Supplement To: Directional limits on persistent gravitational waves using data from Advanced LIGO's first two observing runs

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In this technical supplement we provide additional formulas to support the main text. This closely follows the discussion given in [1].

Following past analyses, we assume that we can factorize $\Omega_{\rm gw}(f,\Theta)$ into frequency and sky-direction-dependent terms

$$\Omega_{\rm gw}(f,\Theta) = \frac{2\pi^2}{3H_0^2} f^3 H(f)\mathcal{P}(\Theta). \qquad \text{(A1)}$$

Note that this quantity has units of the dimensionless energy density parameter per steradian. For the radiometer searches it is useful to define a different representation in terms of energy flux,

$$\mathcal{F}(f,\Theta) = \frac{c^3\pi}{4G} f^2 H(f)\mathcal{P}(\Theta), \qquad \text{(A2)}$$

which has units of $\text{erg}\,\text{s}^{-1}\,\text{cm}^{-2}\,\text{Hz}^{-1}\,\text{sr}^{-1}$, where c is the speed of light and G is Newton's gravitational constant.

We use two different representations to estimate the angular power, $\mathcal{P}(\Theta)$. The *radiometer* method [2, 3] (in both the broadband (BBR) and narrowband (NBR) applications) is optimized for a small number of resolvable, separated point sources on the sky, and so we estimate the angular power in terms of point sources by decomposing onto delta functions

$$\mathcal{P}(\Theta) = \mathcal{P}_{\Theta_0}\delta^2(\Theta,\Theta_0). \qquad \text{(A3)}$$

The radiometer method assumes that the sources are well-localized on the sky (that is, to within one pixel), and so it is not well-suited to sources which are spread over a large solid angle.

To characterize diffuse sources of GWs, we use the *spherical harmonic decomposition* (SHD) [4]. We write the angular power in terms of a sum over spherical harmonics, $Y_{lm}(\Theta)$, with amplitude coefficients, $\mathcal{P}_{lm}$

$$\mathcal{P}(\Theta) = \sum_{l=0}^{l_{\rm max}} \sum_{m=-l}^{l} \mathcal{P}_{lm} Y_{lm}(\Theta). \qquad \text{(A4)}$$

In principle, $l_{\rm max}$ should be infinite. However in practice, we must take a finite value of $l_{\rm max}$. The optimal choice for $l_{\rm max}$ depends upon the spatial separation and the sensitivity curve of the detectors. We use the same choice as in the previous analysis [1], taking $l_{\rm max} = \{3, 4, 16\}$ for the 3 spectral indices.

By construction, the NBR search looks for signals in a narrow range of frequency bins. On the other hand, for the broadband SHD and BBR searches, we must make an additional assumption about the spectral shape of the source. We assume that the GW power spectrum takes a power-law form,

$$H(f) = \left(\frac{f}{f_{\rm ref}}\right)^{\alpha-3}. \qquad \text{(A5)}$$

In this case, the power in each direction is characterized by a spectral index, α, and the amplitude of the energy density or flux at a given reference frequency, $f_{\rm ref}$. As described in the main text, we choose $f_{\rm ref} = 25\,\text{Hz}$ and search for and set limits on spectral indices of $\alpha = (0,\ 2/3,\ 3)$.

Given the spectral shape $H(f)$, we define the following quantities which are used to construct the sky maps. For the BBR search it is convenient to consider the flux F_{α,Θ_0},

$$F_{\alpha,\Theta_0} \equiv \int d^2\Theta\, \mathcal{F}(f_{\rm ref},\Theta) = \frac{c^3\pi}{4G} f_{\rm ref}^2 \mathcal{P}_{\Theta_0}. \qquad \text{(A6)}$$

For the SHD search, we will use the dimensionless energy density per unit sky area

$$\Omega_\alpha(\Theta) \equiv \Omega_{\rm gw}(f_{\rm ref},\Theta) = \frac{2\pi^2}{3H_0^2} f_{\rm ref}^3 \mathcal{P}(\Theta). \qquad \text{(A7)}$$

For more details, see [1].

The starting point of the stochastic analysis is the cross-correlation function $C(f;t)$, which is given by

$$C(f;t) = \frac{2}{T}\tilde{s}_1^*(f;t)\tilde{s}_2(f;t), \qquad \text{(A8)}$$

where $s_i(f;t)$ is the Fourier transform of length T of the data from detector i at time t. To produce a sky map, we convolve $C(f,t)$ with the generalized overlap reduction function $\gamma_\mu(f,t)$, which encodes the time delay between the detectors and the detector response (see [4] for an explicit definition). We construct the dirty-map X_μ (see below)

$$X_\mu = \sum_{f,t} \gamma_\mu^*(f,t) \frac{H(f)}{P_1(f;t)P_2(f;t)} C(f;t). \qquad \text{(A9)}$$

Here, $P_i(f;t)$ is the (one-sided) power spectral density of the noise in detector i and $H(f)$ is the chosen spectral model. We use Greek indices $\mu, \nu, \cdots$ to represent angular degrees of freedom. For the SHD search, μ, ν run over the spherical harmonic coefficients, e.g., $\mu \equiv (lm)$. For the BBR and NBR searches, μ, ν run over individual sky directions (pixels).

The quantity X_μ is called the "dirty map" because it does not faithfully represent the true gravitational-wave power on the sky. In order to obtain the true power, following [4] we introduce the Fisher information matrix, $\Gamma_{\mu\nu}$, which encodes the beam pattern of the detector network

$$\Gamma_{\mu\nu} = \sum_{f,t} \gamma^*_\mu(f,t) \frac{H^2(f)}{P_1(f;t)P_2(f;t)} \gamma_\nu(f;t). \tag{A10}$$

We can get an estimate of the GW power by inverting the Fisher matrix, $\hat{\mathcal{P}}_{\mu\nu} = \Gamma^{-1}_{\mu\nu} X_\nu$. In the case of the BBR and NBR, we ignore correlations between neighboring pixels, and so we don't perform a full matrix inversion, instead taking the inverse of the diagonal elements of the Fisher matrix:

$$\hat{\mathcal{P}}_\Theta = (\Gamma_{\Theta\Theta})^{-1} X_\Theta, \tag{A11}$$

$$\sigma_\Theta = (\Gamma_{\Theta\Theta})^{-1/2}. \tag{A12}$$

In the case of the spherical harmonics, we formally construct an unbiased estimator of the clean map (i.e., the physical map of GW power) using a maximum likelihood estimator [4]

$$\hat{\mathcal{P}}_{lm} = \sum_{l'm'} [\Gamma_{\rm R}^{-1}]_{lm,l'm'} X_{l'm'}. \tag{A13}$$

The Fisher matrix is degenerate because of the existence of blind spots in the detector network, as well as the diffraction limit, [4]. As a result we need to regularize the Fisher matrix to define an inverse.

Construction of a clean map in the pixel basis, as opposed to the spherical harmonics basis, has been proposed [3–5] and implemented on O1 data [6]. This method also suffers from a poorly-conditioned Fisher information matrix, but given that it avoids a spherical harmonics decomposition entirely, it does not use an initial cut-off in $l_{\max}$ as our search does.

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