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DOI
10.7561/SACS.2013.2.169

Publication date
2013

Document Version
Final published version

Published in
Scientific Annals of Computer Science

Scientific Annals of Computer Science vol. 23 (2), 2013, pp. 169–190
doi: 10.7561/SACS.2013.2.169

Timed Tuplix Calculus and the Wesseling and van den Bergh Equation

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Abstract

We develop an algebraic framework for the description and analysis of financial behaviours, that is, behaviours that consist of transferring certain amounts of money at planned times. To a large extent, analysis of financial products amounts to analysis of such behaviours. We formalize the cumulative interest compliant conservation requirement for financial products proposed by Wesseling and van den Bergh by an equation in the framework developed and define a notion of financial product behaviour using this formalization. We also present some properties of financial product behaviours. The development of the framework has been influenced by previous work on the process algebra ACP.

Keywords: timed tuplix calculus, realistic interest calculation axiom, Wesseling and van den Bergh equation, financial product behaviour, signed cancellation meadow

1 Introduction

Analysis of financial products amounts to a large extent to analysis of behaviours that consist of transferring certain amounts of money at planned times. In this paper, such behaviours are called financial behaviours. Mathematically precise analysis of financial products is complicated by the lack of a specialized mathematical framework for the description and analysis of financial behaviours. The main objective of the work presented in this paper is to devise such a framework. We aim at an algebraic framework,

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that is, a framework in which operators enable us to describe a financial behaviour as a behaviour composed of several other financial behaviours and equational axioms enable us to analyze a described financial behaviour by mere algebraic calculations. Our intuitive understanding of the nature of financial behaviours will provide the primary justification of the equations that are taken as axioms.

To achieve our main objective, we develop an extension of the core of tuplix calculus that can deal with the timing of transfers involved in financial behaviours. Tuplix calculus was presented for the first time in [6] and has among other things been applied in modular financial budget design. The extension of the core of tuplix calculus developed in this paper is called timed tuplix calculus. The operators added to the core of tuplix calculus in this extension are comparable to operators introduced earlier in the setting of the process algebra ACP [4]. In the core of tuplix calculus as well as timed tuplix calculus, the mathematical structure for quantities is a signed cancellation meadow [3]. The prime examples of cancellation meadows are the fields of rational and real numbers with the multiplicative inverse operation made total by imposing that the multiplicative inverse of zero is zero. A cancellation meadow is an appropriate mathematical structure for quantities. A signed cancellation meadow is a cancellation meadow expanded with a signum operation.

In [15], Wesseling and van den Bergh formulate a cumulative interest compliant conservation requirement for financial products: the sum of all transfers relating to the product, transposed to some point of time (the focal date) by means of cumulative interest at the effective interest rate of the product, is zero. As an example of the use of timed tuplix calculus, we formalize this conservation requirement by an equation in timed tuplix calculus. Unaware of previous occurrences of the requirement in the financial literature, we call this equation the Wesseling and van den Bergh equation. Using this equation, we define a notion of financial product behaviour. A financial product behaviour can be seen as a financial behaviour for which a financial product can be devised that involves that behaviour.

In addition to that, we adapt the notion of implicit capital of a process introduced in [5] to the current setting. The implicit capital associated with a financial behaviour can be seen as the least amount of money that must be at disposal initially to exhibit that behaviour, taking cumulative interest into account. We use this notion to show that financial behaviours may profit from using some financial product. We also present some other properties of financial product behaviours.

This paper is organized as follows. First, we give a brief summary of signed cancellation meadows (Section 2). Next, we review the core of tuplix calculus (Section 3). Then, we extend the core of tuplix calculus to timed tuplix calculus (Section 4). After that, we formalize the conservation requirement for financial products, define a notion of financial product behaviour, and present some properties of financial product behaviours (Section 5). Following this, we construct the standard model of the timed tuplix calculus (Section 6). Finally, we make some concluding remarks (Section 7).

2 Signed Cancellation Meadows

In the timed tuplix calculus presented in this paper, the mathematical structure for quantities is a signed cancellation meadow. In this section, we give a brief summary of signed cancellation meadows.

A meadow is a field with the multiplicative inverse operation made total by imposing that the multiplicative inverse of zero is zero. A cancellation meadow is a meadow in which the multiplicative inverse operation satisfies the general inverse law (given below). A signed meadow is a meadow expanded with a signum operation. Meadows were defined for the first time in [8] and elaborated in several subsequent papers. The expansion of meadows with a signum operation originates from [3]. In the latter paper, references are made to the key papers on meadows.

The signature of meadows consists of the following constants and operators:

- the constants 0 and 1;
- the binary *addition* operator $+$;
- the binary *multiplication* operator $\cdot$;
- the unary *additive inverse* operator $-$;
- the unary *multiplicative inverse* operator ${}^{-1}$.

We assume that there are infinitely many variables, including u, v and w. Terms are build as usual. We use infix notation for the binary operators $+$ and $\cdot$, prefix notation for the unary operator $-$, and postfix notation for

Table 1: Equations for meadows

$(u+v)+w = u+(v+w)$	$(u \cdot v) \cdot w = u \cdot (v \cdot w)$	$(u^{-1})^{-1} = u$
$u+v = v+u$	$u \cdot v = v \cdot u$	$u \cdot (u \cdot u^{-1}) = u$
$u+0 = u$	$u \cdot 1 = u$	
$u+(-u) = 0$	$u \cdot (v+w) = u \cdot v + u \cdot w$	

the unary operator $^{-1}$. We use the usual precedence convention to reduce the need for parentheses. We introduce subtraction and division as abbreviations: $p - q$ abbreviates $p + (-q)$ and p/q abbreviates $p \cdot q^{-1}$. We use numerals in the common way (2 abbreviates $1 + 1$, etc.). We also use the notation p^n for exponentiation with a natural number as exponent. For each term p over the signature of meadows, the term p^n is defined by induction on n as follows: $p^0 = 1$ and $p^{n+1} = p^n \cdot p$.

The constants and operators from the signature of meadows are adopted from rational arithmetic, which gives an appropriate intuition about these constants and operators.

A meadow is an algebra over the signature of meadows that satisfies the equations given in Table 1. Thus, a meadow is a commutative ring with identity equipped with a multiplicative inverse operation $^{-1}$ satisfying the reflexivity equation $(u^{-1})^{-1} = u$ and the restricted inverse equation $u \cdot (u \cdot u^{-1}) = u$. From the equations given in Table 1, the equation $0^{-1} = 0$ is derivable (see [8]).

In meadows, the multiplicative inverse operation is total. The advantage of working with a total multiplicative inverse operation lies in the fact that conditions like $u \neq 0$ in $u \neq 0 \Rightarrow u \cdot u^{-1} = 1$ are not needed to guarantee meaning.

A *non-trivial meadow* is a meadow that satisfies the *separation axiom*

$$0 \neq 1 \;;$$

and a *cancellation meadow* is a meadow that satisfies the *cancellation axiom*

$$u \neq 0 \wedge u \cdot v = u \cdot w \Rightarrow v = w \;,$$

or equivalently, the *general inverse law*

$$u \neq 0 \Rightarrow u \cdot u^{-1} = 1 \;.$$

Table 2: Equations for signum operation

$\mathsf{s}(u/u) = u/u$	$\mathsf{s}(u^{-1}) = \mathsf{s}(u)$
$\mathsf{s}(1 - u/u) = 1 - u/u$	$\mathsf{s}(u \cdot v) = \mathsf{s}(u) \cdot \mathsf{s}(v)$
$\mathsf{s}(-1) = -1$	$(1 - \frac{\mathsf{s}(u)-\mathsf{s}(v)}{\mathsf{s}(u)-\mathsf{s}(v)}) \cdot (\mathsf{s}(u + v) - \mathsf{s}(u)) = 0$

Important properties of non-trivial cancellation meadows are $u/u = 0 \Leftrightarrow u = 0$ and $u/u = 1 \Leftrightarrow u \neq 0$.

A *signed meadow* is a meadow expanded with a unary *signum* operation s satisfying the equations given in Table 2. In combination with the cancellation axiom, the last equation in this table is equivalent to the conditional equation $\mathsf{s}(u) = \mathsf{s}(v) \Rightarrow \mathsf{s}(u + v) = \mathsf{s}(u)$.

In signed cancellation meadows, the function max is defined as follows:

$$\max(u, v) = \frac{\mathsf{s}(u - v) + 1}{2} \cdot (u - v) + v \ .$$

We will write:

$$p > q \ \text{ for } \ \frac{1 - \mathsf{s}(p - q)}{1 - \mathsf{s}(p - q)} = 0 \ , \qquad p \leq q \ \text{ for } \ \frac{1 - \mathsf{s}(p - q)}{1 - \mathsf{s}(p - q)} = 1 \ .$$

3 Core Tuplix Calculus and Encapsulation

The timed tuplix calculus presented in this paper extends CTC (Core Tuplix Calculus). CTC has been introduced in [6] as the core of TC (Tuplix Calculus). In this section, we give a brief summary of CTC and its extension with encapsulation operators. These operators have been introduced in [6] as well. The operators of the timed tuplix calculus that will be introduced in Section 4 include generalizations of the encapsulation operators.

It is assumed that a fixed but arbitrary set A of *transfer actions* has been given. It is also assumed that a fixed but arbitrary signed non-trivial cancellation meadow $\mathcal{D}$ has been given. $\mathcal{D}$ is adopted as the mathematical structure for quantities.

CTC has two sort: the sort $\mathbf{T}$ of *tuplices* and the sort $\mathbf{Q}$ of *quantities*. To build terms of sort $\mathbf{T}$, it has the following constants and operators:

- the *empty tuplix* constant $\epsilon : \mathbf{T}$;
- the *blocking tuplix* constant $\delta : \mathbf{T}$;

- for each $a \in A$, the unary *transfer action* operator $a : \mathbf{Q} \to \mathbf{T}$;
- the unary *zero test* operator $\gamma : \mathbf{Q} \to \mathbf{T}$;
- the binary *conjunctive composition* operator $\oplus : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$.

To build terms of sort $\mathbf{Q}$, CTC has the constants and operators from the signature of meadows.

We assume that there are infinitely many variables of sort $\mathbf{T}$, including x, y and z, and infinitely many variables of sort $\mathbf{Q}$, including u, v and w. Terms are build as usual for a many-sorted signature (see e.g. [13, 16]). We use infix notation for the binary operator $\oplus$.

A term of sort $\mathbf{T}$ is *tuplix-closed* if it does not contain variables of sort $\mathbf{T}$. A term of sort $\mathbf{T}$ is *closed* if it does not contain variables of any sort.

We look at CTC as a calculus that is concerned with transfers of quantities of something. Let t and t' be closed terms of sort $\mathbf{T}$, and let q be a closed term of sort $\mathbf{Q}$. Intuitively, the constants and operators introduced above can be explained as follows:

- ϵ is a tuplix with no effect;
- δ blocks any joint effect of tuplices;
- the effect of $a(q)$ is performing action a and transferring quantity q on performing that action;
- $\gamma(q)$ is a tuplix with no effect if q equals 0 and blocks any joint effect otherwise;
- the effect of $t \oplus t'$ is the joint effect of t and t'.

In [6], these constants and operators are explained in a different way. We consider that way of explanation less appropriate for the timed extension of CTC that will be presented in Section 4.

We use the following convention: a transfer of a positive quantity is taken as an outgoing transfer and a transfer of a negative quantity is taken as an incoming transfer.

Notice that CTC can be looked upon as a special purpose process algebra in which processes are considered at a level of detail where not even the order in which actions are performed matter. This makes CTC suitable for formalizing budgets: budgets are in fact descriptions of financial

Table 3: Axioms of CTC

$x \oplus y = y \oplus x$	T1	$\gamma(u) = \gamma(u/u)$	T6
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$	T2	$\gamma(0) = \epsilon$	T7
$x \oplus \epsilon = x$	T3	$\gamma(1) = \delta$	T8
$x \oplus \delta = \delta$	T4	$\gamma(u) \oplus \gamma(v) = \gamma(u/u + v/v)$	T9
$a(u) \oplus a(v) = a(u + v)$	T5	$\gamma(u - v) \oplus a(u) = \gamma(u - v) \oplus a(v)$	T10

behaviour at the level of detail where only the actions to be performed and the quantities transferred on performing those actions matter (see also [7]).

The axioms of CTC are given in Table 3. The following proof rule is adopted to lift the valid equations between terms of sort **Q** to CTC:

$$\text{for all terms } p \text{ and } q \text{ of sort } \mathbf{Q}, \quad \mathcal{D} \models p = q \quad \text{implies} \quad \gamma(p) = \gamma(q)\ .$$

We will refer to this proof rule by DE.

To prove a statement for all CTC terms of sort **T**, it is is sufficient to prove it for all CTC canonical terms. A *CTC canonical term* is a CTC term of sort **T** of the form

$$\gamma(p_0) \oplus a_1(p_1) \oplus \ldots \oplus a_k(p_k) \oplus x_1 \oplus \ldots \oplus x_l\ ,$$

where $k, l \geq 0$ and $a_1, \ldots, a_k$ are distinct transfer actions.

Lemma 1 *For all* CTC *terms t of sort* **T**, *there exists a* CTC *canonical term t' such that $t = t'$ is derivable from the axioms of* CTC.

Proof: This proposition is a reformulation of Lemma 1 from [6]. □

Like in [6], we can add the following operators to the operators of CTC to build terms of sort **T**:

- for each $H \subseteq A$, the unary *encapsulation* operator $\partial_H : \mathbf{T} \to \mathbf{T}$.

Let t be a closed term of sort **T**. Intuitively, the encapsulation operators can be explained as follows:

- if, for each $a \in H$, the sum of all quantities transferred by t on performing a equals 0, then $\partial_H(t)$ differs from t in that, for each $a \in H$, the effect of all transfer actions of the form $a(p)$ occurring in t is eliminated; otherwise, $\partial_H(t)$ has the same effect as δ.

Table 4: Axioms for encapsulation

$\partial_H(\epsilon) = \epsilon$	E1	$\partial_H(a(u)) = a(u)$	if $a \notin H$	E4
$\partial_H(\delta) = \delta$	E2	$\partial_H(a(u)) = \gamma(u)$	if $a \in H$	E5
$\partial_H(\gamma(u)) = \gamma(u)$	E3	$\partial_H(x \oplus \partial_H(y)) = \partial_H(x) \oplus \partial_H(y)$		E6
		$\partial_{H \cup H'}(x) = \partial_H(\partial_{H'}(x))$		E7

The name encapsulation was introduced earlier in the setting of the process algebra ACP for similar operations in [4].

The axioms for encapsulation are given in Table 4.

4 Timed Tuplix Calculus

In this section, we extend CTC to TTC (Timed Tuplix Calculus). In the informal explanation of the constants and operators of CTC given in Section 3, we could disregard what it is of which quantities are transferred. Clearly, if CTC is used to formalize budgets, quantities of money are transferred. It happens to be far from obvious to give informal explanations of two of the additional operators of TTC that are not couched in terms of quantities of money, usually called amounts of money. Therefore, we change over in this section to explanations couched in terms of amounts of money. This should not be taken as a suggestion that more abstract explanations are impossible. In Section 5, tuplices are viewed as representations of financial behaviours. The change-over made in this section agrees with this viewpoint.

Like CTC, TTC has two sort: the sort $\mathbf{T}$ of tuplices and the sort $\mathbf{Q}$ of quantities. To build terms of sort $\mathbf{T}$, it has the constants and operators of CTC to build terms of sort $\mathbf{T}$, and in addition the following operators:

- the unary *delay* operator $\sigma : \mathbf{T} \to \mathbf{T}$;
- for each $I \subseteq A$, the unary *pre-abstraction* operator $\mathtt{t}_I : \mathbf{T} \to \mathbf{T}$;
- for each $H \subseteq A$, the binary *interest counting encapsulation* operator $\partial_H : \mathbf{Q} \times \mathbf{T} \to \mathbf{T}$.

To build terms of sort $\mathbf{Q}$, it has the constants and operators from the signature of meadows, and in addition the following operator:

- the binary *implicit capital* operator $\mathrm{Q} : \mathbf{Q} \times \mathbf{T} \to \mathbf{Q}$.

We write $\partial_H^p(t)$ and $\mathrm{Q}^p(t)$, where p is a term of sort $\mathbf{Q}$ and t is a term of sort $\mathbf{T}$, for $\partial_H(p,t)$ and $\mathrm{Q}(p,t)$, respectively. We also use the notation $\sigma^n(t)$. For each term t of sort $\mathbf{T}$, the term $\sigma^n(t)$ is defined by induction on n as follows: $\sigma^0(t) = t$ and $\sigma^{n+1}(t) = \sigma(\sigma^n(t))$.

In TTC, it is assumed that $\mathtt{t} \in A$. A special role is assigned to $\mathtt{t}$: transfer actions of the form $a(p)$ are renamed to $\mathtt{t}(p)$ on pre-abstraction in order to abstract from their identity, but not from their presence.

We look at TTC as a calculus that is concerned with transfers of amounts of money on time. Let t be a closed term of sort $\mathbf{T}$ and let p be a closed term of sort $\mathbf{Q}$. Intuitively, the additional operators introduced above can be explained as follows:

- $\sigma(t)$ differs from t in that the effect of each transfer action occurring in t is delayed one time slice;
- $\mathtt{t}_I(t)$ differs from t in that, for each $a \in I$, the effect of each transfer action of the form $a(p)$ occurring in t is replaced by the effect of $\mathtt{t}(p)$;
- $\partial_H^p(t)$ differs from $\partial_H(t)$ in that, for each $a \in H$, a cumulative interest at the rate of p per time slice is taken into account on the summation of all amounts of money transferred by t on performing a;
- $\mathrm{Q}^p(t)$ is the least amount of money that must be at disposal initially to allow for each transfer action occurring in t to be performed if a cumulative interest at the rate of p per time slice is taken into account.

The delay operator introduced here is comparable to the relative discrete time unit delay operator and the absolute discrete time unit delay operator introduced earlier in the setting of the process algebra ACP in [2]. The pre-abstraction operators introduced here are comparable to the pre-abstraction operators introduced earlier in the setting of the process algebra ACP in [1]. The interest counting encapsulation operators are generalizations of the encapsulation operators introduced in Section 3: $\partial_H(t)$ can be taken as abbreviation of $\partial_H^0(t)$. The implicit capital operator introduced here is comparable to the implicit computational capital operator introduced earlier in the setting of the process algebra ACP in [5].

The implicit capital of a non-blocking tuplix is an amount of money that is non-negative, and the implicit capital of a blocking tuplix is undefined. In order to circumvent the use of algebras with partial operations, -1 is used to represent the undefinedness of the implicit capital of a blocking tuplix.

Table 5: Axioms for delay, pre-abstraction, interest counting encapsulation

$\sigma(\epsilon) = \epsilon$		D1			
$\sigma(\delta) = \delta$		D2			
$\sigma(\gamma(u)) = \gamma(u)$		D3	$\gamma(1 - \frac{1+u}{1+u}) \oplus \partial^u_{\{a\}}(a(v) \oplus x) =$		
$\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$		D4	$\gamma(1 - \frac{1+u}{1+u}) \oplus \partial^u_{\{a\}}(\sigma(a((1+u) \cdot v)) \oplus x)$		RICA
$\mathsf{t}_I(\epsilon) = \epsilon$		PA1	$\partial^u_H(\epsilon) = \epsilon$		ICE1
$\mathsf{t}_I(\delta) = \delta$		PA2	$\partial^u_H(\delta) = \delta$		ICE2
$\mathsf{t}_I(\gamma(u)) = \gamma(u)$		PA3	$\partial^u_H(\gamma(v)) = \gamma(v)$		ICE3
$\mathsf{t}_I(a(u)) = a(u)$	if $a \notin I$	PA4	$\partial^u_H(a(v)) = a(v)$	if $a \notin H$	ICE4
$\mathsf{t}_I(a(u)) = \mathsf{t}(u)$	if $a \in I$	PA5	$\partial^u_H(a(v)) = \gamma(v)$	if $a \in H$	ICE5
$\mathsf{t}_I(x \oplus y) = \mathsf{t}_I(x) \oplus \mathsf{t}_I(y)$		PA6	$\partial^u_H(x \oplus \partial^u_H(y)) = \partial^u_H(x) \oplus \partial^u_H(y)$		ICE6
$\mathsf{t}_I(\sigma(x)) = \sigma(\mathsf{t}_I(x))$		PA7	$\partial^u_H(\sigma(x)) = \sigma(\partial^u_H(x))$		ICE7
$\mathsf{t}_{I \cup I'}(x) = \mathsf{t}_I(\mathsf{t}_{I'}(x))$		PA8	$\partial^u_{H \cup H'}(x) = \partial^u_H(\partial^u_{H'}(x))$		ICE8

Notice that TTC can be looked upon as a special purpose timed process algebra in which processes are considered at a level of detail where the time slices in which actions are performed matter, but not their order within the time slices. This makes TTC suitable for analyzing financial products: financial products involve transfers of amounts of money where the day, week or month in which actions are performed and the amounts of money that are transferred in doing so are relevant, but not their order within the periods concerned.

The axioms of TTC are the axioms of CTC and the additional axioms given in Tables 5 and 6. Like in CTC, the proof rule DE is adopted to lift the valid equations between terms of sort **Q** to TTC.

Axiom RICA (Realistic Interest Calculation Axiom) is equivalent to

$$u \neq -1 \Rightarrow \partial^u_{\{a\}}(a(v) \oplus x) = \partial^u_{\{a\}}(\sigma(a((1+u) \cdot v)) \oplus x) \;.$$

This formula can be paraphrased as follows: when encapsulating a, reckoning with an interest rate u different from -1, an undelayed transfer of an amount v is equivalent to a transfer of an amount $(u+1) \cdot v$ in the next time slice. The exclusion of $u = -1$ prevents that the equation $x = \delta$ can be derived.

Table 6: Axioms for implicit capital

$\mathrm{Q}^u(x) = \mathrm{Q}^u(\mathsf{t}_A(x))$	IC1
$\mathrm{Q}^u(\epsilon) = 0$	IC2
$\mathrm{Q}^u(\delta) = -1$	IC3
$\mathrm{Q}^u(\mathsf{t}(v)) = \max(v, 0)$	IC4
$\frac{1+\mathrm{Q}^u(x)}{1+\mathrm{Q}^u(x)} \cdot \mathrm{Q}^u(\sigma(x)) = \frac{1+\mathrm{Q}^u(x)}{1+\mathrm{Q}^u(x)} \cdot \max(\frac{1}{1+u} \cdot \mathrm{Q}^u(x), 0)$	IC5
$\frac{1+\mathrm{Q}^u(x)}{1+\mathrm{Q}^u(x)} \cdot \mathrm{Q}^u(\mathsf{t}(v) \oplus \sigma(x)) = \frac{1+\mathrm{Q}^u(x)}{1+\mathrm{Q}^u(x)} \cdot \max(v + \frac{1}{1+u} \cdot \mathrm{Q}^u(x), 0)$	IC6

Axioms IC5 and IC6 are equivalent to

$$\mathrm{Q}^u(x) \neq -1 \Rightarrow \mathrm{Q}^u(\sigma(x)) = \max(\tfrac{1}{1+u} \cdot \mathrm{Q}^u(x), 0)\ ,$$

$$\mathrm{Q}^u(x) \neq -1 \Rightarrow \mathrm{Q}^u(\mathsf{t}(v) \oplus \sigma(x)) = \max(v + \tfrac{1}{1+u} \cdot \mathrm{Q}^u(x), 0)\ .$$

These formulas express that, reckoning with an interest rate u, the total contribution of all transfers in the next time slice to the implicit capital equals $\frac{1}{1+u}$ times what their total contribution would be in the current time slice. The exclusion of $\mathrm{Q}^u(x) = -1$ is needed because -1 is used to represent undefinedness.

Example 1 Let p be a closed term of sort $\mathbf{Q}$ such that $\mathcal{D} \models \frac{1+p}{1+p} = 1$. The following is a derivation from the axioms of TTC and the proof rule DE:

$$\begin{aligned}
&\partial^p_{\{a\}}(a(u) \oplus \sigma(a(5)) \oplus \sigma^2(b(u-7)))\\
&= \partial^p_{\{a\}}(a(u) \oplus a(\tfrac{5}{1+p}) \oplus \sigma^2(\partial^p_{\{a\}}(b(u-7))))\\
&= \partial^p_{\{a\}}(a(u + \tfrac{5}{1+p}) \oplus \partial^p_{\{a\}}(\sigma^2(b(u-7))))\\
&= \partial^p_{\{a\}}(a(u + \tfrac{5}{1+p})) \oplus \partial^p_{\{a\}}(\sigma^2(b(u-7)))\\
&= \partial^p_{\{a\}}(a(u + \tfrac{5}{1+p})) \oplus \sigma^2(\partial^p_{\{a\}}(b(u-7)))\\
&= \gamma(u + \tfrac{5}{1+p}) \oplus \sigma^2(b(u-7))\ .
\end{aligned}$$

Because $\mathcal{D} \models \frac{-5}{1+p} + \frac{5}{1+p} = 0$, it follows immediately that

$$\partial^p_{\{a\}}(a(\tfrac{-5}{1+p}) \oplus \sigma(a(5)) \oplus \sigma^2(b(\tfrac{-5}{1+p} - 7))) = \sigma^2(b(\tfrac{-5}{1+p} - 7))\ .$$

Moreover, it follows immediately that

$$\partial^p_{\{a\}}(a(q) \oplus \sigma(a(5)) \oplus \sigma^2(b(q-7))) = \delta$$

for all closed terms q of sort $\mathbf{Q}$ such that not $\mathcal{D} \models q + \frac{5}{1+p} = 0$.

Example 2 Let p and q be closed terms of sort $\mathbf{Q}$. The following is a derivation from the axioms of TTC and the proof rule DE:

$$\begin{aligned}
&\mathrm{Q}^p(a(7) \between \sigma(a'(-8)) \between b(-5) \between \sigma^2(b'((1+q)^2 \cdot 5))) \\
&= \mathrm{Q}^p(\mathsf{t}(7) \between \sigma(\mathsf{t}(-8)) \between \mathsf{t}(-5) \between \sigma^2(\mathsf{t}((1+q)^2 \cdot 5))) \\
&= \mathrm{Q}^p(\mathsf{t}(2) \between \sigma(\mathsf{t}(-8) \between \sigma(\mathsf{t}((1+q)^2 \cdot 5)))) \\
&= \max(2 + \tfrac{1}{1+p} \cdot \mathrm{Q}^p(\mathsf{t}(-8) \between \sigma(\mathsf{t}((1+q)^2 \cdot 5))), 0) \\
&= \max(2 + \tfrac{1}{1+p} \cdot \max(-8 + \tfrac{1}{1+p} \cdot \mathrm{Q}^p(\mathsf{t}((1+q)^2 \cdot 5)), 0), 0) \\
&= \max(2 + \tfrac{1}{1+p} \cdot \max(-8 + \tfrac{1}{1+p} \cdot (1+q)^2 \cdot 5, 0), 0)\ .
\end{aligned}$$

It follows immediately that

$$\mathrm{Q}^p(a(7) \between \sigma(a'(-8)) \between b(-5) \between \sigma^2(b'((1+q)^2 \cdot 5))) = 2$$

for all closed terms p and q of sort $\mathbf{Q}$ such that $\mathcal{D} \models \frac{1}{1+p} \cdot (1+q)^2 \leq \frac{8}{5}$. There are many such p and q, for example, p and q such that $\mathcal{D} \models p = \frac{1}{100}$ and $\mathcal{D} \models q = \frac{10}{100}$, but also p and q such that $\mathcal{D} \models p = \frac{25}{100}$ and $\mathcal{D} \models q = \frac{40}{100}$. We will return to this example in Section 5.

To prove a statement for all tuplix-closed TTC terms of sort $\mathbf{T}$, it is sufficient to prove it for all tuplix-closed TTC canonical terms. The set of *TTC canonical terms* is inductively defined by the following rules:

- if t is a CTC canonical term, then t is a TTC canonical term;
- if t is a CTC canonical term and t' is a TTC canonical term, then $t \between \sigma(t')$ is a TTC canonical term.

Lemma 2 *For all tuplix-closed* TTC *terms t of sort* $\mathbf{T}$*, there exists a tuplix-closed* TTC *canonical term t' such that $t = t'$ is derivable from the axioms of* TTC*.*

Proof: The proof is straightforward by induction on the structure of t, and in the cases $t \equiv \mathsf{t}_I(s)$ and $t \equiv \partial^p_H(s)$ (where we can restrict ourselves to tuplix-closed TTC canonical terms s) by induction on the structure of s. The following easy to prove fact is used in the proof for the case $t \equiv \partial^p_H(s)$: for all TTC terms t_1 of sort $\mathbf{T}$ and all tuplix-closed TTC terms t_2 of sort

$\mathbf{T}$ in which no element of H occurs, $\partial^u_H(t_1 \oplus t_2) = \partial^u_H(t_1) \oplus t_2$ is derivable from the axioms of TTC (cf. Lemma 5 in [6]). □

The following is a useful corollary of Lemma 2.

Corollary 1 *For all tuplix-closed* TTC *terms t of sort* $\mathbf{T}$*, there exists a tuplix-closed* TTC *term t' of the form $\sigma^0(t_0) \oplus \ldots \oplus \sigma^n(t_n)$, where $n \geq 0$ and $t_0, \ldots, t_n$ are tuplix-closed* CTC *canonical terms, such that $t = t'$ is derivable from the axioms of* TTC*.*

5 Financial Product Behaviours

In this section, we formalize the cumulative interest compliant conservation requirement proposed by Wesseling and van den Bergh, use this formalization to introduce the notion of a financial product behaviour, and present some properties of financial product behaviours. We use TTC for this, viewing tuplices as representations of financial behaviours.

Here, the signed cancellation meadow $\mathcal{D}$, which is a parameter of TTC, is confined to the signed meadow of real numbers. The signed meadow of rational numbers would not serve our purpose as will be explained hereafter.

In [15], Wesseling and van den Bergh claim that interest calculations relating to financial products should always be based on cumulative interests. By strictly adhering to the use of cumulative interests, the design of financial products is made symmetric between client and provider and an implicit bias towards either party can be avoided. This is the point of departure of their 'realistic interest calculation approach' and the origin of axiom RICA of TTC. Applying this approach involves a strict separation between transfers related to a financial product proper and transfers related to its costs of delivery. Transfers related to the financial product proper include transfers due to interests. Transfers related to the costs of delivery may include clear profit, general running cost, cost of insurance against non-payment, costs of marketing and communication, etc.

Having made this separation, Wesseling and van den Bergh formulate a cumulative interest compliant conservation requirement for financial products: the sum of all transfers relating to the product, transposed to some point of time (the focal date) by means of cumulative interest at the effective interest rate of the product, is zero. In [15], this requirement is presented in the form of an equation whose left-hand side and right-hand side are informally described. The equation concerned has two unknowns, to wit a financial behaviour and an interest rate. If a financial behaviour and an interest rate make up a solution of the equation, then the interest rate is taken for the effective interest rate of a financial product that involves the financial behaviour.

The cumulative interest compliant conservation requirement for financial products is formalized in TTC by the equation

$$\partial^{u}_{\{\mathtt{t}\}}(\mathtt{t}_A(x)) = \epsilon \;.$$

This equation is called the *Wesseling and van den Bergh equation* or shortly the *W-vdB equation*. In the following definition, we make use of the W-vdB equation. The definition is inspired by the perspective mentioned at the end of the last paragraph. Let t be a closed term of sort $\mathbf{T}$. Then t represents a *financial product behaviour* if

$$\exists u \, (\forall v \, (v > -1 \Rightarrow (\partial^{v}_{\{\mathtt{t}\}}(\mathtt{t}_A(t)) = \epsilon \Leftrightarrow u = v)) \;.$$ [2]

We see that the interest rate v for which the equation $\partial^{v}_{\{\mathtt{t}\}}(\mathtt{t}_A(t)) = \epsilon$ holds must meet the condition that $v > -1$ and the condition that v is the unique interest rate meeting the first condition for which the equation holds. These conditions are healthiness conditions: if they are not met, we have to do with an implausible financial product behaviour. Instead of a uniqueness condition on v, we could have a condition on t based on Descartes' rule of signs or one of its relatives (see e.g. [12]). Thus, we would have replaced the uniqueness condition on v by a condition that is sufficient but not necessary for the uniqueness of v. That is, we would have a less general definition.

Each closed term of sort $\mathbf{T}$ represents a financial behaviour, but not each closed term of sort $\mathbf{T}$ represents a financial product behaviour. A financial product behaviour can be seen as a financial behaviour for which a financial product can be devised that involves the behaviour. However, a financial product behaviour may also have one or more origins different from a financial product. For example, viewed apart, the financial behaviour that is part of a trading behaviour is often a financial product behaviour as well.

The definition of a financial product behaviour given above agrees with the viewpoint that a financial product entails an agreement under which a party gives one or more fixed amounts of money to another party, each

[2] It follows from the decidability of the first-order theory of real numbers with addition, multiplication and order (see [14]) that it is decidable whether a closed term of sort $\mathbf{T}$ represents a financial product behaviour.

of them at a fixed date, with the understanding that the former party will get back one or more fixed amounts of money, each of them at a fixed date (freely cited from [10]).

Consider a loan of €1,000 for which the borrower has to pay back €2,000 after two years. The financial behaviour involved in this loan is a financial product behaviour according to the definition given above only if the equation $-1000 + \frac{2000}{(1+v)^2} = 0$ has a unique solution greater than -1. This equation has a unique solution greater than -1 in the signed meadow of real numbers, to wit $\sqrt{2}-1$, but no solution in the signed meadow of rational numbers. This example shows that there are genuine financial products that involve financial behaviours which would not be financial product behaviours according to the definition given above if interest rates would be restricted to rational numbers. Another matter is that in reality financial institution cannot help but approximate interest rates like $\sqrt{2} - 1$ with some finite accuracy.

Let p be a closed term of sort $\mathbf{Q}$ and t be a closed term of sort $\mathbf{T}$ such that $\partial^{p}_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$. Then t represents a financial product behaviour and p represents the effective interest rate of the underlying financial product. If that financial product is a financial product of credit type, then $\mathrm{Q}^p(t) = 0$. However, if that financial product is a financial product of savings type, then $\mathrm{Q}^p(t) > 0$.

Let p and q be closed terms of sort $\mathbf{Q}$ and t and t' be closed terms of sort $\mathbf{T}$ such that $\partial^{q}_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$, $\mathrm{Q}^q(t) = 0$, and $\mathrm{Q}^p(t') > 0$. Then we say that the financial behaviour t' profits from using the financial product underlying t taking the interest rate p into account if $\mathrm{Q}^p(t \oplus t') < \mathrm{Q}^p(t')$. In any case, we have $\mathrm{Q}^p(t \oplus t') \leq \mathrm{Q}^p(t) + \mathrm{Q}^p(t')$. The important observation is that we may have $\mathrm{Q}^p(t \oplus t') < \mathrm{Q}^p(t')$.

Proposition 1 *There exist closed terms p and q of sort $\mathbf{Q}$ and closed terms t and t' of sort $\mathbf{T}$ with $\partial^{q}_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$, $\mathrm{Q}^q(t) = 0$, and $\mathrm{Q}^p(t') > 0$ such that $\mathrm{Q}^p(t \oplus t') < \mathrm{Q}^p(t')$.*

Proof: Take the case where p and q are such that $\mathcal{D} \models \frac{1}{1+p} \cdot (1+q)^2 \leq \frac{8}{5}$, $t \equiv b(-5) \oplus \sigma^2(b'((1+q)^2 \cdot 5))$, and $t' \equiv a(7) \oplus \sigma(a'(-8))$. We can easily derive that $\partial^{q}_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$, $\mathrm{Q}^q(t) = 0$, and $\mathrm{Q}^p(t') = 7$. Moreover, in Example 2, we have already derived that $\mathrm{Q}^p(t \oplus t') = 2$. Hence, $\mathrm{Q}^p(t \oplus t') < \mathrm{Q}^p(t')$. □

Proposition 1 can be read as follows: there exists an interest rate, a financial product of credit type, and a financial behaviour that profits from that financial product if that interest rate is taken into account.

Proposition 2 *Let t and t' be closed terms of sort* $\mathbf{T}$ *such that t' is t with each subterm of the form $a(p)$ replaced by $a(-p)$, and let q be a closed term of sort* $\mathbf{Q}$ *such that $q \neq -1$. Then $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$ implies $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \epsilon$.*

Proof: Assume that $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$. Then $t \neq \delta$. From this and Corollary 1, it follows that $\mathsf{t}_A(t)$ is of the form $\sigma^0(t_0) \oplus \ldots \oplus \sigma^n(t_n)$, where $t_0, \ldots, t_n$ are of the form $\mathsf{t}(p)$ or ϵ. For each $i \in \{0, \ldots, n\}$, let p_i be such that $\mathsf{t}(p_i) \equiv t_i$ if $t_i \not\equiv \epsilon$ and $p_i \equiv 0$ if $t_i \equiv \epsilon$. Then $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \gamma(\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot p_i)$ and $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \gamma(\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot -p_i)$. Because $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$, we know that $\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot p_i = 0$. From this and the fact that $\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot -p_i = -\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot p_i$, it follows that $\sum_{i=0}^n \frac{1}{(1+q)^i} \cdot -p_i = 0$. Hence, $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \epsilon$. □

Proposition 2 can be read as follows: if we change the incoming transfers of a financial product into outgoing transfers and its outgoing transfers into incoming transfers, then the result is a financial product behaviour as well; and the effective interest rates of the underlying financial products are the same.

Let t and t' be closed terms of sort $\mathbf{T}$. Then t is a *time inverse* of t' if, for some natural number n, there exist closed CTC canonical terms $t_0, \ldots, t_n$ such that $t = \sigma^0(t_0) \oplus \ldots \oplus \sigma^n(t_n)$ and $t' = \sigma^0(t_n) \oplus \ldots \oplus \sigma^n(t_0)$. If follows immediately from the definition that t is a time inverse of t' if and only if t' is a time inverse of t. By Corollary 1, each closed term of sort $\mathbf{T}$ has a time inverse. This time inverse is unique up to derivable equality.

Proposition 3 *Let t and t' be closed terms of sort* $\mathbf{T}$ *such that t is a time inverse of t', and let p and q be closed terms of sort* $\mathbf{Q}$ *such that $p \neq -1$ and $q = \frac{-p}{1+p}$. Then $\partial^p_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$ implies $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \epsilon$.*

Proof: Assume that $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$. Then $t \neq \delta$. From this and Corollary 1, it follows that $\mathsf{t}_A(t)$ is of the form $\sigma^0(t_0) \oplus \ldots \oplus \sigma^n(t_n)$, where $t_0, \ldots, t_n$ are of the form $\mathsf{t}(p)$ or ϵ. For each $i \in \{0, \ldots, n\}$, let p_i be such that $\mathsf{t}(p_i) \equiv t_i$ if $t_i \not\equiv \epsilon$ and $p_i \equiv 0$ if $t_i \equiv \epsilon$. Then $\partial^p_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \gamma(\sum_{i=0}^n \frac{1}{(1+p)^i} \cdot p_i)$ and $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \gamma(\sum_{i=0}^n \frac{1}{(1+q)^{n-i}} \cdot p_i)$. Because $\partial^p_{\{\mathsf{t}\}}(\mathsf{t}_A(t)) = \epsilon$, we know that $\sum_{i=0}^n \frac{1}{(1+p)^i} \cdot p_i = 0$. From this and the fact that $\sum_{i=0}^n \frac{1}{(1+q)^{n-i}} \cdot p_i = (1+p)^n \cdot \sum_{i=0}^n \frac{1}{(1+p)^i} \cdot p_i$, it follows that $\sum_{i=0}^n \frac{1}{(1+q)^{n-i}} \cdot p_i = 0$. Hence, $\partial^q_{\{\mathsf{t}\}}(\mathsf{t}_A(t')) = \epsilon$. □

Proposition 3 can be read as follows: if we reverse the order of time in which the transfers of a financial product behaviour take place, then the result is a financial product behaviour as well; and if the effective interest rate of the former financial products is p then the effective interest rate of the latter financial products is $\frac{-p}{1+p}$.

6 Standard Model of TTC

In this section, we construct the standard model of TTC. The standard model of CTC presented in [6] lies at the root of this model. However, the use of partial functions is circumvented.

We write $\mathcal{D}$ for the domain of the signed cancellation meadow $\mathcal{D}$, and we write $\Diamond$, where $\Diamond$ is a constant or operator from the signature of signed cancellation meadows, for the interpretation of $\Diamond$ in $\mathcal{D}$. To prevent confusion with the constants from the signature of meadows, we write $\underline{0}$ and $\underline{1}$ for the identity elements of addition and multiplication on natural numbers.

We define the set $\mathcal{TE}$ of *tuplix elements*, the set $\mathcal{UT}$ of *untimed tuplices*, and the set $\mathcal{TT}$ of *timed tuplices* as follows:

$$\mathcal{TE} = \bigcup_{A' \subseteq A} (A' \to \mathcal{D}) \;,$$

$$\mathcal{UT} = \{U \subseteq \mathcal{TE} \mid \mathrm{card}(U) \leq \underline{1}\} \;,$$

$$\mathcal{TT} = \{T : \mathbb{N} \to \mathcal{UT} \mid \forall i \in \mathbb{N}\, (\mathrm{card}(T(i)) = \underline{0}) \vee \forall i \in \mathbb{N}\, (\mathrm{card}(T(i)) = \underline{1})\} \;.$$

In the definition of the standard model of TTC, the auxiliary set $\mathcal{TT}^-$ defined by

$$\mathcal{TT}^- = \{T \in \mathcal{TT} \mid \forall i \in \mathbb{N}\, (\mathrm{card}(T(i)) = \underline{1})\}$$

is used as well. We write $el(U)$, where $U \in \mathcal{UT}$, for the unique element $f \in \mathcal{TE}$ such that $f \in U$ if $\mathrm{card}(U) = \underline{1}$, and an arbitrary $f \in \mathcal{TE}$ otherwise.

The *standard model* of TTC, written $\mathcal{M}(\mathcal{D}, A)$, is the expansion of the signed cancellation meadow $\mathcal{D}$ with

- for the sort $\mathbf{T}$, the set $\mathcal{TT}$;
- for each additional constant $\Diamond_0 : \mathbf{T}$ of TTC, the element $\Diamond_0 \in \mathcal{TT}$ defined in Table 7;
- for each additional operator $\Diamond_n : S_1 \times \ldots \times S_n \to S_{n+1}$ of TTC, the operation $\Diamond_n : D_1 \times \ldots D_n \to D_{n+1}$, where $D_i = \mathcal{TT}$ if $S_i \equiv \mathbf{T}$ and

Table 7: Interpretation of constants and operators of TTC

$$
\begin{array}{lll}
\epsilon(i) & = \{[\,]\} & \\
\delta(i) & = \emptyset & \\
a(d)(i) & = \begin{cases} \{[a \mapsto d]\} & \\ \{[\,]\} & \end{cases} & \begin{array}{l} \text{if } i = \underline{0} \\ \text{otherwise} \end{array} \\
\gamma(d)(i) & = \begin{cases} \{[\,]\} & \\ \emptyset & \end{cases} & \begin{array}{l} \text{if } d = 0 \\ \text{otherwise} \end{array} \\
(T \oplus T')(i) & = \{f \,\hat{\oplus}\, f' \mid f \in T(i) \wedge f' \in T'(i)\} & \\
\sigma(T)(i) & = \begin{cases} T(i - \underline{1}) & \\ \{[\,]\} & \\ \emptyset & \end{cases} & \begin{array}{l} \text{if } i > \underline{0} \wedge T(i) \neq \emptyset \\ \text{if } i = \underline{0} \wedge T(i) \neq \emptyset \\ \text{otherwise} \end{array} \\
\mathsf{t}_I(T)(i) & = \{\hat{\mathsf{t}}_I(f) \mid f \in T(i)\} & \\
\partial^d_H(T)(i) & = \{\hat{\epsilon}_H(f) \mid f \in T(i) \wedge \forall a \in H\, (\mathrm{Total}^d_a(T) = 0)\} & \\
\mathrm{Q}^d(T) & = \begin{cases} \hat{\mathrm{Q}}^d(T) & \\ -1 & \end{cases} & \begin{array}{l} \text{if } \exists i \geq \underline{0}\, (T(i) \neq \emptyset) \\ \text{otherwise} \end{array}
\end{array}
$$

$D_i = \mathcal{D}$ if $S_i \equiv \mathbf{Q}$, defined in Table 7.[3]

In Table 7, the following auxiliary functions are used:

- the function $\hat{\oplus} : \mathcal{TE} \times \mathcal{TE} \to \mathcal{TE}$ defined by
  - $\mathrm{dom}(f \,\hat{\oplus}\, f') = \mathrm{dom}(f) \cup \mathrm{dom}(f')$;
  - for each $a \in \mathrm{dom}(f \,\hat{\oplus}\, f')$:

$$
(f \,\hat{\oplus}\, f')(a) = \begin{cases} f(a) + f'(a) & \text{if } a \in \mathrm{dom}(f) \cap \mathrm{dom}(f') \\ f(a) & \text{if } a \in \mathrm{dom}(f) \setminus \mathrm{dom}(f') \\ f'(a) & \text{if } a \in \mathrm{dom}(f') \setminus \mathrm{dom}(f)\ ; \end{cases}
$$

- for each $I \subseteq A$, the function $\hat{\mathsf{t}}_I : \mathcal{TE} \to \mathcal{TE}$ defined by
  - $\mathrm{dom}(\hat{\mathsf{t}}_I(f)) = (\mathrm{dom}(f) \setminus I) \cup \{\mathsf{t} \mid \mathrm{dom}(f) \cap I \neq \emptyset\}$;

[3] We write $[\,]$ for the empty function and $[e \mapsto e']$ for the function f with $\mathrm{dom}(f) = \{e\}$ such that $f(e) = e'$.

– for each $a \in \mathrm{dom}(\hat{\mathsf{t}}_I(f))$:

$$\hat{\mathsf{t}}_I(f)(a) = \begin{cases} f(a) & \text{if } a \neq \mathsf{t} \\ \sum_{a' \in I} f(a') & \text{if } a = \mathsf{t} \;; \end{cases}$$

- for each $H \subseteq A$, the function $\hat{\epsilon}_H : \mathcal{TE} \to \mathcal{TE}$ defined by
  – $\mathrm{dom}(\hat{\epsilon}_H(f)) = \mathrm{dom}(f) \setminus H$;
  – for each $a \in \mathrm{dom}(\hat{\epsilon}_H(f))$:

$$\hat{\epsilon}_H(f)(a) = f(a) \;;$$

- for each $a \in A$, the function $\mathrm{Total}_a : \mathcal{D} \times \mathcal{TT} \to \mathcal{D}$ defined by

$$\mathrm{Total}_a^d(T) = \sum_{i \text{ s.t. } a \in \mathrm{dom}(el(T(i)))} (1+d)^i \cdot el(T(i))(a) \;;$$

- the function $\hat{\mathrm{Q}} : \mathcal{D} \times \mathcal{TT}^- \to \mathcal{D}$ recursively defined by

$$\hat{\mathrm{Q}}^u(T) = \begin{cases} \max(q_0(T), 0) & \text{if } \forall i > \underline{0}\,(T(i) = \{[\,]\}) \\ \max(q_0(T) + \dfrac{1}{1+u} \cdot \hat{\mathrm{Q}}^u(sh(T)), 0) & \text{if } \exists i > \underline{0}\,(T(i) \neq \{[\,]\}) \;, \end{cases}$$

  where:
  – the function $sh : \mathcal{TT}^- \to \mathcal{TT}^-$ is defined by

$$sh(T)(i) = T(i + \underline{1})$$

  for each $i \in \mathbb{N}$;
  – the function $q_0 : \mathcal{TT}^- \to \mathcal{D}$ is defined by

$$q_0(T) = \textstyle\sum_{a \in \mathrm{dom}(el(T(\underline{0})))} el(T(\underline{0}))(a) \;.$$

It is easy to establish the following soundness result: for all terms t and t' of sort $\mathbf{T}$, $t = t'$ is derivable from the axioms of TTC and the proof rule DE only if $\mathcal{M}(\mathcal{D}, A) \models t = t'$. We also have a completeness result.

Theorem 1 *For all closed terms t and t' of sort $\mathbf{T}$, $\mathcal{M}(\mathcal{D}, A) \models t = t'$ only if $t = t'$ is derivable from the axioms of* TTC *and the proof rule* DE.

Proof: By Lemma 2, it is sufficient to show that, for all closed TTC canonical terms t and t', $\mathcal{M}(\mathcal{D}, A) \models t = t'$ only if $t = t'$ is derivable from the axioms of TTC and the proof rule DE. This is easy to prove by induction on the structure of t using Theorem 1 from [6]. □

7 Concluding Remarks

We have developed a timed extension of the core of tuplix calculus in which financial behaviours are considered at a level of detail where the time slices in which actions are performed matter, but not their order within the time slices. This makes it suited for the description and analysis of financial products: financial products exhibit financial behaviours where the day, week or month in which actions are performed and the amounts of money are transferred in doing so are relevant, but not their order within the periods concerned.

We have formalized the cumulative interest compliant conservation requirement for financial products proposed by Wesseling and van den Bergh by an equation in the timed tuplix calculus developed. Thus, a formalization of the starting-point of the material on the mathematics of finance presented in [15] has been achieved. Moreover, we have used this formalization to introduce the notion of a financial product behaviour, and have presented some properties of financial product behaviours. The timed tuplix calculus appears to be a reasonable setting for further work in this area.

In [6], the core of tuplix calculus is among other things extended with a binary alternative composition operator and a variable-binding generalized alternative composition operator for each variable of sort **Q**. The latter operators have proved to be convenient in modular budget design. Extending timed tuplix calculus with these operators would allow for non-deterministic financial behaviours to be described. However, in the presence of non-deterministic financial behaviours it would be less easy to acquire an intuitive understanding of what the implicit capital of a financial behaviour tells us. Moreover, comparison of the implicit capitals of different financial behaviours, like in Section 5, appears to make little sense in the case of non-deterministic financial behaviours.

Like Wesseling and van den Bergh, we consider only financial products of which the interest rate is not dependent on changes in the financial market. If the interest rate of a financial product is made dependent on changes in the financial market, then the expressiveness of the timed tuplix calculus is insufficient. In this more dynamic case, a version of discrete time process algebra [2] looks to be a reasonable setting for the formalization of an adapted version of the cumulative interest compliant conservation requirement.

We remark that we do not have to abandon discrete time if interest is continuously instead of discretely compounded because of the commonly known fact that an interest rate p with continuous compounding is equivalent to an interest rate $\ln(1 + p)$ with discrete compounding.

We mention that the cumulative interest compliant conservation requirement for financial products has been formulated by Wesseling and van den Bergh under the influence of basic ideas on the mathematics of finance presented in [9].

The work to which ours seems to be most closely related is the work on MLFi (Modeling Language for Finance) [11]. MLFi is a language to describe financial products in a mathematically precise, compositional way. A distinctive feature of MLFi is that the descriptions of financial products can be analyzed, manipulated, and translated in many ways. Therefore, MLFi is considered to be the basis of an approach to the application of various formal methods in matters concerning financial products. TTC could find a place among these formal methods.

Acknowledgements

This research was carried out in the framework of the Jacquard-project Symbiosis, which was funded by the Netherlands Organisation for Scientific Research (NWO).

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