $[\varphi]$ is a set of states.

Propositions are viewed as proposals to change the common ground of a dialogue. If $[\varphi]$ contains more than one possibility, it is *inquisitive*: each possibility embodies a possible way to change the common ground, and other dialogue participants are invited to react in such a way that a choice between these possibilities is established in a cooperative way. $[\varphi]$ may also be *informative*: indices that are not contained in any possibility in $[\varphi]$ are eliminated from the common ground, unless other dialogue participants object.

Definition 6 (Inquisitiveness and informativeness)**.**

- φ is *inquisitive* if and only if $[\varphi]$ contains at least two possibilities;
- φ is *informative* if and only if $[\varphi]$ does not cover the set of all indices: $\bigcup[\varphi] \neq \omega$.

Definition 7 (Questions and assertions)**.**

- φ is a *question* if and only if it is not informative;
- φ is an *assertion* if and only if it is not inquisitive.

Definition 8 (Contradictions and tautologies)**.**

- φ is a *contradiction* if and only if $[\varphi] = \{\emptyset\}$
- φ is a *tautology* if and only if $[\varphi] = \{\omega\}$

It is easy to see that a formula is a contradiction iff it is a classical contradiction. This does not hold for tautologies. Classically, a formula is tautological iff it is not informative. In the present system, a formula is tautological iff it is neither informative nor inquisitive. Classical tautologies may well be inquisitive. A prime example of this is the formula $p \vee \neg p$.

Definition 9 (Equivalence)**.** Two formulas φ and ψ are *equivalent*, $\varphi \equiv \psi$, if and only if $[\varphi] = [\psi]$.

It is easy to see that $\varphi \equiv \psi$ if and only if φ and ψ are supported by the same states.

Proposition 3 (Characterization of questions)**.** For any formula φ, the following are equivalent:

1. φ is a question
2. φ is a classical tautology
3. $\neg\varphi$ is a contradiction
4. $\varphi \equiv\ ?\varphi$

Proposition 4 (Characterization of assertions)**.** For any formula φ, the following are equivalent:

1. φ is an assertion
2. for any two states s and t, if $s \models \varphi$ and $t \models \varphi$, then $s \cup t \models \varphi$
3. $|\varphi| \models \varphi$
4. $\varphi \equiv\ !\varphi$
5. $[\varphi] = \{|\varphi|\}$

Note that item 5 states that a formula is an assertion if and only if its meaning consists uniquely of its classical meaning. In this sense, assertions behave classically. The following proposition gives some sufficient syntactic conditions for a formula to be an assertion.

Proposition 5. For any propositional letter p and formulas φ, ψ:

1. p is an assertion;
2. $\bot$ is an assertion;
3. if φ, ψ are assertions, then $\varphi \wedge \psi$ is an assertion;
4. if ψ is an assertion, then $\varphi \rightarrow \psi$ is an assertion.

Using this proposition inductively we obtain the following corollary showing that disjunction is the only source of non-classical, inquisitive behavior in our language: the disjunction-free fragment of the language behaves classically.

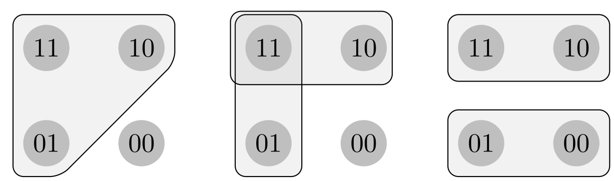
Corollary 1. Any disjunction-free formula is an assertion.

The informative content of a formula φ is embodied by $\bigcup[\varphi]$ (indices that are not in $\bigcup[\varphi]$ are eliminated from the common ground if no other dialogue participant objects). The following proposition guarantees that inquisitive semantics preserves the classical treatment of informative content.

Proposition 6. For any formula φ: $\bigcup[\varphi] = |\varphi|$.

Now let us look at some formulas that are inquisitive, and thus do not behave classically.

Example 1 (Disjunction)**.** To see how the inquisitive treatment of disjunction differs from the classical treatment, consider figures 1(a) and 1(b) below. These pictures assume that $\mathcal{P} = \{p, q\}$; index 11 makes both p and q true, index 10 makes p true and q false, etcetera. Figure 1(a) depicts the classical meaning of $p \vee q$: the set of all indices that make either p or q, or both, true. Figure 1(b) depicts the proposition associated with $p \vee q$ in inquisitive semantics. It consists of two possibilities. One possibility is made up of all indices that make p true, and the other of all indices that make q true. So $p \vee q$ is inquisitive. It invites a response which is directed at chosing between two alternatives. On the other hand, $p \vee q$ also excludes one index, namely the one that makes both p and q false. This illustrates two things: first, that $p \vee q$ is informative, just as in the classical analysis, and second, that formulas can be both informative and inquisitive at the same time.



(a) (b) (c)

Figure 1: (a) the traditional picture of $p \vee q$, (b) the inquisitive picture of $p \vee q$, and (c) polar question $?p$.

Example 2 (Polar questions)**.** Figure 1(c) depicts the proposition expressed by $?p$, which—by definition—abbreviates $p \vee \neg p$. As in the classical analysis, $[?p]$ is not informative. But it *is* inquisitive: it invites a choice between two alternatives, p and $\neg p$. As such, it reflects the essential function of polar questions in natural language.

It should be emphasized that inquisitive semantics does not claim that disjunctive sentences in *natural language* are always inquisitive. Inquisitive semantics primarily introduces a new notion of meaning, which incorporates both informative and inquisitive content. Ultimately, such meanings should of course be associated with expressions in natural language. The present system, which associates inquisitive meanings with expressions in the language of propositional logic, could be seen as a first step in this direction. A proof of concept. The treatment of natural language will evidently be much more intricate, and in particular, the interpretation of disjunction will be affected by several additional factors—including intonation (see, for instance, Groenendijk, 2008b).

We end this section with a definition of entailment and validity, and the logic InqL that inquisitive semantics gives rise to.[1]

Definition 10 (Entailment and validity)**.** A set of formulas Θ *entails* a formula φ in inquisitive semantics, $\Theta \models_{\mathsf{InqL}} \varphi$, if and only if any state that supports all formulas in Θ also supports φ. A formula φ is *valid* in inquisitive semantics, $\models_{\mathsf{InqL}} \varphi$, if and only if φ is supported by all states.

If no confusion arises, we will simply write $\models$ instead of $\models_{\mathsf{InqL}}$. We will also write $\psi_1, \ldots, \psi_n \models \varphi$ instead of $\{\psi_1, \ldots, \psi_n\} \models \varphi$. Note that, as expected, $\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$.

Definition 11 (Logic)**.** Inquisitive logic, InqL, is the set of formulas that are valid in inquisitive semantics.

It is easy to see that a formula is valid in inquisitive semantics if and only if it is both a classical tautology *and* an assertion. In particular, InqL coincides with classical logic as far as assertions are concerned.

Remark. While InqL is closed under the *modus ponens* rule, it is *not* closed under uniform substitution. For instance, $\neg\neg p \to p \in \mathsf{InqL}$ for all proposition letters, but $\neg\neg(p \vee q) \to (p \vee q) \notin \mathsf{InqL}$. We will return to this feature of the logic in section 5.

3 Inquisitive Logic and Intuitionistic Kripke Models

This section explores the connection between inquisitive semantics and intuitionistic Kripke semantics.[2] Both can be conceived of in terms of *information.* In inquisitive semantics, a formula is evaluated with respect to a set of indices. An index assigns truth values to all atomic formulas. Atomic formulas can be thought of as names for atomic facts that may or may not obtain in the world. Thus, an index can be seen as a complete specification of which facts do and do not obtain in the world. In the common parlance of intensional logic, an index specifies a *possible world.* A state, then, specifies a set of possible worlds. Such a set of possible worlds can in principle be conceived of in many different ways. We take an *epistemic* perspective: we think of the possible worlds in a state s as those that are compatible with the information available in s. In this sense, states are conceived of as *information states.* Now, if s is a set of indices, conceived of as an information state, then *subsets* of s can be seen as possible future information states. Whether s supports a formula φ is partly defined in terms of these future information states.

In intuitionistic semantics, a formula is evaluated with respect to a point in a Kripke model, which can be also thought of as an information state. Each point may have access to some other points, which can be seen as future information states. Whether a point u in a model M satisfies a formula φ is again partly defined in terms of these future information states.

This informal analogy has a precise, formal counterpart: inquisitive semantics coincides with intuitionistic semantics on a suitable Kripke model.

Definition 12 (Kripke model for inquisitive semantics)**.** The Kripke model for inquisitive semantics is the model $M_I = \langle W_I, \supseteq, V_I \rangle$ where $W_I := S - \{\emptyset\}$ is the set of all non-empty states and the valuation V_I is defined as follows: for any letter p, $V_I(p) = \{s \in W_I \mid s \models p\}$.

Observe that M_I is an intuitionistic Kripke model: $\supseteq$ is reflexive and transitive, and V_I is persistent.

Proposition 7 (Inquisitive support coincides with Kripke satisfaction on M_I)**.** For every formula φ and every non-empty state s we have:

$$s \models \varphi \iff M_I, s \Vdash \varphi$$

One consequence of this result is that all intuitionistic validities are also inquisitive validities: $\mathsf{IPL} \subseteq \mathsf{InqL}$. In order to obtain a more precise characterization of the relation between IPL and InqL we will distinguish three special properties of M_I, and show that InqL is the logic of the class of Kripke models defined by these three properties. This result will play a crucial role in the proof of the completeness theorem for InqL, which will be presented in section 4.

Now, what are the peculiar properties of M_I? Let us start by analyzing the properties of the underlying frame $F_I := \langle W_I, \supseteq \rangle$. One striking aspect of F_I is that any point in it can see an endpoint. We will call frames with this property E-SATURATED. The mnemonic is that such frames have *enough endpoints.*

Notation. If $F = (W, R)$ is a Kripke frame and s is a point in F, we denote by E_s the set of terminal

[1] For more basic properties of the system, including some normal form and expressive completeness results, we refer to (Ciardelli, 2008).

[2] The ideas laid out below were partly inspired by (van Benthem, 2008) and (Groenendijk, 2008a).

successors of s, that is, $E_s = \{t \in W \mid sRt \text{ \& } t \text{ is an endpoint}\}$.

Definition 13 (E-saturation)**.** A frame $F = (W, R)$ is E-SATURATED iff for any point $s \in W$, $E_s \neq \emptyset$.

A second striking feature of F_I is the following: if s is a point in F_I and E_* is a nonempty set of terminal successors of s, then there is always a "mediating point" t which is a successor of s and has precisely E_* as its set of terminal successors. We will call frames with this property I-SATURATED; the mnemonic is that I-SATURATED frames have *enough intermediate points*.

Definition 14 (I-saturation)**.** A Kripke frame $F = (W, R)$ is I-SATURATED in case for any $s \in W$ and any nonempty $E_* \subseteq E_s$ there is a successor t of s with $E_* = E_t$.

One way of interpreting I-SATURATION is as follows. An endpoint in a Kripke model always satisfies either p or $\neg p$, for all atomic formulas. Thus, an endpoint specifies a possible world, and we may think of E_s as the set of possible worlds that are compatible with the information available in s.

In an I-SATURATED frame, the worlds that are considered possible at a certain point are always taken to be *independent* from one another: each of them can be excluded independently from the others. Any possible world is taken to differ from all the others in some "observable way": there is always some piece of information that has the effect of ruling out that possible world while keeping all the others.

In frames that are not I-SATURATED, possible worlds may be entangled, in the sense that the exclusion of some of them forces the exclusion of others. This is illustrated by the tripod frame in figure 2. The worlds considered possible at u are v, w, and z. But none of these worlds can be excluded independently. For instance, any piece of information that excludes z must also exclude either v or w.

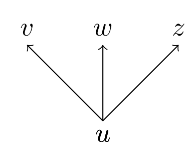


Figure 2: Tripod.

Definition 15 (Saturated frames)**.** We say that a Kripke frame $F = (W, R)$ is saturated in case it is both E-SATURATED and I-SATURATED.

Now let us turn to a special feature of the *valuation function* V_I of the model M_I. Consider a point s in a Kripke model where neither p nor $\neg p$ is satisfied. We could say that p is undecided at this point. Now, in intuitionistic models, p can only be undecided at some point s if there is a successor of s that satisfies p. But there does *not* have to be a successor of s that satisfies $\neg p$. For instance, in the tripod frame in figure 2, p could be satisfied in all the endpoints, but still be undecided at u.

In M_I, this situation never occurs. If p is undecided at s, there is always a successor of s that satisfies $\neg p$. We say that models with this property *ground* undecidedness about atomic formulas.

Definition 16 (Grounded models)**.** We say that a Kripke model $M = (W, R, V)$ is *grounded* in case for any $s \in W$ such that $M, s \nVdash p$, there is a successor t of s with $M, t \Vdash \neg p$.

It should be emphasized that grounded models do not necessarily ground undecidedness about *complex* formulas. For instance, $p \vee q$ may well be undecided even though $\neg(p \vee q)$ is not satisfied at any future point.

Definition 17 (Saturated models)**.** We say that a Kripke model $M = (W, R, V)$ is saturated in case it is grounded and its underlying frame (W, R) is saturated.

Saturated models are models that resemble M_I in certain ways. Obviously M_I itself is a saturated model. The following result states that M_I is the "most general" saturated model, in the sense that any situation arising in a saturated model is already present in M_I. A proof of this proposition is included in appendix A.1.

Proposition 8. For any saturated intuitionistic Kripke model $M = (W, R, V)$, there is a p-morphism η from M to M_I.

In order to compare inquisitive entailment and validity with intuitionistic entailment and validity w.r.t. saturated models, we need the following definitions.

Definition 18 (Entailment and validity)**.** A set of formulas Θ SAT-*entails* a formula φ, $\Theta \models_{\mathsf{SAT}} \varphi$, if and only if every point in every saturated intuitionistic Kripke model that satisfies all formulas in Θ also satsifies φ. A formula φ is SAT-*valid*, $\models_{\mathsf{SAT}} \varphi$, if and only if φ is satisfied by every point in every saturated intuitionistic Kripke model.

Definition 19 (Logic)**.** The logic of saturated intuitionistic Kripke models, SAT, is the set of all formulas that are SAT-valid.

We are now ready to state our first main theorem. A proof of this theorem is included in appendix A.2.

Theorem 1 (Correspondence theorem)**.** For any set of formulas Θ and any formula φ:

$$\Theta \models_{\mathsf{InqL}} \varphi \quad \Longleftrightarrow \quad \Theta \models_{\mathsf{SAT}} \varphi$$

Corollary. $\mathsf{InqL} = \mathsf{SAT}$.

4 Axiomatization of Inquisitive Logic

In this section we establish our second main result: a sound and complete axiomatization of InqL. The axiomatization is obtained from an axiomatization for intuitionistic logic by adding two well-known axiom schemes. First, the law of double negation, $\neg\neg p \to p$, for all proposition letters p. This axiom scheme characterizes the class of grounded intuitionistic Kripke models. Therefore, we will refer to it as **G**. The second ingredient of our axiomatization is the Kreisel-Putnam axiom scheme (Kreisel and Putnam, 1957), to which we will refer as **KP**.

Definition 20 (The system GKP)**.** The system GKP is obtained by adding the following axiom schemes to any sound and complete Hilbert-style system for intuitionistic logic that has *modus ponens* as its only inference rule.

G $\quad \neg\neg p \to p \qquad$ (for any atomic formula $p \in P$)

KP $\quad (\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$

(for all formulas $\varphi, \psi, \chi \in \mathcal{L}_{\mathcal{P}}$)

We write $\Theta \vdash_{\text{GKP}} \varphi$ if and only if φ can be derived from Θ in the system GKP.

We are now ready to state our second main theorem. A proof of this theorem is included in appendix A.3.

Theorem 2 (Completeness theorem)**.** For any set of formulas Θ and any formula φ:

$$\Theta \vdash_{\text{GKP}} \varphi \quad \Longleftrightarrow \quad \Theta \models_{\mathsf{InqL}} \varphi$$

Corollary. GKP axiomatizes InqL.

5 An Intuitive Interpretation of Inquisitive Support

Inquisitive semantics is motivated by conceptual ideas about information exchange through conversation. The link between these ideas and the formal semantics is established at the level of *propositions*. However, propositions are defined indirectly, through the notion of *support*. In this section we suggest an intuitive interpretation of this basic notion of support, which allows for a new perspective on the semantics and the associated logic, and in particular on some of the observations made in previous sections.

We have already seen that states in inquisitive semantics can be conceived of as information states. Now, traditionally an information state s is taken to support a formula φ iff φ is true in all possible worlds that are compatible with the information available in s. In other words, $s \models \varphi$ iff it follows from the information available in s that φ is true.

In inquisitive semantics, $s \models \varphi$ means something slightly different. It does not only require that the information available in s determines *that* φ is true, but also that it determines *how* φ is realized.

The underlying idea is that a formula can be realized *in different ways*. For instance, $p \vee q$ can be realized because p is realized or because q is realized. Thus, in order to know how $p \vee q$ is realized, one must either know that p is realized, or that q is realized.

Atomic formulas are a special case. They can only be realized in *one* way: the atomic fact that they name must obtain. This special character of atomic formulas explains the fact that inquisitive logic is not—and should not be—closed under uniform substitution (see the final remark of section 2).

This intuitive interpretation of inquisitive support also clarifies the relation between inquisitive logic on the one hand, and classical and intuitionistic logic on the other. Inquisitive logic resembles classical logic in that it is concerned with *truth* rather than with *provability*: when it comes to atomic formulas, we just have to go "out there" and check. However, when it comes to complex formulas, inquisitive logic behaves like intuitionistic logic in requiring a justification (proof) for φ in terms of justifications (proofs) for subformulas of φ.

The different ways in which a formula φ may be realized are mirrored by the possibilities for φ. A possibility for φ is a *maximal* state in which φ is known to be realized in one particular way. In other words, all states in which φ is known to be realized in the same way, are included in the same possibility. So every possible way in which φ may be realized corresponds with a possibility for φ, and vice versa.

In the light of this new interpretation, the conversational effect of an utterance can be rephrased as follows. An utterance of φ provides the information *that* φ is true and it raises the issue *how* φ is realized.

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A Proofs of Main Theorems

A.1 Proof of proposition 8

Let $M = (W, R, V)$ be a saturated Kripke model. For any endpoint e of M, denote by δ_e the index defined by: $\delta_e(p) = 1 \iff e \in V(p)$.

Define the map $\eta : W \to W_I$ as follows: for any $a \in W$, $\eta(a) = \{\delta_e \,|\, e \in E_a\}$.

In the first place, since M is E-SATURATED for any $a \in W$ we have $E_a \neq \emptyset$ and so $\eta(a) \neq \emptyset$; this insures that indeed $\eta(a) \in W_I$, so that the map η is well-defined. It remains to check that η is indeed a p-morphism. Fix any $a \in W$:

- **Propositional Letters**. Take any propositional letter p. If $M, a \Vdash p$, then by persistence we have $M, e \Vdash p$ for any $e \in E_a$; this implies $i(p) = 1$ for any index $i \in \eta(a)$ and so $\eta(a) \models p$, whence $M_I, \eta(a) \Vdash p$.

  Conversely, suppose $M, a \nVdash p$. Then since the model M is grounded, there must be a successor b of a with $M, b \Vdash \neg p$. Exploiting again the fact that M is E-SATURATED we can find $e \in E_b$, and by persistence it must be $M, e \Vdash \neg p$, whence $\delta_e(p) = 0$. But by the transitivity of R we also have $e \in E_a$, so $\delta_e \in \eta(a)$: thus $\eta(a) \not\models p$, whence $M_I, \eta(a) \nVdash p$.

- **Forth Condition**. Suppose aRb: then since our accessibility relation is transitive, $E_a \supseteq E_b$.

- **Back Condition**. Suppose $\eta(a) \supseteq t$: we must show that there is some successor b of a such that $\eta(b) = t$.

  Now, since $t \subseteq \eta(a)$, every index $i \in t$ is of the form δ_{e_i} for some terminal successor e_i of a.

  Now, let $E_* := \{e_i \,|\, i \in t\}$: since $E_* \subseteq E_a$ and M is I-SATURATED, there is a successor b of a with $E_b = E_*$. We have: $\eta(b) = \{\delta_e \,|\, e \in E_*\} = \{\delta_{e_i} \,|\, i \in t\} = \{i \,|\, i \in t\} = t$. So we have found a successor b of a with the required properties. $\square$

A.2 Proof of theorem 1

Suppose $\Theta \not\models_{\mathsf{InqL}} \varphi$. Then there is some state s such that $s \models \theta$ for any $\theta \in \Theta$ but $s \not\models \varphi$. Now, s must be non-empty, because the empty state supports everything. So by proposition 7 we have $M_I, s \Vdash \theta$ for all $\theta \in \Theta$ but $M_I, s \nVdash \varphi$. Since M_I is a saturated model, this shows that $\Theta \not\models_{\mathsf{SAT}} \varphi$.

Conversely, suppose $\Theta \not\models_{\mathsf{SAT}} \varphi$. This means that, for some saturated intuitionistic Kripke model M and some point a in M we have $M, a \Vdash \theta$ for all $\theta \in \Theta$, while $M, a \nVdash \varphi$.

Now, according to proposition 8 there is a p-morphism $\eta : M \to M_I$: since satisfaction is invariant under p-morphisms, we have $M_I, \eta(a) \Vdash \theta$ for all $\theta \in \Theta$, while $M_I, \eta(a) \nVdash \varphi$.

Hence, by proposition 7 we have $\eta(a) \models \theta$ for all $\theta \in \Theta$ but $\eta(a) \not\models \varphi$, which shows that $\Theta \not\models_{\mathsf{InqL}} \varphi$. The proof of theorem 1 is thus complete. $\square$

A.3 Proof of theorem 2

Let us start with soundness.

Lemma 1 (Soundness)**.** For any set of formulas Θ and any formula φ, if $\Theta \vdash_{\text{GKP}} \varphi$ then $\Theta \models_{\mathsf{InqL}} \varphi$.

Proof. Suppose $\Theta \vdash_{\text{GKP}} \varphi$: this means that there is a derivation of φ from formulas in Θ, axioms of intuitionistic logic and instances of **G** and **KP** which uses *modus ponens* as only inference rule.

In order to see that $\Theta \models_{\mathsf{InqL}} \varphi$, let s be any state which supports all formulas in Θ. It follows from theorem 1 that s supports any axiom of intuitionistic logic. Moreover, it is obvious by the semantics of implication that the set of formulas which are supported by s is closed under modus ponens. So, lemma 1 will be proved if we can show that s supports any instance of **G** and **KP**.

As for **G**, consider a substate $t \subseteq s$. If $t \not\models p$, then there must be an index $v \in t$ such that $v(p) = 0$; by the classical behaviour of singletons this means that $\{v\} \models \neg p$. Since $\{v\} \subseteq t$, this implies $t \not\models \neg\neg p$. By contraposition, this shows that for any substate $t \subseteq s$, if $t \models \neg\neg p$ then $t \models p$, which means precisely that $s \models \neg\neg p \to p$.

Now, consider an instance $(\neg\psi \to \chi \vee \xi) \to (\neg\psi \to \chi) \vee (\neg\psi \to \xi)$ of the scheme **KP**. Suppose towards a contradiction that s does not support this formula. Then there must be a substate $t \subseteq s$ such that $t \models \neg\psi \to \chi \vee \xi$ but $t \not\models \neg\psi \to \chi$ and $t \not\models \neg\psi \to \xi$.

The fact that $t \not\models \neg\psi \to \chi$ implies that there is a substate $u \subseteq t$ with $u \models \neg\psi$ but $u \not\models \chi$; similarly, since $t \not\models \neg\psi \to \xi$ there is $u' \subseteq t$ with $u' \models \neg\psi$ but $u' \not\models \xi$.

According to proposition 5, $\neg\psi$ is an assertion, so by proposition 4 we have $u \cup u' \models \neg\psi$. But $u \cup u'$ cannot support χ, otherwise by persistency we would have $u \models \chi$, which is not the case; similarly, $u \cup u'$ cannot support ξ, and thus also $u \cup u' \not\models \chi \vee \xi$.

But since $u \subseteq t$ and $u' \subseteq t$ we have $u \cup u' \subseteq t$: this shows that $t \not\models \varphi \to \chi \vee \xi$, contrarily to assumption. This yields the required contradiction and completes the proof of the lemma. □

We can now turn to completeness. We will prove completeness via the construction of a canonical model for GKP.

Definition 21. Let Θ be a set of formulas.

1. We say that Θ is a GKP$-$*theory* if it is closed under deduction in GKP, that is: for any φ, if $\Theta \vdash_{\text{GKP}} \varphi$, then $\varphi \in \Theta$.
2. We say that Θ has the *disjunction property* if whenever a formula $\varphi \vee \psi$ is in Θ, at least one of φ, ψ is in Θ.
3. We say that Θ is GKP-*consistent*, or simply *consistent*, in case $\Theta \nvdash_{\text{GKP}} \bot$.
4. We say that Θ is *complete* in case for any formula φ, exactly one of $\varphi, \neg\varphi$ is in Θ.

We have the usual Lindenbaum-type lemma:

Lemma 2. If $\Theta \nvdash_{\text{GKP}} \varphi$, then there is a consistent GKP$-$theory Γ with the disjunction property such that $\Theta \subseteq \Gamma$ and $\varphi \notin \Gamma$.

Proof. Let $(\psi_n)_{n\in\omega}$ be an enumeration of all formulae. Define:

$$\begin{aligned}
\Gamma_0 &= \Theta \\
\Gamma_{n+1} &= \begin{cases} \Gamma_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \nvdash \varphi \\ \Gamma_n & \text{otherwise} \end{cases} \\
\Gamma &= \textstyle\bigcup_{n\in\omega}\Gamma_n
\end{aligned}$$

Obviously, $\Theta \subseteq \Gamma$. It is immediate to check that by induction that for any n, $\Gamma_n \nvdash \varphi$, whence $\Gamma \nvdash \varphi$ and in particular $\varphi \notin \Gamma$.

Moreover, Γ is an GKP-theory. For, suppose $\psi_n \notin \Gamma$: then $\psi_n \notin \Gamma_{n+1}$, which is only possible in case $\Gamma_n \cup \{\psi_n\} \vdash \varphi$; but then also $\Gamma \cup \{\psi_n\} \vdash \varphi$: therefore $\Gamma \nvdash \psi_n$, since otherwise it would follow $\Gamma \vdash \varphi$, which is not the case.

Finally, Γ has the disjunction property. For, suppose ψ, ψ' are not in Γ. This implies that $\Gamma \cup \{\psi\} \vdash \varphi$ and $\Gamma \cup \{\psi'\} \vdash \varphi$; but then also $\Gamma \cup \{\psi \vee \psi'\} \vdash \varphi$: this entails $\psi \vee \psi' \notin \Gamma$, since otherwise we would have $\Gamma \vdash \varphi$, which is not the case. □

Definition 22. The *canonical model for* GKP is the model $M_{\text{GKP}} = (W_{\text{GKP}}, \subseteq, V_{\text{GKP}})$, where:

- W_{GKP} is the set of all consistent GKP$-$theories with the disjunction property;
- for any propositional letter p, $V_{\text{GKP}}(p) = \{\Gamma \in W_{\text{GKP}} \mid p \in \Gamma\}$.

Note that $\subseteq$ is reflexive and transitive and that the valuation V_{GKP} is persistent, whence M_{GKP} is an intuitionistic Kripke model. As customary in this sort of proofs, the next step is to prove the *truth lemma*, stating that for all points in the canonical model, truth coincides with membership.

Lemma 3 (Truth Lemma)**.** For all formulas φ and points $\Gamma \in W_{\text{GKP}}$ we have $M_{\text{GKP}}, \Gamma \Vdash \varphi \iff \varphi \in \Gamma$.

Proof. By induction on φ, using lemma 2 in the inductive step for implication. $\square$

Since, according to theorem 1, InqL is the logic of *saturated* Kripke models, in order for M_{GKP} to be of any use as a countermodel we have to show that it *is* in fact saturated. However, in order to do so we first need some properties of the canonical model.

Lemma 4 (Endpoints of M_{GKP})**.**

1. The endpoints of M_{GKP} are precisely the complete theories.
2. For any two endpoints Δ, Δ' in M_{GKP}, if Δ and Δ' contain the same propositional letters, then $\Delta = \Delta'$.
3. For any endpoint Δ there is a formula γ_Δ such that for all endpoints Δ', $\Delta' \Vdash \gamma_\Delta \iff \Delta' = \Delta$.

Proof.

1. Let Γ be a complete theory: it is clear that Γ is a consistent GKP-theory with the disjunction property. It remains to see that Γ is an endpoint of M_{GKP}. Suppose towards a contradiction that $\Gamma \subsetneq \Gamma'$ for some $\Gamma' \in W_{\text{GKP}}$. Take a $\gamma \in \Gamma' - \Gamma$; since $\gamma \notin \Gamma$, $\neg\gamma \in \Gamma \subseteq \Gamma'$. But then Γ' contains both γ and $\neg\gamma$ and so it is inconsistent, which contradicts $\Gamma' \in W_{\text{GKP}}$.

   Conversely, let Γ be an endpoint of M_{GKP}. Consider any formula φ; since Γ is consistent, at most one of φ and $\neg\varphi$ can be in Γ.

   Moreover, suppose towards a contradiction that neither φ nor $\neg\varphi$ were in Γ; then $\Gamma \cup \{\varphi\}$ would be consistent and could therefore be extended (by lemma 2) to a point Γ' of M_{GKP}. But then we would have $\Gamma \subsetneq \Gamma'$, so Γ' would be a successor of Γ different from Γ, contradicting the fact that Γ is an endpoint. Thus exactly one of $\varphi, \neg\varphi$ is in Γ, and by the generality of φ this means that Γ is a complete theory.
2. It is easy to check by induction that complete theories behave classically, in the sense that for any φ, whether φ is forced at Δ is determined truth-functionally in the classical way by which propositional letters are true at Δ.

   By the truth lemma, this means that if Δ and Δ' contain the same atoms, then $\Delta = \Delta'$.
3. For any endpoint $\Delta \in W_{\text{GKP}}$, we let $\gamma_\Delta := \bigwedge_{p \in P} {\delta_p}^\Delta$ where:

   $$ {\delta_p}^\Delta = \begin{cases} p & \text{if } p \in \Delta \\ \neg p & \text{if } p \notin \Delta \end{cases} $$

   Obviously, $\Delta \Vdash \gamma_\Delta$. Conversely, if an endpoint Δ' satisfies γ_Δ, then it must satisfy *exactly* the same propositional letters as Δ, and thus by the previous point of this lemma, $\Delta = \Delta'$. $\square$

We are now ready for the core of the completeness proof: showing that M_{GKP} is a saturated Kripke model.

Lemma 5. The model M_{GKP} is saturated.

Proof. Let us start by showing that M_{GKP} is grounded. Consider any $\Gamma \in W_{\text{GKP}}$ and reason by contraposition. Suppose that for no successor Γ' of Γ we have $\Gamma' \Vdash \neg p$. Then $\Gamma \Vdash \neg\neg p$ and so, by the truth lemma, $\neg\neg p \in \Gamma$. By the presence of the double negation axiom $\neg\neg p \to p$ in the system GKP, this implies that $\Gamma \vdash_{\text{GKP}} p$ and so, since Γ is an GKP$-$theory, that $p \in \Gamma$, whence - again by the truth lemma - $\Gamma \Vdash p$. Thus, contrapositively, if $\Gamma \not\Vdash p$ then there is a successor Γ' of Γ with $\Gamma' \Vdash \neg p$. This shows that M_{GKP} is grounded.

Next, consider E-SATURATION. Take any point $\Gamma \in W_{\text{GKP}}$. It is easy to see that Γ can be extended to a complete theory Δ: in order to do so, perform the procedure described in the proof of lemma 2 with $\Theta = \Gamma$ and $\varphi = \bot$. Now, $\Gamma \subseteq \Delta$ and Δ is an endpoint by lemma 4: therefore $E_\Gamma \neq \emptyset$. This shows that M_{GKP} is E-SATURATED.

Finally, let us come to I-SATURATION. Consider a point $\Gamma \in W_{\text{GKP}}$ and let E_* be a non-empty subset of E_Γ where, as usual, E_Γ denotes the set of endpoints of Γ. We must find a consistent GKP-theory $\Gamma' \supseteq \Gamma$ with the disjunction property such that $E_{\Gamma'} = E_*$.

Let Γ' be the deductive closure (in GKP) of the set $\Gamma \cup \{\neg\chi \mid \neg\chi \in \bigcap E_*\}$. We will make use of the following characterization of elements of Γ'.

Lemma 6. A formula φ is in Γ' if and only if there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\text{GKP}} \gamma \wedge \neg\chi \to \varphi$.

Proof. Suppose $\varphi \in \Gamma'$: by the deduction theorem for intuitionistic logic, there are $\gamma_1, \dots, \gamma_n \in \Gamma$, $\neg\chi_1, \dots, \neg\chi_m \in \bigcup E_*$ such that

$$ \vdash_{\text{GKP}} \gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\chi_1 \wedge \cdots \wedge \neg\chi_m \to \varphi $$

But now, since all the γ_i are in Γ and Γ is closed under GKP-deduction, the formula $\gamma := \gamma_1 \wedge \cdots \wedge \gamma_n$ is in Γ. Analogously, for each $\Delta \in E_*$, since all the $\neg\chi_i$ are in Δ and Δ is closed under GKP-deduction, Δ also contains the formula $\neg\chi$ where $\chi := \chi_1 \vee \cdots \vee \chi_n$; so $\neg\chi \in \bigcap E_*$. Finally, since $\neg\chi$ is interderivable with $\neg\chi_1 \wedge \cdots \wedge \neg\chi_n$ in intuitionistic logic (and thus in GKP) we can conclude $\vdash_{\text{GKP}} \gamma \wedge \neg\chi \to \varphi$. The converse implication is trivial. □

Having established lemma 6, we return to the proof of lemma 5. We are going to show that Γ' is a consistent GKP-theory with the disjunction property (hence a point in M_{GKP}), that it is a successor of Γ and has precisely E_* as its set of terminal successors. Thus Γ' is exactly the point whose existence we were required to show in order to establish that M_{GKP} is I-SATURATED.

- Γ' is a GKP−theory by definition.

- Γ' is consistent. For, suppose towards a contradiction $\bot \in \Gamma'$. Then by lemma 6 there would be $\gamma \in \Gamma$, $\neg\chi \in \bigcap E_*$ such that $\vdash_{\text{GKP}} \gamma \wedge \neg\chi \to \bot$. Since E_* is non-empty, consider a $\Delta \in E_*$: Δ is a successor of Γ (because $E_* \subseteq E_\Gamma$), so $\gamma \in \Delta$. But also $\neg\chi \in \Delta$, because $\neg\chi \in \bigcap E_*$ and $\Delta \in E_*$. Therefore, since Δ is an GKP−theory, we would have $\bot \in \Delta$. But this is absurd, since Δ is a point of the canonical model and thus consistent by definition.

- Γ' has the disjunction property. Suppose $\varphi \vee \psi \in \Gamma'$: then there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\text{GKP}} \gamma \wedge \neg\chi \to \varphi \vee \psi$. Since Γ is a GKP−theory, $(\gamma \wedge \neg\chi \to \varphi \vee \psi) \in \Gamma$, and so since $\gamma \in \Gamma$, also $\neg\chi \to \varphi \vee \psi$ is in Γ.

  But now, since GKP contains all instances of the Kreisler-Putnam axiom, and since Γ is closed under GKP−deduction, $(\neg\chi \to \varphi \vee \psi) \in \Gamma$ implies $(\neg\chi \to \varphi) \vee (\neg\chi \to \psi) \in \Gamma$; thus, since Γ has the disjunction property, at least one of $\neg\chi \to \varphi$ and $\neg\chi \to \psi$ is in Γ.

  Suppose the former is the case: then $\Gamma \cup \{\neg\chi\} \vdash_{\text{GKP}} \varphi$, and since $\neg\chi \in \bigcap E_*$ this implies $\varphi \in \Gamma'$, by definition of Γ'. Instead, if it is $\neg\chi \to \psi \in \Gamma$, then reasoning analogously we come to the conclusion $\psi \in \Gamma'$.

  In either case, one of φ, ψ must be in Γ', and this proves that Γ' has the disjunction property.

- Γ' is a successor of Γ, because $\Gamma' \supseteq \Gamma$ by definition.

- $E_* \subseteq E_{\Gamma'}$. To see this, take any $\Delta \in E_*$; we are going to show that $\Gamma' \subseteq \Delta$.

  If $\varphi \in \Gamma'$, there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\text{GKP}} \gamma \wedge \neg\chi \to \varphi$. Since $E_* \subseteq E_\Gamma$, we have $\Gamma \subseteq \Delta$ and thus $\gamma \in \Delta$; on the other hand, since $\Delta \in E_*$, also $\neg\chi \in \Delta$. So, both $\gamma, \neg\chi$ are in Δ which is a GKP-theory, and therefore $\varphi \in \Delta$.

  This shows $\Gamma' \subseteq \Delta$, and thus, since Δ is an endpoint, that $\Delta \in E_{\Gamma'}$.

- $E_{\Gamma'} \subseteq E_*$. Proceed by contraposition. Consider any endpoint $\Delta \notin E_*$. Let γ_Δ be the characteristic formula of the endpoint Δ given by lemma 4: for any $\Delta' \in E_*$, $\Delta' \neq \Delta$ and thus $M_{\text{GKP}}, \Delta' \not\Vdash \gamma_\Delta$; since Δ' is an endpoint, this implies $M_{\text{GKP}}, \Delta' \Vdash \neg\gamma_\Delta$, whence by the truth lemma $\neg\gamma_\Delta \in \Delta'$.

  But since this is the case for any $\Delta' \in E_*$ we have $\neg\gamma_\Delta \in \bigcap E_*$ and therefore $\neg\gamma_\Delta \in \Gamma'$ by definition of Γ'.

  On the other hand, $\Delta \models \gamma_\Delta$ and thus, by the truth lemma, $\neg\gamma_\Delta \notin \Delta$. We conclude that $\neg\gamma_\Delta$ is in $\Gamma' - \Delta$, and thus $\Gamma' \not\subseteq \Delta$, which means that $\Delta \notin E_{\Gamma'}$. □

Proof of theorem 2 (concluded). Suppose $\Theta \not\vdash_{\text{GKP}} \varphi$: then by lemma 2 there is a point $\Gamma \in W_{\text{GKP}}$ with $\Theta \subseteq \Gamma$ and $\varphi \notin \Gamma$. So the truth lemma implies $M_{\text{GKP}}, \Gamma \Vdash \theta$ for all $\theta \in \Theta$ but $M_{\text{GKP}}, \Gamma \not\Vdash \varphi$. But we have seen that M_{GKP} is a saturated model: thus by theorem 1, $\Theta \not\models_{\mathsf{InqL}} \varphi$. □